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OPTIMAL BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS OF RESOLUTION VII, $6 \leqq m \leqq 8$

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Consider the class of balanced fractional 2^m factorial designs of resolution VII. Within this class, optimal designs with respect to the trace criterion are given for any fixed N assemblies, which satisfy (i) $m = 6$, $42 \leqq N \leqq 64$, (ii) $m = 7$, $64 \leqq N \leqq 90$ and (iii) $m = 8$, $93 \leqq N \leqq 128$. The covariance matrices of the estimates of the effects are also given for such designs.

1. Introduction. Balanced designs were first introduced by Chakravarti (1956), who gave them the name "partially balanced arrays," as generalizations of orthogonal designs. An orthogonal design in general requires much more than the desirable number of assemblies or treatment combinations and the possibility of its existence is very small. On the other hand, balanced designs are flexible in the number of assemblies. Unlike an orthogonal design, the estimates of the effects can be correlated in these designs. The balanced designs are often used.

Balanced fractional 2^m factorial (briefly, 2^m-BFF) designs of resolution V have been investigated by Srivastava (1970), Srivastava and Chopra (1971a, b), Chopra and Srivastava (1973a, b), and others. Particularly in Srivastava and Chopra (1971a), and Chopra and Srivastava (1973a, b), optimal 2^m-BFF designs of resolution V with respect to the trace criterion have been given for each m with $4 \leqq m \leqq 7$, and for some numbers of assemblies.

Those investigations, however, have been restricted to designs of resolution V. They have the property that the general mean, the main effects and two-factor interactions are estimable provided the remaining effects are assumed negligible. In general, designs are said to be of resolution $2l + 1$ when in these designs all the effects involving up to l-factor interactions are estimable provided $(l + 1)$-factor and higher order interactions are assumed negligible. As computational techniques have improved, it has been possible to handle an experimental situation with many factors. Among these factors, as a matter of course, there may be ones similar to each other. This indicates that we cannot ignore three-factor interactions, that is, we need to consider designs of resolution VII.

Recently, Yamamoto, Shirakura and Kuwada (1975) have established some general properties of 2^m-BFF designs of resolution $2l + 1$. Furthermore Yamamoto, Shirakura and Kuwada (1974) have succeeded in obtaining an explicit expression for the characteristic polynomial of the information matrix M_T of a

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2^m-BFF design T of resolution $2l+1$. This polynomial is useful for comparing 2^m-BFF designs of higher resolution by the popular criterion such as the trace of M_T^{-1}.

In this paper, by using the above results for the case $l=3$, optimal 2^m-BFF designs of resolution VII are given for any fixed N assemblies, which satisfy (i) $m=6$, $42 \leqq N \leqq 64$, (ii) $m=7$, $64 \leqq N \leqq 90$ and (iii) $m=8$, $93 \leqq N \leqq 128$. The covariance matrices of the estimates of the effects are given for such designs. The smallest number of N given for each case corresponds to a saturated design. On the other hand, for the largest number of N with $m=6$ or 8, there exists an orthogonal fractional 2^m factorial design of resolution VII, for which the information matrix is diagonal. It is well known that this design is optimal with respect to the trace criterion in the class of all designs of resolution VII with N assemblies.

2. 2^m-BFF designs of resolution VII. Consider a factorial experimentation with m factors each at two levels. An assembly or treatment combination will be represented by $(j_1, j_2, \cdots, j_m)$, where j_k, the level of kth factor, equals 0 or 1. Consider the situation where four-factor and higher order interactions are assumed negligible throughout this paper. Then the total number of unknown parameters to be estimated is $p_m = 1 + m + \binom{m}{2} + \binom{m}{3}$. The vector of unknown parameters $\boldsymbol{\theta}(p_m \times 1)$ will be written as

$$\boldsymbol{\theta}' = (\theta_\phi; \theta_1, \theta_2, \cdots, \theta_m; \theta_{12}, \theta_{13}, \cdots, \theta_{m-1\,m}; \theta_{123}, \theta_{124}, \cdots, \theta_{m-2\,m-1\,m}) ,$$

where θ_ϕ, θ_i, θ_{ij} and θ_{ijk} denote the general mean, the main effect of ith factor, the two-factor interaction of ith and jth factors and the three-factor interaction of ith, jth and kth factors, respectively.

Let T be a suitable set of N assemblies (called a fraction), then T can be expressed as a (0, 1) matrix of size $m \times N$ whose columns denote assemblies. Let $\mathbf{y}_T$ be the $N \times 1$ observation vector whose αth element is the observation in αth assembly of T and consider the N observations in $\mathbf{y}_T$ as independent random variables with common variance σ^2. Then the normal equations for estimating $\boldsymbol{\theta}$ based on T are (see, e.g., Bose and Srivastava (1964) or Yamamoto, Shirakura and Kuwada (1975))

$$M_T \hat{\boldsymbol{\theta}} = E_T' \mathbf{y}_T ,$$

where E_T is the $N \times p_m$ design matrix whose elements are 1 or -1 and M_T $(= E_T' E_T)$ is the information matrix of T. A design T is of resolution VII if and only if the information matrix M_T is nonsingular. For any design T of resolution VII, the best linear unbiased estimate of $\boldsymbol{\theta}$ and the covariance matrix of its estimate are given by $\hat{\boldsymbol{\theta}} = V_T E_T' \mathbf{y}_T$ and $\mathrm{Var}\,(\hat{\boldsymbol{\theta}}) = \sigma^2 V_T$, respectively, where $V_T = M_T^{-1}$.

When V_T is invariant under any permutation of m factors, T is called a 2^m-BFF design of resolution VII. In other words, $\mathrm{Var}\,(\hat{\boldsymbol{\theta}})$ is such that $\mathrm{Var}\,(\hat{\theta}_i)$, $\mathrm{Var}\,(\hat{\theta}_{ij})$, $\mathrm{Var}\,(\hat{\theta}_{ijk})$, $\mathrm{Cov}\,(\hat{\theta}_\phi, \hat{\theta}_i)$, $\mathrm{Cov}\,(\hat{\theta}_\phi, \hat{\theta}_{ij})$, $\mathrm{Cov}\,(\hat{\theta}_\phi, \hat{\theta}_{ijk})$, $\mathrm{Cov}\,(\hat{\theta}_i, \hat{\theta}_j)$, $\mathrm{Cov}\,(\hat{\theta}_i, \hat{\theta}_{ij})$, Cov $(\hat{\theta}_i, \hat{\theta}_{jk})$, Cov $(\hat{\theta}_i, \hat{\theta}_{ijk})$, Cov $(\hat{\theta}_i, \hat{\theta}_{jkl})$, Cov $(\hat{\theta}_{ij}, \hat{\theta}_{ik})$, Cov $(\hat{\theta}_{ij}, \hat{\theta}_{kl})$, Cov $(\hat{\theta}_{ij}, \hat{\theta}_{ijk})$, Cov $(\hat{\theta}_{ij}, \hat{\theta}_{ikl})$, Cov $(\hat{\theta}_{ij}, \hat{\theta}_{klp})$, Cov $(\hat{\theta}_{ijk}, \hat{\theta}_{ijl})$, Cov $(\hat{\theta}_{ijk}, \hat{\theta}_{ilp})$ and Cov $(\hat{\theta}_{ijk}, \hat{\theta}_{lpq})$ are independent of distinct integers i, j, k, l, p and q chosen out of the set $\{1, 2, \cdots, m\}$. Let $\varepsilon(x; y)$ denote the element of M_T in the cell corresponding to $(x; y)$ for θ_x and θ_y in $\boldsymbol{\theta}$. Then, for a design T of resolution VII, it is easily shown that a necessary and sufficient condition for T to be a 2^m-BFF design is that M_T has at most seven distinct elements γ_i as follows:

$$\gamma_0 = \varepsilon(\theta_\phi, \theta_\phi) = \varepsilon(\theta_i, \theta_i) = \varepsilon(\theta_{ij}, \theta_{ij}) = \varepsilon(\theta_{ijk}, \theta_{ijk}) ,$$
$$\gamma_1 = \varepsilon(\theta_\phi, \theta_i) = \varepsilon(\theta_j, \theta_{ij}) = \varepsilon(\theta_{ij}, \theta_{ijk}) ,$$
$$\gamma_2 = \varepsilon(\theta_\phi, \theta_{ij}) = \varepsilon(\theta_i, \theta_j) = \varepsilon(\theta_k, \theta_{ijk}) = \varepsilon(\theta_{ik}, \theta_{jk}) = \varepsilon(\theta_{ikl}, \theta_{jkl}) ,$$
$$\gamma_3 = \varepsilon(\theta_\phi, \theta_{ijk}) = \varepsilon(\theta_i, \theta_{jk}) = \varepsilon(\theta_{il}, \theta_{jkl}) , \qquad \gamma_4 = \varepsilon(\theta_{ij}, \theta_{kl}) = \varepsilon(\theta_{ijp}, \theta_{klp}) ,$$
$$\gamma_5 = \varepsilon(\theta_{ij}, \theta_{klp}) , \qquad \gamma_6 = \varepsilon(\theta_{ijk}, \theta_{lpq}) ,$$

where i, j, k, l, p and q $(= 1, 2, \cdots, m)$ are all distinct. Furthermore it is shown that a necessary and sufficient condition for M_T to be expressible by such elements γ_i is that T is a balanced array (B-array) of strength 6, defined below. A (0, 1) matrix T of size $m \times N$ is called a B-array of strength 6, size N, m constraints and index set $\{\mu_0, \mu_1, \cdots, \mu_6\}$ if for every submatrix $T_0(6 \times N)$ of T, every (0, 1) vector with weight (or number of nonzero elements) j occurs exactly μ_j times $(j = 0, 1, \cdots, 6)$ as a column of T_0. To avoid repetition, assume that we are considering such a B-array T throughout this paper. If all μ_i's are equal, then we obtain what is called an orthogonal array of strength 6. It is well known that an orthogonal fractional 2^m factorial design of resolution VII is identical with an orthogonal array of strength 6.

A connection between the elements γ_i of M_T and the indices μ_i of a B-array T is given as follows:

$$\gamma_0 = N = \mu_6 + \mu_0 + 6(\mu_5 + \mu_1) + 15(\mu_4 + \mu_2) + 20\mu_3 ,$$
$$\gamma_1 = \mu_6 - \mu_0 + 4(\mu_5 - \mu_1) + 5(\mu_4 - \mu_2) ,$$
$$\gamma_2 = \mu_6 + \mu_0 + 2(\mu_5 + \mu_1) - (\mu_4 + \mu_2) - 4\mu_3 ,$$
$$\gamma_3 = \mu_6 - \mu_0 - 3(\mu_4 - \mu_2) ,$$
$$\gamma_4 = \mu_6 + \mu_0 - 2(\mu_5 + \mu_1) - (\mu_4 + \mu_2) + 4\mu_3 ,$$
$$\gamma_5 = \mu_6 - \mu_0 - 4(\mu_5 - \mu_1) + 5(\mu_4 - \mu_2) ,$$
$$\gamma_6 = \mu_6 + \mu_0 - 6(\mu_5 + \mu_1) + 15(\mu_4 + \mu_2) - 20\mu_3 .$$

The following two theorems are obtained from the characteristic polynomial of the information matrix M_T of a 2^m-BFF design of resolution VII which has been given by Yamamoto, Shirakura and Kuwada (1974):

THEOREM 2.1. *For a* 2^m-BFF *design* T *of resolution* VII,

$$\text{(2.1)} \qquad \operatorname{tr} V_T = \operatorname{tr} \mathrm{K}_0^{-1} + (m-1) \cdot \operatorname{tr} \mathrm{K}_1^{-1} + \frac{m(m-3)}{2} \cdot \operatorname{tr} \mathrm{K}_2^{-1} + \frac{m(m-1)(m-5)}{6} \cdot \mathrm{K}_3^{-1} ,$$

where

$$\underset{(4\times 4)}{\mathrm{K}_0} = \begin{bmatrix} \gamma_0 & m^{\frac{1}{2}}\gamma_1 & \binom{m}{2}^{\frac{1}{2}}\gamma_2 & \binom{m}{3}^{\frac{1}{2}}\gamma_3 \\ & \gamma_0 + (m-1)\gamma_2 & \{(m-1)/2\}^{\frac{1}{2}}\{2\gamma_1 + (m-2)\gamma_3\} & \{\binom{m-1}{2}/3\}^{\frac{1}{2}}\{3\gamma_2 + (m-3)\gamma_4\} \\ & & \gamma_0 + 2(m-2)\gamma_2 + \binom{m-2}{2}\gamma_4 & \{(m-2)/3\}^{\frac{1}{2}}\{3\gamma_1 + 3(m-3)\gamma_3 + \binom{m-3}{2}\gamma_5\} \\ (\mathrm{Sym.}) & & & \gamma_0 + 3(m-3)\gamma_2 + 3\binom{m-3}{2}\gamma_4 + \binom{m-3}{3}\gamma_6 \end{bmatrix},$$

$$\underset{(3\times 3)}{\mathrm{K}_1} = \begin{bmatrix} \gamma_0 - \gamma_2 & (m-2)^{\frac{1}{2}}(\gamma_1 - \gamma_3) & \binom{m-2}{2}^{\frac{1}{2}}(\gamma_2 - \gamma_4) \\ & \gamma_0 + (m-4)\gamma_2 - (m-3)\gamma_4 & \{(m-3)/2\}^{\frac{1}{2}}\{2\gamma_1 + (m-6)\gamma_3 - (m-4)\gamma_5\} \\ (\mathrm{Sym.}) & & \gamma_0 + (2m-9)\gamma_2 + (m-4)(m-9)/2 \cdot \gamma_4 - \binom{m-4}{2}\gamma_6 \end{bmatrix},$$

$$\underset{(2\times 2)}{\mathrm{K}_2} = \begin{bmatrix} \gamma_0 - 2\gamma_2 + \gamma_4 & (m-4)^{\frac{1}{2}}(\gamma_1 - 2\gamma_3 + \gamma_5) \\ (\mathrm{Sym.}) & \gamma_0 + (m-7)\gamma_2 - (2m-11)\gamma_4 + (m-5)\gamma_6 \end{bmatrix} \quad \textit{and}$$

$$\mathrm{K}_3 = \gamma_0 - 3\gamma_2 + 3\gamma_4 - \gamma_6 = 2^6\mu_3 .$$

THEOREM 2.2. *A necessary condition for the existence of a* 2^m-BFF *design* T *of resolution* VII *is that the matrices* K_0, K_1 *and* K_2 *are positive definite and* $\mu_3 \geqq 1$.

Using the elements of the inverse matrices K_i^{-1} ($i = 0, 1, 2, 3$), Shirakura and Kuwada (1976) have obtained an explicit expression for all the distinct elements of V_T. It will be very useful for calculating the covariance matrices Var $(\hat{\boldsymbol{\theta}})$ for optimal designs. Note that Theorem 2.2 includes a necessary condition for nonsingularity of M_T that the number of distinct columns in T must be at least p_m. Also note that for a 2^m-BFF design T of resolution VII, we have tr $V_T =$ tr $V_{\bar{T}}$, where $\bar{T}$ is the complementary design obtained from T by an interchange of 0 and 1. This implies that since if T is a B-array with index set $\{\mu_0, \mu_1, \cdots, \mu_6\}$, then $\bar{T}$ is a B-array with index set $\{\mu_6, \mu_5, \cdots, \mu_0\}$, we may restrict to B-arrays such that (a) $\mu_2 > \mu_4$, if $\mu_2 \neq \mu_4$, (b) $\mu_1 > \mu_5$, if $\mu_2 = \mu_4$ and $\mu_1 \neq \mu_5$, or (c) $\mu_0 \geqq \mu_6$, if $\mu_2 = \mu_4$ and $\mu_1 = \mu_5$.

3. Constructions of B-arrays. In this section, we state some known results for later use. Let τ^m $(i_1, i_2, \cdots, i_k)$ denote the number of times the vector $\mathbf{x}$ occurs as a column of a B-array T, where $\mathbf{x}$ contains 1 exactly at i_1th, i_2th, $\cdots$, i_kth positions and 0 elsewhere. Particularly, let $\tau^m(\phi)$ denote the number of times the vector of weight 0 occurs as a column of T. The following three theorems are due to Srivastava (1972):

THEOREM 3.1. *Let* $m = 7$. *A necessary and sufficient condition for the existence* *of a* B-*array T is that there exists an integer d such that*

$$(3.1)\qquad \begin{aligned} d \geqq \psi_{11} &= \max(0, \mu_0 - \mu_1, \mu_0 - \mu_1 + \mu_2 - \mu_3, \\ &\qquad \mu_0 - \mu_1 + \mu_2 - \mu_3 + \mu_4 - \mu_5)\,, \\ d \leqq \psi_{12} &= \min(\mu_0, \mu_0 - \mu_1 + \mu_2, \mu_0 - \mu_1 + \mu_2 - \mu_3 + \mu_4, \\ &\qquad \mu_0 - \mu_1 + \mu_2 - \mu_3 + \mu_4 - \mu_5 + \mu_6)\,. \end{aligned}$$

Also if there exists an integer d which satisfies (3.1), *then, as a construction of T,*

$$(3.2)\qquad \begin{aligned} &\tau^7(i_1, i_2, \cdots, i_k) = \textstyle\sum_{q=1}^{k} (-1)^{k+q}\mu_{q-1} + (-1)^k d \qquad \text{for} \quad 1 \leqq k \leqq 7\,, \\ &\tau^7(\phi) = d\,. \end{aligned}$$

THEOREM 3.2. *Let $m = 8$. A necessary and sufficient condition for the existence of a* B-*array T is that there exist integers $d_1, d_2, \cdots, d_8$ and d_0 such that*

$$(3.3)\qquad \begin{aligned} &\psi_{12} \geqq d_1 \geqq d_2 \geqq d_3 \geqq d_4 \geqq d_5 \geqq d_6 \geqq d_7 \geqq d_8 \geqq \psi_{11}\,, \\ &d_0 \geqq \psi_{21} = \max(0, d_1 + d_2 - \mu_0, d_1 + d_2 + d_3 + d_4 - 3\mu_0 + 2\mu_1 - \mu_2, \\ &\qquad d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \\ &\qquad - 5\mu_0 + 4\mu_1 - 3\mu_2 + 2\mu_3 - \mu_4, \\ &\qquad d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 \\ &\qquad - 7\mu_0 + 6\mu_1 - 5\mu_2 + 4\mu_3 - 3\mu_4 + 2\mu_5 - \mu_6)\,, \\ &d_0 \leqq \psi_{22} = \min(d_8, d_8 + d_7 + d_6 - 2\mu_0 + \mu_1, \\ &\qquad d_8 + d_7 + d_6 + d_5 + d_4 - 4\mu_0 + 3\mu_1 - 2\mu_2 + \mu_3, \\ &\qquad d_8 + d_7 + d_6 + d_5 + d_4 + d_3 + d_2 \\ &\qquad - 6\mu_0 + 5\mu_1 - 4\mu_2 + 3\mu_3 - 2\mu_4 + \mu_5)\,. \end{aligned}$$

Also if there exist integers $d_0, d_1, \cdots, d_8$ which satisfy (3.3), *then, as a construction of T,*

$$(3.4)\qquad \begin{aligned} &\tau^8(i_1, i_2, \cdots, i_k) = \textstyle\sum_{q=1}^{k-1} (-1)^{k-1+q}(k-q)\mu_{q-1} + (-1)^{k+1}\sum_{\alpha=1}^{k} d_{i_\alpha} \\ &\qquad + (-1)^k d_0 \qquad \text{for} \quad 2 \leqq k \leqq 8\,, \\ &\tau^8(i_1) = d_{i_1} - d_0\,, \\ &\tau^8(\phi) = d_0\,. \end{aligned}$$

THEOREM 3.3. *Let $x_q (q = 0, 1, \cdots, m)$ denote the number of columns of a* B-*array T each of which is of weight q. Then*

$$(3.5)\qquad \textstyle\sum_{q=0}^{m} \binom{q}{j}\binom{m-q}{6-j} x_q = \binom{m}{6}\binom{6}{j}\mu_j \qquad \text{for} \quad j = 0, 1, \cdots, 6\,,$$

where $\binom{a}{b} = 0$ if $b > a \geqq 0$ or $b < 0$.

In Theorem 3.3, particularly, a B-array T is said to be "trim" if $x_0 = x_m = 0$.

Let $C(j; m)$ be the (0, 1) matrix of size $m \times \binom{m}{j}$ whose columns are all the distinct vectors of weight j ($0 \leqq j \leqq m$). Then it can be easily shown that $C(j; m)$ is a B-array with indices $\binom{m-6}{j-i}$ ($i = 0, 1, \cdots, 6$). Therefore an array obtained by juxtaposing each $C(j; m)$ ($j = 0, 1, \cdots, m$), λ_j ($\geqq 0$) times yields a B-array with indices $\mu_i = \sum_{j=0}^{m} \lambda_j \binom{m-6}{j-i}$ ($i = 0, 1, \cdots, 6$). Such an array is called a simple array (S-array). An S-array is obviously expressible by $m+1$ non-negative integers λ_j ($j=0,1,\cdots,m$). Fortunately, as will be seen later, all B-arrays required in this paper are S-arrays except only two arrays for $m=8$.

4. Optimal 2^m-BFF designs with $m=8$. In Table 3, the optimal 2^8-BFF designs of resolution VII for each number of N satisfying $93\ (=p_m) \leqq N \leqq 128$ are given with the distinct elements of V_T. For each optimal design, its constructions are indicated in Table 5. We make certain investigations on B-arrays with $m=8$ which are helpful in obtaining such designs.

First consider trim B-arrays with $x_0=x_8=0$. Also define $\nu_0=\mu_0+\mu_6$, $\nu_1=\mu_1+\mu_5$ and $\nu_2=\mu_2+\mu_4$. Then it follows from (3.5) that for a trim B-array T^*, the following hold:

$$\text{(4.1)}\qquad \begin{aligned}
&\text{(a)}\quad x_1+x_7=8(-\mu_3+\nu_2-\nu_1+\nu_0)\geqq 0\\
&\text{(b)}\quad x_2+x_6=28(2\mu_3-2\nu_2+2\nu_1-\nu_0)\geqq 0\\
&\text{(c)}\quad x_3+x_5=56(-3\mu_3+3\nu_2-2\nu_1+\nu_0)\geqq 0\\
&\text{(d)}\qquad\quad x_4=35(4\mu_3-3\nu_2+2\nu_1-\nu_0)\geqq 0\,.
\end{aligned}$$

THEOREM 4.1. *For a trim* B-*array* T^* *with* $m=8$, *the following inequalities hold*:

$$\text{(4.2)}\qquad \begin{aligned}
&\text{(a)}\quad \mu_3\leqq\nu_2\\
&\text{(b)}\quad 21\nu_2+14\mu_3\leqq N\\
&\text{(c)}\quad 8\nu_1+35\mu_3\leqq N\,.
\end{aligned}$$

PROOF. From (4.1b, c) it is clear that $\nu_2\geqq\mu_3$ and $\nu_0+15\nu_2\geqq 2\nu_1+15\mu_3$ hold. Since $N=\nu_0+6\nu_1+15\nu_2+20\mu_3$, we have $8\nu_1+35\mu_3\leqq N$. From (4.1a, b, c) it is clear that $6\nu_1+\nu_0\geqq 6(\nu_2-\mu_3)$ holds. Similarly we have $21\nu_2+14\mu_3\leqq N$.

Since we are interested in designs with $N\leqq 128$, it follows from (4.2c) that B-arrays with $\mu_3\geqq 4$ need not be considered.

THEOREM 4.2. *If there exists a trim* B-*array* T^* *with* $m=8$, $N\leqq 128$ *and* $\mu_3=1$, *then* $2\leqq\nu_2\leqq 5$, $\nu_1\leqq 11$ *and* $3\nu_2-2\nu_1+\nu_0=4$ (i.e., $x_4=0$) *hold.*

PROOF. The inequalities (4.2a, b, c), immediately, give $1\leqq\nu_2\leqq 5$ and $\nu_1\leqq 11$. From (4.1b, c, d) it follows that $\nu_2=1$ implies $x_4=35$, $x_3+x_5=0$ and $x_2+x_6=0$. Furthermore $x_3+x_5=0$ or $x_2+x_6=0$ implies $d_1=d_2=\cdots=d_8$ in (3.4). Hence $x_4=35$ contradicts that x_4 is a multiple of $\binom{m}{4}=70$. Hence we have $2\leqq\nu_2$. The inequalities (4.1c, d) give $0\leqq 4-3\nu_2+2\nu_1-\nu_0\leqq 1$. It follows that $4-3\nu_2+2\nu_1-\nu_0=1$ implies $x_4=35$ and $x_3+x_5=0$. Similarly $4-3\nu_2+2\nu_2-\nu_0=1$ gives a contradiction. Hence we have $3\nu_2-2\nu_1+\nu_0=4$.

Similarly we have

THEOREM 4.3. *If there exists a trim* B-*array* T^* *with* $m=8$, $N\leqq 128$ *and* $\mu_3=2$, *then* $2\leqq\nu_2\leqq 4$, $\nu_1\leqq 7$ *and* $2\nu_2-4\leqq 2\nu_1-\nu_0\leqq 3\nu_2-6$ *hold.*

THEOREM 4.4. *If there exists a trim* B-*array* T^* *with* $m=8$, $N\leqq 128$ *and* $\mu_3=3$, *then* $\nu_2=4$, $\nu_1\leqq 2$ *and* $2\leqq 2\nu_1-\nu_0\leqq 3$ *hold.*

Using Theorem 3.2, we can obtain trim B-arrays with $m = 8$ and $93 \leqq N \leqq 128$ which satisfy a necessary condition of Theorem 4.2, 4.3 or 4.4. In the same time, we find that all B-arrays obtained so are S-arrays with $\lambda_0 = \lambda_8 = 0$ except the following only one array with index set $\{2, 2, 2, 2, 2, 2, 1\}$ and $N = 127$:

$$\left[\begin{array}{c:c:c:c:c:c:c} 0\ 0 \cdots 0 & 1\ 1 \cdots 1 & 0\ 0 \cdots 0 & 1\ 1 \cdots 1 & 0\ 0 \cdots 0 & 1\ 1 \cdots 1 & 0 \\ \hdashline & & & & & & 1 \\ C(1;7) & C(1;7) & C(3;7) & C(3;7) & C(5;7) & C(5;7) & 1 \\ & & & & & & \vdots \\ & & & & & & 1 \end{array}\right]. \tag{4.3}$$

General B-arrays can be easily obtained from trim B-arrays by adding column vectors, each being of weight 0 or 8. Consider the set of index sets which satisfy $N = \mu_0 + \mu_6 + 6(\mu_1 + \mu_5) + 15(\mu_2 + \mu_4) + 20\mu_3$, among all the B-arrays for each N with $93 \leqq N \leqq 128$. Then, using Theorem 2.1, we can find the required optimal designs which minimize tr V_T in this set. For a given index set, there are in general more than one distinct (nonisomorphic) B-arrays. That is, note that optimal designs cannot always be determined uniquely.

EXAMPLE 1. The following is an S-array with $\lambda_0 = \lambda_2 = \lambda_4 = \lambda_6 = 1$ and $\lambda_1 = \lambda_3 = \lambda_5 = \lambda_7 = \lambda_8 = 0$:

$$\left[\begin{array}{c:c:c:c} 0 & & & \\ 0 & & & \\ \vdots & C(2;8) & C(4;8) & C(6;8) \\ 0 & & & \end{array}\right].$$

This array is a B-array with index set $\{2, 2, 2, 2, 2, 2, 1\}$ and $N = 127$, which is distinct from the array in (4.3). From Table 3, it is seen that the B-arrays with this index set are the optimal designs for $N = 127$.

5. Optimal 2^m-BFF designs with $m = 6$ and 7.

(i) The case $m = 6$. Here, we are interested in B-arrays with $p_m\,(= 42) \leqq N \leqq 64$. It follows from the definition of a B-array that there exists a B-array for any index set $\{\mu_0, \mu_1, \cdots, \mu_6\}$ and that it is also an S-array (i.e., $\lambda_i = \mu_i$ $(i = 0, 1, \cdots, 6)$). Therefore, among all solutions $\{\mu_0, \cdots, \mu_6\}$ satisfying $N = \mu_0 + \mu_6 + 6(\mu_1 + \mu_5) + 15(\mu_2 + \mu_4) + 20\mu_3$ for each N $(42 \leqq N \leqq 64)$, we find the optimal designs which minimize tr V_T in (2.1). In Table 1, these are listed with the distinct elements of V_T. As will be seen in this table, for each number of $N = 48$, 50 and 52, there are two optimal designs in which their index sets are distinct from each other.

(ii) The case $m = 7$. Here, we are interested in B-arrays with $p_m\,(= 64) \leqq N \leqq 90$. Using the same method as the case $m = 8$, we make certain investigations on B-arrays with $m = 7$. For a trim B-array T^* with $x_0 = x_7 = 0$,

(a) $x_1 + x_6 = 7\nu_0$

TABLE 1

N	index set	tr V_T	$V(\hat\theta_\phi)$	Cov $(\hat\theta_\phi, \hat\theta_i)$	Cov $(\hat\theta_\phi, \hat\theta_{ij})$	Cov $(\hat\theta_\phi, \hat\theta_{ijk})$	$V(\hat\theta_i)$	Cov $(\hat\theta_i, \hat\theta_j)$	Cov $(\hat\theta_i, \hat\theta_{ij})$	Cov $(\hat\theta_i, \hat\theta_{jk})$	Cov $(\hat\theta_i, \hat\theta_{ijk})$
42	1011010	1.46417	0.02583	0.00153	0.00028	−0.00146	0.02583	0.00153	0.00028	−0.00146	0.00028
43	1011011	1.41525	0.02580	0.00150	0.00008	−0.00166	0.02580	0.00150	0.00008	−0.00166	0.00008
44	2011011	1.40310	0.02447	0.00234	−0.00057	−0.00121	0.02527	0.00097	0.00048	−0.00125	−0.00020
45	2011012	1.39491	0.02445	0.00234	−0.00064	−0.00128	0.02527	0.00097	0.00047	−0.00126	−0.00021
46	3011012	1.39068	0.02398	0.00263	−0.00087	−0.00113	0.02509	0.00079	0.00062	−0.00112	−0.00030
47	0111010	1.33854	0.03125	0.00000	0.00000	−0.00781	0.02474	0.00130	0.00260	−0.00130	0.00260
48	0111011	1.21875	0.02832	0.00195	0.00098	−0.00391	0.02344	0.00000	0.00195	−0.00195	0.00000
48	1111010	1.21875	0.02466	0.00293	−0.00073	−0.00195	0.02344	0.00000	0.00293	−0.00098	0.00000
49	1111011	1.17909	0.02464	0.00300	−0.00060	−0.00180	0.02314	−0.00030	0.00240	−0.00150	−0.00060
50	1111012	1.17188	0.02464	0.00302	−0.00058	−0.00178	0.02308	−0.00036	0.00231	−0.00160	−0.00071
50	2111011	1.17188	0.02397	0.00320	−0.00089	−0.00142	0.02308	−0.00036	0.00249	−0.00142	−0.00071
51	2111012	1.16507	0.02397	0.00319	−0.00090	−0.00143	0.02304	−0.00040	0.00239	−0.00151	−0.00080
52	2111013	1.16217	0.02397	0.00319	−0.00091	−0.00144	0.02302	−0.00042	0.00235	−0.00155	−0.00083
52	3111012	1.16217	0.02368	0.00326	−0.00104	−0.00129	0.02302	−0.00042	0.00243	−0.00148	−0.00083
53	3111013	1.15929	0.02368	0.00326	−0.00106	−0.00130	0.02300	−0.00043	0.00239	−0.00152	−0.00087
54	1111020	1.14148	0.02144	0.00095	−0.00177	−0.00234	0.01963	−0.00071	0.00084	−0.00089	−0.00086
55	1111021	1.10695	0.02068	0.00107	−0.00107	−0.00138	0.01961	−0.00073	0.00073	−0.00101	−0.00101
56	1111022	1.09942	0.02051	0.00110	−0.00092	−0.00117	0.01961	−0.00073	0.00070	−0.00103	−0.00105
57	0111101	0.87629	0.01994	0.00119	0.00211	−0.00101	0.01994	0.00119	0.00211	−0.00101	0.00211
58	1111101	0.84375	0.01855	0.00195	0.00098	0.00000	0.01953	0.00078	0.00273	−0.00039	0.00156
59	1111102	0.83144	0.01752	0.00114	0.00039	−0.00036	0.01890	0.00015	0.00228	−0.00085	0.00128
60	2111102	0.82489	0.01718	0.00129	0.00012	−0.00015	0.01884	0.00009	0.00239	−0.00074	0.00119
61	2111103	0.82061	0.01680	0.00100	−0.00010	−0.00027	0.01863	−0.00012	0.00223	−0.00090	0.00110
62	0111110	0.73958	0.01660	0.00000	0.00098	0.00000	0.01823	0.00260	0.00000	0.00000	0.00260
63	1111110	0.68608	0.01634	0.00071	0.00071	0.00071	0.01634	0.00071	0.00071	0.00071	0.00071
64	1111111	0.65625	0.01563	0.00000	0.00000	0.00000	0.01563	0.00000	0.00000	0.00000	0.00000

Optimal 2^6-BFF *designs*

Cov $(\hat\theta_i, \hat\theta_{jkl})$	$V(\hat\theta_{ij})$	Cov $(\hat\theta_{ij}, \hat\theta_{ik})$	Cov $(\hat\theta_{ij}, \hat\theta_{kl})$	Cov $(\hat\theta_{ij}, \hat\theta_{ijk})$	Cov $(\hat\theta_{ij}, \hat\theta_{ikl})$	Cov $(\hat\theta_{ij}, \hat\theta_{klp})$	$V(\hat\theta_{ijk})$	Cov $(\hat\theta_{ijk}, \hat\theta_{ikl})$	Cov $(\hat\theta_{ijk}, \hat\theta_{ilp})$	Cov $(\hat\theta_{ijk}, \hat\theta_{lpq})$
−0.00146	0.03667	0.00194	−0.00153	0.00194	−0.00153	0.01063	0.03667	0.00194	−0.00153	0.01063
−0.00166	0.03527	0.00055	−0.00292	0.00055	−0.00292	0.00923	0.03527	0.00055	−0.00292	0.00923
−0.00194	0.03496	0.00024	−0.00323	0.00077	−0.00271	0.00945	0.03513	0.00040	−0.00307	0.00908
−0.00195	0.03470	−0.00002	−0.00350	0.00053	−0.00294	0.00921	0.03492	0.00019	−0.00328	0.00888
−0.00204	0.03458	−0.00014	−0.00362	0.00060	−0.00287	0.00928	0.03487	0.00015	−0.00332	0.00883
0.00260	0.02865	−0.00065	0.00130	0.00521	−0.00260	0.00521	0.03646	0.00521	0.00521	0.02083
0.00000	0.02832	−0.00098	0.00098	0.00391	−0.00391	0.00391	0.03125	0.00000	0.00000	0.01563
0.00000	0.02856	−0.00073	0.00122	0.00586	−0.00195	0.00586	0.03125	0.00000	0.00000	0.01563
−0.00060	0.02764	−0.00165	0.00030	0.00481	−0.00300	0.00481	0.03005	−0.00120	−0.00120	0.01442
−0.00071	0.02748	−0.00182	0.00013	0.00462	−0.00320	0.00462	0.02983	−0.00142	−0.00142	0.01420
−0.00071	0.02752	−0.00178	0.00018	0.00497	−0.00284	0.00497	0.02983	−0.00142	−0.00142	0.01420
−0.00080	0.02732	−0.00198	−0.00003	0.00478	−0.00303	0.00478	0.02966	−0.00159	−0.00159	0.01403
−0.00083	0.02723	−0.00207	−0.00011	0.00470	−0.00311	0.00470	0.02958	−0.00167	−0.00167	0.01396
−0.00083	0.02725	−0.00205	−0.00009	0.00485	−0.00296	0.00485	0.02958	−0.00167	−0.00167	0.01396
−0.00087	0.02715	−0.00214	−0.00019	0.00477	−0.00304	0.00477	0.02951	−0.00174	−0.00174	0.01389
0.00038	0.02620	−0.00157	0.00190	0.00487	−0.00208	0.00660	0.03046	−0.00029	0.00020	0.01632
0.00023	0.02556	−0.00222	0.00126	0.00399	−0.00295	0.00573	0.02926	−0.00150	−0.00100	0.01512
0.00019	0.02542	−0.00235	0.00112	0.00380	−0.00315	0.00553	0.02900	−0.00176	−0.00126	0.01486
−0.00101	0.02105	0.00230	−0.00083	0.00230	−0.00083	−0.00395	0.02105	0.00230	−0.00083	−0.00395
−0.00156	0.02012	0.00137	−0.00176	0.00313	0.00000	−0.00313	0.02031	0.00156	−0.00156	−0.00469
−0.00184	0.01979	0.00104	−0.00209	0.00292	−0.00020	−0.00333	0.02019	0.00144	−0.00169	−0.00481
−0.00193	0.01957	0.00082	−0.00230	0.00309	−0.00004	−0.00316	0.02005	0.00130	−0.00182	−0.00495
−0.00202	0.01945	0.00070	−0.00243	0.00302	−0.00010	−0.00323	0.02002	0.00127	−0.00186	−0.00498
0.00260	0.01660	0.00098	0.00098	0.00000	0.00000	0.00000	0.01823	0.00260	0.00260	0.00260
0.00071	0.01634	0.00071	0.00071	0.00071	0.00071	0.00071	0.01634	0.00071	0.00071	0.00071
0.00000	0.01563	0.00000	0.00000	0.00000	0.00000	0.00000	0.01563	0.00000	0.00000	0.00000

TABLE 2

N	index set	tr V_T	$V(\hat{\theta}_\phi)$	Cov $(\hat{\theta}_\phi, \hat{\theta}_i)$	Cov $(\hat{\theta}_\phi, \hat{\theta}_{ij})$	Cov $(\hat{\theta}_\phi, \hat{\theta}_{ijk})$	$V(\hat{\theta}_i)$	Cov $(\hat{\theta}_i, \hat{\theta}_j)$	Cov $(\hat{\theta}_i, \hat{\theta}_{ij})$	Cov $(\hat{\theta}_i, \hat{\theta}_{jk})$	Cov $(\hat{\theta}_i, \hat{\theta}_{ijk})$
64	1111111	1.00000	0.01563	0.00000	0.00000	0.00000	0.01563	0.00000	0.00000	0.00000	0.00000
65	2111111	0.99219	0.01550	0.00012	−0.00012	0.00012	0.01550	−0.00012	0.00012	0.00012	−0.00012
66	2111112	0.98477	0.01534	0.00000	−0.00029	0.00000	0.01541	−0.00021	0.00000	0.00000	−0.00021
67	3111112	0.98221	0.01528	0.00004	−0.00035	0.00004	0.01538	−0.00024	0.00004	0.00004	−0.00024
68	3111113	0.97964	0.01521	0.00000	−0.00041	0.00000	0.01536	−0.00027	0.00000	0.00000	−0.00027
69	4111113	0.97833	0.01518	0.00002	−0.00044	0.00002	0.01534	−0.00028	0.00002	0.00002	−0.00028
70	1211111	0.96759	0.01478	0.00063	−0.00040	0.00018	0.01480	−0.00033	0.00053	0.00004	−0.00023
71	2211111	0.94531	0.01477	0.00061	−0.00037	0.00012	0.01477	−0.00037	0.00061	0.00012	−0.00037
72	2211112	0.93750	0.01465	0.00049	−0.00049	0.00000	0.01465	−0.00049	0.00049	0.00000	−0.00049
73	3211112	0.93348	0.01465	0.00049	−0.00049	0.00001	0.01465	−0.00049	0.00049	0.00001	−0.00049
74	3211113	0.93097	0.01461	0.00045	−0.00054	−0.00003	0.01461	−0.00053	0.00045	−0.00004	−0.00053
75	4211113	0.92917	0.01460	0.00045	−0.00055	−0.00002	0.01461	−0.00053	0.00045	−0.00004	−0.00053
76	4211114	0.92789	0.01458	0.00043	−0.00057	−0.00004	0.01459	−0.00055	0.00042	−0.00006	−0.00055
77	2211121	0.91627	0.01397	−0.00001	−0.00073	−0.00004	0.01405	−0.00076	0.00008	0.00008	−0.00053
78	2211122	0.89406	0.01397	0.00000	−0.00071	0.00000	0.01402	−0.00078	0.00000	0.00000	−0.00065
79	3211122	0.88947	0.01397	0.00000	−0.00071	−0.00001	0.01401	−0.00079	0.00002	0.00002	−0.00067
80	3211123	0.88561	0.01397	0.00000	−0.00070	0.00000	0.01401	−0.00079	0.00000	0.00000	−0.00069
81	4211123	0.88385	0.01397	0.00000	−0.00070	0.00000	0.01401	−0.00079	0.00001	0.00001	−0.00070
82	4211124	0.88217	0.01397	0.00000	−0.00069	0.00000	0.01401	−0.00080	0.00000	0.00000	−0.00070
83	5211124	0.88119	0.01397	0.00000	−0.00069	0.00000	0.01401	−0.00080	0.00000	0.00000	−0.00071
84	5211125	0.88023	0.01397	0.00000	−0.00069	0.00000	0.01401	−0.00080	0.00000	0.00000	−0.00071
85	1221111	0.83594	0.01306	0.00110	−0.00012	−0.00037	0.01306	−0.00012	0.00110	−0.00037	−0.00012
86	2221111	0.82813	0.01294	0.00122	−0.00024	−0.00024	0.01294	−0.00024	0.00122	−0.00024	−0.00024
87	2221112	0.82105	0.01291	0.00115	−0.00032	−0.00029	0.01278	−0.00041	0.00106	−0.00041	−0.00035
88	3221112	0.81849	0.01286	0.00119	−0.00037	−0.00025	0.01275	−0.00043	0.00110	−0.00037	−0.00038
89	3221113	0.81597	0.01285	0.00116	−0.00040	−0.00027	0.01270	−0.00049	0.00104	−0.00042	−0.00042
90	4221113	0.81467	0.01282	0.00118	−0.00043	−0.00025	0.01269	−0.00050	0.00106	−0.00041	−0.00043

Optimal 2^7-BFF *designs*

$V(\hat\theta_{ij})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{kl})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ikl})$	$V(\hat\theta_{ijk})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{ilp})$	$\mathrm{Cov}(\hat\theta_{i}, \hat\theta_{jkl})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ik})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ijk})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{klp})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{ikl})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{lpq})$
0.01563	0.00000	0.00000	0.01563	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.01550	−0.00012	0.00012	0.01550	−0.00012	−0.00012	−0.00012	0.00012	0.00012	−0.00012	−0.00012
0.01534	−0.00029	0.00000	0.01541	−0.00021	−0.00021	−0.00029	0.00000	0.00000	−0.00021	−0.00021
0.01528	−0.00035	0.00004	0.01538	−0.00024	−0.00024	−0.00035	0.00004	0.00004	−0.00024	−0.00024
0.01521	−0.00041	0.00000	0.01536	−0.00027	−0.00027	−0.00041	0.00000	0.00000	−0.00027	−0.00027
0.01518	−0.00044	0.00002	0.01534	−0.00028	−0.00028	−0.00044	0.00002	0.00002	−0.00028	−0.00028
0.01497	0.00032	−0.00020	0.01528	0.00063	0.00025	−0.00016	0.00029	−0.00069	0.00014	0.00112
0.01477	0.00012	0.00012	0.01477	0.00012	0.00012	−0.00037	0.00061	−0.00037	−0.00037	0.00061
0.01465	0.00000	0.00000	0.01465	0.00000	0.00000	−0.00049	0.00049	−0.00049	−0.00049	0.00049
0.01458	−0.00007	0.00007	0.01458	−0.00007	−0.00001	−0.00056	0.00056	−0.00042	−0.00056	0.00042
0.01452	−0.00012	0.00003	0.01455	−0.00010	−0.00004	−0.00061	0.00052	−0.00046	−0.00059	0.00039
0.01449	−0.00016	0.00006	0.01452	−0.00013	−0.00004	−0.00065	0.00055	−0.00042	−0.00062	0.00036
0.01446	−0.00019	0.00004	0.01450	−0.00014	−0.00006	−0.00068	0.00053	−0.00044	−0.00063	0.00034
0.01382	0.00059	0.00033	0.01468	0.00070	0.00029	−0.00061	0.00033	0.00033	−0.00012	0.00152
0.01360	0.00038	0.00000	0.01418	0.00020	0.00017	−0.00082	0.00000	0.00000	−0.00062	0.00102
0.01356	0.00034	0.00007	0.01408	0.00010	0.00015	−0.00086	0.00007	0.00007	−0.00073	0.00092
0.01350	0.00028	0.00000	0.01400	0.00002	0.00013	−0.00093	0.00000	0.00000	−0.00080	0.00085
0.01347	0.00025	0.00003	0.01397	−0.00001	0.00012	−0.00096	0.00003	0.00003	−0.00083	0.00081
0.01343	0.00021	0.00000	0.01394	−0.00004	0.00012	−0.00099	0.00000	0.00000	−0.00086	0.00079
0.01341	0.00019	0.00002	0.01393	−0.00005	0.00011	−0.00101	0.00002	0.00002	−0.00087	0.00077
0.01339	0.00017	0.00000	0.01391	−0.00007	0.00011	−0.00103	0.00000	0.00000	−0.00089	0.00076
0.01306	0.00037	−0.00037	0.01306	0.00037	0.00037	−0.00012	0.00110	0.00012	−0.00012	−0.00110
0.01294	0.00024	−0.00024	0.01294	0.00024	0.00024	−0.00024	0.00122	0.00024	−0.00024	−0.00122
0.01278	0.00008	−0.00035	0.01287	0.00017	0.00014	−0.00041	0.00111	0.00014	−0.00032	−0.00129
0.01272	0.00002	−0.00031	0.01284	0.00014	0.00011	−0.00046	0.00116	0.00018	−0.00035	−0.00132
0.01266	−0.00004	−0.00035	0.01281	0.00012	0.00007	−0.00053	0.00112	0.00014	−0.00037	−0.00135
0.01262	−0.00007	−0.00033	0.01280	0.00010	0.00006	−0.00056	0.00114	0.00016	−0.00038	−0.00136

TABLE 3

N	index set	tr V_T	$V(\hat\theta_\phi)$	Cov $(\hat\theta_\phi, \hat\theta_{ij})$	$V(\hat\theta_i)$	Cov $(\hat\theta_i, \hat\theta_{ij})$	Cov $(\hat\theta_i, \hat\theta_{ijk})$	Cov $(\hat\theta_\phi, \hat\theta_i)$	Cov $(\hat\theta_\phi, \hat\theta_{ijk})$	Cov $(\hat\theta_i, \hat\theta_j)$	Cov $(\hat\theta_i, \hat\theta_{jk})$
93	2221122	1.17184	0.01306	−0.00082	0.01306	0.00082	−0.00082	0.00069	−0.00043	−0.00069	−0.00043
94	2221123	1.16531	0.01265	−0.00061	0.01305	0.00079	−0.00084	0.00074	−0.00028	−0.00070	−0.00046
95	3221123	1.16255	0.01264	−0.00064	0.01301	0.00084	−0.00086	0.00077	−0.00026	−0.00074	−0.00041
96	3221124	1.16035	0.01251	−0.00057	0.01301	0.00083	−0.00087	0.00078	−0.00022	−0.00074	−0.00042
97	4221124	1.15911	0.01250	−0.00058	0.01299	0.00085	−0.00088	0.00079	−0.00021	−0.00076	−0.00040
98	4221125	1.15798	0.01244	−0.00054	0.01299	0.00085	−0.00088	0.00079	−0.00019	−0.00076	−0.00040
99	5221125	1.15727	0.01244	−0.00055	0.01298	0.00086	−0.00089	0.00080	−0.00019	−0.00077	−0.00039
100	5221126	1.15658	0.01240	−0.00052	0.01298	0.00086	−0.00089	0.00080	−0.00017	−0.00077	−0.00039
101	4321122	1.11959	0.01301	−0.00088	0.01261	0.00070	−0.00066	0.00078	−0.00041	−0.00083	−0.00024
102	4321123	1.11306	0.01260	−0.00068	0.01260	0.00068	−0.00068	0.00084	−0.00026	−0.00084	−0.00026
103	4321124	1.11088	0.01247	−0.00061	0.01260	0.00067	−0.00068	0.00085	−0.00021	−0.00084	−0.00027
104	4321125	1.10980	0.01240	−0.00057	0.01260	0.00066	−0.00069	0.00086	−0.00018	−0.00084	−0.00027
105	5321125	1.10874	0.01240	−0.00057	0.01260	0.00067	−0.00069	0.00086	−0.00019	−0.00084	−0.00027
106	5321126	1.10808	0.01236	−0.00055	0.01260	0.00067	−0.00069	0.00087	−0.00017	−0.00084	−0.00027
107	2222112	1.04927	0.01130	0.00015	0.01077	0.00117	−0.00003	0.00150	−0.00058	0.00030	−0.00034
108	3222112	1.04241	0.01066	−0.00001	0.01076	0.00115	0.00000	0.00142	−0.00034	0.00029	−0.00036
109	3222113	1.03944	0.01066	−0.00003	0.01075	0.00115	−0.00001	0.00142	−0.00035	0.00028	−0.00037
110	4222113	1.03736	0.01043	−0.00009	0.01075	0.00114	0.00000	0.00139	−0.00028	0.00028	−0.00037
111	4222114	1.03614	0.01043	−0.00011	0.01075	0.00113	−0.00001	0.00139	−0.00029	0.00028	−0.00038
112	5222114	1.03509	0.01031	−0.00014	0.01075	0.00113	0.00000	0.00137	−0.00025	0.00028	−0.00038
113	5222115	1.03440	0.01030	−0.00015	0.01075	0.00112	0.00000	0.00137	−0.00026	0.00028	−0.00039
114	6222115	1.03376	0.01022	−0.00018	0.01074	0.00112	0.00000	0.00136	−0.00024	0.00027	−0.00039
115	2222124	0.99811	0.01048	−0.00006	0.00941	0.00043	−0.00032	0.00104	−0.00063	0.00019	−0.00024
116	3222124	0.99126	0.00985	−0.00022	0.00940	0.00041	−0.00029	0.00096	−0.00039	0.00018	−0.00026
117	3222125	0.98886	0.00977	−0.00018	0.00938	0.00043	−0.00027	0.00092	−0.00034	0.00016	−0.00025
118	4222125	0.98693	0.00959	−0.00024	0.00938	0.00043	−0.00026	0.00091	−0.00028	0.00016	−0.00025
119	4222126	0.98581	0.00957	−0.00023	0.00937	0.00044	−0.00025	0.00089	−0.00026	0.00015	−0.00024
120	5222126	0.98487	0.00948	−0.00026	0.00937	0.00043	−0.00025	0.00089	−0.00023	0.00015	−0.00024
121	5222127	0.98421	0.00947	−0.00025	0.00936	0.00044	−0.00024	0.00088	−0.00022	0.00014	−0.00024
122	6222127	0.98364	0.00942	−0.00027	0.00936	0.00044	−0.00024	0.00088	−0.00021	0.00014	−0.00024

Optimal 2^8-BFF *designs*

$V(\hat\theta_{ij})$	Cov $(\hat\theta_{ij}, \hat\theta_{kl})$	Cov $(\hat\theta_{ij}, \hat\theta_{ikl})$	$V(\hat\theta_{ijk})$	Cov $(\hat\theta_{ijk}, \hat\theta_{ilp})$	Cov $(\hat\theta_{i}, \hat\theta_{jkl})$	Cov $(\hat\theta_{ij}, \hat\theta_{ik})$	Cov $(\hat\theta_{ij}, \hat\theta_{ijk})$	Cov $(\hat\theta_{ij}, \hat\theta_{klp})$	Cov $(\hat\theta_{ijk}, \hat\theta_{ikl})$	Cov $(\hat\theta_{ijk}, \hat\theta_{lpq})$
0.01255	0.00033	−0.00033	0.01255	0.00033	0.00043	−0.00050	0.00050	0.00057	−0.00050	−0.00057
0.01245	0.00023	−0.00041	0.01249	0.00027	0.00041	−0.00061	0.00043	0.00050	−0.00056	−0.00063
0.01240	0.00018	−0.00038	0.01247	0.00025	0.00039	−0.00065	0.00046	0.00053	−0.00058	−0.00065
0.01236	0.00014	−0.00040	0.01246	0.00024	0.00038	−0.00070	0.00043	0.00050	−0.00060	−0.00067
0.01234	0.00011	−0.00039	0.01245	0.00023	0.00037	−0.00072	0.00045	0.00051	−0.00060	−0.00067
0.01231	0.00009	−0.00040	0.01244	0.00022	0.00037	−0.00074	0.00043	0.00050	−0.00061	−0.00068
0.01230	0.00008	−0.00039	0.01244	0.00022	0.00036	−0.00076	0.00044	0.00051	−0.00062	−0.00069
0.01228	0.00006	−0.00040	0.01244	0.00021	0.00036	−0.00077	0.00043	0.00050	−0.00062	−0.00069
0.01200	0.00041	−0.00023	0.01196	0.00036	0.00028	−0.00074	0.00092	0.00036	−0.00079	−0.00023
0.01190	0.00030	−0.00030	0.01190	0.00030	0.00026	−0.00084	0.00084	0.00029	−0.00084	−0.00029
0.01187	0.00027	−0.00033	0.01188	0.00028	0.00026	−0.00088	0.00082	0.00026	−0.00086	−0.00031
0.01185	0.00025	−0.00034	0.01187	0.00027	0.00025	−0.00089	0.00080	0.00025	−0.00087	−0.00032
0.01183	0.00023	−0.00033	0.01186	0.00027	0.00025	−0.00091	0.00082	0.00026	−0.00088	−0.00032
0.01182	0.00022	−0.00034	0.01186	0.00026	0.00025	−0.00092	0.00081	0.00025	−0.00089	−0.00033
0.01090	−0.00031	−0.00049	0.01155	−0.00050	−0.00040	0.00009	0.00140	0.00022	0.00031	0.00129
0.01086	−0.00035	−0.00043	0.01146	−0.00059	−0.00037	0.00005	0.00146	0.00028	0.00022	0.00120
0.01083	−0.00039	−0.00047	0.01142	−0.00063	−0.00038	0.00001	0.00143	0.00025	0.00019	0.00116
0.01081	−0.00041	−0.00044	0.01140	−0.00065	−0.00037	−0.00001	0.00145	0.00027	0.00017	0.00114
0.01079	−0.00043	−0.00046	0.01139	−0.00066	−0.00037	−0.00003	0.00143	0.00025	0.00015	0.00113
0.01077	−0.00044	−0.00045	0.01138	−0.00067	−0.00037	−0.00004	0.00144	0.00026	0.00014	0.00112
0.01076	−0.00045	−0.00046	0.01137	−0.00068	−0.00037	−0.00005	0.00144	0.00026	0.00014	0.00111
0.01075	−0.00046	−0.00045	0.01137	−0.00068	−0.00037	−0.00006	0.00144	0.00026	0.00013	0.00111
0.01002	−0.00009	−0.00057	0.01128	−0.00049	−0.00027	−0.00024	0.00104	0.00042	0.00019	0.00144
0.00998	−0.00012	−0.00051	0.01119	−0.00058	−0.00024	−0.00028	0.00110	0.00048	0.00010	0.00135
0.00996	−0.00014	−0.00053	0.01116	−0.00061	−0.00022	−0.00030	0.00108	0.00046	0.00007	0.00132
0.00995	−0.00016	−0.00051	0.01114	−0.00063	−0.00021	−0.00031	0.00110	0.00048	0.00005	0.00130
0.00994	−0.00017	−0.00052	0.01113	−0.00064	−0.00020	−0.00032	0.00109	0.00046	0.00003	0.00128
0.00993	−0.00018	−0.00051	0.01112	−0.00065	−0.00020	−0.00033	0.00110	0.00048	0.00002	0.00127
0.00992	−0.00019	−0.00052	0.01111	−0.00066	−0.00019	−0.00034	0.00109	0.00047	0.00002	0.00127
0.00991	−0.00019	−0.00052	0.01110	−0.00067	−0.00019	−0.00035	0.00110	0.00047	0.00001	0.00126

TABLE 3

N	index set	tr V_T	$V(\hat\theta_\phi)$	Cov $(\hat\theta_\phi, \hat\theta_i)$	Cov $(\hat\theta_\phi, \hat\theta_{ij})$	Cov $(\hat\theta_\phi, \hat\theta_{ijk})$	$V(\hat\theta_i)$	Cov $(\hat\theta_i, \hat\theta_j)$	Cov $(\hat\theta_i, \hat\theta_{ij})$	Cov $(\hat\theta_i, \hat\theta_{jk})$	Cov $(\hat\theta_i, \hat\theta_{ijk})$
123	4322124	0.97306	0.00991	0.00123	−0.00024	−0.00049	0.00931	0.00014	0.00058	−0.00021	−0.00041
124	5322124	0.96878	0.00964	0.00111	−0.00030	−0.00036	0.00925	0.00008	0.00056	−0.00023	−0.00035
125	5322125	0.96638	0.00957	0.00107	−0.00026	−0.00032	0.00923	0.00006	0.00058	−0.00022	−0.00032
126	6322125	0.96484	0.00948	0.00103	−0.00029	−0.00027	0.00921	0.00004	0.00056	−0.00023	−0.00030
127	2222221	0.74732	0.00804	0.00022	0.00022	0.00022	0.00804	0.00022	0.00022	0.00022	0.00022
128	2222222	0.72656	0.00781	0.00000	0.00000	0.00000	0.00781	0.00000	0.00000	0.00000	0.00000

TABLE 4
Constructions of optimal 2^7-BFF *designs*

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
64	1	1	1	1	1	1	1	1	0	1	0	1	0	1	0
65	2	1	1	1	1	1	1	a 2	0	1	0	1	0	1	0
								b 1	1	0	1	0	1	0	1
66	2	1	1	1	1	1	2	2	0	1	0	1	0	1	1
67	3	1	1	1	1	1	2	a 3	0	1	0	1	0	1	1
								b 2	1	0	1	0	1	0	2
68	3	1	1	1	1	1	3	3	0	1	0	1	0	1	2
69	4	1	1	1	1	1	3	a 4	0	1	0	1	0	1	2
								b 3	1	0	1	0	1	0	3
70	1	2	1	1	1	1	1	0	1	1	0	1	0	1	0
71	2	2	1	1	1	1	1	a 1	1	1	0	1	0	1	0
								b 0	2	0	1	0	1	0	1
72	2	2	1	1	1	1	2	a 1	1	1	0	1	0	1	1
								b 0	2	0	1	0	1	0	2
73	3	2	1	1	1	1	2	a 2	1	1	0	1	0	1	1
								b 1	2	0	1	0	1	0	2
74	3	2	1	1	1	1	3	a 2	1	1	0	1	0	1	2
								b 1	2	0	1	0	1	0	3
75	4	2	1	1	1	1	3	a 3	1	1	0	1	0	1	2
								b 2	2	0	1	0	1	0	3
76	4	2	1	1	1	1	4	a 3	1	1	0	1	0	1	3
								b 2	2	0	1	0	1	0	4
77	2	2	1	1	1	2	1	0	2	0	1	0	1	1	0
78	2	2	1	1	1	2	2	0	2	0	1	0	1	1	1
79	3	2	1	1	1	2	2	1	2	0	1	0	1	1	1
80	3	2	1	1	1	2	3	1	2	0	1	0	1	1	2
81	4	2	1	1	1	2	3	2	2	0	1	0	1	1	2
82	4	2	1	1	1	2	4	2	2	0	1	0	1	1	3
83	5	2	1	1	1	2	4	3	2	0	1	0	1	1	3
84	5	2	1	1	1	2	5	3	2	0	1	0	1	1	4
85	1	2	2	1	1	1	1	a 0	1	1	1	0	1	0	1
								b 1	0	2	0	1	0	1	0
86	2	2	2	1	1	1	1	a 1	1	1	1	0	1	0	1
								b 2	0	2	0	1	0	1	0
87	2	2	2	1	1	1	2	a 1	1	1	1	0	1	0	2
								b 2	0	2	0	1	0	1	1
88	3	2	2	1	1	1	2	a 2	1	1	1	0	1	0	2
								b 3	0	2	0	1	0	1	1
89	3	2	2	1	1	1	3	a 2	1	1	1	0	1	0	3
								b 3	0	2	0	1	0	1	2
90	4	2	2	1	1	1	3	a 3	1	1	1	0	1	0	3
								b 4	0	2	0	1	0	1	2

(continued)

$\mathrm{Cov}(\hat\theta_i, \hat\theta_{jkl})$	$V(\hat\theta_{ij})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ik})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{kl})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ijk})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{ikl})$	$\mathrm{Cov}(\hat\theta_{ij}, \hat\theta_{klp})$	$V(\hat\theta_{ijk})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{ikl})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{ilp})$	$\mathrm{Cov}(\hat\theta_{ijk}, \hat\theta_{lpq})$
−0.00029	0.00954	−0.00044	−0.00001	0.00127	−0.00049	0.00035	0.01110	0.00009	−0.00051	0.00149
−0.00024	0.00953	−0.00045	−0.00002	0.00130	−0.00046	0.00038	0.01104	0.00003	−0.00057	0.00143
−0.00021	0.00951	−0.00047	−0.00003	0.00128	−0.00049	0.00035	0.01101	0.00000	−0.00060	0.00140
−0.00019	0.00951	−0.00047	−0.00004	0.00129	−0.00047	0.00037	0.01099	−0.00002	−0.00062	0.00138
0.00022	0.00804	0.00022	0.00022	0.00022	0.00022	0.00022	0.00804	0.00022	0.00022	0.00022
0.00000	0.00781	0.00000	0.00000	0.00000	0.00000	0.00000	0.00781	0.00000	0.00000	0.00000

TABLE 5

Constructions of optimal 2^8-BFF *designs*

N	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
93	2	2	2	1	1	2	2	0	1	0	1	0	0	1	0	1
94	2	2	2	1	1	2	3	0	1	0	1	0	0	1	0	2
95	3	2	2	1	1	2	3	1	1	0	1	0	0	1	0	2
96	3	2	2	1	1	2	4	1	1	0	1	0	0	1	0	3
97	4	2	2	1	1	2	4	2	1	0	1	0	0	1	0	3
98	4	2	2	1	1	2	5	2	1	0	1	0	0	1	0	4
99	5	2	2	1	1	2	5	3	1	0	1	0	0	1	0	4
100	5	2	2	1	1	2	6	3	1	0	1	0	0	1	0	5
101	4	3	2	1	1	2	2	0	2	0	1	0	0	1	0	1
102	4	3	2	1	1	2	3	0	2	0	1	0	0	1	0	2
103	4	3	2	1	1	2	4	0	2	0	1	0	0	1	0	3
104	4	3	2	1	1	2	5	0	2	0	1	0	0	1	0	4
105	5	3	2	1	1	2	5	1	2	0	1	0	0	1	0	4
106	5	3	2	1	1	2	6	1	2	0	1	0	0	1	0	5
107	2	2	2	2	1	1	2	1	0	1	0	1	0	0	1	0
108	3	2	2	2	1	1	2	2	0	1	0	1	0	0	1	0
109	3	2	2	2	1	1	3	2	0	1	0	1	0	0	1	1
110	4	2	2	2	1	1	3	3	0	1	0	1	0	0	1	1
111	4	2	2	2	1	1	4	3	0	1	0	1	0	0	1	2
112	5	2	2	2	1	1	4	4	0	1	0	1	0	0	1	2
113	5	2	2	2	1	1	5	4	0	1	0	1	0	0	1	3
114	6	2	2	2	1	1	5	5	0	1	0	1	0	0	1	3
115	2	2	2	2	1	2	4	1	0	1	0	1	0	0	2	0
116	3	2	2	2	1	2	4	2	0	1	0	1	0	0	2	0
117	3	2	2	2	1	2	5	2	0	1	0	1	0	0	2	1
118	4	2	2	2	1	2	5	3	0	1	0	1	0	0	2	1
119	4	2	2	2	1	2	6	3	0	1	0	1	0	0	2	2
120	5	2	2	2	1	2	6	4	0	1	0	1	0	0	2	2
121	5	2	2	2	1	2	7	4	0	1	0	1	0	0	2	3
122	6	2	2	2	1	2	7	5	0	1	0	1	0	0	2	3
123	4	3	2	2	1	2	4	1	1	1	0	1	0	0	2	0
124	5	3	2	2	1	2	4	2	1	1	0	1	0	0	2	0
125	5	3	2	2	1	2	5	2	1	1	0	1	0	0	2	1
126	6	3	2	2	1	2	5	3	1	1	0	1	0	0	2	1
127	2	2	2	2	2	2	1	a 1	0	1	0	1	0	1	0	0
								b see (4.3)								
128	2	2	2	2	2	2	2	a 1	0	1	0	1	0	1	0	1
								b add a column (1, 1, ⋯, 1)′ to (4.3).								

(b) $x_2 + x_5 = 21(\nu_1 - \nu_0) \geqq 0$

(c) $x_3 + x_4 = 35(\nu_2 + \nu_0 - \nu_1) = 35\mu_3$.

THEOREM 5.1. *For a trim* B-*array* T^* *with* $m = 7$, *the inequalities* (a) $\mu_3 \leqq \nu_2$, (b) $21\nu_2 + 14\mu_3 \leqq N$ *and* (c) $7\nu_1 + 35\mu_3 \leqq N$ *hold.*

THEOREM 5.2. *If there exists a trim* B-*array* T^* *with* $m = 8$, $N \leqq 90$ *and* $\mu_3 = 1$, *then* $1 \leqq \nu_2 \leqq 3$, $\nu_1 \leqq 7$ *and* $\nu_0 = 1 + \nu_1 - \nu_2$ *hold.*

THEOREM 5.3. *If there exists a trim* B-*array* T^* *with* $m = 8$, $N \leqq 90$ *and* $\mu_3 = 2$, *then* $\nu_2 = 1$, $\nu_1 \leqq 5$ *and* $\nu_0 = 2 + \nu_1 - \nu_2$ *hold.*

Using Theorem 3.1, we can obtain trim B-arrays with $m = 7$ and $64 \leqq N \leqq 90$ which satisfy a necessary condition of Theorem 5.2 or 5.3. However, since $\tau^7(i_1, i_2, \cdots, i_k)$ in (3.2) do not depend on the positions in which 1 occurs, it follows that all B-arrays with $m = 7$ are S-arrays. That is, every B-array is expressible by $\lambda_0 = \tau^7(\phi)$ and $\lambda_k = \tau^7(i_1, i_2, \cdots, i_k)$ for $k = 1, 2, \cdots, 7$.

As in Section 4, the required optimal designs which minimize tr V_T for each N ($64 \leqq N \leqq 90$) can be obtained. In Table 2, these are listed with the distinct elements of V_T. For each design in this table, its constructions are indicated in Table 4.

EXAMPLE 2. Consider a B-array with index set $\{1, 2, 2, 1, 1, 1, 1\}$ which is an optimal design for $N = 85$. From Theorem 3.1, there exist two distinct values of d (say $d = 0$ and 1). Therefore we have the following two distinct S-arrays:

(a) For $d = 0$,

$$\lambda_0 = \lambda_4 = \lambda_6 = 0 \quad \text{and} \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_7 = 1 .$$

(b) For $d = 1$,

$$\lambda_1 = \lambda_3 = \lambda_5 = \lambda_7 = 0 , \quad \lambda_0 = \lambda_4 = \lambda_6 = 1 \quad \text{and} \quad \lambda_2 = 2 .$$

Tables 1, 2 and 3 are prepared at the Hiroshima University Computing Center, TOSBAC 3400/41.

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