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PROPERTIES OF GENERALIZED SEQUENTIAL PROBABILITY RATIO TESTS

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We consider generalized sequential probability ratio tests (GSPRT's), which are not necessarily based on independent or identically distributed observations, to distinguish between probability measures P and Q. It is shown that if T is any test in a wide class of GSPRT's, including all SPRT's, and T' is any rival test possessing error probabilities and sample sizes no greater than those of T, then T' must be equivalent to T. This notion of optimality of T is weaker than that of Kiefer and Weiss but the results are stronger than theirs. It is also shown that, if an SPRT T' has at least one error probability strictly less than that of another SPRT T with the other error probability no larger, T' requires strictly more observations than T some of the time, under both P and Q, and never fewer observations. This assertion generalizes Wijsman's conclusions. The methods used in this paper are quite general, and are different from those of the earlier authors.

1. Introduction. There now exist many important examples of sequential probability ratio tests (SPRT's) for which the classical i.i.d. model is inappropriate. These include a number of invariant sequential tests, such as the sequential t, t^2, χ^2, F, T^2 and rank tests (cf., Hall, Wijsman and Ghosh (1965), and B. K. Ghosh (1970)). More recently, Robbins and Siegmund (1974) have introduced a family of SPRT's in a sequential design context. Even though all of these examples arise in the presence of random samples, the i.i.d. model (defined explicitly below) is inappropriate because the likelihood ratios, employed in these tests, are not defined *directly* from random samples. For the most part, the properties of these tests have been investigated individually. In contrast, the objective of this paper is to examine what can and cannot be said about SPRT's in general, when all assumptions about how the data is generated and utilized are dropped. Most of the results are more generally described for generalized sequential probability ratio tests (GSPRT's), which are defined formally in Section 2. Such generality should be of some interest because there are a number of GSPRT's recommended in the literature (e.g., Armitage (1957) and Anderson (1960)) including at least one for which the i.i.d. model is unsuitable (Meyers, Schneiderman and Armitage (1966)).

Specifically, the phrase "i.i.d. model" ("independent model") will be used to

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describe the situation in which the potential data is i.i.d. (independent) under each of two probability measures P and Q, and, for each $n \geqq 1$, the nth likelihood ratio is the likelihood ratio of the first n observations of the potential data, as opposed to being the likelihood ratio of some convenient statistic (e.g., a maximal invariant) of the first n observations. The phrase "general model," whose full generality is described in Section 2, includes the i.i.d. and independent models within its purview as well as situations in which the potential data has arbitrary specified distributions under P and Q.

The Wald–Wolfowitz (1948) optimality property holds for any SPRT under the i.i.d. model, but not in general under the other two models. It says that one cannot reduce either error probability of an SPRT without resorting to a test of larger expected sample size (under both hypotheses). B. K. Ghosh (1970) has described a weaker form of optimality which holds for any SPRT under the general model. His optimality states that for any given SPRT there is no other SPRT with smaller error probabilities and a smaller expected sample size. Simons (1974) refines Ghosh's results and circumvents an error in his proof (cf., page 88, line 12). This paper includes and extends Simons' result (Corollary 3.2) and simplifies his proof.

Following Kiefer and Weiss (1957), a sequential test for deciding between P and Q, whose stopping variable is N and whose error probabilities are α and β, will be called *inadmissible* if there exists an alternative test T', with stopping variable N' and error probabilities α' and β', such that

$$\alpha' \leqq \alpha\,, \qquad \beta' \leqq \beta\,, \qquad P(N' > n) \leqq P(N > n)\,,$$
$$Q(N' > n) \leqq Q(N > n)\,, \quad n = 1, 2, \cdots,$$

with at least one of these inequalities strict. T is *admissible* if no such T' exists. Weiss (1953) and Le Cam (1954) show that, under this notion of admissibility, the family of closed GSPRT's is an essentially complete class for the independent model when none of the likelihood ratios assumes a single value (including 0 or ∞) with positive (P or Q) probability. The result is more generally true if the family of GSPRT's is expanded by the introduction of a mild form of randomization (cf., Le Cam (1954)). Their restriction to closed tests (tests which stop with probability one under both measures) can be removed. Their result cannot be extended to the general model, as our Counterexample 1 shows. Such an extension would appear to be ruled out by the fact that the sequence of likelihood ratios, which is always transitive under the independent model, need not be transitive under the general model. (For a definition and discussion of transitivity, see Bahadur (1954).) When transitivity fails to hold, it is not even clear that one should restrict one's attention to stopping rules which depend only on the values in the sequence of likelihood ratios. However, since transitivity holds in Counterexample 1, there is no hope that the Weiss–Le Cam result could hold even for a "transitive model."

A test T will be called *strongly inadmissible* if there exists an alternative test

T' which not only improves upon T in the sense of admissibility (i.e., it renders T inadmissible) but stops no later than T does under P and Q. T is *weakly admissible* if no such T' exists. While Counterexample 1 shows that not every SPRT is admissible, Corollary 2.1 below shows that every SPRT is weakly admissible.

Not every GSPRT is weakly admissible. However, a GSPRT is weakly admissible under the general model if $0 \leqq A_{n+1} \leqq A_n \leqq B_n \leqq B_{n+1} \leqq \infty$ for each n, where the A_n's (B_n's) are the lower (upper) boundaries of the test.

Kiefer and Weiss (1957) demonstrate conclusively that no simple characterization of admissibility is possible under the i.i.d. model. We suspect that the situation is no better for weak admissibility under the general model. Counterexample 1 demonstrates that, in at least one respect, the situation is worse. One must be concerned with weakly admissible tests which are not GSPRT's (or even nearly so). This means, of course, that the sufficient conditions for weak admissibility mentioned in the previous paragraph are not necessary. However, Theorem 4 states that, for a GSPRT to be weakly admissible, it must be equivalent to a GSPRT for which $\sup A_n \leqq \inf B_n$, where $\sup A_n$ is the supremum of the lower boundaries of the GSPRT and $\inf B_n$ is the infimum of the upper boundaries. This same condition is shown by Kiefer and Weiss to be necessary for admissibility under the i.i.d. model. Their result follows from Theorem 4.

The condition $\sup A_n \leqq \inf B_n$ occurs as well in Theorem 1, which essentially states that $c(\alpha' - \alpha) + (\beta' - \beta) \geqq 0$ for each real $c \in [\sup A_n, \inf B_n]$ (and somewhat more), where α and β are the error probabilities of the GSPRT, and α' and β' are the error probabilities of any competing test which stops no later than the GSPRT. This result generalizes in several directions Theorem 2 of Wijsman (1960) and shows that the competing test can improve upon one of the GSPRT's error probabilities only at the expense of the other. Actually, Wijsman's theorem claims, but his proof does not support (cf., Wijsman (1975)), *strict* inequalities. The possibility of equality is demonstrated in our Counterexample 2. Wijsman (1963) leaves open this possibility in a subsequent paper (Theorem 3), but in a more general context.

If $\sup A_n < \inf B_n$, $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$, then $\alpha' = \alpha$, $\beta' = \beta$ and the two tests make the same decision (or "indecision" when sampling continues forever). This is described in Theorem 2.

It should be pointed out that J. K. Ghosh (1960) is probably the first one to study GSPRT's under the general model. He shows that every GSPRT is an unbiased test. Lemma 2 generalizes his result.

It seems appropriate to announce a simple but useful result, which is in the spirit of this paper, concerning the question of closure. For any GSPRT, under the general model, the following system of inequalities holds:

$$A_n P(N > n) \leqq Q(N > n) \leqq B_n P(N > n)\,, \qquad n \geqq 1\,. \tag{1}$$

These inequalities imply (when $\inf A_n > 0$ and $\sup B_n < \infty$) that either a GSPRT stops with probability one under both probability measures or under neither one. Similar implications follow concerning the finiteness of the moments of N. Results of this type will be the subject of another paper.

2. Preliminaries. Let P and Q be fixed probability measures on a measurable space $(\Omega, \mathscr{F})$. Usually at this point, one introduces a sequence of random variables $X_1, X_2, \cdots$, representing potential data, and defines a sequence of likelihood ratios $\lambda_n = q_n(X_1, \cdots, X_n)/p_n(X_1, \cdots, X_n)$, $n \geqq 1$, where p_n and q_n are densities for $X_1, \cdots, X_n$, with respect to a common measure, under P and Q respectively. However, this model is too restrictive since it does not (naturally) allow for likelihood ratios based on maximal invariants (as with invariant sequential tests) nor allow for sequentially designed experiments (as with Robbins and Siegmund (1974)). Savage and Savage (1965) have solved this problem in a particular instance by working with "data" of increasing dimensionality. A general solution is obtained by introducing the following "general model": Let $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \cdots \subset \mathscr{E}_\infty \subset \mathscr{F}$ be a nested sequence of σ-fields where $\mathscr{E}_n$ represents the "information" available, or to be used, at the nth stage of an experiment. The likelihood ratio λ_n associated with the nth stage of an experiment is defined directly in terms of $\mathscr{E}_n$: An extended nonnegative random variable λ will be called a *likelihood ratio* for a σ-field $\mathscr{E} \subset \mathscr{F}$ if it is $\mathscr{E}$-measurable and

$$Q(E, \lambda \neq \infty) = \int_E \lambda \, dP, \qquad P(E, \lambda \neq 0) = \int_E \lambda^{-1} \, dQ, \qquad E \in \mathscr{E}, \tag{2}$$

where $\lambda^{-1} = 0$ when $\lambda = \infty$. It is easy to verify that λ, which is little more than a Radon–Nikodym derivative, always exists and is unique up to a P and Q equivalence. Moreover, if $\mathscr{E}$ is generated by a random mapping X, then λ becomes the likelihood ratio of X in the usual ratio of densities sense.

A *stopping variable* N is an extended random variable taking values in the set $\{1, 2, \cdots, \infty\}$ such that the event $[N = n] \in \mathscr{E}_n$ for each $n = 1, 2, \cdots, \infty$. Denote by $\mathscr{E}_n$ the σ-field of events up to time N, that is of events E such that $E \cap [N = n] \in \mathscr{E}_n$ for each n. It easily follows that the extended random variable λ_N, which agrees with λ_n on $[N = n]$, is in fact the likelihood ratio of $\mathscr{E}_N$.

A *decision* D is a random mapping taking values in the "action space" $\{P, Q, \Delta\}$ such that the event $[D = a] \in \mathscr{E}_N$ for each action a, and such that $[D = \Delta] = [N = \infty]$. The occurrence of the event $[D = P]$ ($[D = Q]$) will be interpreted to mean that the null (alternative) hypothesis, that $P(Q)$ is the true probability measure, has been accepted. The occurrence of the event $[D = \Delta]$ simply means that neither hypothesis is accepted.

A *test* T is a pair (N, D). The *error probabilities* for T are defined by $\alpha = P(D = Q)$ and $\beta = Q(D = P)$. Two tests $T = (N, D)$ and $T' = (N', D')$ are said to be *equivalent* if $P(N = N', D = D') = Q(N = N', D = D') = 1$.

A *generalized sequential probability ratio test* (GSPRT) is a test $T = (N, D)$ of the form:

$$\begin{aligned} N &= \text{first} \quad n \geqq 1 \quad \text{such that} \quad \lambda_n \notin (A_n, B_n)\,, \\ &= \infty \quad \text{if no such} \quad n \quad \text{exists,} \end{aligned}$$

$$\begin{aligned} D &= P \quad \text{on} \quad [N < \infty, \lambda_N \leqq A_N]\,, \qquad \text{and} \\ &= Q \quad \text{on} \quad [N < \infty, \lambda_N \geqq B_N]\,, \end{aligned}$$

where A_n and B_n, $0 \leqq A_n \leqq B_n \leqq \infty$, are fixed constants for $n = 1, 2, \cdots$. D is defined as P if $\lambda_N = A_N = B_N = 0$ and as Q if $\lambda_N = A_N = B_N = \infty$. If $0 < \lambda_N = A_N = B_N < \infty$, D can be defined as P or Q in an arbitrary manner, or by randomization. (The restriction $[D = a] \in \mathscr{E}_N$ is no barrier to randomization.) This test will be denoted as $S(A_n, B_n)$ for short. If each $A_n = A$ and each $B_n = B$, with $A < B$, the test $T = S(A, B)$ is the *sequential probability ratio test* (SPRT).

For the remainder of this paper $T = (N, D)$ denotes a GSPRT, unless otherwise stated, and, when relevant, $T' = (N', D')$, with error probabilities α' and β', denotes a competing test.

3. Theory. There is a wealth of information contained in (2). For instance, if $T = (N, D)$ is the GSPRT $S(A_n, B_n)$, then $A_n < \lambda_n < B_n$ when $N > n$, and (1) is immediately obtainable upon setting $\mathscr{E} = \mathscr{E}_n$ and $E = [N > n]$ in (2). Alternatively, if T is the SPRT $S(A, B)$ and one sets $\mathscr{E} = \mathscr{E}_N$ and $E = [D = P]$, then $\lambda_N \leqq A$ on E and the well-known inequality $\beta \leqq A(1 - \alpha)$ directly follows. The related inequality $\alpha \leqq B^{-1}(1 - \beta)$ is similarly derivable. In fact, it is possible to derive the celebrated "fundamental identity" of sequential analysis from (2) (cf., Bahadur (1958)). The identities of (2) are at the heart of the following lemma.

LEMMA 1. *Let* $T = (N, D)$ *be the* GSPRT $S(A_n, B_n)$. *Then for each event* $E \in \mathscr{E}_N$,

(4) $$Q(E \cap [D = Q] \cap [\lambda_N \neq \infty]) \geqq (\inf B_n) P(E \cap [D = Q])$$

and

(5) $$P(E \cap [D = P] \cap [\lambda_N \neq 0]) \geqq (\sup A_n)^{-1} Q(E \cap [D = P])\,.$$

REMARK. With this lemma and the theory which follows, the product $\infty \cdot c$ will be interpreted as ∞ for $c > 0$, as 0 for $c = 0$ and $-\infty$ for $c < 0$. For instance, if each $B_n = \infty$, (4) implies $\alpha = P(D = Q) = 0$.

PROOF. Let $\mathscr{E} = \mathscr{E}_N$ in (2). Then $E \cap [D = Q] \in \mathscr{E}$ and

$$\begin{aligned} Q(E \cap [D = Q] \cap [\lambda_N \neq \infty]) &= \int_{E \cap [D=Q]} \lambda_N \, dP \\ &\geqq (\inf B_n) P(E \cap [D = Q])\,. \end{aligned}$$

The proof of (5) is similar. □

THEOREM 1. *Let* $T = (N, D)$ *be the* GSPRT $S(A_n, B_n)$ *and* $T' = (N', D')$ *be any other test for which* $N' \leqq N$ a.s. *under* P *and* Q. *Further, suppose* $\sup A_n \leqq \inf B_n$. *Then for every* $c \in [\sup A_n, \inf B_n]$,

(6a) $$c(\alpha' - \alpha - p) + (\beta' - \beta - q) \geqq 0$$

and

(6b) $$(\alpha' - \alpha - p) + c^{-1}(\beta' - \beta - q) \geqq 0\,,$$

where

$$p = P(D = \Delta, D' = Q) + P(D = P, \lambda_N = 0, D' = Q)\,,$$
$$q = Q(D = \Delta, D' = P) + Q(D = Q, \lambda_N = \infty, D' = P)\,.$$

REMARK. Because of the convention described in the previous remark, the two inequalities (6a) and (6b) are not always equivalent. Inequality (6b) is more informative than (6a) when $c = 0$ (so that $c^{-1} = \infty$), and (6a) is the more informative when $c = \infty$ (so that $c^{-1} = 0$).

PROOF. $\alpha = P(D = Q) = P(D = D' = Q) + P(D = Q \neq D')$ and $\alpha' = P(D' = Q) = P(D = D' = Q) + P(D \neq Q = D')$. Since $N' \leqq N$, $D' \neq \Delta$ when $D = Q$. Thus

$$\alpha' - \alpha = P(D = P, D' = Q) + P(D = \Delta, D' = Q) - P(D = Q, D' = P) \tag{7}$$

and

$$\alpha' - \alpha - p = P(D = P, \lambda_N \neq 0, D' = Q) - P(D = Q, D' = P)\,. \tag{8}$$

Likewise

$$\beta' - \beta = Q(D = Q, D' = P) + Q(D = \Delta, D' = P) - Q(D = P, D' = Q) \tag{9}$$

and

$$\beta' - \beta - q = Q(D = Q, \lambda_N \neq \infty, D' = P) - Q(D = P, D' = Q)\,. \tag{10}$$

Set $E = [D' = P]$ in (4) and $E = [D' = Q]$ in (5). Then (4) and (5) yield

$$(\beta' - \beta - q) + Q(D = P, D' = Q) \geqq (\inf B_n)P(D = Q, D' = P)$$

and

$$(\alpha' - \alpha - p) + P(D = Q, D' = P) \geqq (\sup A_n)^{-1}Q(D = P, D' = Q)\,,$$

from which (6a) easily follows for $0 \leqq c < \infty$, and (6b) follows for $0 < c = \infty$. Now suppose $c = \infty$. Then necessarily $\inf B_n = \infty$, $\alpha = 0$ and from (8) $\alpha' - \alpha - p = P(D = P, \lambda_N \neq 0, D' = Q) \geqq 0$. But when $\alpha' - \alpha - p > 0$, (6a) is obvious (for $c = \infty$). On the other hand, if $\alpha' - \alpha - p = 0$, then $P(D = P, \lambda_N \neq 0, D' = Q) = 0$ and, using (2) for $\mathscr{E} = \mathscr{E}_N$,

$$Q(D = P, D' = Q) = Q(D = P, D' = Q, \lambda_N \neq \infty) = \int_{[D=P,D'=Q]} \lambda_N \, dP = 0\,.$$

Consequently, (10) becomes $\beta' - \beta - q = Q(D = Q, \lambda_N \neq \infty, D' = P) \geqq 0$, and, again, (6a) follows. A similar proof establishes (6b) when $c = 0$. □

Theorem 1 has the following two simple but important corollaries:

COROLLARY 1.1. *Under the assumptions of Theorem* 1, *if* $\alpha' \leqq \alpha$ *and* $\beta' \leqq \beta$, *then* $\alpha' = \alpha$ *and* $\beta' = \beta$.

COROLLARY 1.2. *If, in addition to the assumptions of Theorem* 1, $\sup A_n \leqq 1 \leqq \inf B_n$, *then*

$$\alpha' + \beta' \geqq \alpha + \beta\,. \tag{11}$$

It is pointed out in the introduction that Theorem 1 substantially generalizes a theorem of Wijsman's (1960). In addition, it is a close analog, in a more general setting, of Simons' (1974) Proposition 1. Corollary 1.1 can be used to circumvent the error, previously referred to, in B. K. Ghosh's (1970) proof. (Alternatively, his proof can be patched up with the use of somewhat stronger assumptions.) Corollary 3.2 below generalizes his theorem. Corollary 1.1 means, among other things, that, under the stated assumptions,

$$\alpha' < \alpha \Rightarrow \beta' > \beta \qquad \text{and} \qquad \beta' < \beta \Rightarrow \alpha' > \alpha .$$

Counterexample 3 shows that these implications need not hold if the assumption $\sup A_n \leqq \inf B_n$ is dropped. Note that, if $P(N = n) = Q(N = n) = 1$, Corollary 1.1 reduces to the Neyman–Pearson lemma.

The assertions of Corollary 1.1 do not imply that the competing test T' cannot be superior to the GSPRT T. It is possible that $\alpha' = \alpha$, $\beta' = \beta$, and $N' < N$ with (P and Q) probability one. (See Counterexample 4.) Furthermore, the equalities $\alpha' = \alpha$ and $\beta' = \beta$ (and even $N' \equiv N$) do not imply that T' makes the same decision as T. (See Counterexample 5.) Theorem 2 below addresses itself to the latter issue, and its corollary addresses itself to the former. A lemma is needed first.

LEMMA 2. *Let $T = (N, D)$ be a* GSPRT. *Then for each integer $m \geqq 1$ and any event $E \in [N \geqq m]\mathscr{E}_m$,*

$$Q(E \cap [D = Q])P(E) \geqq P(E \cap [D = Q])Q(E) \tag{12}$$

and

$$P(E \cap [D = P])Q(E) \geqq Q(E \cap [D = P])P(E) . \tag{13}$$

REMARK. This lemma generalizes J. K. Ghosh's (1960) Theorem 1. If $P(E) > 0$ and $Q(E) > 0$, then the conclusions in (12) and (13) can also be expressed in terms of conditional probabilities, i.e., $Q(D = Q \mid E) \geqq P(D = Q \mid E)$ and $P(D = P \mid E) \geqq Q(D = P \mid E)$.

PROOF. It suffices to show for $n \geqq m$ that

$$\gamma_n = P(E \cap [D = P, N \leqq n])Q(E) - Q(E \cap [D = P, N \leqq n])P(E) \geqq 0$$

and

$$\delta_n = Q(E \cap [D = Q, N \leqq n])P(E) - P(E \cap [D = Q, N \leqq n])Q(E) \geqq 0 .$$

In turn, it suffices to show for $n \geqq m$ that

$$\gamma_n Q(E \cap [N \geqq n]) \geqq \gamma_{n-1} Q(E \cap \{[N \geqq n] - C_n\}) + \delta_{n-1} Q(E \cap C_n) \tag{14}$$

and

$$\delta_n P(E \cap [N \geqq n]) \geqq \delta_{n-1} P(E \cap \{[N \geqq n] - D_n\} + \gamma_{n-1} P(E \cap D_n) , \tag{15}$$

where $C_n = [D = P, N = n]$, $D_n = [D = Q, N = n]$ and (since $E \in [N \geqq m]\mathscr{E}_m$) $\gamma_{m-1} = \delta_{m-1} = 0$. Because, if $Q(E \cap [N \geqq n]) > 0$, then the inequalities

$\gamma_{n-1} \geqq 0$, $\delta_{n-1} \geqq 0$ and (14) imply $\gamma_n \geqq 0$; and if $Q(E \cap [N \geqq n]) = 0$, then $\gamma_n = P(E \cap [D = P, N \leqq n])Q(E) - Q(E \cap [D = P, N \leqq n-1])P(E) \geqq \gamma_{n-1} \geqq 0$. In either case, the inequalities $\gamma_{n-1} \geqq 0$ and $\delta_{n-1} \geqq 0$ imply $\gamma_n \geqq 0$. Similarly, they imply $\delta_n \geqq 0$. Now by direct computation,

$$\gamma_n - \gamma_{n-1} = P(E \cap C_n)Q(E) - Q(E \cap C_n)P(E)$$

and

$$\delta_{n-1} - \gamma_{n-1} = P(E \cap [N \geqq n])Q(E) - Q(E \cap [N \geqq n])P(E) .$$

It follows that (14) is equivalent to

$$\text{(16)} \quad Q(E \cap \{[N \geqq n] - C_n\})P(E \cap C_n)Q(E) \geqq P(E \cap \{[N \geqq n] - C_n\})Q(E \cap C_n)Q(E) .$$

But on C_n, $\lambda_n \leqq A_n$ (the lower boundary of the GSPRT at sample size n), and on $[N \geqq n] - C_n$, $\lambda_n \geqq A_n$. Using these facts, (16) follows upon two applications of (2) with $\mathscr{E} = \mathscr{E}_n$:

$$\begin{aligned} Q(E \cap \{[N \geqq n] - C_n\})P(E \cap C_n) &\geqq \int_{E \cap \{[N \geqq n] - C_n\}} \lambda_n \, dP \cdot P(E \cap C_n) \\ &\geqq A_n P(E \cap \{[N \geqq n] - C_n\})P(E \cap C_n) \\ &\geqq P(E \cap \{[N \geqq n] - C_n\}) \int_{E \cap C_n} \lambda_n \, dP \\ &= P(E \cap \{[N \geqq n] - C_n\})Q(E \cap C_n) . \end{aligned}$$

The derivation of inequality (15) is similar. □

THEOREM 2. *Let $T = (N, D)$ be the* GSPRT $S(A_n, B_n)$ *and $T' = (N', D')$ be any other test for which $N' \leqq N$ a.s. under P and Q. Further, suppose $0 \leqq \sup A_n < \inf B_n \leqq \infty$. If $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$ then $P(D' = D) = Q(D' = D) = 1$.*

PROOF. It will be shown that (a) $P(D \neq D' = \Delta) = Q(D \neq D' = \Delta) = 0$, (b) $P(D \neq D' = Q) = Q(D \neq D' = P) = 0$, and (c) $P(D \neq D' = P) = Q(D \neq D' = Q) = 0$. Now (a) is an immediate consequence of the fact that $N' \leqq N$ a.s. under P and Q.

To show (b): From (7) and (9), it is seen that $P(D \neq D' = Q) \leqq P(D = Q, D' = P)$ and $Q(D \neq D' = P) \leqq Q(D = P, D' = Q)$ since $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$. Consequently, $(\inf B_n)Q(D \neq D' = P) \leqq (\inf B_n)Q(D = P, D' = Q)$ (which by Lemma 1) $\leqq (\inf B_n)(\sup A_n)P(D = P, D' = Q) \leqq (\inf B_n)(\sup A_n)P(D = Q, D' = P)$ (which by Lemma 1) $\leqq (\sup A_n)Q(D = Q, D' = P) \leqq (\sup A_n)Q(D \neq D' = P)$. Since $\sup A_n < \inf B_n$, $Q(D \neq D' = P) = 0$. A similar proof shows that $P(D \neq D' = Q) = 0$.

To show (c): To show $Q(D \neq D' = Q) = 0$, it suffices to show that

$$\text{(17)} \quad Q(D = D' = Q, N' = m) = Q(D' = Q, N' = m) , \qquad m = 1, 2, \cdots .$$

Now the event $E = [D' = Q, N' = m] \in [N \geqq m]\mathscr{E}_m$. Thus, by Lemma 2,

$$\text{(18)} \quad Q(D = D' = Q, N' = m)P(D' = Q, N' = m) \geqq P(D = D' = Q, N' = m)Q(D' = Q, N' = m) .$$

But, by part (b), $P(D = D' = Q, N' = m) = P(D' = Q, N' = m)$. Consequently, (18) implies (17) for any m for which $P(D' = Q, N' = m) > 0$. Furthermore, (2) (with $\mathscr{E} = \mathscr{E}_m$) implies (17) for any m for which $P(D' = Q, N' = m) = 0$:

$$\begin{aligned} Q(D = D' = Q, N' = m) & \\ &\geqq Q(D' = Q, \lambda_m = \infty, N' = m) \\ &= Q(D' = Q, N' = m) - Q(D' = Q, \lambda_m \neq \infty, N' = m) \\ &= Q(D' = Q, N' = m) - \int_{[D'=Q,N'=m]} \lambda_m \, dP \\ &= Q(D' = Q, N' = m) . \end{aligned}$$

This shows $Q(D \neq D' = Q) = 0$. The relation $P(D \neq D' = P) = 0$ is shown similarly. □

COROLLARY 2.1. *If, in addition to the assumptions of Theorem 2, the sequence A_n is nonincreasing and the sequence B_n is nondecreasing, then the test T' is equivalent to T.*

PROOF. In view of Theorem 2, it suffices to show (a) $P(D' = P, N > N') = 0$, (b) $Q(D' = P, N > N') = 0$, and (c) $P(D' = Q, N > N') = Q(D' = Q, N > N') = 0$. The proof of (c) is similar to the proofs of (a) and (b). Also, it is a simple consequence of Lemma 3 below that (a) and (b) are equivalent. Consequently, only (a) will be shown. Theorem 2 will be used in several places and (2) will be used twice, once with $\mathscr{E} = \mathscr{E}_{N'}$ and once with $\mathscr{E} = \mathscr{E}_N$:

$$\begin{aligned} 0 = Q(D' = P) - Q(D = P) &\geqq Q(D' = P, \lambda_{N'} \neq \infty) - Q(D = P, \lambda_N \neq \infty) \\ = \int_{[D'=P]} \lambda_{N'} \, dP - \int_{[D=P]} \lambda_N \, dP &= \int_{[D=D'=P,N>N']} (\lambda_{N'} - \lambda_N) \, dP . \end{aligned}$$

But on $[D = D' = P, N > N']$, $\lambda_{N'} > A_{N'} \geqq A_N \geqq \lambda_N$ because of the monotonicity of the A_n sequence. Consequently, $P(D' = P, N > N') = P(D = D' = P, N > N') = 0$. □

Corollary 2.1 shows that *a* GSPRT $S(A_n, B_n)$ *is weakly admissible if* $0 \leqq A_{n+1} \leqq A_n < B_n \leqq B_{n+1} \leqq \infty$ *for* $n \geqq 1$. In particular, *every* SPRT *is weakly admissible*. On the other hand, a fixed-sample likelihood ratio test, treated as a GSPRT, does not satisfy the boundary conditions of Corollary 2.1, and, in fact, it is easily shown that not every fixed-sample likelihood ratio test is weakly admissible.[1]

A stopping variable N will be called *regular* if, for each $n \geqq 1$, $N > n$ implies $\lambda_n \neq 0$ or ∞. Every GSPRT has a regular stopping variable, as does every sensible test.

LEMMA 3. *Let N be a regular stopping variable and N' be any other stopping variable. Then, for any event $E \in [N' < N]\mathscr{E}_{N'}$, $P(E) = 0$ if, and only if, $Q(E) = 0$.*

PROOF. From (2), it easily follows that $P(E) = 0$ implies $Q(E) = 0$ for

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[1] For example, suppose a coin is tossed twice and one is testing p (= probability of a head) $= \frac{1}{2}$ versus $p = \frac{2}{3}$. The test which rejects $p = \frac{1}{2}$ if both tosses are heads is a strongly inadmissible likelihood ratio test; if the first toss is a tail the second toss is unnecessary.

$E \in [\lambda \neq \infty]\mathscr{E}$, and that $Q(E) = 0$ implies $P(E) = 0$ for $E \in [\lambda \neq 0]\mathscr{E}$. Thus, it suffices to observe that $[N' < N]$ is a subevent of $[\lambda_{N'} \neq 0 \text{ or } \infty]$ when N is regular. □

Theorems 1 and 2 discuss a GSPRT $T = (N, D) = S(A_n, B_n)$ and a competing test T' which must stop by the time T does. A different kind of competitor will now be considered. For purposes of introduction, suppose $T' = (N', D')$ is a second GSPRT $S(A_n', B_n')$ with boundaries "above" those of T, that is, with $A_n' \geqq A_n$ and $B_n' \geqq B_n$ for each n. Then, by geometrical considerations,

$$D = P \Rightarrow D' = P \quad \text{and} \quad N' \leqq N, \tag{19}$$

$$D' = Q \Rightarrow D = Q \quad \text{and} \quad N \leqq N'. \tag{20}$$

(The special case $N = N'$, $\lambda_N = A_N = B_N = A_N' = B_N'$, $D = P$, $D' = Q$ must be excluded.) The next theorem discusses competitors $T' = (N', D')$, not necessarily GSPRT's, which satisfy (19) and (20).

THEOREM 3. *Let $T = (N, D)$ be the* GSPRT $S(A_n, B_n)$ *and $T' = (N', D')$ be any other test which satisfies* (19) *and* (20). *Then*

(i) $\alpha - \alpha' = P(D \neq D') - P(D = \Delta, D' = P) = P(D' \neq D = Q)$

(ii) $\beta' - \beta = Q(D \neq D') - Q(D = Q, D' = \Delta) = Q(D \neq D' = P)$

(iii) *If $\beta' = \beta$, then $\alpha - \alpha' = P(D \neq D') = P(D = Q, N' = \infty) + P(D = Q, D' = P, N < N')$.*

(iv) *If $\beta' = \beta$ and the sequence A_n is nonincreasing, then $P(N' \geqq N) = Q(N' \geqq N) = 1$.*

PROOF. Statements (i) and (ii) follow directly from (19) and (20).

To show (iii): In view of (i) and (ii), it suffices to show that $Q(D \neq D' = P) = 0$ implies $P(D \neq D' = P, N \geqq N') = 0$ or, what is equivalent,

$$\begin{aligned} &P(D = D' = P, N \geqq N' = m) \\ &\quad = P(D' = P, N \geqq N' = m)\,, \qquad m = 1, 2, \cdots . \end{aligned}$$

With the aid of (13) of Lemma 2, this can be deduced in the same way the conditions in (c), appearing in the proof of Theorem 2, are deduced.

To show (iv): This follows from the conditions in (a), (b) and (c), appearing in the proof of Corollary 2.1. Step (c) follows immediately from (20), and the equivalence of (a) and (b) follows from Lemma 3. The remaining step (a) is shown as in the proof of Corollary 2.1 with minor modifications. Because of (19),

$$\int_{[D'=P]} \lambda_{N'}\, dP - \int_{[D=P]} \lambda_N\, dP \geqq \int_{[D=D'=P, N>N']} (\lambda_{N'} - \lambda_N)\, dP\,.$$

Also, the equation $P(D' = P, N > N') = P(D = D' = P, N > N')$, appearing in that proof, is a consequence of (iii). □

Counterexample 6 demonstrates that the condition $Q(D \neq D' = P) = 0$ does *not* imply $P(D \neq D' = P) = 0$. It shows that the last probability in (iii) is not superfluous. It can be positive.

Notice that (iii) above says, among other things, that

$$\alpha = \alpha' \quad \text{and} \quad \beta = \beta' \Rightarrow P(D = D') = 1 .$$

However, the implication $Q(D = D') = 1$ is not warranted in general. See Counterexample 7. But if T' is a GSPRT $S(A_n', B_n')$ which satisfies (19) and (20), then the implication $Q(D = D') = 1$ is warranted. This can be seen by interchanging the roles of T and T', and of P and Q, in Theorem 3. Moreover, if, in addition, the sequence B_n' is nondecreasing, then $P(N' \leqq N) = Q(N' \leqq N) = 1$. Consequently, we have:

COROLLARY 3.1. *Let* $T = (N, D)$ *be a* GSPRT $S(A_n, B_n)$ *and* $T' = (N', D')$ *be another* GSPRT $S(A_n', B_n')$ *which is above* T (i.e., *which satisfies* (19) *and* (20)). *If* $\alpha = \alpha'$ *and* $\beta = \beta'$, *then* $P(D = D') = Q(D = D') = 1$. *If, in addition, the sequence* A_n *is nonincreasing and the sequence* B_n' *is nondecreasing, then* T *and* T' *are equivalent tests.*

COROLLARY 3.2. *Suppose* $T = (N, D)$ *and* $T' = (N', D')$ *are* SPRT'*s for which* $\alpha' \leqq \alpha$ *and* $\beta' \leqq \beta$. *Then*:

(i) $P(N \leqq N') = Q(N \leqq N') = 1, P(N = N', D \neq D') = Q(N = N', D \neq D') = 0$.
(ii) *If* $\alpha' = \alpha$ *and* $\beta' = \beta$, *then* T *and* T' *are equivalent.*
(iii) *If* $\alpha' < \alpha$ *or* $\beta' < \beta$, *then* $P(N < N')$ *and* $Q(N < N')$ *are both positive.*

PROOF. If the boundaries of T' are between those of T, then T and T' are equivalent tests according to Corollary 2.1. Alternatively, if the boundaries A' and B' of T' are "above" the boundaries A and B of T (i.e., $A' \geqq A$ and $B' \geqq B$), then, by (ii) of Theorem 3, $\beta' = \beta$. If, in addition, $\alpha' = \alpha$, then Corollary 3.1 implies the conclusion in (ii). On the other hand, if $\alpha' < \alpha$, then (iii) of Theorem 3 implies $P(N < N') > 0$. Likewise, $Q(N < N') > 0$, as a consequence of Lemma 3 (with the roles of N and N' reversed). Thus (iii) holds. To see that (i) holds, observe that $P(N \leqq N') = Q(N \leqq N') = 1$ according to (iv) of Theorem 3, and that $P(N = N', D \neq D') = 0$ according to (iii) of the same theorem. Likewise, $Q(N = N', D \neq D') = 0$ since $[N = N', D \neq D']$ is a subevent of $[\lambda_N \neq 0$ or $\infty]$. (See the proof of Lemma 3.) Alternatively, if the boundaries of T are above those of T', a similar argument applies. The remaining alternative to consider is that the boundaries of T are between those of T'. Then (i) is obvious on geometrical grounds, and (ii) is a consequence of Corollary 2.1. Now $P(N < N')$ and $Q(N < N')$ are positive together or zero together according to Lemma 3. Assume, for purposes of contradiction, that $P(N < N') = Q(N < N') = 0$. Then (i) implies that $P(N = N') = Q(N = N') = 1$ and, in turn, that $P(D \neq D') = Q(D \neq D') = 0$. Consequently, $\alpha' = \alpha$ and $\beta' = \beta$. This establishes (iii). □

In the remainder of this section, it is shown that only a certain class of GSPRT's can be weakly admissible. The approach is based upon the Neyman–Pearson fundamental lemma and is quite elementary. Kiefer and Weiss (1957) use a more sophisticated approach, utilizing the theory of sequential optimization, to establish a related result for admissible GSPRT's. We see no way to prove our more general result using their approach.

Let N be any stopping variable. For each point $k \in [0, \infty]$ and $\gamma \in [0, 1]$, define a randomized test $T(k, \gamma) = (N, D(k, \gamma))$, where

$$\begin{aligned} D(k, \gamma) &= P \quad \text{if} \quad N < \infty \quad \text{and} \quad \lambda_N < k\,, \\ &= Q \quad \text{if} \quad N < \infty \quad \text{and} \quad \lambda_N > k\,, \\ &= P \quad \text{with probability} \quad \gamma \quad \text{if} \quad N < \infty \quad \text{and} \quad \lambda_N = k\,, \\ &= Q \quad \text{with probability} \quad (1-\gamma) \quad \text{if} \quad N < \infty \quad \text{and} \quad \lambda_N = k\,, \\ &= \Delta \quad \text{if} \quad N = \infty\,. \end{aligned}$$

Let $\alpha(k, \gamma)$ and $\beta(k, \gamma)$ denote the error probabilities of $T(k, \gamma)$. Finally, for ease of exposition, linearly order the points (k, γ) in the rectangle $[0, \infty] \times [0, 1]$ lexicographically. Then

(i) *$\alpha(k, \gamma)(\beta(k, \gamma))$ is a nonincreasing (nondecreasing) function of (k, γ) and assumes every value in the interval $[0, P(N < \infty)]([0, Q(N < \infty)])$.*

(ii) *For a given value $\alpha \in [0, P(N < \infty)]$, the equation $\alpha(k, \gamma) = \alpha$ holds for a least value (k_0, γ_0).*

(iii) *$T(k_0, \gamma_0)$ is a most powerful test of size α among all tests whose stopping variable is N.*

This last result is a version of the Neyman–Pearson fundamental lemma. It follows from (2). In particular, if $T = (N, D)$ is a test with error probabilities α and β, then: when $k_0 < \infty$,

$$\beta - \beta(k_0, \gamma_0) \geqq \int (\lambda_N - k_0)(I(D = P) - I(D(k_0, \gamma_0) = P))\, dP \geqq 0 \tag{21}$$

(where $I(E) = 1$ if the event E occurs, and $= 0$ otherwise), and when $k_0 = \infty$, $\alpha(k_0, \gamma_0) = \alpha = \gamma_0 = 0$ and

$$\beta(k_0, \gamma_0) = Q(N < \infty, \lambda_N \neq \infty) = \int_{[N<\infty]} \lambda_N\, dP = \int_{[D=P]} \lambda_N\, dP \leqq \beta\,.$$

If T is weakly admissible and $k_0 < \infty$, the integrand in (21) must be zero almost surely (P), and it follows that

$$P(\lambda_N > k_0, D = P) = P(\lambda_N < k_0, D = Q) = 0\,. \tag{22}$$

This holds as well for $k_0 = \infty$. Suppose, further, that T is a GSPRT $S(A_n, B_n)$. Define $T' = (N', D') = S(A_n', B_n')$ where $A_n' = \min(A_n, k_0)$, $B_n' = \max(B_n, k_0)$ and $D' = D$ if $\lambda_N = A_N = B_N = k_0$. Clearly $N' \geqq N$. Since the event $[N' > N]$ is contained in the union of the two P-null events described in (22), $P(N = N') = 1$. It follows from Lemma 3 that $Q(N = N') = 1$ as well. Thus, we have demonstrated:

THEOREM 4. *Every weakly admissible* GSPRT *is equivalent to a* GSPRT $S(A_n, B_n)$ *for which* $\sup A_n \leqq \inf B_n$.

4. Counterexamples. The counterexamples below are referred to by number in previous sections. These points of reference explain their relevance. Only needed information is given. Thus, only the random variables X_1 and X_2 are described in several counterexamples where N and N' are no larger than two. Everywhere $T = (N, D)$ is a GSPRT $S(A_n, B_n)$, sometimes an SPRT. $T' = (N', D')$ is a competing test.

COUNTEREXAMPLE 1. $T = S(.98, 1.05)$

X_1	X_2	P	Q	λ_1	λ_2	N	D	N'	D'
-1	-1	.33	0	.970	0	1	P	2	P
-1	1	0	.32	.970	∞	1	P	2	Q
0	-1	.18	.16	1.000	.889	2	P	1	P
0	1	.16	.18	1.000	1.125	2	Q	1	P
1	-1	.33	0	1.030	0	2	P	2	P
1	1	0	.34	1.030	∞	2	Q	2	Q

(Here, the third and fourth columns give the P and Q probabilities of the various X_1, X_2 pairs. The values of λ_1 and λ_2 are computed from these to three decimal places.)

$$\alpha = P(D = Q) = .16\,, \qquad \alpha' = P(D' = Q) = 0\,, \qquad \beta = Q(D = P) = .48\,,$$
$$\beta' = Q(D' = P) = .34\,, \qquad P(N = 1) = .33\,, \qquad P(N' = 1) = .34\,,$$
$$Q(N = 1) = .32\,, \qquad Q(N' = 1) = .34\,.$$

REMARKS.

1. Since $\alpha' < \alpha$, $\beta' < \beta$, $P(N' = 1) > P(N = 1)$ and $Q(N' = 1) > Q(N = 1)$, T' is better than T in essentially every respect and T *is an inadmissible* SPRT.
2. It is easily shown that T' *is admissible* and, consequently, *weakly admissible.*
3. T' is not equivalent to any GSPRT.
4. Since λ_1 determines X_1 and (λ_1, λ_2) determines (X_1, X_2), *transitivity holds* if, for instance, $X_n \equiv 0$ for $n \geqq 3$.

COUNTEREXAMPLE 2. Let $X_1, X_2, \cdots$ be i.i.d. Bernoulli variables with mean $\frac{1}{3}$ under P and mean $\frac{2}{3}$ under Q. For $T = S(\frac{1}{4}, 2)$, $\alpha = \frac{3}{7}$ and $\beta = \frac{1}{7}$. For $T' = S(\frac{1}{2}, 2)$, $\alpha' = \beta' = \frac{1}{3}$. Then $2(\alpha' - \alpha) + (\beta' - \beta) = 0$.

COUNTEREXAMPLE 3. $A_1 = 1$, $B_1 = 2$, $A_2 = 0$, $B_2 = \frac{1}{2}$.

X_1	X_2	P	Q	λ_1	λ_2	$N = N'$	D	D'
-1	-1	$\frac{1}{4}$	0	0	0	1	P	P
0	-1	$\frac{1}{3}$	$\frac{1}{6}$	$1\frac{1}{2}$	$\frac{1}{2}$	2	Q	P
0	1	$\frac{1}{6}$	$\frac{7}{12}$	$1\frac{1}{2}$	$3\frac{1}{2}$	2	Q	Q
1	1	$\frac{1}{4}$	$\frac{1}{4}$	1	1	1	P	Q

$\alpha' = \frac{5}{12} < \frac{1}{2} = \alpha$ and $\beta' = \frac{1}{6} < \frac{1}{4} = \beta$. $\max(A_1, A_2) = 1 > \frac{1}{2} = \min(B_1, B_2)$. Thus $\sup A_n > \inf B_n$. There exist more complicated examples of this type within the independent model.

COUNTEREXAMPLE 4. Let X_1 be a Bernoulli variable with mean $\frac{1}{3}$ under P and mean $\frac{2}{3}$ under Q, and let $X_2 \equiv 0$. Then $\lambda_1 = \lambda_2 = \frac{1}{2}$ if $X_1 = 0$ and $\lambda_1 = \lambda_2 = 2$ if $X_1 = 1$. Let $A_1 = \frac{1}{4}$, $B_1 = 4$, $A_2 = \frac{1}{2}$, $B_2 = 2$, and let $T' = S(\frac{1}{2}, 2)$. Then $\alpha = \alpha' = \beta = \beta' = \frac{1}{3}$, $P(D = D') = Q(D = D') = 1$ and $\max(A_1, A_2) = \frac{1}{2} < 2 = \min(B_1, B_2)$. But $N' \equiv 1$ and $N \equiv 2$.

COUNTEREXAMPLE 5. $A_1 = 1$, $B_1 = 2$, $A_2 = 0$, $B_2 = 1$.

X_1	X_2	P	Q	λ_1	λ_2	$N = N'$	D	D'
-1	-1	$\frac{1}{4}$	0	0	0	1	P	P
0	-1	$\frac{1}{4}$	$\frac{1}{4}$	$1\frac{1}{2}$	1	2	Q	P
0	1	$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{2}$	2	2	Q	Q
1	1	$\frac{1}{4}$	$\frac{1}{4}$	1	1	1	P	Q

Here, $\alpha = \alpha' = \frac{1}{2}$, $\beta = \beta' = \frac{1}{4}$, $N \equiv N'$ and $\max(A_1, A_2) = \min(B_1, B_2) = 1$. But $P(D \neq D')$ and $Q(D \neq D')$ are both positive.

COUNTEREXAMPLE 6. Let X_1 (X_2) be a Bernoulli variable with mean $\frac{1}{3}$ (zero) under P and mean $\frac{2}{3}$ (one) under Q. Further, let $T = S(\frac{1}{2}, 2)$ and $T' = S(\frac{1}{2}, 3)$, so that T' is "above" T. Then $Q(D \neq D' = P) = 0$. But $P(D \neq D' = P) = P(D \neq D' = P, N < N') = P(X_1 = 1) = \frac{1}{3}$. As the proof of Theorem 3 asserts, $P(D \neq D' = P, N \geqq N') = 0$.

COUNTEREXAMPLE 7. Let $X_1, X_2, \cdots$ be i.i.d. Bernoulli variables with mean $\frac{1}{3}$ under P and mean $\frac{2}{3}$ under Q. Let $T = S(\frac{1}{2}, 2)$ and define $N' = \inf\{n \geqq 1 : \lambda_n \leqq 1\}$, $[D' = P] = [X_1 = 0]$, $[D' = Q] = [X_1 = 1, \lambda_n = 1 \text{ for some } n \geqq 2]$ and $[D' = \Delta] = [N' = \infty] = [\lambda_n > 1, n \geqq 1]$. Then (19) and (20) are satisfied, $\alpha = \alpha' = \beta = \beta' = \frac{1}{3}$ and $P(D = D') = 1$. But $Q(D \neq D') = Q(D = Q, D' = \Delta) = \frac{1}{3}$.

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