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UPPER CONFIDENCE AND FIDUCIAL LIMITS TO THE RANGE OF SEVERAL MEANS

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The problem of determining upper limits to the range of means of several independent normal variables with possibly different variances is considered. Simultaneous confidence regions based on others' work yield confidence limits which are rather conservative. We then approach the problem from a fiducial viewpoint and derive the equation which yields the exact fiducial upper limit.

1. The problem. Suppose that X_i, $i = 1, \cdots, n$, for $n > 1$ are mutually independent random variables with X_i distributed as $N(\mu_i, \sigma_i^2)$ where the σ_i^2 are known. Let R denote the range of the μ_i, i.e., $\max|\mu_i - \mu_j|$. For a prescribed probability level P^* we wish to determine an upper limit $B = B(X_1, \cdots, X_n)$ for R so that $P(R \leqq B) \geqq P^*$ for all $\boldsymbol{\mu}$, where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)$. Two different upper confidence limits are discussed in Section 2; in Section 3 we consider an upper fiducial limit, or primary result. We derive this in Section 4, and in Section 6 present a few extensions; Section 5 provides numerical illustrations.

Although we have assumed normality, we see wide possibilities for practical use of the technique described here, since the X's may well represent means for large samples. By the Central Limit Theorem such X's would tend to be normally distributed, and the variance estimates would be essentially the true variances. Information concerning R then provides a valuable starting point for the analysis of several populations.

2. Confidence limits based on previous work. For a given P^* let $\chi = t^{\frac{1}{2}}$ be defined by $P^* = P(u < t)$ where u is distributed $\chi^2(n-1)$. Then, for each $\boldsymbol{\mu}$, $P(R \leqq B_1) \geqq P^*$ where

$$B_1 = \max_{i \neq j}(|X_i - X_j| + \chi(\sigma_i^2 + \sigma_j^2)^{\frac{1}{2}}) . \tag{2.1}$$

To derive this result, we may copy the proof of Scheffé (1959, pages 66–67) with the quantities I, β_i, $y_{i\cdot}$, and $\operatorname{Var}(y_{i\cdot})$ (equal to σ^2/J_i) and α there, corresponding to our n, μ_i, X_i, σ_i^2, and P^* respectively, and with because our variances σ_i^2 are known, infinitely many denominator degrees of freedom for the F-value associated with S there. In our notation, S thus becomes $(n-1)\chi/n - 1$, or χ. Let E_1 be the event that for all sets of fixed scalars c_i with $\sum c_i = 0$ the inequalities

$$\sum c_i X_i - \chi \operatorname{Var}^{\frac{1}{2}}(\sum c_i X_i) \leqq \sum c_i \mu_i \leqq \sum c_i X_i + \chi \operatorname{Var}^{\frac{1}{2}}(\sum c_i X_i) \tag{2.2}$$

are simultaneously satisfied; from Scheffé, we have that $P(E_1) \geqq P^*$. Let E_2 be

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the event that for all $i \neq j$ the inequalities

$$(2.3) \qquad X_i - X_j - \chi(\sigma_i^2 + \sigma_j^2)^{\frac{1}{2}} \leqq \mu_i - \mu_j \leqq X_i - X_j + \chi(\sigma_i^2 + \sigma_j^2)^{\frac{1}{2}}$$

are simultaneously satisfied. Then $E_1 \Rightarrow E_2$, since (2.2) is based on all contrasts and (2.3) only on pairwise differences. Occurrence of E_2 in turn implies that $R \leqq B_1$, so that $P^* \leqq P(E_1) \leqq P(E_2) \leqq P(R \leqq B_1)$, and B_1 is a valid P^*-level upper confidence limit.

Where the σ_i are all equal to a common value σ, we obtain from Scheffé (1959, pages 73–75) a P^*-level upper confidence limit $B_2 = Q\sigma + \max|X_i - X_j|$ where Q is the upper-P^* point of the Studentized range for degrees of freedom n and ∞ ("standardized range") with $B_2 \leqq B_1$.

For each of B_1 and B_2 there is an accompanying test—the first developed by Scheffé and the second by Tukey—of the null hypothesis that $R = 0$. Further tests of this hypothesis, by Newman and Kuels, may be used for equal σ's, but without any associated B's other than B_2; these methods and similar ones may simultaneously be used also to divide means into "nonsignificantly different" groups. Fraser (1952) constructed for equal σ_i a single confidence interval for all the μ_i, Dudewicz and Tong (1971) constructed confidence limits on $\max \mu_i$; Bechhofer (1954) considered the problem of ranking the μ_i's.

3. An upper fiducial limit. Our final value for B is as follows. Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cdf and density for $N(0, 1)$. The value obtained is the solution to the equation

$$(3.1) \qquad P^* = \int_{-\infty}^{\infty} \Big[\sum_k \prod_{i \neq k} \Big(\Phi\Big(\frac{X_k - X_i + B}{\sigma_i} + \frac{\sigma_k}{\sigma_i} z\Big) - \Phi\Big(\frac{X_k - X_i}{\sigma_i} + \frac{\sigma_k}{\sigma_i} z\Big)\Big)\Big] \phi(z)\, dz\,.$$

We base our result on a fiducial—see Fisher (1935) and Fraser (1958, pages 289–291)—distribution of the μ_i given the X_i—namely, that the μ_i are mutually independent $N(X_i, \sigma_i^2)$, as follows. The quantities $Z_i = X_i - \mu_i$ are mutually independent $N(0, \sigma_i^2)$. Let $\mathbf{Z}$ and $\mathbf{X}$ denote the vectors of Z's and X's and, for each region C in R^n, $C^*(\mathbf{X})$ denote $\{\mathbf{X} - \mathbf{Y} \mid \mathbf{Y} \in C\}$; we have that $\mathbf{Z} \in C \Leftrightarrow \boldsymbol{\mu} \in C^*(\mathbf{X})$ so that $P(\mathbf{Z} \in C) = P(\boldsymbol{\mu} \in C^*(\mathbf{X}))$, and the above distribution of $\boldsymbol{\mu}$, based on all C, arises.

This fiducial distribution also has a meaningful Bayesian interpretation as a posterior distribution, in the following sense. Suppose the prior distribution of $\boldsymbol{\mu}$ is multivariate normal with fixed mean and covariance matrix $a\boldsymbol{\Sigma}$ for fixed positive definite matrix $\boldsymbol{\Sigma}$ and positive scalar a; it may be shown that as $a \to \infty$, for any $\boldsymbol{\Sigma}$, the posterior distribution of $\boldsymbol{\mu}$ converges to our fiducial distribution. An interpretation is that the latter corresponds to the limiting form of a prior distribution which is nearly uniform over an increasingly wide region: that is, it corresponds to a prior assumption that no one value, or set of values of given measure, for $\boldsymbol{\mu}$ is more likely than another.

4. Derivation of results. Derivation is as follows. Suppose B satisfies (3.1). Let E be the event that $R \leqq B$, E_k the event that $0 \leqq \mu_i - \mu_k \leqq B$ for all $i \neq k$. Then, fiducially, $P(E) = \sum P(E_k)$, as follows.

(a) For $i \neq j$, $E_i \cap E_j \Rightarrow \mu_j = \mu_i$; but $P(\mu_k = \mu_i) = 0$, so that $P(E_i \cap E_j) = 0$.

(b) We have $E \Rightarrow E_k$ for one k, namely the k—unique with probability 1 by (a)—satisfying $\min \mu_i = \mu_k$, so that $E \Rightarrow \bigcup E_k$.

(c) If E_k occurs for any k, $0 \leqq \mu_i - \mu_k \leqq B$ for all $i \neq k$; and thus, for $i \neq k \neq j$, $0 - B \leqq (\mu_i - \mu_k) - (\mu_j - \mu_k) \leqq B - 0$, so that $-B \leqq \mu_i - \mu_j \leqq B$, and $|\mu_i - \mu_j| \leqq B$. Hence $E_k \Rightarrow E$, and $\bigcup E_k \Rightarrow E$.

(d) Combining (b) and (c), we have $E \Leftrightarrow \bigcup E_k$, and thus $P(E) = P(\bigcup E_k)$, $= \sum P(E_k)$ by (a).

We have the conditional probability $P(E_k \mid \mu_k)$ equal to $\prod_{i \neq k} P(\mu_k \leqq \mu_i \leqq \mu_k + B)$, because of the mutual independence of the μ_i's for $i \neq k$. This expression is

$$\prod_{i \neq k} \left(\Phi\left(\frac{\mu_k - X_i + B}{\sigma_i} \right) - \Phi\left(\frac{\mu_k - X_i}{\sigma_i} \right) \right).$$

Letting $g_k(\mu_k)$ denote the density function for μ_k, we have

$$P(E_k) = \int_{-\infty}^{\infty} P(E_k \mid \mu_k) g_k(\mu_k)\, e\mu_k .$$

Since μ_k has the distribution $X_k + \sigma_k z$ where z is $N(0, 1)$ (and X_k, σ_k are fixed), we have that

$$P(E_k) = \int_{-\infty}^{\infty} \sum_{i \neq k} \left(\Phi\left(\frac{X_k - X_i + B}{\sigma_i} + \frac{\sigma_k}{\sigma_i} z \right) - \Phi\left(\frac{X_k - X_i}{\sigma_i} + \frac{\sigma_k}{\sigma_i} z \right) \right) \phi(z)\, dz .$$

Hence, summing over k, we have, since B satisfies (3.1) by assumption, that $P(E) = P^*$.

Hence, fiducially, $P(R \leqq B) = P^*$. We may conveniently think of $P^* = P^*(B)$ as a fiducial cdf for B.

5. Numerical examples. An example of a set of values of X_i and σ_i^2 to which the above analysis has been applied are:

i	1	2	3	4	5
X	8.9016	8.6823	9.2805	8.2784	9.5238
σ^2	.15939	.16336	.15730	.21720	.49717 .

The values of X represent Census Bureau coder error rates (expressed in percent) for different areas of the country, with the σ^2's the variances associated with these estimates.

First we considered the first four areas alone (the fifth, Puerto Rico, lying outside the U.S. proper). The upper 95% confidence limit B is (to 2 places) 2.71; the lower bound B_0 is 2.00; and (3.1) gives 2.08. Fiducial probabilities P^* corresponding to B are

B	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
P^*	.7779	.8297	.8725	.9068	.9335	.9537	.9685	.9790	.9864	.9914,

With all five areas considered, B_1 is 3.85, and (3.1) gives 2.71. Fiducial probabilities are:

B	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1
P^*	.8326	.8653	.8928	.9155	.9342	.9492	.9613	.9708	.9782	.9839·

Thus, along with the overly generous 95% upper confidence limits $B_1 = 2.71$ and 3.85, we have the much smaller 95% upper fiducial limits (with corresponding interpretation) $B = 2.08$ and 2.71, respectively.

Computation of P^* is based on integration according to (3.1); possibly the most efficient general method is first to determine a finite interval $[c_1, c_2]$ outside which the contributions to the integral may be considered negligible (these contributions are easily seen to be no greater than $n\Phi(c_1)$ and $n(1 - \Phi(c_2))$), and second, to apply Gauss's quadrature formula to $[c_1, c_2]$ (possibly divided into several subintervals). In the second stage, cruder techniques such as division of $[c_1, c_2]$ into small increments will work (though less efficiently) to any desired accuracy, and we have proceeded accordingly in our above example.

6. Generalizations and variations. All the above results are based on exact knowledge of the σ's. Similar results may be obtained when the σ's are not known but estimated by, say, quantities s_i, and the statistics $(X_i - \mu_i)/s_i$ are known to have, approximately, mutually independent t-distributions (instead of standard normal) with, say, m_i degrees of freedom (for all values of μ_i and σ_i). More generally, suppose that these statistics are mutually independent with known densities $h_i(z)$ and cdf's $H_i(z)$, Then, working as in Section 4, (3.1) becomes

$$P^* = \int_{-\infty}^{\infty} \sum_k \left(h_k(z) \prod_{i\neq k} \left(H_i\left(\frac{X_k - X_i + B}{s_i} + \frac{s_k}{s_i} z\right) - H_i\left(\frac{X_k - X_i}{s_i} + \frac{s_k}{s_i} z\right)\right)\right) dz \,. \tag{6.1}$$

If the X's have independent distributions each of which is known up to a constant translation, we may work out (independent) fiducial distributions for the corresponding parameters, and obtain results resembling (6.1).

Also, if we have X's no longer mutually independent but multivariate normal with known covariance matrix, the fiducial distribution of the μ's has the same covariance matrix and we are still able to express P^* in terms of B. If the joint distribution of the X's is known up to a constant translation, we may likewise find and make use of the fiducial distribution of the corresponding parameters.

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