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RESOLVABILITY OF BLOCK DESIGNS

BY SANPEI KAGEYAMA[1]

Osaka University

The concept of resolvability of a balanced incomplete block (BIB) design, introduced by Bose (1942), was generalized to μ-resolvability of an incomplete block design by Shrikhande and Raghavarao (1964). As a further generalization of these, the concept of $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvability is here introduced for an incomplete block design. This concept may be useful from both combinatorial and practical points of view. Furthermore, some methods of constructing BIB designs with the generalized concept are discussed with illustrations. One method is based on a finite geometry over a Galois field.

1. Introduction and background. The concept of resolvability introduced by Bose [2] was generalized to μ-resolvability by Shrikhande and Raghavarao [12] in a combinatorial sense. The concept of μ-resolvability can be further generalized to $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvability as follows.

DEFINITION. A block design is called $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable if the blocks can be separated into t sets of m_i ($\geqq 2$) blocks such that the set consisting of m_i blocks contains every treatment exactly μ_i ($\geqq 1$) times, i.e., the set of m_i blocks forms a μ_i-replication set of each treatment ($i = 1, 2, \cdots, t$). Furthermore, when $\mu_1 = \mu_2 = \cdots = \mu_t$ ($= \mu$, say), it is called μ-resolvable for $\mu \geqq 1$.

Note that this definition of μ-resolvability corresponds to that of μ-resolvability introduced by Shrikhande and Raghavarao [12]. Then a 1-resolvable block design is simply called resolvable in a sense of Bose [2].

Further, note that the existence of a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable incomplete block design with parameters v (number of treatments) and k (block size) implies the existence of t incomplete block designs with parameters $v^* = m_i$ (number of treatments), $b^* = v$ (number of blocks), $r^* = k$ (replication of each treatment) and $k^* = \mu_i$ (block size), $i = 1, 2, \cdots, t$.

We now consider a BIB design with parameters $v = 16$, $b = 40$, $r = 15$, $k = 6$ and $\lambda = 5$. This design is not resolvable since v ($= 16$) is not divisible by k ($= 6$), but this design is (3, 6, 6)-resolvable since this design can be generated by the blocks [(0, 1, 3, 8, 9, 11)(1, 2, 4, 9, 10, 12)(2, 3, 5, 10, 11, 13)(3, 4, 6, 11, 12, 14)(4, 5, 7, 12, 13, 15)(5, 6, 8, 13, 14, 0)(6, 7, 9, 14, 15, 1)(7, 8, 10, 15, 0, 2)][(0, 1, 3, 5, 9, 12)(0, 1, 2, 3, 6, 12) mod 16]. This example shows the practical usefulness of $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB designs.

As a practical application we can consider an experiment of comparing washing

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[1] Now at Hiroshima University.

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power of sixteen washes by means of the dishwashing test, quoting in part from [4]. In this testing procedure, plates soiled with a standard soil are washed one at a time until they are clean. Furthermore, six basins shall be used, i.e., six washes are tested at the same time. The six operators wash at the same speed during the test, and the "yield" reported is the number of plates washed before the foam disappears. If we consider each set of six such concurrent trials as a block, it is clear that variation between the blocks can be expected, due to some factors. However, in order to make the number of such factors as less as possible, we would like to wash plates together according to kinds of a soil of plates, for example, plates soiled by the meat, fish, oil, vegetable and so on, being a kind of balancing in a combinatorial sense for this experiment. As an arrangement with sixteen treatments in some blocks of six plots each satisfying those requirements we may use the above BIB design. Then the meaning of this experiment is as follows: For example, for plates soiled by the fish we test each wash three times (i.e., we form a 3-replication set of each wash), and for plates soiled by the meat we test each wash six times (i.e., we form a 6-replication set) and so on. Thus, $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB designs may be useful and be utilized for a practical experiment.

For the combinatorial parameters v, b, r and k of a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable incomplete block design we clearly have from definition

$$b = \sum_{i=1}^{t} m_i\,, \qquad r = \sum_{i=1}^{t} \mu_i\,, \quad vr = bk\,,$$
$$v\mu_i = m_i k\,, \qquad b\mu_i = m_i r\,, \quad i = 1, 2, \cdots, t\,.$$

Moreover, as inequalities to hold for these designs, it is known (cf. [6], [9]) that for a μ-resolvable BIB design with parameters v, b, r, k and λ, i.e., $\mu_1 = \mu_2 = \cdots = \mu_t = \mu$, an inequality $b \geqq v + t - 1$ holds. Now we can show that for a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB design with parameters v, $b = \sum_{i=1}^{t} m_i$, $r = \sum_{i=1}^{t} \mu_i$, k and λ such that $l\mu$'s $(1 \leqq l \leqq t)$ among $\mu_1, \mu_2, \cdots, \mu_t$ are equal, an inequality $b \geqq v + l - 1$ holds. For, from assumption, let for example $\mu_1 = \mu_2 = \cdots = \mu_l\ (= \mu^{(l)}$, say) and N be the incidence matrix of the $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB design. Then we have $m_1 = m_2 = \cdots = m_l\ (= m^{(l)}$, say). In each of l sets of $m^{(l)}$ blocks (or columns) each in N, where a set of the $m^{(l)}$ columns is such that each treatment occurs exactly $\mu^{(l)}$ times, adding the 1st, 2nd, $\cdots$, $(m^{(l)} - 1)$th columns to the $m^{(l)}$th column of a set, we obtain a column consisting of $\mu^{(l)}$ only. As there are such l sets evidently $v = \operatorname{Rank} N \leqq b - (l - 1)$. Therefore we have $b \geqq v + l - 1$. Finally, in this case note that if there exists a divisibility between $\mu^{(l)}$ and μ_i $(i = l + 1, \cdots, t)$, the inequality $b \geqq v + l - 1$ can be improved further.

The above definition of $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvability can also be applied to that for a general block design which is balanced in a certain sense [8].

2. Construction. For constructions of μ-resolvable BIB designs the reader is referred to, for example, [5], [6], [11] and [12]. Here we mainly consider constructions of $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB designs with $b = \sum_{i=1}^t m_i$ and $r = \sum_{i=1}^t \mu_i$ in which $\mu_i \neq \mu_j$ for some $i, j\,(\neq) = 1, 2, \cdots, t$.

First, some obvious results can be observed.

(i) The existence of a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable design implies the existence of an $(m_1 - \mu_1, m_2 - \mu_2, \cdots, m_t - \mu_t)$-resolvable design by the complementary method, and vice versa.

(ii) If there exists a $\mu\,(\geqq 1)$-resolvable BIB design with more than two μ-replication sets, then grouping of some μ-replication sets in the design leads to a $(p_1\mu, p_2\mu, \cdots, p_{t'}\mu)$-resolvable BIB design for some positive integers p_i, $i = 1, 2, \cdots, t'\,(\leqq t)$.

(iii) If there are $(\mu_1^{(i)}, \mu_2^{(i)}, \cdots, \mu_{t_i}^{(i)})$-resolvable BIB designs ($i = 1, 2$) with common parameters v and k, then a form of juxtaposition of incidence matrices of the designs leads to a $(\mu_1^{(1)}, \mu_2^{(1)}, \cdots, \mu_{t_1}^{(1)}, \mu_1^{(2)}, \mu_2^{(2)}, \cdots, \mu_{t_2}^{(2)})$-resolvable BIB design.

(iv) If there are BIB designs with parameters v, k, b_i, r_i and λ_i ($i = 1, 2, \cdots, t$), then there exists an $(r_1, r_2, \cdots, r_t)$-resolvable BIB design with parameters v, k, $b = \sum_{i=1}^t b_i$, $r = \sum_{i=1}^t r_i$ and $\lambda = \sum_{i=1}^t \lambda_i$.

(v) If there are m-associate PBIB designs with parameters v, k, $b^{(i)}$, $r^{(i)}$, $\lambda_j^{(i)}$, $j = 1, 2, \cdots, m$ ($i = 1, 2, \cdots, t$) based on the same association scheme such that

$$\sum_{i=1}^t \lambda_j^{(i)} = \text{constant } (= \lambda,\ \text{say})\,, \qquad \text{for all}\quad j = 1, 2, \cdots, m\,,$$

then there exists an $(r^{(1)}, r^{(2)}, \cdots, r^{(t)})$-resolvable BIB design with parameters v, k, $b = \sum_{i=1}^t b^{(i)}$, $r = \sum_{i=1}^t r^{(i)}$ and $\lambda = \sum_{i=1}^t \lambda_j^{(i)}$.

Examples of observations (i), (ii), (iii) and (iv) can be easily given. The following two examples illustrate observation (v).

EXAMPLE 1. If there exists a PBIB design with incidence matrix N and parameters $v = mn$, b, r, $k = n$, $\lambda_1 = 0$, $\lambda_2 = 1$, based on an N_2 type association scheme of $v = mn$ treatments (m groups of n treatments), then juxtaposition $[N : I_m \otimes E_{n\times 1}]$ is an $(r, 1)$-resolvable BIB design with parameters $v' = v$, $k' = k = n$, $b' = b + m$, $r' = r + 1$ and $\lambda' = 1$, where $E_{n\times 1}$ is an $n \times 1$ matrix with elements all unity and $I_m \otimes E_{n\times 1}$ is the Kronecker product of the unit matrix I_m of order m and $E_{n\times 1}$. Some of 2-associate PBIB designs of this type can be found in [3]. Furthermore, if N is $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable, then $[N : I_m \otimes E_{n\times 1}]$ is $(\mu_1, \mu_2, \cdots, \mu_t, 1)$-resolvable.

EXAMPLE 2. Consider triangular type PBIB designs with parameters $v = 10$, $k = 6$, $b = 5$, $r = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$; $v = 10$, $k = 6$, $b = 10$, $r = 6$, $\lambda_1 = 3$, $\lambda_2 = 4$, $n_1 = 6$, $n_2 = 3$ [3]. Then there exists a $(3, 6)$-resolvable BIB design with parameters $v = 10$, $k = 6$, $b = 15$, $r = 9$ and $\lambda = 5$.

From Theorems 2.2 and 2.3 in [6] we respectively have

(vi) Let N_1 be a BIB design with parameters v_1, b_1, r_1, k_1, λ_1, and $N_1' = (\mathbf{n}_1', \mathbf{n}_2', \cdots, \mathbf{n}_{v_1}')$, where $\mathbf{n}_i\mathbf{n}_j' = r_1$ ($i = j$) or λ_1 ($i \neq j$), N_1' is the transpose of an incidence matrix N_1. Let N_2 be a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB design with parameters v_2, $b_2 = \sum_{i=1}^{t} m_i$, $r_2 = \sum_{i=1}^{t} \mu_i$, $k_2 = v_1$, λ_2. Substitute v_1 distinct row vectors $\mathbf{n}_i(1 \times b_1)$ in place of v_1 distinct units and $\mathbf{0}(1 \times b_1)$ in place of $v_2 - v_1$ distinct 0 (zero) in every block of an incidence matrix N_2. Then the resulting matrix is an $(r_1\mu_1, r_1\mu_2, \cdots, r_1\mu_t)$-resolvable BIB design with parameters $v = v_2$, $b = b_1 b_2$, $r = r_1 r_2$, $k = k_1$ and $\lambda = \lambda_1 \lambda_2$.

(vii) If N_1 is a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB design with parameters v_1, $b_1 = \sum_{i=1}^{t} m_i$, $r_1 = \sum_{i=1}^{t} \mu_i$, k_1, λ_1 satisfying $b_1 = 4(r_1 - \lambda_1)$, and N_2 is a BIB design with parameters $v_2, b_2, r_2, k_2, \lambda_2$ satisfying $b_2 = 4(r_2 - \lambda_2)$, then $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ is an $(\alpha_1, \alpha_2, \cdots, \alpha_t)$-resolvable BIB design with parameters $v = v_1 v_2$, $b = b_1 b_2$, $r = r_1 r_2 + (b_1 - r_1)(b_2 - r_2)$, $k = k_1 k_2 + (v_1 - k_1)(v_2 - k_2)$, $\lambda = r - b/4$, $\alpha_i = \mu_i r_2 + (m_i - \mu_i)(b_2 - r_2)$, $i = 1, 2, \cdots, t$, where N_j^* is the complement of a BIB design N_j $(j = 1, 2)$ and $A \otimes B = ||a_{ij} B||$ denotes the Kronecker product of matrices $A = ||a_{ij}||$ and B.

(viii) From Takeuchi's table [13] which gives all possible combinations of BIB designs with $v \leqq 100$ and $r \leqq 20$ and of symmetric BIB designs with $v \leqq 100$ and $r \leqq 30$ and difference sets generating them, it is clear that if difference sets listed in the table do not include the symbols ∞ and A, B, C (fixed varieties) which remain unaltered in the developed blocks, then a BIB design generated by the difference set is $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable. For example, a BIB design with parameters $v = 9$, $k = 4$, $b = 18$, $r = 8$ and $\lambda = 3$, generated by (0, 1, 2, 4)(0, 3, 4, 7) mod 9, is 4-resolvable. As mentioned in Section 1, a BIB design with parameters $v = 16$, $k = 6$, $b = 40$, $r = 15$ and $\lambda = 5$, generated by (0, 1, 3, 8, 9, 11) PC (8)(0, 1, 3, 5, 9, 12)(0, 1, 2, 3, 6, 12) mod 16, is (3, 6, 6)-resolvable which is not resolvable, where symbol PC(θ) stands for a partial cycle of θ, i.e., by adding successively $1, 2, \cdots, \theta - 1$ to the integers of the block, other blocks are obtained.

Observation (viii) may show the possibility of constructing a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB design by the method of differences. One example of this approach will be discussed hereinafter.

A finite projective t-dimensional geometry over a Galois field GF (q), where q is a prime or a prime power, is denoted by PG (t, q). It is known that a BIB design with parameters $v = \phi(t, 0, q)$, $b = \phi(t, d, q)$, $r = \phi(t-1, d-1, q)$, $k = \phi(d, 0, q)$, $\lambda = \phi(t-2, d-2, q)$ is obtained by choosing the points as treatments and all d-dimensional linear subspaces (d-flats) as blocks from PG (t, q), where $\phi(t, d, q) = (q^{t+1} - 1)(q^t - 1) \cdots (q^{t-d+1} - 1)/(q^{d+1} - 1)(q^d - 1) \cdots (q - 1)$ is the number of d-flats in PG (t, q) [1]. The design so obtained is denoted by PG $(t, q) : d$. Furthermore, a μ-fold spread S is defined as a collection of d-flats in $PG(t, q)$ such that each point of PG (t, q) occurs in exactly μ d-flats of S [10]. Then we obviously have

LEMMA 1. *A* BIB *design* PG (t, q) : *d is* $(\mu_1, \mu_2, \cdots, \mu_l)$-*resolvable if and only if all the d-flats in* PG (t, q) *are decomposed into l disjoint* μ_i-*fold spreads*, $i = 1, 2, \cdots, l$.

Concerning conditions for the existence of spreads, we have

LEMMA 2 (cf. [10], [14]). *When $t+1$ and $d+1$ have $i+1$ as a common factor, a μ-fold spread of d-flats in* PG (t, q) *exists, where $\mu = \phi(d, 0, q)/\phi(i, 0, q)$ which assumes the unity when $\phi(d, 0, q)$ divides $\phi(t, 0, q)$.*

Lemma 2 shows that when $t+1$ and $d+1$ have $i+1$ as a common factor, PG $(t, q): d$ is a $(\mu, r-\mu)$-resolvable BIB design with $r = \phi(t-1, d-1, q)$ and $\mu = \phi(d, 0, q)/\phi(i, 0, q)$, which is not resolvable provided $i \neq d$.

LEMMA 3. *A necessary condition for the existence of a μ-fold spread of d-flats in* PG (t, q) *is that μ is a multiple of an integer $(q^{d+1}-1)/(q^g-1)$, where $g = (t+1, d+1)$.*

PROOF. If a μ-fold spread exists, then we have $v\mu = xk$ (x being the number of d-flats of the μ-fold spread). Since x is integral, we obtain $v\mu \equiv 0 \pmod{k}$. Now, $g = (t+1, d+1)$ implies $t+1 = t_1 g$, $d+1 = d_1 g$ and $(t_1, d_1) = 1$. Since $(q^{t_1 g}-1, q^{d_1 g}-1) = q^g - 1$, we have $(1 + q^g + \cdots + q^{g(t_1-1)}, 1 + q^g + \cdots + q^{g(d_1-1)}) = 1$, i.e., $(\theta_1, \theta_2) = 1$, $\theta_1 = (q^{t+1}-1)/(q^g-1)$ and $\theta_2 = (q^{d+1}-1)/(q^g-1)$. Furthermore, from $v/k = \theta_1/\theta_2$ and $v\mu \equiv 0 \pmod{k}$, we get $\theta_1\mu \equiv 0 \pmod{\theta_2}$, which leads to $\mu \equiv 0 \pmod{\theta_2}$ by $(\theta_1, \theta_2) = 1$. Hence μ is a multiple of $\theta_2 = (q^{d+1}-1)/(q^g-1)$.

Note that when $(t+1, d+1) = 1$, there does not exist an l-fold spread of d-flats in PG (t, q) for $1 \leqq l < k = \phi(d, 0, q)$. Further, note that $\mu = \phi(d, 0, q)/\phi(i, 0, q)$ in Lemma 2 is a multiple of $\theta_2 = (q^{d+1}-1)/(q^g-1)$.

Lemmas 2 and 3 imply the existence of many μ-fold spreads of d-flats in PG (t, q). Thus, some $(\mu_1, \mu_2, \cdots, \mu_l)$-resolvable BIB designs PG $(t, q): d$ can be obtained by use of the concept of a μ-fold spread.

It is well known that starting with a given d-flat (which is called an initial d-flat in PG (t, q)) represented by a set of integers, we can generate other d-flats by adding successively integers $1, 2, \cdots$ to each element of the d-flat and reducing the integers mod v $(= \phi(t, 0, q))$. However, not all d-flats so generated may be distinct. Now, the smallest integer θ such that at the θ-stage the initial d-flat is reproduced is called the cycle of the initial d-flat. We call the minimum value of θ the minimum cycle (m.c.) of the initial d-flat. The maximum value of θ is clearly v. Then it is obvious (cf. [10]) that all the d-flats of PG (t, q) can be generated cyclically from a set of initial d-flats which may be of different cycles. As a more explicit form of this fact we have

LEMMA 4 (cf. [14]). (1) *If $(t+1, d+1) = 1$, then all d-flats in* PG (t, q) *have the* m.c. *v and can be generated from $\phi(t, d, q)/v$ initial d-flats.* (2) *If $(t+1, d+1) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$ $(> 1$, p's are primes such that $p_i < p_{i+1})$, then the number of different* m.c. *is $\prod_{i=1}^{l}(1+\alpha_i)$. Let*

$$\theta[x_1, x_2, \cdots, x_l] = (q^{t+1}-1)/(q^{p_1^{x_1}\cdots p_l^{x_l}}-1)\,,$$
$$t[x_1, x_2, \cdots, x_l] = (t+1)/(p_1^{x_1}\cdots p_l^{x_l}) - 1\,,$$
$$d[x_1, x_2, \cdots, x_l] = (d+1)/(p_1^{x_1}\cdots p_l^{x_l}) - 1\,,$$
$$m[x_1, x_2, \cdots, x_l] = q^{p_1^{x_1}\cdots p_l^{x_l}}\,.$$

Then the numbers of d-flats having the cycle $\theta[x_1, \cdots, x_l]$ *and* m.c. $\theta[x_1, \cdots, x_l]$ *are respectively*

$$n(x_1, \cdots, x_l) = \phi(t[x_1, \cdots, x_l], d[x_1, \cdots, x_l], m[x_1, \cdots, x_l]) ,$$
$$n^*(\alpha_1, \cdots, \alpha_l) = n(\alpha_1, \cdots, \alpha_l) ,$$
$$n^*(x_1, \cdots, x_l) = n(x_1, \cdots, x_l) - \sum_{x_j \leqq y_j \leqq \alpha_j ; \exists_j, x_j < y_j} n^*(y_1, \cdots, y_l) .$$

The number of initial d-flats of any m.c. $\theta[x_1, \cdots, x_l]$ *is* $n^*(x_1, \cdots, x_l)/\theta[x_1, \cdots, x_l]$, *from which the totality of d-flats having the* m.c. $\theta[x_1, \cdots, x_l]$ *can be generated.*

From Lemmas 1, 2, 3 and 4 and grouping of some spreads we have

THEOREM. *Design* PG $(t, q) : d$ *is a* $(\mu_1, \mu_2, \cdots, \mu_l)$*-resolvable* BIB *design with parameters* $v = \phi(t, 0, q)$, $b = \phi(t, d, q)$, $r = \phi(t-1, d-1, q)$, $k = \phi(d, 0, q)$, $\lambda = \phi(t-2, d-2, q)$ *for some* $\mu_1, \mu_2, \cdots, \mu_l$.

COROLLARY. *When* $(t+1, d+1) = 1$, PG $(t, q) : d$ *is a* $(\mu_1, \mu_2, \cdots, \mu_l)$*-resolvable* BIB *design with* $\mu_1 = \mu_2 = \cdots = \mu_l = k = \phi(d, 0, q)$ *and* $l = \phi(t, d, q)/\phi(t, 0, q)$ *which is not resolvable.*

The corollary and observation (ii) yield many $(\mu_1, \mu_2, \cdots, \mu_{l'})$-resolvable BIB designs with $l' < l$ and $\mu_i \neq \mu_j$ for some i, $(\neq) j$.

EXAMPLE 3. PG (3, 3) : 1 is a (1, 4, 4, 4)-resolvable BIB design with parameters $v = 40$, $b = 130$, $r = 13$, $k = 4$ and $\lambda = 1$ having a solution (0, 10, 20, 30) PC (10)(0, 1, 26, 32)(0, 7, 19, 36)(0, 3, 16, 38) mod 40 [13].

EXAMPLE 4. PG $(5, q)$: 3 is a $(\mu_1, \mu_2, \cdots, \mu_l)$-resolvable BIB design with parameters $v = (q^6-1)/(q-1)$, $b = (q^6-1)(q^5-1)/(q^2-1)(q-1)$, $r = (q^5-1)(q^4-1)/(q^2-1)(q-1)$, $k = (q^4-1)/(q-1)$, $\lambda = (q^4-1)(q^3-1)/(q^2-1)(q-1)$ and $\mu_i = (q^4-1)/(q-1)$ or (q^2+1), $i = 1, 2, \cdots, l$, since 3-flats in PG $(5, q)$ have the cycles $\theta[1] = (q^6-1)/(q^2-1)$ and $\theta[0] = (q^6-1)/(q-1)$. This design is not resolvable.

For a finite affine t-dimensional geometry over a Galois field we may also have an argument similar to that of PG (t, q), since a BIB design can be obtained by choosing the points as treatments and all d-flats as blocks from a finite affine geometry.

Thus, combining some results described here yields the many of $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable BIB designs. For $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable PBIB designs we may have an argument similar to that of BIB designs. However, it is omitted here.

Finally, as an interesting example of a $(\mu_1, \mu_2, \cdots, \mu_t)$-resolvable incomplete block design, we have the following: Let $A_0, A_1, \cdots, A_m$ be the association matrices of an m-class association scheme with parameters v, n_i, p^i_{jk}, $i, j, k = 0, 1, \cdots, m$; and $i_1, i_2, \cdots, i_t$ be distinct integers such that $i_l \in \{0, 1, \cdots, m\}$ for $l = 1, 2, \cdots, t\ (\leqq m)$. Then $N = [A_{i_1} : A_{i_2} : \cdots : A_{i_t}]$ is an $(n_{i_1}, n_{i_2}, \cdots, n_{i_t})$-resolvable incomplete block design. The case where this design N becomes a BIB design or a PBIB design can be seen in [7].

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Mathematical Institute
Faculty of Education
Hiroshima University
Shinonome, Hiroshima
Japan