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UNIFORMLY MINIMUM VARIANCE ESTIMATION IN LOCATION PARAMETER FAMILIES

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Let $x_1, \cdots, x_n$ be a sample of size n of an rv with df $F(x-\theta)$, where F is known but θ unknown. In this paper we make a Fourier approach to the problem of existence of a statistic $g(x_1, \cdots, x_n)$ which is a uniformly minimum variance (UMV) estimator of its own mean value. We mention only some of the results. If $n = 1$ we find an NASC for an estimator $g(x_1)$ to be, in a restricted sense, UMV. This condition is given in terms of the zeroes of the ch.f. of F and the support of the Fourier transform of g. If $n \geqq 2$, it is shown that a statistic of the form $g(\bar{x})$, where $\bar{x}$ is the sample mean, cannot be UMV, unless g is periodic or F is a normal df. We prove the non-existence of a UMV-estimator of θ, provided that the tail of F tends to zero rapidly enough. Finally, it is proved that no polynomial $P(x_1, \cdots, x_n)$ can be a UMV-estimator, unless F is a normal df.

1. Introduction and summary. Let $x_1, \cdots, x_n$ be n independent observations of an rv X having df $F(x-\theta)$, where θ is unknown but F is known. The content of this paper circles around the question of existence of a statistic $g(x_1, \cdots, x_n)$ which is a uniformly minimum variance (UMV) estimator of its own mean value. The basic tool for these investigations is a well-known UMV-criterion saying roughly that g is UMV iff g is uncorrelated with every unbiased (maybe complex) estimator of zero (see e.g. Rao [21] page 257). Due to the fact that θ is a location parameter, it is possible to apply Fourier methods. To give a hint, if ζ is a zero of the characteristic function (ch.f.) of F, then $\exp\{i\zeta x_1\}$ becomes an unbiased estimator of zero.

In Section 3 we treat the case $n = 1$. It will prove possible, by using some results from harmonic analysis given in Section 2, to obtain a condition which is both necessary and sufficient for an estimator to be, in a restricted sense, UMV. If the tail of F tends to zero rapidly enough, it turns out that $g(x_1)$ cannot be a UMV-estimator, unless g is a periodic function or F is a normal df. In Section 4 we consider the case $n \geqq 2$ and find a necessary condition for $g(x_1, \cdots, x_n)$ to be UMV which will enable us to derive unrigorously an old operator UMV-condition given by Stein [23] and others. In Section 5 we give a simple new proof of a result due to Lehmann and Scheffé [15], saying that it is impossible to UMV-estimate a nonconstant parametric function in the case of the uniform distribution. A somewhat more general result of the same type is also proved.

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In Section 6 we derive two UMV-characterizations. The normal distribution is shown to be the only one admitting a UMV-estimator of the form $g(\bar{x})$, where g is aperiodic and not increasing too fast, and where $\bar{x}$ is the sample mean. This generalizes a result by Kagan [11]. When g is periodic it is only possible to conclude that the df is a convolution of a normal df with a lattice-df. If a statistic $g(x_1) + \cdots + g(x_n)$ is a UMV-estimator, then, under slight assumptions, F is normal, the df of the logarithm of some power of a gamma-distributed rv, or a lattice-df. In Section 7 we prove that, under some conditions, a UMV-estimator must also be a sufficient statistic. Similar results have earlier been obtained by other authors. UMV-estimation of the location parameter itself is treated in Section 8. Very recently, this subject has also been studied by Takeuchi [24]. If the tail of F tends to zero rapidly enough, it turns out that no UMV-estimator of θ exists. In Section 9 we prove that no polynomial can be a UMV-estimator, unless the distribution is normal. The final section contains other more traditional aspects of the problem.

2. Some results from harmonic analysis. For the theory that will follow we need certain results from harmonic analysis.

The class of Borel measurable functions g on R, satisfying $\sup_x |g(x)|/(1 + |x|)^N < \infty$, where N is an integer $\geqq 0$, is called B_N. We set $B_\infty = \lim_{N\to\infty} B_N$. If g coincides almost everywhere with respect to the Lebesgue measure (a.e. (L)) with a function belonging to B_N, we write $g \in_L B_N$. If $g \in_L B_N$ but $g \notin_L B_{N-1}$, we say that g belongs effectively to B_N.

For $g \in_L B_\infty$, the real set $S(g)$, called the spectrum of g, is defined as the support of the Fourier transform of g. The Fourier transform is usually a generalized function (distribution). Concerning these concepts, see Donoghue [3] Chapter II and Chapter III, Section 46. However, note that in our paper the Fourier transform of an integrable function f is defined by $\hat{f}(\zeta) = \int \exp\{i\zeta x\} f(x)\, dx$.

LEMMA 2.1. *Let $g \in B_N$ and let $d\mu$ be a measure, not certainly positive or real, satisfying $\int (1 + |x|)^N |d\mu(x)| < \infty$. If the convolution $g * d\mu$ is zero everywhere, then $\int \exp\{i\zeta x\}\, d\mu(x) = 0$ for all $\zeta \in S(g)$.*

A set of the form $\{ja;\ j \in Z\}$, where $0 < a < \infty$, is called arithmetic.

LEMMA 2.2. *Let g be a function belonging to B_N, effectively belonging to B_M $(M \leqq N)$, and having spectrum contained in an arithmetic set. Then a point $\zeta \in S(g)$ exists such that $\int x^k \exp\{i\zeta x\}\, d\mu(x) = 0$, $k = 0, \cdots, M$, for all measures $d\mu$, satisfying $\int (1 + |x|)^N |d\mu(x)| < \infty$ and $g * d\mu = 0$.*

LEMMA 2.3. *Let $g \in_L B_N$. Then $S(g)$ is contained in the arithmetic set $\{ja;\ j \in Z\}$* iff *the $N + 1$th difference $\Delta_{2\pi/a}^{N+1} g$ is zero* a.e. (L). *Further*, $S(g) = \{0\}$ iff *g coincides* a.e. (L) *with a polynomial of degree at most N.*

LEMMA 2.4. *For $g, h \in_L B_\infty$, we have $S(gh) \subset \overline{S(g) + S(h)}$. (The bar denotes closure.)*

LEMMA 2.5. *Let $g \in_L B_N$ and let $d\mu$ be an absolutely continuous measure for which $\int (1 + |x|)^{N+\frac{1}{2}}|d\mu(x)| < \infty$. If, for all $\zeta \in S(g)$ and for all $k = 0, \cdots, N$, $\int x^k \exp\{i\zeta x\}\, d\mu(x) = 0$, then $g * d\mu = 0$.*

Lemma 2.2 is very essential for us and a proof will be given in an appendix. The arguments used there also prove Lemma 2.3. Concerning the other lemmas we only make some remarks. Lemma 2.1 is for $N = 0$ only a slight extension of the Wiener Tauberian theorem. For a proof, see Donoghue [3] page 232. However, it is assumed there that $d\mu$ is absolutely continuous but that is not necessary. L. Gårding has communicated a way of generalizing this proof to cover also the case $N > 0$. Lemma 2.4 is quite elementary. Lemma 2.5 is for $N = 0$ a famous unpublished theorem of Beurling. After certain changes the arguments given on page 233 in [3] yield a proof. The book by Donoghue is based on lectures by L. Hörmander and the lecture notes (in Swedish) contain an explicit proof for $N = 0$. With certain modifications this proof also works in the case $N > 0$.

3. The one observation case. In this section we consider the case when there is given only a single observation x from a population with df $F(x - \theta)$, where F is considered to be known.

Suppose only $2N$ moments of F are known to be finite. Then, since we measure the goodness of an estimator by the variance, it is to some extent reasonable to consider only estimators $g(x)$, where $g \in B_N$. Such estimators are called B_N-estimators. If $g(x)$ is a UMV-estimator in the class of B_N-estimators, we say that $g(x)$ is B_NUMV. A B_N-estimator $g(x)$, having finite variance, is B_NUMV iff $E_\theta[g(x)h(x)] \equiv 0$ for all (complex) B_N-estimators $h(x)$ such that $E_\theta[h(x)] \equiv 0$ and $E_\theta[|h(x)|^2] < \infty$. This is just a variant of the UMV-criterion.

Letting φ stand for the ch.f. of F, i.e. $\varphi(\zeta) = \int \exp\{i\zeta x\}\, dF(x)$, we have

THEOREM 3.1A. *If $g(x)$ is a B_NUMV-estimator of its mean value, then $S(g) \subset D$, where $D = \{\zeta;\ \varphi(\eta - \zeta) = 0$ for all zeroes η of $\varphi\}$.*

PROOF. Let η be any zero of φ. We have $E_\theta[\exp\{i\eta x\}] \equiv 0$. Therefore, from the UMV-criterion, $E_\theta[g(x)\exp\{i\eta x\}] \equiv 0$. Equivalently, $g * \exp\{-i\eta x\}\, dF^* = 0$, where $dF^*(x) = dF(-x)$. So it follows from Lemma 2.1 that, for all $\zeta \in S(g)$, $\int \exp\{i\zeta x\} \cdot \exp\{-i\eta x\}\, dF^*(x) = 0$, i.e. $\varphi(\eta - \zeta) = 0$. The theorem is proved.

Let $N(\varphi)$ be the set of zeroes of φ. Since $N(\varphi)$ is symmetric, it is easy to see that D is equal to the set of periods of $N(\varphi)$ (i.e. periods of the indicator function of $N(\varphi)$). The set D is a closed subgroup of R and hence D can only be of three different types, namely:

(i) $D = R$
(ii) $D = \{ja;\ j \in Z\}$, $a > 0$
(iii) $D = \{0\}$.

The first alternative holds iff $N(\varphi) = \varnothing$, the second one iff $N(\varphi)$ is a periodic set, and the third one otherwise. The ch.f. $\varphi(\zeta) = (\sin c\zeta)/c\zeta$ (uniform distribution over $[-c, c]$) provides an example of case (iii) and $\varphi(\zeta) = \cos\zeta$ (two-point distribution) an example of case (ii).

An immediate consequence of Theorem 3.1A and Lemma 2.3 is that in case (ii) only estimators $g(x)$, where $\Delta_{2\pi/a}^{N+1}g$ is zero a.e. (L), can be B_NUMV. For $g(x)$ to have this property it is in case (iii) necessary that g is a.e. (L) equal to a polynomial of degree at most N. So, for e.g. the uniform distribution only constants are B_0UMV-estimators. In fact, the next theorem shows that also a B_NUMV-estimator $(N > 0)$ must be constant for this distribution.

THEOREM 3.1B. *Let F have at least $2N$ moments. Further, let g belong to B_N and effectively to B_M $(0 < M \leqq N)$. If $g(x)$ is a B_NUMV-estimator, then every zero of φ is of multiplicity at least $N + M + 1$.*

If $g(x) = x$, this theorem is almost immediate. For, if $\varphi(\eta) = 0$, then $E_\theta[\exp\{i\eta x\}] \equiv 0$, and hence, by the UMV-criterion, $E_\theta[x \exp\{i\eta x\}] \equiv 0$. Repeated use of the criterion yileds $E_\theta[x^k \exp\{i\eta x\}] \equiv 0$, $k = 0, 1, \cdots, N + 1$. Setting $\theta = 0$, we have the theorem.

PROOF. If $N(\varphi)$ is empty, there is nothing to prove. In the contrary case we have shown above that $S(g)$ is contained in an arithmetic set. Let η be any zero of φ. From the proof of Theorem 3.1A we have

$$g * \exp\{-i\eta x\}\, dF^* = 0 \,. \tag{3.1}$$

Now, let ζ be the point in $S(g)$ mentioned in Lemma 2.2. It follows from (3.1) and this lemma that $\eta - \zeta$ is a zero of $\varphi, \varphi', \cdots, \varphi^{(M)}$. Since $\varphi(-\xi) = \overline{\varphi(\xi)}$, also $\zeta - \eta$ is a zero of these derivatives. Hence $x^k \exp\{i(\zeta - \eta)x\}$, $k = 0, \cdots, M$, are unbiased estimators of zero. Therefore

$$E_0[g(x + \theta)(x + \theta)^k \exp\{i(\zeta - \eta)(x + \theta)\}] \equiv 0\,, \qquad k = 0, \cdots, M\,.$$

By induction it then follows that

$$E_0[g(x + \theta)x^k \exp\{i(\zeta - \eta)x\}] \equiv 0\,, \qquad k = 0, \cdots, M\,,$$

i.e.

$$g * x^k \exp\{i(\eta - \zeta)x\}\, dF^* = 0\,, \qquad k = 0, \cdots, M\,.$$

Using again Lemma 2.2, we find that $-\eta$ is a zero of $\varphi, \varphi', \cdots, \varphi^{(2M)}$, and so is η. If $M = N$, nothing remains to prove. If $M < N$, it follows as above that

$$g * x^k \exp\{-i\eta x\}\, dF^* = 0\,, \qquad k = 0, \cdots, \min(2M, N)\,.$$

Then iterating sufficiently many times the argumentation above, the rest of the proof is easily accomplished.

Now we state a converse of Theorems 3.1A and B.

THEOREM 3.2. *Let F be absolutely continuous and have moments up to order $2N + \frac{1}{2}$. If $g \in B_M$ $(M \leqq N)$, if $S(g) \subset D$ (D is defined above), and if all real zeroes of φ have multiplicity at least $N + M + 1$, then $g(x)$ is a B_NUMV-estimator of its mean value.*

PROOF. Let $h(x)$ be any unbiased B_N-estimator of zero. It follows from Lemma 2.1 that $S(h) \subset N(\varphi)$. Using also Lemma 2.4, the assumption that $S(g) \subset D$, the definition of D, and the fact that $N(\varphi)$ is a closed symmetric set, we get

$$S(gh) \subset \overline{S(g) + S(h)} \subset \overline{S(g) + N(\varphi)} \subset \overline{D + N(\varphi)} = \overline{N(\varphi)} = N(\varphi) .$$

Evidently, $gh \in B_{N+M}$. Since $M \leqq N$, F has at least $N + M + \frac{1}{2}$ finite moments. An application of Lemma 2.5 yields $gh * dF^* = 0$, i.e. $E_\theta[g(x)h(x)] \equiv 0$, and the proof is complete. Observe that if $N(\varphi) = \emptyset$ or if $M < N$, it is only necessary to assume the existence of $2N$ finite moments of F. □

Obviously it is possible to let N take the value infinity in the theorems given.

Let F be absolutely continuous and have at least $2N$ ($N > 0$) finite moments and mean zero. If $N = 1$, we also suppose $\int |x|^{\frac{5}{2}}\, dF(x) < \infty$. Consider $g(x) = x$. Of course, $S(g) = \{0\}$. From Theorem 3.1B and Theorem 3.2 it follows that x is a B_NUMV-estimator of θ iff every zero of φ has multiplicity at least $N + 2$.

It is not hard to construct a ch.f. which is zero outside an interval and real-valued and strictly positive inside it, and furthermore infinitely differentiable. For this purpose we take a symmetric infinitely differentiable function ψ with $\psi(\zeta) = 0$ when $|\zeta| \geqq 1$, and $\psi(\zeta) > 0$ when $|\zeta| < 1$. We then form $\varphi = c(\psi * \psi)$, where c is chosen such that $\varphi(0) = 1$. This φ is a ch.f. and fulfills the requirements. (Cp. Donoghue [3] pages 183–184.) The corresponding df F is absolutely continuous and has moments of all orders and mean value zero. Theorem 3.2 for $N = \infty$ shows that x is a B_∞UMV-estimator of θ. However, the family $dF(x - \theta)$ is not B_∞-complete.

If we just know that $\int \exp\{|x|^\alpha\}\, dF(x) < \infty$, $0 < \alpha < 1$, then it is not so unnatural to consider only estimators $g(x)$ which satisfy

$$\sup_x |g(x)|/\exp\{|x|^{\beta(g)}\} < \infty , \qquad 0 \leqq \beta(g) < \alpha .$$

We only mention that it is possible to extend (at least partly) the theory above to cover also this situation. This depends mostly on the fact that $\int \log(\exp\{|x|^\alpha\})/(1 + x^2)\, dx < \infty$ (non-quasianalytic case).

Let now F satisfy e.g. $\int \exp\{|x|\}\, dF(x) < \infty$ and have mean zero. Suppose x is a UMV-estimator of θ in the class K of estimators $g(x)$ satisfying, for all $k \geqq 0$, $\sup_x |x^k g(x)|/\exp\{|x|/2\} < \infty$. It follows that $\int x^k \cdot h(x + \theta)\, dF(x) \equiv 0$, $k = 0, 1, \cdots$, for all $h \in K$ such that $\int h(x + \theta)\, dF(x) \equiv 0$. Since, easily checked, $h(x + \theta)\, dF(x)$ has an analytic Fourier transform, we get $h(x + \theta)\, dF(x) = 0$. Hence $h = 0$ a.s. $dF(x - \theta)$ for all θ. So from the UMV-assumption for x we have proved K-completeness for $dF(x - \theta)$. (The essential point is that $\int \log(\exp\{|x|\})/(1 + x^2) = \infty$; quasianalytic case.)

Next we give a characterization of the normal distribution. For the concepts and results used, see Lukacs [17] or Ramachandran [22].

THEOREM 3.3. *Suppose that φ is an entire analytic function of finite order. If, for some function g belonging to B_∞ and not being* a.e. (L) *equal to a periodic function,* *the statistic $g(x)$ is a* UMV-*estimator of its mean value, then the distribution is normal* (*or degenerate*).

REMARK. Let $T(x) = 1 - F(x) + F(-x)$. Then φ has order $1 + 1/\alpha$ ($\alpha > 0$) iff

$$\liminf_{x\to+\infty} \log\{\log(1/T(x))\}/\log x = 1 + \alpha .$$

PROOF. Of course, the proof of Theorem 3.1A shows that $S(g) \subset D'$, where $D' = \{\zeta \in R;\ \varphi(\eta - \zeta) = 0$ for all complex zeroes η of $\varphi\}$. It is easy to check that D' is a closed subgroup of R. Assume that (complex) zeroes of φ exist. Hence D' is arithmetic or contains the point zero only. By the assumption that g is not periodic and by Lemma 2.3, we see that g cannot effectively belong to B_0. So g effectively belongs to some class B_M, where $M > 0$. Using the proof of Theorem 3.1B with certain changes (notice that $\varphi(\eta) = 0 \Rightarrow \varphi(-\bar{\eta}) = 0$), we find that every zero of φ is also a zero of all the derivatives of φ. As φ is analytic, this is an obvious contradiction, proving that φ vanishes nowhere. By the Hadamard factorization theorem for functions of finite order, we have $\varphi(\zeta) = \exp\{P(\zeta)\}$, where P is a polynomial. Then a well-known theorem of Marcinkiewicz asserts that the degree of P cannot exceed two, and the theorem stands proved.

REMARK. The aperiodicity assumption is necessary. For, let F be a lattice-df with span h (i.e. dF vanishes outside some set $\{a + jh,\ j \in Z\}$) and let g have period h. Then $g(x) = g(\theta + a)$ a.s. $dF(x - \theta)$. Hence $g(x)$ estimates its mean value without error and is therefore a UMV-estimator.

We end this section with an odd result.

THEOREM 3.4. *If dF has compact support but is not degenerate, no strictly monotone statistic $g(x)$ can be a* UMV-*estimator.*

PROOF. The ch.f. φ is an entire function of order 1. Hence it has a zero η somewhere in the complex plane. We have $E_\theta[\exp\{i\eta x\}] \equiv 0$. Using that $g(x)$ is a UMV-estimator, we find that $E_\theta[(g(x))^j \exp\{i\eta x\}] = 0$, for all nonnegative integers j and for all θ. Hence, setting $g(x) = y$, $E_\theta[y^j E_\theta[\exp\{i\eta x\} \mid g(x) = y]] = 0$, $j = 0, 1, 2, \cdots$. Now, since $E_\theta[\exp\{i\eta x\} \mid g(x) = y]$ is (or can be defined to be) zero outside a bounded set (depending on θ), the Weierstrass approximation theorem can be applied, showing that $E_\theta[\exp\{i\eta x\} \mid g(x) = y] = 0$ a.s. for all θ. As g is injective, this is a contradiction ending the proof.

4. The case of two or more observations. Here we suppose that n ($n \geqq 2$) observations $x_1, \cdots, x_n$ are available. To abbreviate the formulas we write as though $n = 2$.

The following proposition is a basic result for the rest of this paper, even though it will not always be explicitly utilized.

THEOREM 4.1. *If $g(x_1, x_2)$ is a* UMV-*estimator of its mean value, then there exists a function $C(\zeta, \theta)$ such that the relation*

$$E_0[g(x_1 + \theta, x_2 + \theta)\exp\{i\zeta_1 x_1 + i\zeta_2 x_2\}] = C(\zeta_1 + \zeta_2, \theta)\varphi(\zeta_1)\varphi(\zeta_2) \tag{4.1}$$

holds for all (*real*) ζ_1, ζ_2 *and* θ.

PROOF. It is equivalent to show that a function $C_1(\zeta, \theta)$ exists such that

$$E_\theta[g(x_1, x_2)\exp\{i\zeta_1 x_1 + i\zeta_2 x_2\}] = C(\zeta_1 + \zeta_2, \theta)\varphi(\zeta_1)\varphi(\zeta_2) . \tag{4.2}$$

Consider the points on the line $\zeta_1 + \zeta_2 = \zeta =$ constant. Two cases arise depending on whether:

(i) no point (ζ_1, ζ_2) on the line exists such that $\varphi(\zeta_1) \neq 0$ and $\varphi(\zeta_2) \neq 0$; or:
(ii) at least one such point exists.

In case (i) $\exp\{i\zeta_1 x_1 + i\zeta_2 x_2\}$ is an unbiased estimator of zero, and hence any choice of $C_1(\zeta, \theta)$ suits. In case (ii), we fix one point (η_1, η_2) such that $\varphi(\eta_1) \neq 0$, $\varphi(\eta_2) \neq 0$, and define

$$C_1(\zeta, \theta) = E_\theta[g(x_1, x_2)\exp\{i\eta_1 x_1 + i\eta_2 x_2\}]/\varphi(\eta_1)\varphi(\eta_2) .$$

Since the statistic

$$h(x_1, x_2) = \exp\{i\zeta_1 x_1 + i\zeta_2 x_2\} - (\varphi(\zeta_1)\varphi(\zeta_2)/\varphi(\eta_1)\varphi(\eta_2))\exp\{i\eta_1 x_1 + i\eta_2 x_2\}$$

has mean value zero for all θ, it follows that $E_\theta[g(x_1, x_2)h(x_1, x_2)] = 0$, which is exactly the desired result. □

Using formal calculations, we are also able to show that (4.1) is a sufficient condition for g to be UMV. We introduce a family of linear operators L_θ, operating on functions with argument y, by $L_\theta(\exp\{i\zeta y\}) = C(\zeta, \theta)$. This means that L_θ is the Fourier transform of $C(\zeta, \theta)$. By extension the L_θ's will be defined on an appropriately large space of functions. Supposing that the df has a density function f, and letting $\widehat{\ }$ stand for the two-dimensional Fourier transform, we see that (4.1) can be written as

$$\widehat{g(x_1 + \theta, x_2 + \theta)f(x_1)f(x_2)}(\zeta_1, \zeta_2) = L_\theta[\exp\{i(\zeta_1 + \zeta_2)y\}] \cdot \widehat{f(x_1)f(x_2)}(\zeta_1, \zeta_2) .$$

Since both L_θ and the Fourier transform are linear,

$$\begin{aligned} L_\theta[\exp\{i(\zeta_1 + \zeta_2)y\}] \cdot \widehat{f(x_1)f(x_2)}(\zeta_1, \zeta_2) &= L_\theta[\exp\{i(\zeta_1 + \zeta_2)y\} \cdot \widehat{f(x_1)f(x_2)}(\zeta_1, \zeta_2)] \\ &= L_\theta[\widehat{f(x_1 - y)f(x_2 - y)}(\zeta_1, \zeta_2)] \\ &= \widehat{L_\theta[f(x_1 - y)f(x_2 - y)]}(\zeta_1, \zeta_2) . \end{aligned}$$

The uniqueness theorem for the Fourier transform therefore implies

$$g(x_1 + \theta, x_2 + \theta)f(x_1)f(x_2) = L_\theta[f(x_1 - y)f(x_2 - y)] . \tag{4.3}$$

Let now h be an unbiased estimator of zero. Then

$$\begin{aligned} E_\theta[gh] &= \int\int g(x_1 + \theta, x_2 + \theta)h(x_1 + \theta, x_2 + \theta)f(x_1)f(x_2)\,dx_1\,dx_2 \\ &= \int\int h(x_1 + \theta, x_2 + \theta)L_\theta[f(x_1 - y)f(x_2 - y)]\,dx_1\,dx_2 \\ &= L_\theta[\int\int h(x_1 + \theta, x_2 + \theta)f(x_1 - y)f(x_2 - y)\,dx_1\,dx_2] = L_\theta[0] = 0 . \end{aligned}$$

Hence g is a UMV-estimator.

It should be observed that for a fixed θ (4.1) is instead a necessary (and formally a sufficient) condition for g to be a locally best (at the point θ) unbiased estimator of its mean value. The formal calculations above indicate equivalence between (4.1) and the operator condition (4.3) given by Stein [23] and others.

5. A theorem of Lehmann and Scheffé. In this section we give a new proof of a result first proved by Lehmann and Scheffé [15]. Then we try to find generalizations. We write as though $n = 2$.

THEOREM 5.1. *For the uniform distribution over* $[\theta - c, \theta + c]$, *no nonconstant parametric function can be* UMV-*estimated.*

PROOF. It is no restriction to suppose $c = \pi$. Then $\varphi(\zeta) = (\sin \pi\zeta)/\pi\zeta$. Let $g(x_1, x_2)$ be a UMV-estimator of $g^*(\theta)$. As all integers $\neq 0$ are zeroes of φ, it follows from Theorem 4.1 that

$$(1/2\pi)^2 \int\int_{-\pi \leqq x_1, x_2 \leqq \pi} g(x_1 + \theta, x_2 + \theta) \exp\{ijx_1 + ikx_2\}\, dx_1\, dx_2 = 0\,, \qquad j, k \in Z,\ (j, k) \neq (0, 0)\,.$$

But this means that all the Fourier coefficients c_{jk}, $(j, k) \neq (0, 0)$, of the function $g(x_1 + \theta, x_2 + \theta)$ are zero, and hence it must be constant a.e. $(L \times L)$ in the interval $-\pi \leqq x_1, x_2 \leqq \pi$. As this is true for all θ, g is constant a.e. $(L \times L)$ in the strip $|x_1 - x_2| \leqq 2\pi$. Therefore also $g^*(\theta)$ is constant and the theorem is proved.

Observe that the proof also applies to the case $n = 1$. Lehmann and Scheffé's proof does not.

For what other distributions does the conclusion of Theorem 5.1 hold? It is tempting to believe that if for $n = 1$ only constants can be UMV-estimated, then the same must be true for $n \geqq 2$. The next theorem is a partial solution of this problem. (Cp. Theorem 3.1.A).

THEOREM 5.2. *Let F have a strictly positive density function f. Further, let at least $2N$ ($N \geqq 0$) moments be finite. Suppose also that*

$$D = \{\zeta;\ \varphi(\eta - \zeta) = 0 \ \textit{ for all } \ \eta \in N(\varphi)\} = \{0\}\,.$$

Then, if a statistic $g(x_1, x_2)$, satisfying $\sup_{x_1, x_2} |g(x_1, x_2)|/(1 + x_1^2 + x_2^2)^{N/2} < \infty$, *is a* UMV-*estimator of its mean value, this mean value must be constant.*

PROOF. Let ζ_1 be any (real) zero of φ and ζ_2 any (real) number. Applying Fubini's theorem to (4.2), we find

$$\int \exp\{i\zeta_2 x_2\} f(x_2 - \theta)\left(\int g(x_1, x_2) \exp\{i\zeta_1 x_1\} f(x_1 - \theta)\, dx_1\right) dx_2 = 0\,.$$

Since this is true for all ζ_2, the uniqueness theorem for the Fourier transform and the fact that f is strictly positive yield

$$\int g(x_1, x_2) \exp\{i\zeta_1 x_1\} f(x_1 - \theta)\, dx_1 = 0 \qquad \text{for almost all } (L)\ x_2\,. \tag{5.1}$$

The possible exceptional set may depend on θ. However, (5.1) holds simultaneously for all rational θ for almost all (L) x_2. It is not difficult to show that the left-hand side of (5.1) is a continuous function of θ. Hence, after multiplication by $\exp\{-i\zeta_1\theta\}$,

$$g(\cdot, x_2) * \exp\{-i\zeta_1(\cdot)\}f^* = 0 \qquad \text{for almost all } (L)\ x_2 .$$

Comparison with the proof of Theorem 3.1A and the theory following it shows that, for almost all (L) x_2, $g(x_1, x_2)$ is a.e. (L) equal to a polynomial in x_1. In view of the fact that this is true also if x_1 and x_2 are interchanged, it easily follows that $g(x_1, x_2)$ is equal to a polynomial in (x_1, x_2) a.e. $(L \times L)$. This polynomial is then also a UMV-estimator with the same mean value as $g(x_1, x_2)$. If the polynomial is constant, the theorem is immediate. If not, Theorem 9.1 to be proved later asserts that F is a normal df. This is a contradiction since the normal ch.f. has no zeroes. The proof is finished.

As seen earlier, $D = \{0\}$ iff φ has zeroes and $N(\varphi)$ is not periodic. If $F = G * H$, where G is a lattice df whose ch.f. has (real) zeroes, and H is a normal df, then $g(\bar{x})$, which will be shown in Section 6, is a UMV-estimator, provided that g is periodic with an appropriate period. This proves that the condition $D = \{0\}$ in Theorem 5.2 cannot be replaced by the weaker condition $N(\varphi) \neq \emptyset$.

6. UMV-characterizations. In this section we shall UMV-characterize certain distributions. Kagan, Linnik, and Rao [10] proved that the sample mean $\bar{x}$ $(n \geqq 3)$ is a UMV-estimator of θ only if F is normal. The same conclusion holds if instead $P(\bar{x})$, where P is a polynomial, is supposed to be a UMV-estimator of its mean value (see Kagan [11]). The following proposition contains a more general result.

THEOREM 6.1. *Let F have at least $2N$ $(N \geqq 0)$ moments. Suppose that, for some function g, belonging to B_N and not being* a.e. (L) *equal to a periodic function, the statistic $g(\bar{x})$, where $n \geqq 2$, is a* UMV*-estimator. Then F is normal.*

PROOF. It suffices to consider the case $n = 2$ only. Let $h(x_1, x_2)$ be an unbiased bounded estimator of zero. Then also $h(x_1 + \lambda, x_2 + \lambda)$, where λ is arbitrary, is an unbiased estimator of zero. So from the suppositions it easily follows that

$$E_\theta[g(\bar{x} + \theta)h(x_1, x_2)] = 0 ,$$

and hence

$$\int g(x + \theta)E_\theta[h(x_1, x_2) \mid \bar{x} = x]\, dF_{\bar{x}}(x) = E_0[g(\bar{x} + \theta)E_0[h(x_1, x_2) \mid \bar{x}]] = 0 . \tag{6.1}$$

So, using Lemma 2.1, for each point $\zeta \in S(g)$, we find

$$E_0[\exp\{-i\zeta\bar{x}\}h(x_1, x_2)] = E_0[\exp\{-i\zeta\bar{x}\}E_0[h(x_1, x_2) \mid \bar{x}]] = \int \exp\{-i\zeta x\}E_0[h(x_1, x_2) \mid \bar{x} = x]\, dF_{\bar{x}}(x) = 0 . \tag{6.2}$$

Now, first we take into consideration the case when $S(g)$ is not contained in any arithmetic set. In this case the ch.f. φ has no zeroes. For, if the contrary holds, also the ch.f. of the variable $\bar{x}$ has a zero. Then, considering $\bar{x}$ as one observation of a variable with df $F_{\bar{x}}(x - \theta)$, we find through Theorem 3.1A and the theory following it that $S(g)$ is included in an arithmetic set, which is a contradiction.

For points $\xi_1, \xi_2, \eta_1, \eta_2$, satisfying $\xi_1 + \xi_2 = \eta_1 + \eta_2$, we set

$$h(x_1, x_2) = \frac{\exp\{i\xi_1 x_1 + i\xi_2 x_2\}}{\varphi(\xi_1)\varphi(\xi_2)} - \frac{\exp\{i\eta_1 x_1 + i\eta_2 x_2\}}{\varphi(\eta_1)\varphi(\eta_2)} .$$

From (6.2) we have, for $\zeta \in S(g)$,

$$\frac{E_0[\exp\{-i\zeta\bar{x}\} \cdot \exp\{i\xi_1 x_1 + i\xi_2 x_2\}]}{\varphi(\xi_1)\varphi(\xi_2)} = \frac{E_0[\exp\{-i\zeta\bar{x}\} \cdot \exp\{i\eta_1 x_1 + i\eta_2 x_2\}]}{\varphi(\eta_1)\varphi(\eta_2)} .$$

Since this is true whenever $\xi_1 + \xi_2 = \eta_1 + \eta_2$, the left-hand side depends only on the sum $\xi_1 + \xi_2$. After some manipulations we find

$$\log\left[\frac{\varphi(\xi_1 - \tau)}{\varphi(\xi_1)}\right] + \log\left[\frac{\varphi(\xi_2 - \tau)}{\varphi(\xi_2)}\right] = C(\xi_1 + \xi_2) ,$$

where $\tau = \zeta/2$ and C is some function. This is Cauchy's functional equation, and it follows that there exist constants A_ζ and B_ζ such that

$$\log\left[\frac{\varphi(\xi - \tau)}{\varphi(\xi)}\right] = A_\zeta \xi + B_\zeta . \tag{6.3}$$

Hence the second derivative of $\log \varphi$ is a periodic function with period τ. This is true for any $\zeta \in S(g)$. It is easy to show that then also the numbers $j_1\zeta_1/2 + j_2\zeta_2/2$, where $j_1, j_2 \in Z$ and $\zeta_1, \zeta_2 \in S(g)$, must be periods. Since it has been assumed that $S(g)$ is not contained in any arithmetic set, infinitely small numbers of this type exist. Therefore $d^2/d\xi^2 \log \varphi(\xi)$ has infinitely small periods and must then be constant. This is what we wanted to prove. (If two derivatives do not exist, finite differences can be used instead of derivatives.)

It remains to give a proof of the theorem for the case that $S(g)$ is included in an arithmetic set. As g is not a.e. (L) equal to a periodic function, g effectively belongs to some class B_M $(M > 0)$. Therefore an application of Lemma 2.2 to (6.1) shows that a point ζ exists such that

$$E_0[\exp\{-i\zeta\bar{x}\}h(x_1, x_2)] = 0 ; \tag{6.4}$$

$$E_0[\bar{x} \cdot \exp\{-i\zeta\bar{x}\}h(x_1, x_2)] = 0 . \tag{6.5}$$

Let $\tau = \zeta/2$. If $\varphi(\tau) = 0$, we may set $h(x_1, x_2) = \exp\{i\tau(x_1 + x_2)\}$. For this choice of h we find that $\exp\{-i\zeta\bar{x}\}h(x_1, x_2) = 1$, and this evidently contradicts (6.4). Thus $\varphi(\tau) \neq 0$. By employing instead the expression of $h(x_1, x_2)$ used in the first part of this proof, (6.4) and (6.5) lead, after some argumentation, to the equations

$$\varphi(\xi_1 - \tau)\varphi(\xi_2 - \tau)/\varphi(\xi_1)\varphi(\xi_2) = C_1(\xi_1 + \xi_2)$$

$$\varphi'(\xi_1 - \tau)\varphi(\xi_2 - \tau)/\varphi(\xi_1)\varphi(\xi_2) + \varphi(\xi_1 - \tau)\varphi'(\xi_2 - \tau)/\varphi(\xi_1)\varphi(\xi_2) = C_2(\xi_1 + \xi_2) ,$$

where C_1 and C_2 are two functions. The equations are valid for all ξ_1, ξ_2 in some neighborhood of the point τ. By division we find

$$\varphi'(\xi_1 - \tau)/\varphi(\xi_1 - \tau) + \varphi'(\xi_2 - \tau)/\varphi(\xi_2 - \tau) = C_2(\xi_1 + \xi_2)/C_1(\xi_1 + \xi_2) .$$

This is Cauchy's functional equation. Hence $\varphi'(\xi)/\varphi(\xi)$ is linear in some neighborhood of the origin. The rest of the proof proceeds as in Kagan, Linnik, and Rao [10].

If g is periodic, also other distributions than the normal one can appear. To see this, assume that φ vanishes nowhere. By recalling (6.3), it is not hard to verify that a quadratic polynomial P (depending on A_ζ and B_ζ) can be found such that $\varphi(\xi)\exp\{-P(\xi)\}$ is a periodic function ψ with period τ. Hence

$$\varphi(\xi) = \psi(\xi)\exp\{P(\xi)\}\,.$$

We may set $P(0) = 0$ and $\psi(0) = 1$, and therefore $\exp\{P(\xi)\}$ is the ch.f. of a normal df (we neglect the fact that e.g. $P''(0)$ may be complex-valued). If e.g. two finite moments of F exist, then φ is twice continuously differentiable and so is ψ. It follows from general theory for Fourier series that we have

$$\psi(\xi) = \sum p_k \exp\{i2\pi k\xi/\tau\}\,,$$

where $\sum |p_k| < \infty$ and $\sum p_k = 1$. Even though some of the numbers p_k are negative, it is possible that φ is a ch.f. Let X be the rv of which the x_j's are observations. Thus, from the UMV-assumption and some regularity conditions we have shown that we can write

$$X = Y + Z\,,$$

where Z is a normally distributed rv with unknown location θ (+ constant), where Y is an rv taking values (we suppose $\tau = 2\pi$) $0, \pm 1, \pm 2, \cdots$ (with possibly negative probability), and where Y and Z are independent.

It is more remarkable that the converse is true, i.e. if $X = Y + Z$, Y and Z as above, and x_j, $j = 1, \cdots, n$, are independent observations of X, then $g(\bar{x})$ is a UMV-estimator of its mean value, provided that g has period $1/n$.

To prove this, let y_j, z_j, $j = 1, \cdots, n$, be imagined independent observations of Y and Z such that $x_j = y_j + z_j$. As g has period $1/n$, $g(\bar{x}) = g(\bar{z})$. Let $h(x_1, \cdots, x_n) = h(y_1 + z_1, \cdots, y_n + z_n)$ be any unbiased estimator of zero. We then have

$$0 = E_\theta[h(y_1 + z_1, \cdots, y_n + z_n)] = E_\theta[E_\theta[h(y_1 + z_1, \cdots, y_n + z_n) \,|\, \bar{z}]]\,.$$

Since the distribution of Y does not depend on θ and Z has a normal df, $\bar{z}$ is a complete sufficient statistic for θ. Therefore $h^*(\bar{z}) = E_\theta[h(y_1 + z_1, \cdots, y_n + z_n) \,|\, \bar{z}]$ is independent of θ and $h^*(\bar{z}) = 0$. Hence

$$E_\theta[g(\bar{x})h(x_1, \cdots, x_n)] = E_\theta[g(\bar{z})h^*(\bar{z})] = 0\,.$$

So from the UMV-criterion the desired result follows. (The author does not know whether this result has been obtained before.) Observe that if Z is degenerate, estimation without error is possible.

REMARK. Omitting the proof, we mention that if $g(\bar{x})$, $g \in B_\infty$ arbitrary but not constant a.e. (L), is a UMV-estimator for all sample sizes, then the distribution must be normal.

Above we have found all df's admitting UMV-estimators of a certain form, namely $g(\bar{x})$. Now the same will be done for estimators of another simple form.

THEOREM 6.2. *Let g be a Borel measurable function which is locally integrable (L) on R and not constant* a.e. (L). *Suppose $[g(x_1) + \cdots + g(x_n)]/n$ is a* UMV-*estimator of $E_\theta[g(x_1)]$ for all sample sizes $n \geqq 2$. Then F is a normal* df (*and g is a linear function*), *the* df *of the logarithm of some power of a gamma variable* (*and $g(x) = c_1 a \cdot \exp\{ax\} + c_2$, where a, c_1, c_2 are constants*), *or a lattice*-df (*and g is a periodic function*).

PROOF. Let φ be the ch.f. of F. Using Theorem 4.1, we get

$$\begin{aligned}(6.6)\qquad E_0[g(x_1+\theta)\exp\{i\zeta_1 x_1\}] + \frac{E_0[g(x_2+\theta)\exp\{i\zeta_2 x_2\}]\cdot\varphi(\zeta_1)}{\varphi(\zeta_2)} + \cdots \\ + \frac{E_0[g(x_n+\theta)\exp\{i\zeta_n x_n\}]\cdot\varphi(\zeta_1)}{\varphi(\zeta_n)} = C_n(\zeta_1+\cdots+\zeta_n, \theta)\cdot\varphi(\zeta_1)\,,\end{aligned}$$

provided that $\varphi(\zeta_2), \cdots, \varphi(\zeta_n)$ are all different from zero. Let I_0 be the largest open interval containing the origin on which φ does not vanish. Let $\zeta_1, \cdots, \zeta_n \in I_0$ and divide both sides of (6.6) by $\varphi(\zeta_1)$. We get Cauchy's functional equation and hence functions $A(\theta)$ and $B(\theta)$ exist so that for all $\zeta \in I_0$

$$(6.7)\qquad E_0[g(x_1+\theta)\exp\{i\zeta x_1\}] = (i\zeta A(\theta) + B(\theta))\cdot\varphi(\zeta)\,.$$

Of course, $B(\theta) = E_\theta[g(x_1)]$. However, (6.7) holds for all ζ. To see this, let ζ be arbitrary and choose n so large that $-\zeta/(n-1) \in I_0$ and set $\zeta_1 = \zeta$, $\zeta_2 = \cdots = \zeta_n = -\zeta/(n-1)$. Using (6.6) and (6.7) and observing that $B(\theta) = C_n(0, \theta)/n$, we then get

$$\begin{aligned}&E_0[g(x_1+\theta)\exp\{i\zeta x_1\}] \\ &\qquad = n\cdot B(\theta)\varphi(\zeta) - [(i\zeta_2 A(\theta) + B(\theta))\varphi(\zeta) + \cdots + (i\zeta_n A(\theta) + B(\theta))\varphi(\zeta)] \\ &\qquad = (i\zeta A(\theta) + B(\theta))\varphi(\zeta)\,,\end{aligned}$$

which is the desired result. Equivalently,

$$(6.8)\qquad \int (g(x+\theta) - B(\theta))\exp\{i\zeta x\}\,dF(x) = i\zeta A(\theta)\cdot\int \exp\{i\zeta x\}\,dF(x)\,.$$

It is easy to see that $A(\theta)$ and $B(\theta)$ are real-valued. Let now for a moment θ be fixed and suppose $A(\theta) \neq 0$. Then Khatri and Rao [13] have shown that the solution dF of (6.8) must have a strictly positive density function given by

$$(6.9)\qquad f(x) = \exp\left\{C(\theta) - \int_0^x \frac{g(y+\theta) - B(\theta)}{A(\theta)}\,dy\right\}, \qquad -\infty < x < +\infty\,,$$

where $C(\theta)$ is a normalization constant. In their proof they assumed g to be continuous. However, a new proof based on the theory of generalized functions (distributions) will show that it suffices to assume that g is locally integrable (L) on R (see [1]). If $A(\theta)$ is zero for some $\theta = \theta_0$ but not identically equal to zero, then a uniqueness theorem for Fourier transforms yields

$$g(x+\theta_0) = B(\theta_0) \qquad \text{a.s.} \quad (dF)\,.$$

Since dF has a strictly positive density function, this relation also holds a.e. (L) and therefore g is constant a.e. (L). In view of the assumptions, this is a contradiction. Hence $A(\theta)$ vanishes either nowhere or everywhere. Consider first the first alternative. Then (6.9) holds for all θ and it follows that both $C(\theta)$ and the continuous function

$$H(x) = \int_0^x \frac{g(y+\theta) - B(\theta)}{A(\theta)}\, dy$$

are independent of θ. Let $G(x) = \int_0^x g(y)\, dy$. Hence, for all θ and x,

$$G(x+\theta) - G(\theta) = B(\theta)x + A(\theta)H(x)\ . \tag{6.10}$$

Let δ be any number. Equation (6.10) yields

$$\begin{aligned} G(x+\theta+2\delta) - 2G(x+\theta+\delta) + G(x+\theta) \\ = A(\theta)(H(x+2\delta) - 2H(x+\delta) + H(x))\ , \end{aligned}$$

i.e.

$$\Delta_\delta{}^2 G(x+\theta) = A(\theta) \cdot \Delta_\delta{}^2 H(x)\ . \tag{6.11}$$

Now, H cannot be a linear function, for linearity of H implies via (6.11) that also G is linear, and hence g is constant a.e. (L), which is a contradiction. Thus, since H is not linear, there exists a sequence $(\delta_n)_1^\infty$ decreasing to zero such that for every δ_n an x_n can be found with property $\Delta_{\delta_n}^2 H(x_n) \neq 0$. So $\Delta_{\delta_n}^2 G(x_n+\theta) \neq 0$ for all θ and n (as $A(\theta)$ never vanishes). Hence $\Delta_{\delta_n}^2 G(y) \neq 0$ for all y and n. Thus it is permitted to take the logarithms of both sides of (6.11). We obtain

$$\log \Delta_{\delta_n}^2 G(x+\theta) = \log A(\theta) + \log \Delta_{\delta_n}^2 H(x)\ .$$

As this is just a variant of Cauchy's functional equation, we easily get

$$\Delta_{\delta_n}^2 G(x) = \exp\{ax + b_n\}\ , \tag{6.12}$$

where $a, b_1, b_2, \cdots$ are constants. If $a = 0$, (6.12) means that, for x and n fixed, all points $(x + k\delta_n, G(x + k\delta_n))$, $k \in Z$, lie on some parabola. Letting n tend to infinity and recalling that G is continuous, we find that G is a quadratic polynomial, and hence $g(x) = c_1 + c_2 x$ a.e. (L), where the constant $c_2 \neq 0$. Inserting this result in (6.9) and putting $\theta = 0$, we immediately conclude that F is normal. If $a \neq 0$, similar arguments will show that

$$G(x) = c_1 \exp\{ax\} + c_2 x + c_3\ .$$

The corresponding df is that of the logarithm of some power of a gamma variable.

It remains to study the case $A(\theta) \equiv 0$. For every θ we then have

$$g(x+\theta) = B(\theta) \qquad \text{a.s.} \quad (dF)\ . \tag{6.13}$$

This means that it is possible to estimate the parametric function $B(\theta)$ without any error. An application of Fubini's theorem gives

$$G(x+\theta) - G(x) = \int_0^\theta B(\theta')\, d\theta' \qquad \text{a.s.} \quad (dF)\ .$$

Calling the function on the right-hand side β, we thus have

$$G(x + \theta) = G(x) + \beta(\theta) \qquad \text{a.s.} \quad (dF) .$$

Translating the measure dF if necessary, we see that 0 can be supposed to be a point of increase of F. This means that every open interval containing the origin has positive measure with respect to dF. Since G is continuous and $G(0) = 0$, it therefore follows that $G(\theta) \equiv \beta(\theta)$, and hence, for all θ,

$$G(x + \theta) = G(x) + G(\theta) \qquad \text{a.s.} \quad (dF) . \tag{6.14}$$

It follows from (6.14) by induction that for all θ and all N

$$G(\textstyle\sum_1^N x_j + \theta) = G(\sum_1^N x_j) + G(\theta) \qquad \text{a.s.} \quad (dF \times \cdots \times dF) . \tag{6.15}$$

If F is not degenerate, it is further no restriction to assume that F has both positive and negative points of increase and we do so. If F is not a lattice-df, Lemma 2 in Feller [5] page 144 shows that, for any $y \in R$, there exist points of increase x_j, $j = 1, 2, \cdots$, of F such that $\sum_1^N x_j \to y$ when $N \to \infty$. Equation (6.15) therefore yields

$$G(y + \theta) = G(y) + G(\theta) \qquad \text{for all } \ y, \theta \in R .$$

Hence G is linear and g is constant a.e. (L). This contradiction shows that F must be a lattice-df. In this case (6.13) is equivalent to

$$g(x + \theta) = B(\theta)$$

for all θ and for all x that belong to the minimal lattice that supports dF. Hence g is a periodic function. The proof is complete.

REMARK 1. Also certain non-lattice-df's can admit an estimator which estimates a nonconstant parametric function without any error. Consider e.g. a discrete non-lattice-df. Let then θ and θ' be equivalent if there exist finitely many points $x_1, \cdots, x_N, y_1, \cdots, y_{N'}$ belonging to the support of dF so that

$$\theta + \textstyle\sum_1^N x_j = \theta' + \sum_1^{N'} y_j .$$

This is an equivalence relation which yields a nontrivial partition of R. Put then $B(\theta) = 1$ when θ belongs to the same class as 0 and $= 0$ otherwise. Put also $g(x) = B(x)$. It is easy to check that $g(x_1)$ estimates $B(\theta)$ without error. However, observe that $g(x)$ is constant a.e. (L).

REMARK 2. It would be desirable to replace the condition "for all n" by "for some $n \geqq 2$" in the formulation of Theorem 6.2. This is possible if one instead imposes some weak a priori condition on F, e.g. that the set of zeroes of the ch.f. φ has no interior points (cp. [1]).

7. A general theorem. In this section we do not assume that θ is a location parameter. It is well known that if a complete sufficient statistic exists, then all estimable parametric functions admit UMV-estimators. Here a partial converse is given. Similar results have been discussed by several authors, see e.g. Rao [20] and Klebanov et al. [14].

THEOREM 7.1. *Let T be a* UMV-*estimator. If $E_\theta[T^k] < \infty$ for all positive integers k and all θ, and further, these moments uniquely determine the distribution of T, then T and every ancillary statistic Y are independent for all θ.*

PROOF. Let ψ be any bounded measurable function of Y. Then $\psi(Y) - E_\theta[\psi(Y)]$ is an unbiased estimator of zero. (Observe that $E_\theta[\psi(Y)]$ does not depend on θ since Y is ancillary.) Using the UMV-criterion iteratively, we get

$$E_\theta[T^k(\psi(Y) - E_\theta[\psi(Y)])] = 0\,, \qquad k = 1, 2, \cdots.$$

Equivalently,

$$E_\theta[T^k\psi(Y)] = E_\theta[T^k] \cdot E_\theta[\psi(Y)]\,, \qquad k = 1, 2, \cdots. \tag{7.1}$$

Let ψ be positive and $\psi(Y)$ not zero a.s. It is easy to check that

$$\gamma(\zeta) = E_\theta[\exp\{i\zeta T\}\psi(Y)]/E_\theta[\psi(Y)]$$

is a characteristic function. Since (7.1) shows that the derivatives of γ coincide with the derivatives of $E_\theta[\exp\{i\zeta T\}]$ at $\zeta = 0$ and since these derivatives uniquely determine the distribution (and the corresponding characteristic function), we have for all ζ and all positive ψ

$$E_\theta[\exp\{i\zeta T\}\psi(Y)] = E_\theta[\exp\{i\zeta T\}] \cdot E_\theta[\psi(Y)]\,.$$

Obviously, this relation also holds for all negative ψ. The decomposition $\psi = \psi^+ - \psi^-$ therefore shows that it is valid for all real ψ and hence also for all complex-valued ψ. This concludes the proof.

COROLLARY. *If T satisfies the assumptions of Theorem* 7.1, *and if further, for some ancillary statistic Y, the mapping: $(x_1, \cdots, x_n) \to (T, Y)$ is injective (with measurable inverse), then T is a sufficient statistic.*

PROOF. It must be proved that $E_\theta[g(x, \cdots, x_n) \mid T = t]$ is independent of θ for all bounded functions g. We set $g^*(T, Y) = g(x_1, \cdots, x_n)$. Using Fubini's theorem and Theorem 7.1, we easily see that $E_\theta[g^*(t, Y)]$ is a possible representation of the conditional mean value not depending on θ. □

No example seems to have been constructed where a UMV-estimator and an ancillary statistic are not independent.

8. The Pitman estimator. Here we return to the location parameter families and study UMV-estimators of the location parameter θ itself. We set $Y = (x_1 - \bar{x}, \cdots, x_n - \bar{x})$. Note that Y is ancillary.

An estimator θ^* of θ is said to be translative if $\theta^*(x_1 + \lambda, \cdots, x_n + \lambda) = \lambda + \theta^*(x_1, \cdots, x_n)$ for all $\lambda \in R$ and all $(x_1, \cdots, x_n) \in R^n$. The Pitman estimator $\theta_P{}^*$ is the best, with respect to square error loss, translative estimator of θ. We have $\theta_P{}^* = \bar{x} - E_0[\bar{x} \mid Y]$ (see e.g. Rao [21] page 259).

THEOREM 8.1. *If a* UMV-*estimator of θ exists, then $\theta_P{}^*$ is also a* UMV-*estimator of θ.*

PROOF. It suffices to show that there is a translative UMV-estimator of θ.

Let θ^* be UMV. For any λ we set

$$\theta_\lambda^*(x_1, \cdots, x_n) = \theta^*(x_1 + \lambda, \cdots, x_n + \lambda) - \lambda .$$

Clearly θ_λ^* is also a UMV-estimator of θ. In view of the uniqueness of a UMV-estimator, we have for each fixed λ

$$\theta_\lambda^* = \theta^* \quad \text{a.s.} \qquad \text{for all } \theta .$$

With $\psi(x_1, \cdots, x_n) = \theta^*(x_1, \cdots, x_n) - x_1$ this is equivalent to

$$\psi(x_1, \cdots, x_n) = \psi(x_1 + \lambda, \cdots, x_n + \lambda) \quad \text{a.s.} \qquad \text{for all } \theta ,$$

i.e. ψ is almost translation invariant with respect to all the measures $dF(x - \theta)$ and especially with respect to the measure $dF(x - \theta_0)$, where θ_0 is fixed. Then by Theorem 4 in Lehmann [16] page 225 there is a translation invariant function ψ^0 such that $\psi = \psi^0$ a.s. for $\theta = \theta_0$. Using again that ψ is almost translation invariant, we easily see that then also $\psi = \psi^0$ a.s. for all θ. Therefore the translative statistic $x_1 + \psi^0$ is a UMV-estimator of θ and the proof is finished.

REMARK. A referee has pointed out that Theorem 8.1 has earlier been proved by Ghosh [8]. The idea can partly be found in Ghosh and Singh [7]. Another proof was very recently given by Takeuchi [24].

THEOREM 8.2. *If F is symmetric and a* UMV-*estimator of θ exists for $n = 2$, F must be normal or degenerate.*

PROOF. When $n = 2$ the symmetry of F implies that $\theta_P^* = \bar{x}$. An application of Theorem 8.1 and Theorem 6.1 completes the proof.

When do the moments of θ_P^* uniquely determine its distribution so that Theorem 7.1 and its corollary can be applied (with Y as above)? Observe that the mapping: $(x_1, \cdots, x_n) \to (\theta_P^*, Y)$ is injective.

LEMMA 8.1.

$$E_0[|\theta_P^*|^j] \leqq 2^j \cdot E_0[|x_1|^j] , \qquad j = 0, 1, 2, \cdots .$$

PROOF. The lemma follows from the elementary inequalities below.

$$\begin{aligned} E_0[|\theta_P^*|^j] &= E_0[|\bar{x} - E_0[\bar{x} \mid Y]|^j] \leqq 2^{j-1} \cdot (E_0[|\bar{x}|^j] + E_0[|E_0[\bar{x} \mid Y]|^j]) \\ &\leqq 2^{j-1}(E_0[|\bar{x}|^j] + E_0[E_0[|\bar{x}|^j \mid Y]]) = 2^j \cdot E_0[|\bar{x}|^j] \leqq 2^j \cdot E_0[|x_1|^j] . \end{aligned}$$

Due to a theorem of Carleman a sequence of moments μ_j uniquely determines the df if $\sum_{j=1}^{\infty} (\mu_{2j})^{-1/(2j)} = \infty$. From Lemma 8.1 we find that if the moments of F satisfy the Carleman condition so do (for all θ) the moments of θ_P^*.

Next we see what happens if the tail of F tends to zero rapidly.

THEOREM 8.3. *If the* ch.f. *φ of F is an entire analytic function of finite order < 2, no* UMV-*estimator of θ exists, unless F is degenerate.*

PROOF. Assume that θ_P^* is a UMV-estimator of θ (recall Theorem 8.1). An analytic function $H(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$ is of order η ($\eta \geqq 0$) if

$$\liminf_{j\to\infty} \frac{\log (1/|a_j|)}{j \log j} = 1/\eta . \tag{8.1}$$

If the left-hand side is zero, (8.1) shall be interpreted as: H is not entire, or H is entire but not of finite order. (See Titchmarsh [25] page 246.) From Lemma 8.1, (8.1), and the fact that $(E_0[|\theta_P{}^*|^j])^{1/j}$ increases when j does, we easily find that the order of the ch.f. of $\theta_P{}^*$ (for $\theta = 0$) is less or equal to the order of φ. The estimator $\theta_P{}^*$ has a df of the form $G(x - \theta)$. We now consider $\theta_P{}^*$ as one single observation from a population with df $G(x-\theta)$. Then, in view of Theorem 3.3, G must be normal or degenerate. Since the order is less than 2, G is not normal. Obviously, G is degenerate iff F is degenerate. This ends the proof.

REMARK. It is reasonable to believe that normality of G implies normality of F. However, without assuming some regularity properties of F (e.g. the existence of a strictly positive density function) this seems hard to show. That F must be continuous is obvious.

EXAMPLE. Consider a df F with density function $C_0 \exp\{-x^4\}$. The order of φ is $\frac{4}{3}$. Theorem 8.3 then shows that no UMV-estimator of θ exists. Also the corollary of Theorem 7.1 proves the non-existence, for the moments of F satisfy the Carleman condition, and there exists no single sufficient statistic.

The main theorem in Takeuchi [24] states that if $\theta_P{}^*$ is a UMV-estimator, then $\theta_P{}^*$ is independent of Y and hence a sufficient statistc. The remarkable thing is that no additional conditions on $\theta_P{}^*$ are used (cp. Theorem 7.1). However, there is a gap in the proof which seems hard to fill.

Let $\varphi_{\theta_P*}(\zeta) = E_0[\exp\{i\zeta\theta_P{}^*\}]$. The (real) zeroes of φ_{θ_P*} form a closed set N. The complement N^c is open and therefore it admits the representation: $N^c = \bigcup_{-\infty}^{+\infty} I_k$, where I_k are open disjoint intervals. We may suppose $0 \in I_0$. If φ_{θ_P*} has no zeroes, then $I_0 = R$. We end this section by giving a correct variant of Takeuchi's theorem and a simplified proof.

THEOREM 8.4. *Suppose φ_{θ_P*} is uniquely determined by its values on I_0. If $\theta_P{}^*$ is a* UMV-*estimator of θ, then $\theta_P{}^*$ and Y are independent for all θ and $\theta_P{}^*$ is a sufficient statistic.*

PROOF. We set:

$$H = \{\psi;\ \psi \text{ complex-valued measurable function of } Y \text{ and } E_0[|\psi(Y)|^2] < \infty\}.$$

Then H is a Hilbert space with scalar product $(\psi_1, \psi_2) = E_0[\overline{\psi_1(Y)}\psi_2(Y)]$. Let ζ be a fixed real number. We put

$$\varphi_\zeta(Y) = E_0[\exp\{i\zeta\theta_P{}^*\} \mid Y] \qquad \text{and} \qquad \varphi_\zeta{}'(Y) = E_0[i\theta_P{}^* \exp\{i\zeta\theta_P{}^*\} \mid Y].$$

As $E_0[|\theta_P{}^*|^2] < \infty$, both φ_ζ and $\varphi_\zeta{}'$ belong to H. Let $(\psi, \varphi_\zeta) = 0$, i.e. $E_0[\exp\{i\zeta\theta_P{}^*\}\overline{\psi(Y)}] = 0$. Since $\theta_P{}^*$ is translative and Y translation invariant, it follows that $E_\theta[\exp\{i\zeta\theta_P{}^*\}\overline{\psi(Y)}] = 0$ for all θ, i.e. $\exp\{i\zeta\theta_P{}^*\}\overline{\psi(Y)}$ is an unbiased estimator of zero. The UMV-criterion yields

$$E_\theta[\theta_P{}^* \exp\{i\zeta\theta_P{}^*\}\overline{\psi(Y)}] = 0 \qquad \text{for all } \theta.$$

For $\theta = 0$ we get $(\psi, \varphi_\zeta') = 0$. Thus

$$\psi \perp \varphi_\zeta \Rightarrow \psi \perp \varphi_\zeta' . \tag{8.2}$$

Let H_1 and H_2 be closed linear subspaces of H. A well-known theorem from linear algebra says: $H_1^\perp \subset H_2^\perp \Rightarrow H_2 \subset H_1$. So, (8.2) proves the existence of a constant $a(\zeta)$ such that $\varphi_\zeta' = a(\zeta)\varphi_\zeta$. Hence, for all $\psi \in H$,

$$E_0[i\theta_P^* \exp\{i\zeta\theta_P^*\}\psi(Y)] = a(\zeta) \cdot E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] .$$

Equivalently, (from now on we let ζ vary)

$$\frac{d}{d\zeta} E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] = a(\zeta) \cdot E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] . \tag{8.3}$$

Setting $\psi = 1$, we see that $a(\zeta)$ is continuous on each component I_k and hence locally integrable over I_k. So an integration of (8.3) yields

$$E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] = \exp\{\textstyle\int_{\zeta_k}^{\zeta} a(z)\, dz\} \cdot B_k(\psi) , \qquad \zeta \in I_k ,$$

where ζ_k is any fixed point in I_k and where $B_k(\psi)$ is a constant depending on ψ. The left factor on the right-hand side can be calculated by setting $\psi = 1$, and hence we obtain

$$E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] = E_0[\exp\{i\zeta\theta_P^*\}] \cdot C_k(\psi) , \qquad \zeta \in I_k ,$$

where $C_k(\psi) = B_k(\psi)/B_k(1)$. Setting $\zeta = 0$, we find $C_0(\psi) = E_0[\psi(Y)]$. To prove that also $C_k(\psi) = E_0[\psi(Y)]$ when $k \neq 0$ and that $E_0[\exp\{i\zeta\theta_P^*\}\psi(Y)] = 0$ when $\zeta \in N$ without using any additional conditions on $\varphi_{\theta_P^*}$ seems to be impossible. (Easily verified, $a(\zeta)$ is locally integrable over R iff $I_0 = R$.) However, now we suppose that $\varphi_{\theta_P^*}$ is uniquely determined by its values on I_0. Arguments similar to those that were used at the end of the proof of Theorem 7.1 then show that θ_P^* and Y are independent for $\theta = 0$. As θ_P^* is translative and Y translation invariant, they must be independent for all θ, and hence θ_P^* is a sufficient statistic (cp. the corollary of Theorem 7.1).

9. Polynomial UMV-estimation. It turns out that only the normal df admits a polynomial nontrivial UMV-estimator. More exactly, we have

THEOREM 9.1. *Let $P(x_1, \cdots, x_n)$ be a nonconstant polynomial in the observations $x_1, \cdots, x_n$ $(n \geqq 2)$. If P is a* UMV-*estimator of its mean value, then F is normal or dF has finite support. In the first case P depends only on $\bar{x}$. In the second case P is constant* a.s.

To prove this theorem we need a lemma.

LEMMA 9.1. *Suppose that dF has not finite support and let $P(x_1, \cdots, x_n)$ be a polynomial* UMV-*estimator. Then, for every choice of $(x_{k+1}^0, \cdots, x_n^0)$, $0 \leqq k \leqq n-1$, $P(x_1, \cdots, x_k, x_{k+1}^0, \cdots, x_n^0)$ is a* UMV-*estimator based on $(x_1, \cdots, x_k)$.*

PROOF. Using induction, we realize that it suffices to prove the lemma for $k = n - 1$. We write as though $n = 3$, $k = 2$. Let $h(x_1, x_2)$ be any unbiased estimator of zero. Then also $\exp\{i\zeta x_3\}h(x_1, x_2)$ has mean value zero for all θ. Hence, by the UMV-criterion,

$$\int \exp\{i\zeta x_3\}[\int \int P(x_1, x_2, x_3)h(x_1, x_2)\, dF(x_1 - \theta)\, dF(x_2 - \theta)]\, dF(x_3 - \theta) = 0 .$$

As this is true for all ζ,

$$\int \int P(x_1, x_2, x_3)h(x_1, x_2)\, dF(x_1 - \theta)\, dF(x_2 - \theta) = 0 \tag{9.1}$$

for almost all x_3 with respect to the measure $dF(x - \theta)$. Since dF has not finite support, (9.1) holds for at least infinitely many x_3. However, the left-hand side of (9.1) is a polynomial in x_3, and therefore (9.1) is an identity. In view of the UMV-criterion, this proves the lemma.

PROOF OF THEOREM 9.1. The proof is long and is given in some stages.

Suppose first down to something else is said that dF has not finite support. Then, in view of Lemma 9.1, it suffices to consider the case $n = 2$ in order to conclude that F is normal.

Assume that $P(x_1, x_2) = \sum_{j,k} a_{jk} x_1{}^j x_2{}^k$ is a nonconstant UMV-estimator. Let $J = \max\{j;\, a_{jk} \neq 0 \text{ for some } k\}$ and $K = \max\{k;\, a_{jk} \neq 0 \text{ for some } j\}$. Since $P(x_1, x_2)$ has finite variance, it is not hard to see that F must have at least $\max(2J, 2K)$ finite moments.

I. *The* ch.f. φ *of* F *is analytic in some neighborhood of the origin.* To prove this, we may suppose $J > 0$. By Theorem 4.1 (for $\theta = 0$), we have for real points ζ_1, ζ_2

$$\frac{\varphi^{(J)}(\zeta_1) \cdot \sum_k a_{Jk}^* \varphi^{(k)}(\zeta_2)}{\varphi(\zeta_1)\varphi(\zeta_2)} + \frac{\sum'_{j,k} a_{jk}^* \varphi^{(j)}(\zeta_1)\varphi^{(k)}(\zeta_2)}{\varphi(\zeta_1)\varphi(\zeta_2)} = C(\zeta_1 + \zeta_2) , \tag{9.2}$$

where $a_{jk}^* = (-i)^{j+k} a_{jk}$, and where $'$ on the second summation sign indicates that there is no summation over $j = J$. It is no restriction to suppose that F has mean zero and variance 1 (for F is not degenerate). Then according to an inequality for ch.f.'s (see e.g. Feller [5] page 487), $|\varphi(\zeta) - 1| < \frac{1}{2}$ when $|\zeta| < 1$, and therefore (9.2) holds for all ζ_1, ζ_2 in this region. We differentiate (9.2) separately once with respect to ζ_1 and once with respect to ζ_2 (which is permitted). The left-hand sides obtained must then be equal. We then fix ζ_2 so that $\sum_k a_{Jk}^* \varphi^{(k)}(\zeta_2) \neq 0$ (if this were impossible, dF would have finite support) and solve for $\varphi^{(J+1)}(\zeta_1)$ which we write as

$$\varphi^{(J+1)}(\zeta_1) = H(\varphi(\zeta_1), \varphi'(\zeta_1), \cdots, \varphi^{(J)}(\zeta_1)) ,$$

where H is analytic (indeed rational) in the region $|\varphi - 1| < \frac{1}{2}$. For any given initial conditions there is always an analytic solution of this differential equation (see e.g. Cartan [2] Chapter 7). The result wanted then follows from the uniqueness part of the Picard–Lindelöf theorem (see e.g. Hale [9] page 18).

We now write $P(x_1, x_2) = \sum_k a_k(Y)(\bar{x})^k$. Here $a_k(Y)$ are polynomials in $Y = (x_1 - x_2)$. Let M be the largest integer such that $a_M(Y)$ is not equal to zero a.s.

II. *If* $M > 0$, *then the Pitman estimator* $\theta_P{}^*$ *coincides* a.s. *with a polynomial*

UMV-*estimator*. To prove this, we first note that the translated estimator $P(x_1 + \lambda, x_2 + \lambda)$, $\lambda \in R$, and all the derivatives of this estimator with respect to λ have the UMV-property if P has.

Obviously, $\sum_{k=0}^{M} a_k(Y)(\bar{x})^k$ is a UMV-estimator. Translating this estimator λ unities, then differentiating M respectively $M - 1$ times with respect to λ, and after that setting $\lambda = 0$, we find that both $M!\, a_M(Y)$ and $M!\, a_M(Y) \cdot \bar{x} + (M - 1)!\, a_{M-1}(Y)$ are UMV-estimators. Then $a_M(Y)$ is uncorrelated with every function of Y and must therefore be equal to a nonzero constant a_M a.s. (In stage III we use that this is true also if $M = 0$.) Hence, also $\bar{x} + R(Y)$, where $R(Y) = a_{M-1}(Y)/Ma_M$ is a UMV-estimator. Clearly, $\theta_P^* = \bar{x} + R(Y)$ (+constant) a.s.

III. *Necessarily $M > 0$, for otherwise dF would have finite support.* To prove this, suppose M is not larger than zero (this does not mean that $M = 0$). We know that, for every $k \geqq 0$, $a_k(Y)$ is equal to a constant a_k a.s. (cp. stage II). Here $a_k = 0$ when $k \geqq 1$. All these equalities cannot hold identically. So, there exists an integer m such that $a_m(Y) - a_m$ is zero a.s. but not identically zero. Since the polynomial $a_m(Y) - a_m$ has only a finite number of zeroes, all the probability mass of $Y = (x_1 - x_2)$ must be concentrated at these points. Hence dF has finite support.

IV. *The* df *F must be normal.* As φ is analytic and as $\theta_P^* = \bar{x} + R(Y)$ a.s., it follows from Lemma 8.1 that the distribution of $\bar{x} + R(Y)$ is uniquely determined by its moments. Hence, by Theorem 7.1, $\bar{x} + R(Y)$ and Y are independent. Let $\bar{x} + R(Y)$ be of degree p. The coefficient of x_1^p (or x_2^p) is not zero. Then a theorem due to Zinger (see e.g. Lukacs and Laha [18] Theorem 5.3.2, page 89, or Ramachandran [22] Theorem 8.1.2, page 166) guarantees that φ is an entire function of finite order. Recalling Lemma 9.1, we know that also $P(x_1, x_2^0)$ (or if necessary $P(x_1^0, x_2)$) for an appropriately chosen number $x_2^0(x_1^0)$ is a nonconstant UMV-estimator based upon one single observation. An application of Theorem 3.3 then gives the result wanted.

From now on we consider the general case with n ($n \geqq 2$) observations, and we let $P(x_1, \cdots, x_n)$ be a polynomial UMV-estimator. It is possible to write $P(x_1, \cdots, x_n) = \sum_{k=0}^{N} a_k(Y)(\bar{x})^k$, where now $Y = (x_1 - \bar{x}, \cdots, x_n - \bar{x})$.

V. *If F is normal, P depends on $\bar{x}$ only.* Using the same method as in stage II, we find that $a_N(Y)$ is constant a.s. and therefore identically equal to a constant a_N. Since for the normal distribution $(\bar{x})^N$ is a UMV-estimator, so is $\sum_{k=0}^{N} a_k(Y)(\bar{x})^k - a_N(\bar{x})^N$. By induction, $a_{N-1}(Y), a_{N-2}(Y), \cdots, a_0(Y)$ must also be constants.

VI. *Let now dF have finite support.* Then P is constant a.s. Exactly as in stages II and III it follows that either $\bar{x} + R(Y)$ is a UMV-estimator or $a_N(Y), \cdots$ $a_1(Y)$ are all zero a.s. and $a_0(Y)$ constant a.s. In case of the second possibility, P is trivially constant a.s. In case of the first one, we find from Theorem 8.3 that $\bar{x} + R(Y)$ has a degenerate df. Hence also F is degenerate, and P is constant a.s.

The theorem is completely proved.

10. Additional remarks. The purpose of this paper has been to see what new light Fourier methods are able to cast on the UMV-problem. Finally, we therefore touch on another aspect which deliberately has been neglected before.

Suppose there exists a UMV-estimator T such that the mapping: $(x_1, \cdots, x_n) \to (T, x_1 - \bar{x}, \cdots, x_n - \bar{x})$ is injective (in the sequel this is called Property I). Under the conditions of Theorem 7.1 we know that T is a sufficient statistic. Under the further condition that F has a sufficiently smooth density function, it is then well known that $F(x - \theta)$ must constitute an exponential family. This fact implies that F is either

(i) a normal df, or
(ii) the df of the logarithm of some power of a gamma variable.

The normal df's can be considered as limits of the distributions in (ii). Also other limits are possible, namely:

(iii) the translates of exponential (negative exponential) df's.

If F is of type (iii), $\min_i x_i$ ($\max_i x_i$) is a sufficient statistic for θ. It can also be shown that if F is a one-sided df having a sufficiently smooth density function, then a single sufficient statistic exists only if F is of type (iii). There exist also discrete analogues of the distributions in (iii) admitting single sufficient statistics. For more details concerning the statements we have made, see Dynkin [4] and Ferguson [6]. An elementary treatment can be found in Takeuchi [24] (where, however, the theorems partly remain unproved; cp. Section 8). See also Kagan et al. [12]. These mentioned location parameter families are the only known ones which admit single sufficient statistics. It may therefore be conjectured that UMV-estimators having Property I can be found in these cases only. (But a strict proof seems to be far away.) In view of the facts above the results given in Theorems 6.1, 6.3, and 9.1 may seem somewhat poor. However, there we have not assumed Property I and no regularity for the df. The assumptions concern instead the form of the estimator (cp. Pfanzagl [19], Section 3).

APPENDIX

Here we give proofs of Lemmas 2.2 and 2.3. The reader is assumed to be familiar with the stuff treated in Chapter II of the book by Donoghue [3], and therefore we freely use the terminology and the results given there. However, recall our definition of the Fourier transform. First some new lemmas are stated and proved.

LEMMA 1. *Let $g \in_L B_N$ and suppose ζ_0 is an isolated point in $S(g)$. Then the Fourier transform $\hat{g}$ is locally at ζ_0 a linear combination $\sum_{k=0}^{N} c_k \delta_{\zeta_0}^{(k)}$ of the Dirac measure at ζ_0 and its derivatives up to order at most N.*

PROOF. The fact that only finitely many derivatives, say L, are needed is just the corollary on page 103 in [3]. Our point is that $L \leqq N$. It is no restriction to assume $\zeta_0 = 0$. Let φ be any testfunction (i.e. $\varphi \in C_0^\infty$) such that $\varphi^{(L)}(0) \neq 0$.

We set $\varphi_\varepsilon(\zeta) = \varphi(\zeta/\varepsilon)$. Let here $\varepsilon > 0$ be so small that $\operatorname{supp}(\varphi_\varepsilon) \cap S(g) = \{0\}$. Obviously, $\hat{\varphi}_\varepsilon(x) = \varepsilon\hat{\varphi}(\varepsilon x)$. By the definition of $\hat{g}$ we have

(1) $$\hat{g}(\varphi_\varepsilon) = g(\hat{\varphi}_\varepsilon) .$$

We now find

(2) $$\begin{aligned}\hat{g}(\varphi_\varepsilon) &= \textstyle\sum_{k=0}^{L} c_k \varphi_\varepsilon{}^{(k)}(0) \\ &= \textstyle\sum_{k=0}^{L} c_k \varepsilon^{-k}\varphi^{(k)}(0) \sim c_L \varepsilon^{-L}\varphi^{(L)}(0) \qquad \text{when } \varepsilon \to 0 .\end{aligned}$$

Further,

$$g(\hat{\varphi}_\varepsilon) = \int g(x)\hat{\varphi}_\varepsilon(x)\,dx = \int g(x)\varepsilon\hat{\varphi}(\varepsilon x)\,dx = \int g(x/\varepsilon)\hat{\varphi}(x)\,dx .$$

As $|g(x)| \leqq \text{constant } (1 + |x|)^N$ a.e. (L), we easily get

(3) $$|g(\hat{\varphi}_\varepsilon)| \leqq 0(\varepsilon^{-N}) .$$

The lemma therefore follows from (1), (2), and (3).

LEMMA 2. *Let g be any function such that $\hat{g}$ is defined and such that $S(g) \subset \{ja;\ j \in Z\}$, $a > 0$. Then $\hat{g} = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{L_j} c_{jk}\delta_{ja}^{(k)}$. If $\max_j L_j \leqq N$, then $\Delta_{2\pi/a}^{N+1} g = 0$ a.e. (L).*

PROOF. The first statement follows again from the corollary in [3]. Then it is easy to show by induction that $(\exp\{-i2\pi\zeta/a\} - 1)^{N+1}\hat{g} = 0$. Since

(4) $$\widehat{\Delta_{2\pi/a}^{N+1} g} = \text{constant} \times (\exp\{-i2\pi\zeta/a\} - 1)^{N+1}\hat{g} ,$$

the rest is immediate.

LEMMA 3. *Let g be any locally bounded function. If $\Delta_{2\pi/a}^{N+1} g = 0$ a.e. (L), then $g \in_L B_N$.*

We omit the proof. Hint: Use induction, telescope sums, and the fact that $\sum_{k=0}^{j} k^{N-1} = O(j^N)$.

LEMMA 4. *Let g be any function such that $\hat{g}$ is defined. If $\Delta_{2\pi/a}^{N+1} g = 0$ a.e. (L), then $S(g) \subset \{ja;\ j \in Z\}$.*

PROOF. This lemma follows from (4) and the fact that $\exp\{-i2\pi\zeta/a\} - 1$ is zero only when ζ is an integral multiple of a.

PROOF OF LEMMA 2.2. Let $g \in B_N$ belong effectively to B_M and let $S(g) \subset \{ja;\ j \in Z\}$. Lemmas 1, 2, and 3 (where N has different meanings) together yield

$$\hat{g} = \textstyle\sum_{j=-\infty}^{\infty} \sum_{k=0}^{L_j} c_{jk}\delta_{ja}^{(k)} ,$$

where, which is the point, $\max\{L_j;\ c_{jL_j} \neq 0\} = M$. Choose j such that $L_j = M$. Take $\hat{\varphi} \in C_0^\infty$ such that $\operatorname{supp}(\hat{\varphi}) \cap S(g) = \{ja\}$ and such that $\hat{\varphi} \equiv 1$ in some neighborhood of the point ja. Then

$$\widehat{g * \varphi} = \hat{\varphi} \cdot \hat{g} = \textstyle\sum_{k=0}^{M} c_{jk}\delta_{ja}^{(k)} = \text{constant} \times \widehat{P(x)\exp\{-ijax\}} ,$$

where P is a polynomial of degree exactly M. Hence

$$g * \varphi = \text{constant} \times P(x)\exp\{-ijax\} .$$

As $g * d\mu = 0$ implies $g * \varphi * d\mu = 0$, it then easily follows that

$$\int x^k \exp\{ijax\}\, d\mu(x) = 0\,, \qquad k = 0, \cdots, M\,,$$

which ends the proof.

PROOF OF LEMMA 2.3. Concerning the first statement, the necessity follows from Lemmas 1 and 2 and the sufficiency from Lemma 4. The second statement is essentially only a consequence of the corollary in [3].

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