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CONFIDENCE INTERVALS FOR LINEAR FUNCTIONS OF THE NORMAL PARAMETERS

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Uniformly most accurate level $1 - \alpha$ confidence procedures for a linear function $\mu + \lambda\sigma^2$ with known λ for the parameters of a normal distribution defined by Land were previously shown for both the one-sided and two-sided procedures to be always intervals for $\nu \geqq 2$, ν being the number of degrees of freedom for estimating σ^2. These results are shown in this paper to hold also in the case $\nu = 1$. During the course of the argument a new inequality is obtained relating to the modified Bessel functions which is of independent interest.

1. Introduction. Uniformly most accurate unbiased level $1 - \alpha$ confidence procedures for linear functions of the mean μ and variance σ^2 of a normal distribution have been defined [2] in terms of the uniformly most powerful unbiased level α tests of null hypotheses of the form $H_0: \mu + \lambda\sigma^2 = m$ against the usual one- and two-sided alternatives. It is shown ([2] and [3] respectively) that the confidence sets defined by these procedures are respectively one- and two-sided intervals provided that $\nu \geqq 2$, ν being the number of degrees of freedom for estimating σ^2. In this note these results are shown to hold also in the case $\nu = 1$, which had remained open.

In the course of the argument a new inequality relating to the modified Bessel functions is proved (Lemma 3.4) which appears to be of independent interest.

2. The problem. Y is distributed normally with mean μ and variance σ^2/γ where γ is known; S^2/σ^2 is distributed χ^2 independently of Y with ν degrees of freedom; λ is a known constant. The null hypothesis is $H_0: \mu + \lambda\sigma^2 = m$. The two-sided alternative is $H_1: \mu + \lambda\sigma^2 \neq m$. The one-sided alternative is $H_2: \mu + \lambda\sigma^2 < m$. The uniformly most accurate, unbiased level $1 - \alpha$ confidence sets for m in the sense of Lehmann [4, page 177] for a given data point (y, s^2) have been derived [2] from the UMP tests of H_0 vs H_1 and H_0 vs H_2. The confidence sets for any given values γ', λ' of γ and λ and the data point (y, s^2) are identical with the confidence sets for $\gamma = 1$, $\lambda = \lambda'\gamma'$ and the data point $(y, s^2/\gamma')$. Hence to simplify the formulae we put $\gamma = 1$ w.l.g. The confidence sets for m for the data point (y, s^2) derived from the tests of H_0 vs H_1 and H_0 vs H_2 are then respectively given by

$$(1) \qquad (\nu + 1)q_1 \leqq \frac{y - m}{z} \leqq (\nu + 1)q_2 \, ,$$

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and

$$(\nu + 1)q_0 \leqq \frac{y - m}{z} . \tag{2}$$

Here z, q_1, q_2, q_0 satisfy

$$z = [s^2 + (y - m)^2]^{\frac{1}{2}}/(\nu + 1) , \tag{3}$$

$$\int_{q_1}^{q_2} h(v)\, dv = (1 - \alpha) \int_{-1}^{1} h(v)\, dv , \tag{4}$$

$$\int_{q_1}^{q_2} v h(v)\, dv = (1 - \alpha) \int_{-1}^{1} v h(v)\, dv , \tag{5}$$

$$\int_{q_0}^{1} h(v)\, dv = (1 - \alpha) \int_{-1}^{1} h(v)\, dv \tag{6}$$

in which $h(v) = (1 - v^2)^{\nu/2-1} \exp(\zeta v)$, $\zeta = \beta z$, $\beta = -\lambda(\nu + 1)$, $-1 \leqq v \leqq 1$. The problem is to show that these confidence sets are always intervals (the intervals being one-sided for the set defined by (2)). In equations (4) to (6), q_i for $i = 0, 1, 2$, are for given α, functions of z. Let $p_i(z) = zq_i$, $i = 0, 1, 2$. The conditions that $|p_i(z)| \leqq 1$ for $i = 1, 2$, and for $i = 0$ are respectively sufficient for the confidence sets defined by (1) and (2) to be intervals and they are satisfied provided $\nu \geqq 2$ ([3], [2]). But for $\nu = 1$ this argument fails as the conditions in question are not always satisfied. (A proof of this assertion for the one-sided case is in Section 6 of [2].) The argument in this note consists of (i) obtaining a weaker condition which for $\nu \geqq 1$ is sufficient for the confidence sets in question to be intervals and (ii) verifying that it is satisfied for $\nu = 1$. The verification requires protracted computations which are outlined. (The weaker sufficient condition is also satisfied when $\nu \geqq 2$ as may easily be verified.)

3. Main result.

PROPOSITION 3.1. *For $\nu = 1$, the confidence sets defined by* (1) *are intervals and those defined by* (2) *are one-sided intervals open at the lower end.*

PROOF. We shall deal only with the general case and exclude trivial special cases by specifying that $\alpha \neq 0$ or 1 and $\lambda \neq 0$. Hence $\beta \neq 0$. The cases $\beta > 0$ and $\beta < 0$ being symmetrical it suffices to prove the result for $\beta > 0$. Let for $i = 0, 1, 2$

$$f_i(\zeta) = \zeta^2(1 - q_i^2) , \tag{7}$$

where $q_i = q_i(\zeta)$ are the functions of ζ, defined for fixed α, $0 < \alpha < 1$, for $i = 1, 2$ by (4) and (5) and for $i = 0$ by (6), and ζ varies in $(0, \infty)$ by (3) since $\beta > 0$. For convenience of presentation the following argument is divided into independent lemmas.

LEMMA 3.1. *For $\nu \geqq 1$,* (i) *if the functions $f_i(\zeta)$ in* (7), *$i = 1, 2$, increase strictly with ζ for $\zeta \in (0, \infty)$, then the confidence set for m defined by* (1) *is an interval,* (ii) *if $f_0(\zeta)$ increases strictly with ζ for $\zeta \in (0, \infty)$, the confidence set defined by* (2) *is a one-sided interval open at the lower end.*

PROOF. Consider the set of values of m which satisfy the first inequality in (1):

$$(\nu + 1)q_1 \leqq \frac{y - m}{z} . \tag{8}$$

At the boundary point of this set $(\nu + 1)q_1 = (y - m)/z$ and hence $f_1(\zeta) = K$ where

$$K = s^2\beta^2(\nu + 1)^{-2} . \tag{9}$$

From equations (4) to (6), it can be shown that for $i = 1, 2$ and 0, as $\zeta \to \infty$, $1 - q_i(\zeta) = 0(1/\zeta)$; hence as ζ increases in $(0, \infty)$ $f_i(\zeta)$ increases strictly from 0 to ∞. In particular, for any nonnegative K, $f_1(\zeta) = K$ has a unique solution, ζ_1^0 say. Let $q_1^0 = q_1(\zeta_1^0)$. The boundary value m_1^0 of the set in (8) is given by

$$(\nu + 1)q_1^0 = \frac{y - m_1^0}{z} . \tag{10}$$

Let m' denote any value of m such that $m' \neq m_1^0$, and let z' and q_1' denote the corresponding values for given y, s^2 of z and q_1 respectively. The following argument shows that the set defined by (8) is identical with the set $m \leqq m_1^0$. Consider points $m' < m_1^0$. Either (i) $|y - m'| > |y - m_1^0|$ or (ii) $|y - m'| \leqq |y - m_1^0|$. If (i) holds then $\zeta' > \zeta_1^0$ by (3), hence since $f_1(\zeta)$ is strictly increasing $\zeta'^2(1 - q_1'^2) > \zeta_1^{02}(1 - q_1^{02}) = K$ and hence by (9), $q_1' < (1 - K/\zeta'^2)^{\frac{1}{2}} = (y - m')/z \cdot 1/(\nu + 1)$ as alternative (i) with $m' < m_1^0$ implies that $y - m' > 0$. If (ii) holds, then $\zeta' \leqq \zeta_1^0$. Alternative (ii) with $m' < m_1^0$ implies that $y - m_1^0 < 0$ and hence by (10) $q_1^0 < 0$. As by (4) and (5), $q_1(\zeta)$ increases strictly with ζ, $q_1' \leqq q_1^0 < 0$. Hence the strict increasing of $f_1(\zeta)$ yields that $\zeta'^2(1 - q_1'^2) \leqq \zeta_1^{02}(1 - q_1^{02}) = K$ which as $q_1' < 0$, yields $q_1' \leqq -(1 - K/\zeta'^2)^{\frac{1}{2}} \leqq (y - m')/z' \times 1/(\nu + 1)$ in which at most one of the two equality signs holds. Thus whether the alternative (i) holds or (ii), all points $m' < m_1^0$ belong to the set in (8). By the converse argument no $m' > m_1^0$ belongs to the set, thus proving that the set is given by $m \leqq m_1^0$.

The set defined by the second inequality in (1) is similarly shown to be the set $m \geqq m_2^0$ where

$$(\nu + 1)q_2^0 = \frac{y - m_2^0}{z} , \qquad q_2^0 = q_2(\zeta_2^0) , \tag{11}$$

ζ_2^0 being the unique value of ζ satisfying $f_2(\zeta) = K$ where K is given by (9). Since for each ζ, $q_1(\zeta) < q_2(\zeta)$, $f_1(\zeta) > f_2(\zeta)$ which implies $\zeta_1^0 < \zeta_2^0$ and hence $q_1^0 = q_1(\zeta_1^0) < q_2(\zeta_1^0) < q_2(\zeta_2^0) = q_2^0$. It follows by (10) and (11) that $m_2^0 < m_1^0$. Thus the confidence set defined by (1) is the interval $m_2^0 \leqq m \leqq m_1^0$.

The proof of (ii) of the lemma follows from the above argument.

LEMMA 3.2. *For* $\nu \geqq 1$, $f_i(\zeta)$ *in* (7) *for* $i = 1, 2$ *increase strictly with* ζ, *for* $\zeta > 0$ *if the following inequality always holds*:

$$q_1(1 - q_2^2)h(q_2) - q_2(1 - q_1^2)h(q_1) < 0 . \tag{12}$$

PROOF. Differentiating both sides of (4) with respect to ζ and using (5) gives $q_1'h(q_1) = q_2'h(q_2)$ where the dash indicates the derivative with respect to ζ. Using this result and (4) after differentiating (5) yields $q_i' = [(q_2 - q_1)h(q_i)]^{-1}\{\int_{q_1}^{q_2}(1 - v^2)h(v)\,dv - (1-\alpha)\int_{-1}^{1}(1 - v^2)h(v)\,dv\}$ which by partial integration in the right-hand side taking the antiderivative of $\exp(\zeta v)$ yields after some reduction

$$q_i' = \frac{1}{\zeta}[(q_2 - q_1)h(q_i)]^{-1}\{(1 - q_2^2)h(q_2) - (1 - q_1^2)h(q_1)\} \tag{13}$$

for $i = 1, 2$. Since $f_i'(\zeta) = 2\zeta(1 - q_i q_i')$ by (7), substituting for q_i' by (13) yields after some reduction that $f_i'(\zeta) > 0$ for $\zeta > 0$ if (12) holds, which proves the lemma.

NOTE 3.1. The argument up to this point holds for $\nu \geqq 1$. The further argument is restricted to the case of interest $\nu = 1$. (But the results can be shown to hold for $\nu \geqq 2$ also.)

LEMMA 3.3. *For $\nu = 1$, the inequality (12) holds for all $\zeta > 0$.*

PROOF. (12) obviously holds if $q_1 \leqq 0$. For $q_1 > 0$, $q_2 > 0$. Hence, dividing (12) by $q_1 q_2$, it suffices to prove that for $q_1 > 0$,

$$\frac{1}{q_2}(1 - q_2^2)h(q_2) - \frac{1}{q_1}(1 - q_1^2)h(q_1) < 0 \tag{14}$$

in which $h(v) = (1 - v^2)^{-\frac{1}{2}}\exp(\zeta v)$. In (4) and (5) q_i for $i = 1, 2$ were treated so far as functions of ζ for given α. In the following q_1 and ζ are treated as independent variates, q_2 and α being their functions defined by (4) and (5). Let $F(q_1, \zeta)$ denote the left-hand side of (14). Partial integration of $\int_{q_1}^{q_2}(1/v)(1 - v^2)^{\frac{1}{2}}\exp(\zeta v)\,dv$ by taking the antiderivative of $\exp(\zeta v)$ yields

$$F(q_1, \zeta) = \int_{q_1}^{q_2}\frac{1}{v^2}[\zeta(v - v^3) - 1]h(v)\,dv\,. \tag{15}$$

Since $v - v^3 \leqq 2(3)^{-\frac{3}{2}}$, $F(q_1, \zeta) < 0$ if $\zeta \leqq \frac{1}{2}(3)^{\frac{3}{2}}$. The proof is completed by showing that $\partial F/\partial\zeta < 0$ for $\zeta > \frac{1}{2}(3)^{\frac{3}{2}}$. Let

$$A(\zeta) = \int_{-1}^{1} vh(v)\,dv \div \int_{-1}^{1} h(v)\,dv\,. \tag{16}$$

By standard formulae $A(\zeta) = I_1(\zeta)/I_0(\zeta)$, where I_0 and I_1 are the modified Bessel functions of orders 0 and 1. Eliminating α between (4) and (5) gives

$$\int_{q_1}^{q_2}(v - A)h(v)\,dv = 0\,. \tag{17}$$

Since $0 < \alpha < 1$, (17) implies

$$-1 < q_1 < A(\zeta) < q_2 < 1\,. \tag{18}$$

Partial differentiation of (17), with respect to ζ for fixed q_1, yields on using (18)

$$(q_2 - A)h(q_2)\frac{\partial q_2}{\partial\zeta} = \int_{q_1}^{q_2}(A^2 + A' - v^2)h(v)\,dv\,. \tag{19}$$

Differentiating (15) with respect to ζ for fixed q_1, substituting for $\partial q_2/\partial\zeta$ by (19) and using that $q_2^{-2}(q_2 - A)^{-1}[\zeta(q_2 - q_2^3) - 1] = -\zeta + q_2^{-2}(q_2 - A)^{-1}[\zeta q_2(1 - Aq_2) - 1]$, yields after some reduction

$$(20)\qquad \frac{\partial F}{\partial \zeta} = [\zeta(1 - A^2 - A') - A] \int_{q_1}^{q_2} h(v)\,dv + q_2^{-2}(q_2 - A)^{-1}[\zeta q_2(1 - Aq_2) - 1] \int_{q_1}^{q_2} (A^2 + A' - v^2)h(v)\,dv\,.$$

Since $\int_{q_1}^{q_2} v^2 h(v)\,dv < (1 - \alpha) \int_{-1}^{1} v^2 h(v)\,dv$ (cf. Lemma in [3]), in the second term in the right-hand side of (20), by (4) and (16)

$$\int_{q_1}^{q_2} (A^2 + A' - v^2)h(v)\,dv > (1 - \alpha) \int_{-1}^{1} (A^2 + A' - v^2)h(v)\,dv = (1 - \alpha)\frac{\partial}{\partial\zeta} \int_{-1}^{1} (A - v)h(v)\,dv = 0\,.$$

For given A, $t(1 - At)$ decreases as t increases for $t \geqq 1/2A$. From the tables of the modified Bessel functions (Table 9.8, [1]) for $\zeta = 2.5$, $A(\zeta) = .765 > 1/2A(\zeta)$. Also $A(\zeta)$ is an increasing function of ζ and $q_2 > A(\zeta)$ by (18). Hence for $\zeta \geqq 2.5$, $\zeta q_2(1 - Aq_2) < \zeta A(1 - A^2) < 1$ by Lemma 3.4. Thus the second term in the right-hand side of (20) is negative, and its first term vanishes by Lemma 3.5. Hence $\partial F/\partial\zeta < 0$ for $\zeta \geqq 2.5$. As $2.5 < \frac{1}{2}(3)^{\frac{3}{2}}$ this proves the lemma.

LEMMA 3.4. *$A(\zeta) = I_1(\zeta)/I_0(\zeta)$, where I_0 and I_1 are the modified Bessel functions of orders 0 and 1, satisfies for all ζ, $\zeta A(1 - A^2) < 1$ where $A = A(\zeta)$.*

PROOF. Let $\phi(\zeta) = \zeta A(1 - A^2)$. By the asymptotic formulae for I_0 and I_1 (Formulae 9.7.1, [1]), $\phi(\zeta) = 1 - 1/2|\zeta| + O(1/\zeta^2) < 1$ for large $|\zeta|$. Since $\phi(0) = 1$, $\phi(\zeta) \geqq 1$ for some ζ iff it has a local maximum $\geqq 1$. Let ϕ_0 be a local maximum of ϕ attained for $\zeta = \zeta_0$. Let $[A(\zeta_0)] = A_0$. Since $\phi'(\zeta_0) = 0$, differentiation of $\log\phi(\zeta)$ at $\zeta = \zeta_0$ gives $\zeta_0 A'(\zeta_0) = A_0(1 - A_0^2)(3A_0^2 - 1)^{-1}$. Substituting $[\zeta_0(1 - A_0^2) - A_0]$ for $\zeta_0 A'(\zeta_0)$ by Lemma 3.5, and multiplying both sides by A_0 gives $\phi_0 = 2A_0^4(3A_0^2 - 1)^{-1}$. The quadratic in A_0^2, $2A_0^4 - 3A_0^2 + 1 = 0$ has the roots $\frac{1}{2}$ and 1. Hence if $\frac{1}{2} < A_0^2 < 1$, $\phi_0 < 1$. By (16) $A_0(\zeta) < 1$. If $A_0^2 \leqq \frac{1}{2}$, $A_0 < .765 = A(2.5)$, so that $|\zeta_0| < 2.5 < \frac{1}{2}(3)^{\frac{3}{2}}$. Since $A_0 - A_0^3 \leqq 2(3)^{-\frac{3}{2}}$, $\phi_0 < 1$ for $A_0^2 \leqq \frac{1}{2}$ also. Thus for all ζ_0, $\phi_0 < 1$ which proves the lemma.

LEMMA 3.5. *The function $A(\zeta)$ in Lemma 3.4 satisfies $\zeta(1 - A^2 - A') = A$ where A' is the derivative of $A(\zeta)$.*

PROOF. Differentiation of $A(\zeta)$ yields $1 - A^2 - A' = \int_{-1}^{1} vh(v)\,dv / \int_{-1}^{1} h(v)\,dv$. Partial integration in the numerator by taking the antiderivative of $\exp(\zeta v)$ proves the result.

LEMMA 3.6. *For $\nu = 1$ and $\zeta > 0$, $f_0(\zeta)$ in (7) increases strictly with ζ if the following inequality always holds*:

$$(21)\qquad (1 - q_0^2)^{\frac{1}{2}} \exp(\zeta q_0) - q_0\zeta \int_{q_0}^{1} (v - A)h(v)\,dv > 0\,.$$

PROOF. Differentiation of (6) gives $q_0'h(q_0) = \int_{q_0}^1 (v - A)h(v)\,dv$. Using this, $f_0'(\zeta) =$ left-hand side of (21), multiplied by $2\zeta[h(q_0)]^{-1}$ which proves the result.

LEMMA 3.7. *For* $\nu = 1$, (21) *holds for all* $\zeta > 0$.

PROOF. In (21)

$$\int_{q_0}^1 (v - A)h(v)\,dv > 0 \tag{22}$$

as the integral in (22) is greater than $(q_0 - A) \int_{q_0}^1 h(v)\,dv$ if $q_0 > A$ and than $\int_{-1}^1 (v - A)h(v)\,dv = 0$ if $q_0 \leqq A$. Thus (21) holds if $q_0 \leqq 0$. For $q_0 > 0$, (21) is equivalent to

$$\int_{q_0}^1 \frac{1}{v^2}[1 - \zeta(v - Av^2)]h(v)\,dv > 0\,. \tag{23}$$

(23) is obtained by partial integration of $\int_{q_0}^1 (1 - v^2)^{\frac{1}{2}}/v \exp(\zeta v)\,dv$, by taking the antiderivative of $\exp(\zeta v)$ and making some reduction. Treating q_0 and ζ as independent variates in (6), let $F(q_0, \zeta)$ denote the left-hand side of (23). Since $v - Av^2 \leqq 1/4A$, (23) holds for $\zeta \leqq \zeta_0$ where $\zeta_0 = 1/4A(\zeta_0)$. The proof is completed by showing that $\partial F/\partial\zeta > 0$. By (23), $\partial F/\partial \zeta = A \int_{q_0}^1 h(v)\,dv + \zeta \int_{q_0}^1 (A' - 1 + Av)\,dv$, which on substituting in the first term $\zeta(1 - A^2 - A')$ for A by Lemma 3.5 reduces to $\int_{-1}^1 (v - A)h(v)\,dv > 0$ by (22), thus proving the result.

And Lemmas 3.1 to 3.7 combined prove the Proposition 3.1.

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