The Annals of Statistics
1975, Vol. 3, No. 3, 717-727

ON MOST EFFECTIVE TOURNAMENT PLANS WITH FEWER GAMES THAN COMPETITORS[1]

BY WILLI MAURER

University of Florida

Let Ω_n denote a set of n players, p_{ij} the probability that player i defeats player j and Γ the class of preference matrices (p_{ij}) with $p_{1j} \geqq \frac{1}{2}$, $j > 2$. Under the assumption that the outcomes of games are independent and distributed according to $(p_{ij}) \in \Gamma$, the *effectiveness* (relative to (p_{ij})) of a tournament plan, together with a rule to select a winner, is measured by the probability that player 1 (the "best" player) wins the tournament. A k.o. plan is a tournament plan in which a player is eliminated from the tournament if he loses one game. It is shown that there are no plans on Ω_n with $n-1$ games that are more effective than k.o. plans relative to all matrices contained in certain reasonable subclasses of Γ. Among the k.o. plans for $2^m + k$, $0 \leqq k < 2^m$, players, those which consist of a preliminary round of k games followed by a "symmetric" k.o. tournament on the remaining 2^m players are more effective than all other plans relative to the preference matrices contained in two large subclasses of Γ. In order to prove these assertions, the tournament plans are interpreted as mappings with directed digraphs as domain and range.

1. Introduction. General random tournament designs have been introduced by Narayana and Zidek (1969). The definition of nonrandom tournament plans proposed in the author's thesis (1972) is consistent with this more general conception: A tournament plan on a set of players is a rule that generates a sequence of games with the game to be played or termination of the sequence depending only on the outcomes of previous games. In a k.o. plan a player is eliminated from the tournament if he loses a game. The interpretation of tournament plans as mappings with digraphs as domain and range proves to be useful to show the superiority of k.o. plans over other plans with the same fixed number of games in finding the "best" among n players.

David (1963) proposes to measure the effectiveness of a plan in selecting the "best" out of n players by the probability that this player wins the tournament under the assumption of an appropriate probability model. Glenn (1960) calculated and compared the effectiveness of generalized knock-out plans with varying numbers of games and Round Robin plans on four players using fixed but representative preference matrices. Searls (1963) extended these results to eight players.

Received September 1973; revised April 1974.

[1] Work was done while the author was a postdoctoral research fellow at the Department of Statistics, Yale University, and a visiting postdoctoral associate at the Statistical Laboratory, Iowa State University. Financial support was provided by the Swiss National Foundation and by Army Research Office—Durham.

AMS 1970 *subject classifications.* Primary 62K05; Secondary 05B30, 05C20.

Key words and phrases. Tournament plan, knockout plan, winner rule, dominance structure, preference matrix, complete digraph.

It is our aim to show that for two different probability models there exists a class of k.o. plans that is more effective than any other class of tournament plans with the same fixed number of games. A corollary of our main theorem gives an application to random k.o. plans, defined by Narayana (1968). It shows that for $2^m + k$, $0 \leqq k < 2^m$, players no random plan is more effective than the following one: First a preliminary round consisting of k games between k randomly selected disjoint pairs of players is held; then a standard random symmetric plan is played among the k winners of the first round and the $2^m - k$ players who drew a "bye". In general, this plan is more effective than the random plan which has as many games as possible in each round.

Throughout the paper we use the language of sport competition together with that of graph theory used by Harary, Norman and Cartwright (1965). It is, however, obvious that in the context of pairwise comparisons, finding the most effective plan is a problem in the area of the designs of experiments. Many extensions and generalizations of this problem are possible, among them the use of other probability models, the selection of a subset of players containing the best one and the determination of the most effective plans that have more games than players and allow repetition of games. If we restrict ourselves to plans with fewer than n games, however, it is sufficient to consider plans without repeated games.

2. Preference structures and tournament plans. Let Ω_n denote a set of n players $\{1, 2, \cdots, n\}$. A *preference matrix* $P_n = (p_{ij})$ is an $n \times n$ matrix with nonnegative elements p_{ij} such that $p_{ij} + p_{ji} = 1$ for $i, j \in \Omega_n$. The quantity p_{ij} is the probability that player i beats player j. If all elements of a preference matrix P_n are either 0 or 1 (except the elements in the diagonal which are $\frac{1}{2}$) we call it a *dominance matrix* $D_n = (d_{ij})$. A dominance matrix is equivalent to a complete asymmetric relation $(\rightarrow)$ in Ω_n by $p_{ij} = 1 \Leftrightarrow i \rightarrow j$ (read: i dominates j). If D_n is an adjacency matrix of a graph G_n with point set Ω_n, then G_n has to be complete and directed. Such a digraph is usually, but unfortunately, called a tournament. In order to avoid ambiguities we call it here *dominance graph* and denote it (as well as the associated adjacency matrix) by D_n.

Given a fixed but "unknown" dominance graph D, a *game* $\mathscr{G}_{ij}$ between players i and j reveals if either $i \rightarrow j$ or $j \rightarrow i$ in D. A game $\mathscr{G}_{ij}$, therefore, can be interpreted as a mapping from D onto the subdigraph of D with point set $\{i, j\}$. $\mathscr{G}_{ij}(D)$ is then called the *outcome of game* $\mathscr{G}_{ij}$. If $\mathscr{G}_{ij}(D) = i \rightarrow j$, we say: i has won the game against j. A *tournament plan* $\mathscr{P}_n$ on Ω_n is a rule that generates a sequence of games depending on the underlying dominance structure D_n such that at each stage in this sequence the decision of which game to play or whether to terminate the tournament depends only on the outcomes of the preceding games. The union of the point set Ω_n and all outcomes of the games generated by $\mathscr{P}_n$ and D_n is called the *outcome of the plan* $\mathscr{P}_n$ *given* D_n and is denoted by $\mathscr{P}_n(D_n)$. This allows us to interpret $\mathscr{P}_n$ as a mapping of D_n onto $C_n = \mathscr{P}_n(D_n)$

with the following two properties:

(2.1) $$\mathscr{P}(D) = C \subset D$$

i.e., C is a partial digraph of the dominance graph D and

(2.2) if $\mathscr{P}(D) = C$ and $C \subset D^*$ then $\mathscr{P}(D^*) = C$,

D^* being a dominance graph, since at each stage in the sequence of games played simultaneously on D and D^* the same partial digraph of C is revealed; hence the same decision of which game follows is made. We are here only interested in the outcomes C of the tournaments and not in the sequence of games itself. We consider, therefore, two plans $\mathscr{P}_n$ and $\mathscr{P}_n^*$ as *equal* ($\mathscr{P}_n = \mathscr{P}_n^*$) if $\mathscr{P}_n(D_n) = \mathscr{P}_n^*(D_n)$ for all dominance structures D_n on Ω_n. In this sense all Round–Robin tournament plans $\mathscr{R}_n$ are equal (but not necessarily identical) because $\mathscr{R}_n(D_n) = D_n$ for all D_n on Ω_n.

Given any plan $\mathscr{P}_n$ and a permutation σ of Ω_n, the plan generated from $\mathscr{P}_n$ by interchanging the players' names according to σ is denoted by $\mathscr{P}_n^\sigma$. We call two plans $\mathscr{P}$ and $\mathscr{P}^*$ *equivalent* ($\mathscr{P} \sim \mathscr{P}^*$) if there exists a permutation σ such that $\mathscr{P}^* = \mathscr{P}^\sigma$ and refer to a full class $\tilde{\mathscr{P}}$ of equivalent plans as the *plan type* $\tilde{\mathscr{P}}$. It can be shown, that the automorphisms of a plan $\mathscr{P}_n$, i.e., the permutations for which $\mathscr{P} = \mathscr{P}^\alpha$, form a subgroup $A_{\mathscr{P}}$ of the symmetric group S_n and that the size $|\tilde{\mathscr{P}}|$ of the full equivalence class containing $\mathscr{P}$ is $n!/\mathrm{order}\,(A_{\mathscr{P}})$.

3. K.o. plans. A plan $\mathscr{P}_n$ is called a *k.o. plan* and denoted by $\mathscr{K}_n$ if a player who loses a game is eliminated, i.e., may not play any further game and the tournament ends when all players except one are eliminated.

A k.o. plan $\mathscr{K}_n$ consists of $n - 1$ games because in every game one player is eliminated. The plan can be described by a binary tree consisting of $2n - 1$ nodes, n of them "pendant" nodes representing the players and $n - 1$ nodes representing games between directly designated players or winners of former games. The players' names are assigned to the pendant nodes and in permuting them one produces all possible equivalent plans of the same type. We shall label the pendant nodes with neutral names $\nu_1, \nu_2, \cdots, \nu_n$ if we want to refer to specific ones.

It is typographically easier to describe k.o. plans by bracket clusters. The relation between this and the tree-method is clarified by Figure 1. Some examples will illustrate the definitions and concepts presented so far. The k.o. plan $\mathscr{K} = ((1\ 2)3)$ is equal to $(3(2\ 1))$ and equivalent to $\mathscr{K}' = ((1\ 3)2)$ and $\mathscr{K}'' = ((2\ 3)1)$. All three plans are pairwise different because e.g., for $D = \{1 \to 2 \to 3, 3 \to 1\}$ the tournament outcomes are $\mathscr{K}(D) = (3 \to 1 \to 2)$, $\mathscr{K}'(D) = (2 \to 3 \to 1)$ and $\mathscr{K}''(D) = (1 \to 2 \to 3)$. Together they form the only equivalence class of k.o. plans on three players, $\tilde{\mathscr{K}}_3 = ((**)*)$. Clearly $A_{\mathscr{K}} = \{\varepsilon, \langle 1, 2\rangle\}$, where $\langle i, j\rangle$ is the transposition of i and j and $\mathscr{K}' = \mathscr{K}^{\langle 2,3\rangle}$, $\mathscr{K}'' = \mathscr{K}^{\langle 1,3\rangle}$. For four players there exist two types of k.o. plans namely $\tilde{\mathscr{R}}_4 = ((**)(**))$ and $\tilde{\mathscr{U}}_4 = (((**)*)*)$. We call the first one *symmetric*. Symmetric plans exist for 2^m, $m = 2, 3, \cdots$, players.

TABLE 1
Some values of a and b

i	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_i	1	2	3	6	11	23	46	98	207	451	983	2179	4850	10905
b_i	1	1	1	2	1	1	1	3	3	5	3	3	1	1
$a_{24} = 8436379$, $b_{24} = 35$; $a_{32} = 7862958391$, $b_{32} = 1$														

We call a k.o. plan on $n = 2^m + k$, $0 \leqq k < 2^m$, players *balanced* if it consists of a preliminary round of k games among $2k$ players, followed by a symmetric plan on the k winners of these games and the remaining $2^m - k$ players.

Let $l(\nu_i)$ be the *level* of the pendant node ν_i in the tree representation of a k.o. plan $\mathscr{K}_n$, i.e., $l(\nu_i)$ is the number of games a player assigned to this node has to play in order to win the tournament. It is then obvious that in a balanced k.o. plan the maximum difference between all levels is one, and some reflection shows that a k.o. plan with this property is balanced.

The number a_n of different k.o. plan types and the number b_n of different balanced k.o. plan types for n players may be computed by recursion formulas that we give without proof.

$$(3.1) \qquad a_1 = a_2 = 1\,; \qquad a_n = \tfrac{1}{2} \textstyle\sum_{i=1}^{n-1} a_i(a_{n-i} + \delta_{i,n-i})\,;$$

where δ_{ij} denotes Kronecker's delta.

$$(3.2) \qquad b_1 = b_2 = 1\,; \qquad b_n = \tfrac{1}{2} \textstyle\sum_{i=l}^{u} b_i(b_{n-i} + \delta_{i,n-i})$$

where for

$$n = 2^m + k\,, \quad 0 \leqq k < 2^m: \qquad l = \max(2^{m-1}, k)\,, \quad u = \min(2^{m-1} + k, 2^m)\,.$$

4. The winner rule. It has been shown by Maurer (1972) that k.o. plans may be characterized by the property that all their outcomes are rooted trees. The root of the tree; i.e., that player from which every other player can be reached by exactly one directed path, is called winner of the tournament. In general, a *winner rule* w associated with a plan $\mathscr{P}_n$ is a many-to-one mapping of the range of $\mathscr{P}_n$ onto the set of players Ω_n. We call w *unbiased* if $w(\mathscr{P}_n(D_n)) = b$ for all D_n in which player b dominates all other players. The usual winner rules for k.o. plans and the one for Round–Robin plans which maps each dominance graph on one of the players with maximum out-degree are both unbiased. For k.o. plans there exists no other unbiased winner rule.

LEMMA 4.1. *Let $\mathscr{P}_n$ be any plan with maximum number of games less than n and w a winner rule associated with $\mathscr{P}_n$. Then w is unbiased* iff *$\mathscr{P}_n$ is a knock-out plan and w its usual winner rule.*

PROOF. One direction of the assertion is evident. Assume that $\mathscr{P}_n$ is not a k.o. plan. Then there exists an outcome $\mathscr{P}_n(D) = C$ that is not a rooted tree.

If C is connected it has to be a tree possessing at least two nodes with only outgoing branches. Therefore, there exist two transitive dominance graphs D' and D'' with different dominating players b' and b'' such that $C \subset D'$, $C \subset D''$ and hence by (2.2) $\mathscr{P}(D') = \mathscr{P}(D'') = C$. Hence $\mathscr{P}_n$ cannot have an unbiased winner rule. If C is not connected, but has at least two acyclic components, the same argument holds. This proves the lemma because one can show that if C is not connected and has cycles there must exist an outcome C^* of $\mathscr{P}_n$ with at least two acyclic components.

5. Probability models. Let $p\{D_n\}$ be a probability distribution over the set of all dominance graphs on Ω_n and let $\mathbf{D}_n$ be a random dominance graph with this distribution, so that $\Pr(\mathbf{D}_n = D_n) = p(D_n)$. Then

$$\Pr(\mathscr{P}_n(\mathbf{D}_n) = C_n) = \textstyle\sum_{D_n} p(D_n) \cdot I[\mathscr{P}_n(D_n) = C_n]\,, \tag{5.1}$$

where I is the indicator function.

We assume that a preference matrix P_n is given with $p(i \to j) = p_{ij}$ and that all dominances in D_n are independent. We then have

$$p(D_n) = \textstyle\prod_{i \neq j} p_{ij}^{d_{ij}}\,, \tag{5.2}$$

(d_{ij}) being the dominance matrix associated with D_n.

We consider the following classes of preference matrices:

Γ_0: The class of all preference matrices,
Γ_1: $\{P; p_{1j} \geqq \frac{1}{2}$ for $j \geqq 2\}$,
Γ_2: $\{P; p_{1j} = p \geqq \frac{1}{2}$ for $j \geqq 2\}$,
L_3: $\{P; p_{1j} \geqq \frac{1}{2}, p_{ij} = \frac{1}{2}$ for $i, j \geqq 2\}$.

In all classes except Γ_0 player 1 is distinguished from the others and will be called the "best" player. If a plan is chosen at random from an equivalence class $\tilde{\mathscr{P}}$ and w is the winner rule associated with this plan, then the probability that player 1 wins this random tournament is

$$\pi(\tilde{\mathscr{P}}(P)) = \frac{1}{n!} \textstyle\sum_{\sigma \in S} \sum_D I[1 = w(\mathscr{P}^\sigma(D))]p(D) = \sum_D \pi(\tilde{\mathscr{P}}(D))p(D) \tag{5.3}$$

where $\mathscr{P} \in \tilde{\mathscr{P}}$, n is the number of players and P is a preference matrix on Ω_n.

By the random choice of a plan from an equivalence class we can express our ignorance of the best player. As a special case given an equivalence class $\tilde{\mathscr{K}}_n$ of k.o. plans represented by a tree or bracket clusters with labeled nodes $\nu_1, \nu_2, \cdots, \nu_n$, we can assign the players at random to the nodes in order to generate a random plan out of $\tilde{\mathscr{K}}$.

6. The chance to win from a fixed place. If player i is assigned to node ν_k and the other competitors are placed at random on the remaining nodes, we denote the probability that player 1 wins the tournament given P is the underlying preference matrix by $\pi(\tilde{\mathscr{K}}(P), i \text{ on } \nu_k)$ or shorter $\pi(i \text{ on } \nu_k)$.

LEMMA 6.1. *Let $\tilde{\mathscr{K}}_n$ be a* k.o. *plan type, ν_i and ν_j nodes of $\tilde{\mathscr{K}}_n$ with levels $l(\nu_i) < l(\nu_j)$ and P_n an underlying preference matrix.*
Then the inequality

$$\pi(1 \text{ on } \nu_i) \geqq \pi(1 \text{ on } \nu_j)$$

holds if any one of the following assumptions is true:

$$\text{(i)} \quad P_n \in \Gamma_2 \cup \Gamma_3 \qquad \textit{for} \quad n \geqq 3$$
$$\text{(ii)} \quad P_n \in \Gamma_0 \qquad \textit{for} \quad n = 3, 4, 5, 6 .$$

For $n \geqq 7$ there exist plans, nodes and preference matrices for which the inequality does not hold.

PROOF. (i) If $P_n \in \Gamma_2$, the probability for player 1 to win a tournament starting from node ν_i in a fixed plan from equivalence class $\tilde{\mathscr{K}}_n$ is $p^{l(\nu_i)}$; therefore, $\pi(1 \text{ on } \nu_i) = p^{l(\nu_i)}$.

In model Γ_3 all players except number 1 are assumed to be of equal strength. If player 1 wins the tournament, starting from node ν_i, he is likely to meet each subset of $l(\nu_i)$ players out of the remaining $n-1$ ones with equal probability, given they are assigned with equal probability to the remaining nodes.

With $l = l(\nu_i)$ and $\{k_1, k_2, \cdots, k_l\}$ being a subset of $(\Omega_n - \{1\})$ we thus have

$$\pi(1 \text{ on } \nu_i) = [1/\tbinom{n-1}{l}] \textstyle\sum_{\{k_1,\cdots,k_l\}} p_{1k_1}p_{1k_2} \cdots p_{1k_l} . \tag{6.1}$$

Under the assumption: $m = l(\nu_j) > l$, we can write

$$\begin{aligned} \pi(1 \text{ on } \nu_j) = {} & ([1/\tbinom{n-1}{l}] \textstyle\sum_{\{k_1,\cdots,k_l\}} p_{1k_1}p_{1k_2} \cdots p_{1k_l}) \\ & \times ([1/\tbinom{n-l-1}{m-l}] \textstyle\sum_{\{k_{l+1},\cdots,k_m\}} p_{1k_{l+1}} \cdots p_{1k_m}) \end{aligned} \tag{6.2}$$

where $\{k_{l+1}, \cdots, k_m\}$ is a subset of $\Omega_n - \{1, k_1, \cdots, k_l\}$.

The second factor represents the probability that player 1 wins the remaining games of the tournament given he already won against $k_1, k_2, \cdots, k_l$. From (6.2) the inequality follows easily.

It is sufficient to prove (ii) for dominance matrices. This can be shown by the following representation:

$$\pi(\tilde{\mathscr{K}}_n(P_n), 1 \text{ on } \nu_i) = \textstyle\sum_{D_n} p(D_n)\pi(\tilde{\mathscr{K}}_n(D_n), 1 \text{ on } \nu_i) . \tag{6.3}$$

Let V_i denote the number of tournaments player 1 wins if he starts from node ν_i; the other players are permuted on the remaining nodes and the underlying dominance matrix is D_n. Then

$$\pi(\tilde{\mathscr{K}}_n(D_n), 1 \text{ on } \nu_i) = \frac{1}{(n-1)!} V_i . \tag{6.4}$$

In order to prove the assertion for the simplest case, the k.o. plan type $\tilde{\mathscr{K}}_3 = ((\nu_1, \nu_2), \nu_3)$, we have to show that $V_3 \geqq V_1$. Assume that for a given D_3 player 1 wins a tournament starting from node ν_1. Then by transposing the player on node ν_3 and player 1, one obtains a plan with 1 starting from ν_3 and winning the tournament. The possible permutations of the remaining players while 1 is fixed on node ν_1 are transformed by that transposition into the permutations of the players with 1 fixed on ν_3; hence $V_3 \geqq V_1$.

This type of proof works for any plan if one compares a node of level one with a node of higher level.

In order to prove (ii) for $n = 4, 5, 6$ we have to check every plan type separately. The following more general methods help to verify the inequality for all possible pairs of nodes in these plans:

1) Compare only nodes ν_i, ν_j with $l(\nu_i) < l(\nu_j)$ for which no node ν_k exists such that $l(\nu_i) < l(\nu_k) < l(\nu_j)$.

2) It is sufficient to consider only the smallest subplan in which both nodes are contained.

3) Instead of interchanging only player 1 on ν_i with the player on node ν_j for a given plan in order to get a comparable plan with 1 on ν_j, interchange simultaneously a whole set of players.

4) Make use of symmetries.

All these methods are sufficient, too, to verify the inequality for 10 out of the 11 plans for $n = 7$. They fail for $\tilde{\mathscr{E}}_7 = ((\nu_1, \nu_2)((\nu_3\ \nu_4)(\nu_5(\nu_6\ \nu_7))))$ in comparing the nodes $\nu_2(l(\nu_2) = 2)$ and $\nu_5(l(\nu_5) = 3)$. In fact a simple counterexample can be found:

Let Q_7 be the strong dominance graph on Ω_7 with $i \to j$ for $i < j$ except that player 7 dominates 1 ($7 \to 1$). It is then easy to verify that $\pi(\tilde{\mathscr{E}}_7(Q_7)$, 1 on $\nu_5) = 1$ and $\pi(\tilde{\mathscr{E}}_7(Q_7)$, 1 on $\nu_2) = \frac{5}{6}$. □

7. Most effective plans with $n - 1$ games for n players. We say a plan $\mathscr{P}$ of type $\tilde{\mathscr{P}}$ with associated winner rule w is *more effective* than plan $\mathscr{P}^*$ of type $\tilde{\mathscr{P}}^*$ with winner rule w^* relative to one of the classes Γ_i, $i = 1, 2, 3$, if $\pi(\tilde{\mathscr{P}}(P)) \geqq \pi(\tilde{\mathscr{P}}^*(P))$ for all $P \in \Gamma_i$ and if for at least one $P \in \Gamma_i$ this inequality holds strictly.

For all Γ_i, $i = 1, 2, 3$ clearly no winner rule is more effective for k.o. plans than the usual one, because it is the only rule with $\pi(\tilde{\mathscr{K}}(P)) = 1$ if $p_{1j} = 1$, $j > 2$, in P. In the sequel, k.o. plans are always understood to be associated with the usual winner rule.

THEOREM 7.1. (i) *With any winner rule, no plan $\mathscr{P}_n$ on Ω_n with fewer than n games is more effective than any* k.o. *plan $\mathscr{K}_n$ on Ω_n relative to Γ_1, Γ_2 and Γ_3 for $n \geqq 3$.*

(ii) *Balanced* k.o. *plans $\mathscr{B}_n$ are more effective than unbalanced* k.o. *plans $\mathscr{U}_n$ relative to Γ_2 and Γ_3 for $n \geqq 4$.*

PROOF. (i) The preference matrix P_n with $p_{1j} = 1, j = 2, \cdots, n$, and $p_{ij} = \frac{1}{2}$ for $i, j = 2, \cdots, n$ is an element of Γ_1, Γ_2 and Γ_3, $n = 2, 3, \cdots$. For a k.o. plan $\mathscr{K}_n$ we have for this matrix: $\pi(\tilde{\mathscr{K}}_n(P_n)) = 1$; $\mathscr{K}_n \in \tilde{\mathscr{K}}_n$. Thus, for any $\mathscr{P}_n$ in order to be more effective than $\mathscr{K}_n$, we must have $\pi(\tilde{\mathscr{P}}_n(P)) = \sum_{D_n} \pi(\tilde{\mathscr{P}}_n(D_n))p(D_n) = 1$, $\mathscr{P}_n \in \tilde{\mathscr{P}}_n$. Hence $\pi(\tilde{\mathscr{P}}_n(D_n)) = 1$ for all dominance

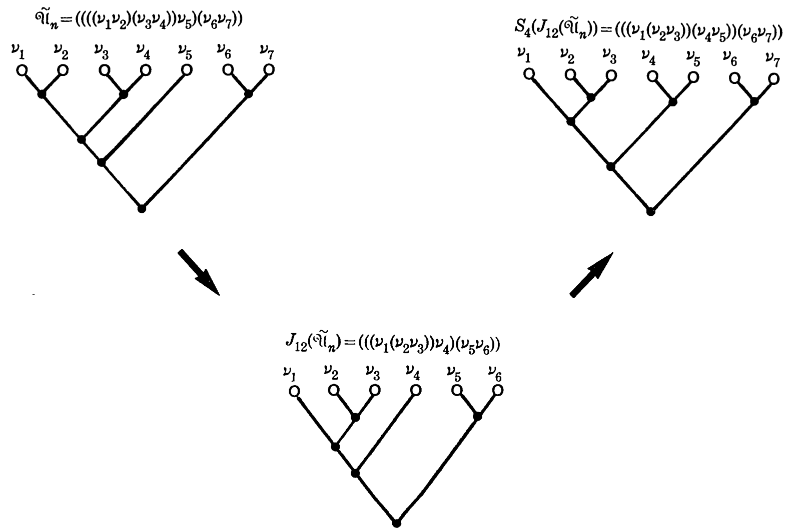

FIG. 1.

matrices D_n with $p(D_n) > 0$, i.e., for all D_n with $d_{1j} = 1, j = 2, \cdots, n$. This implies: $w(\mathscr{P}_n(D_n)) = 1$ for all D_n where player 1 dominates all other players. According to Lemma 4.1 this is only the case if $\mathscr{P}_n$ is a k.o. plan.

(ii) Let P_{n+1} be any preference matrix on Ω_{n+1} and $P_{n+1;-j}$ denote the $n \times n$ matrix obtained from P_{n+1} by deleting the jth row and column. Let $\tilde{\mathscr{K}}_n$ be any k.o. plan type with nodes $\nu_1, \nu_2, \cdots, \nu_n$. We now define two operators S (split) and J (join) which increase and decrease, respectively, the number of nodes (see Figure 1): $S_i\tilde{\mathscr{K}}_n$ is the k.o. plan type with $n+1$ nodes obtained from $\tilde{\mathscr{K}}_n$ by replacing node ν_i by a game (**); if $\tilde{\mathscr{K}}_n$ contains a game $(\nu_i\nu_j)$ then $J_{ij}\tilde{\mathscr{K}}_n$ denotes the plan type with $n-1$ nodes obtained from $\tilde{\mathscr{K}}_n$ by replacing $(\nu_i\nu_j)$ by a single node.

In addition to these definitions we will use a further abbreviation: $\pi_{-j}(i \text{ on } \nu_k) = \pi(\tilde{\mathscr{K}}_n(P_{n+1;-j}); i \text{ on } \nu_k)$, the probability that player 1 wins the random tournament $\tilde{\mathscr{K}}_n$ if player i is assigned to node ν_k and the remaining players $(\Omega_{n+1} - \{1, i, j\})$ are assigned at random to the remaining nodes. (For $i = j$ we define: $\pi_{-j}(j \text{ on } \nu_k) = 0$.)

We need the following two basic formulas:

$$\pi(\tilde{\mathscr{K}}_n(P_n)) = \frac{1}{n} \sum_{i=1}^{n} \pi(i \text{ on } \nu_k) \qquad \text{for } k = 1, \cdots, n \tag{7.1}$$

and

$$\pi(S_k\tilde{\mathscr{K}}_n(P_{n+1})) = 1/\tbinom{n+1}{2} \sum_{i=1}^{n+1} \sum_{j=2}^{n+1} p_{ij}\pi_{-j}(i \text{ on } \nu_k) \qquad \text{for } k = 1, \cdots, n\,. \tag{7.2}$$

Using now $p_{ji} = 1 - p_{ij}$ we can transform (7.2) into

$$\begin{aligned}(7.3)\qquad &\pi(S_k \tilde{\mathscr{K}}_n(P_{n+1})) \\ &= 1/\tbinom{n+1}{2}\{\textstyle\sum_{j=2}^{n+1} p_{1j}\pi_{-j}(1 \text{ on } \nu_k) + \frac{1}{2}\sum_{i,j=2}^{n+1}\pi_{-j}(i \text{ on } \nu_k) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{i=2}^{n}\sum_{j=i+1}^{n+1}(p_{ij} - p_{ji})(\pi_{-1}(i \text{ on } \nu_k) - \pi_{-1}(j \text{ on } \nu_k))\}\,.\end{aligned}$$

The third sum disappears for $P_{n+1} \in \Gamma_2 \cup \Gamma_3$ because for $P_{n+1} \in \Gamma_2$ we have $\pi_{-j}(i \text{ on } \nu_k) = \pi_{-i}(j \text{ on } \nu_k)$, $i, j = 2, \cdots, n$ and for $P_{n+1} \in \Gamma_3$ $p_{ij} = p_{ji}$, $i, j = 2, \cdots, n$.

The second of the three sums in the bracket is by (7.1) equal to $\sum_{j=2}^{n+1}((n/2) \times \pi(\tilde{\mathscr{K}}_n(P_{n+1;-j})) - \frac{1}{2}\pi_{-j}(1 \text{ on } \nu_k))$. Assuming $P_n \in \Gamma_2 \cup \Gamma_3$ in the sequel (until otherwise stated) we finally have

$$\begin{aligned}(7.4)\qquad \pi(S_k \tilde{\mathscr{K}}_n(P_{n+1})) = 1\Big/\binom{n+1}{2}\Big\{&\frac{n}{2}\textstyle\sum_{j=2}^{n+1}\pi(\tilde{\mathscr{K}}_n(P_{n+1;-1})) \\ &+ \textstyle\sum_{j=2}^{n+1}(p_{1j} - \frac{1}{2})\pi_{-j}(1 \text{ on } \nu_k)\Big\}\,.\end{aligned}$$

Because of $(p_{1j} - \frac{1}{2}) \geqq 0$ for $j = 2, \cdots, n$ and Lemma 6.1 it follows from (7.4) that

$$\begin{aligned}(7.5)\qquad l(\nu_h) < l(\nu_k) &\Rightarrow \pi_{-j}(1 \text{ on } \nu_h) \geqq \pi_{-j}(1 \text{ on } \nu_k) \qquad \text{for } j = 2, \cdots, n \\ &\Rightarrow \pi(S_h \tilde{\mathscr{K}}_n(P_{n+1})) \geqq \pi(S_k \tilde{\mathscr{K}}_n(P_{n+1}))\,.\end{aligned}$$

We now show that for any unbalanced plan $\mathscr{U}_n$ there exists always a more effective balanced plan $\mathscr{B}_n$.

In $\tilde{\mathscr{U}}_n$ there exist two nodes, say ν_1 and ν_2 with maximum level m and another node say ν_n such that $l(\nu_1) \geqq l(\nu_n) + 2$ (otherwise $\tilde{\mathscr{U}}_n$ would be balanced). If operator J_{12} joins game (ν_1, ν_2) to ν_1 in $\tilde{\mathscr{U}}_n$ we have in $\tilde{\mathscr{K}}_{n-1} = J_{12}\tilde{\mathscr{U}}_n$ the inequality $l(\nu_1) \geqq l(\nu_n) + 1$ and thus according to (7.5):

$$(7.6)\qquad \pi(S_n \tilde{\mathscr{K}}_{n-1}(P_n)) \geqq \pi(S_1 \tilde{\mathscr{K}}_{n-1}(P_n)) = \pi(\tilde{\mathscr{U}}_n(P_n))\,. \qquad \text{(See Figure 1.)}$$

This means that plan type $S_n J_{12}\tilde{\mathscr{U}}_n$ is more effective than $\tilde{\mathscr{U}}_n$. If $S_n J_{12}\tilde{\mathscr{U}}_n$ is not already a balanced plan type we iterate this procedure until it stops; the final plan has to be balanced. □

This proof also shows that the relation "more effective than" is a simply structured half order on all k.o. plan types. Furthermore, all balanced plans are equally effective for $P_n \in \Gamma_2 \cup \Gamma_3$; this is easy to see using proof (i) of Lemma 6.1. By a linearity argument that we owe to J. A. Hartigan, we can check the assertion (ii) for the class Γ_1 of preference matrices for every n by a finite number of computations: $\pi(\tilde{\mathscr{K}}_n(P_n))$ is linear in every element p_{ij} of P_n; so is $\pi(\tilde{\mathscr{K}}_n(P_n)) - \pi(\tilde{\mathscr{U}}_n(P_n))$. If, therefore, this difference is positive for every preference matrix Q_n with q_{1j} taking only values 1 or $\frac{1}{2}$ and q_{ij} only values 1 or 0 for $i, j = 2, \cdots, n$, the same has to be true for any preference matrix $P_n \in \Gamma_1$. We verified that for $n = 4$ the balanced plan type is more effective than the unbalanced one for all $P_4 \in \Gamma_1$. Checks for $n = 5$ with "extreme" matrices produced no contradiction either. However, result (ii) in Lemma 6.1 together with (7.3) seem to indicate that (probably for relatively large n) there exist unbalanced plans which are for some $P_n \in \Gamma_1$ more effective than balanced ones.

8. Application to random k.o. designs. Narayana (1968) defines a random k.o. "design" $\mathscr{R}(m_1, m_2, \cdots, m_k)$ as follows. Let m_i, $i = 1, \cdots, k$, be non-negative integers summing to $n - 1$ and such that $2m_j \leqq n - \sum_{i=1}^{j-1} m_i$ for $j = 1, \cdots, k$. The random tournament consists of k rounds. In the first round m_1 pairs of players are chosen with equal probability to form m_1 games, the losers of which are eliminated. In the jth round m_j pairs of players are selected at random from the remaining winners of former games and all players who have not yet played a game. For $n = 2^m + l$, $0 \leqq l < 2^m$, the sample space of $\tilde{\mathscr{R}}(l, 2^{m-1}, 2^{m-2}, \cdots, 1)$ is the class of all balanced plans on Ω_n.

From Theorem 7.1 we thus gain the

COROLLARY 8.1. *For* $P_n \in \Gamma_2 \cup \Gamma_3$ *and* $n = 2^m + l, 0 \leqq l < 2^m$, *the random* k.o. *design* $\tilde{\mathscr{R}}(l, 2^{m-1}, \cdots, 1)$ *is at least as effective as any other random* k.o. *design.*

At first glance this result is especially surprising for the random design $\tilde{\mathscr{R}}([n/2], [(n - [n/2])/2], \cdots) = \tilde{\mathscr{R}}([n/2], [(n + 1)/4], [(n + 3)/8], [(n + 7)/16], \cdots)$ (cf. Moon (1968), page 49).

In this design in every round as many games as possible are played among the remaining competitors. For $n = 2^m$ and $n = 2^m - 1$ this is equivalent to playing the balanced design; for other n (as e.g., for $n = 6$) the sample space of this design is a proper subset of the class of balanced plans, but for $n = 2^m + 1$ it contains unbalanced plans and, therefore, is not most effective.

Acknowledgment. The author wishes to thank Professors J. A. Hartigan and H. A. David for their help and encouragement. Thanks are also due to the referee for suggesting improvements to an earlier version of this paper.

REFERENCES

[1] DAVID, H. A. (1963). *The Method of Paired Comparisons.* Griffin, London.

[2] GLENN, W. A. (1960). A comparison of the effectiveness of tournaments. *Biometrika* **47** 253–262.

[3] HARARY, F., NORMAN, R. Z. and CARTWRIGHT, D. (1965). *Structural Models.* Wiley, New York.

[4] MAURER, W. (1972). *Optimales Rangieren nach Turnieren.* Thesis (ETH). Juris Druck und Verlag, Zurich.

[5] MOON, J. W. (1968). *Topics on Tournaments.* Holt, Rinehart, and Winston, New York.

[6] NARAYANA, T. V. (1968). Quelques résultats relatifs aux tournois "knock-out" et leurs applications aux comparaisons aux paires. *C. R. Acad. Sci. Paris* **267** *Série A* 32–33.

[7] NARAYANA, T. V. and ZIDEK, J. (1969). Contributions to the theory of tournaments, Part I, The combinatorics of knock-out tournaments. *Cahiers du Bureau Universitaire de Recherche Opérationelle, Université de Paris* **13** 1–18.

[8] NARAYANA, T. V. and ZIDEK, J. (1969). Contributions to the theory of tournaments, Part II, statistical inference in random tournaments. *Rev. Roum. Math. Pures et Appl.* **10** 1563–1576.

[9] SEARLS, D. T. (1963). On the probability of winning with different tournament procedures. *J. Amer. Statist. Assoc.* **58-2** 1064–1081.

WIRTSCHAFTS-MATHEMATIK AG ZÜRICH
MÜHLEBACHSTRASSE 38
POSTFACH
8032 ZÜRICH, SWITZERLAND