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ON THE REDUCTION OF ASSOCIATE CLASSES FOR THE PBIB DESIGN OF A CERTAIN GENERALIZED TYPE

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For BIB designs N_i and their complements N_i^* $(i = 1, 2, \cdots, n)$, Kageyama (1972) gave necessary and sufficient conditions for a PBIB design $N = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ with at most three associate classes having the rectangular association scheme to be reducible to a PBIB design with only two distinct associate classes having the L_2 association scheme. In this paper similar results for the PBIB design $N_1 \otimes N_2 \otimes \cdots \otimes N_n + N_1^* \otimes N_2^* \otimes \cdots \otimes N_n^*$, which is in a sense a generalization of the Kronecker products of the above type, are described.

1. Introduction and summary. The Kronecker product of designs and reduced designs were defined by Vartak [4], but the association schemes concerning these designs were not considered explicitly. Kageyama [2] dealt with the reduced designs together with the association schemes matching these designs, and gave necessary and sufficient conditions for a PBIB design with l associate classes constructed by the Kronecker product of some BIB designs to be reducible to a PBIB design with l_1 associate classes for a positive integer l_1 satisfying $l_1 < l$. When an arrangement with the parameters of a PBIB design is given, it is important to determine the association scheme matching its design in relation to the problem showing the uniqueness of the association scheme. It should also be remarked that an association scheme can be defined and characterized independently of treatment-block incidence of the design (cf. [1]). The axioms of an association scheme, however, have been derived from describing the relation among treatments in terms of the structure of treatment-block incidence of the design, in particular, the numbers λ_i. This paper is based on this thinking, that is, when the parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ of a PBIB design are not all different, the m associate classes of the PBIB design based on a certain association scheme may not be all distinct. The problem considered here will be of both theoretical and practical importance with regard to constructing certain PBIB designs.

Vartak's approach and the approach given in this paper differ; in his approach the coincidence numbers λ_i and the second kind of parameters p^i_{jk} of a PBIB design N are used, while in the approach considered here, the coincidence numbers λ_i and the latent roots ρ_i of the matrix NN' are used.

In Section 2, necessary and sufficient conditions for a PBIB design with at most seven associate classes constructed by a sum of the Kronecker products of

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two sets of BIB designs to be reducible to a PBIB design with only three distinct associate classes are given. In Section 3, a generalization of this type is shown.

The algebraic structure concerning an association scheme of a PBIB design can be found in Bose and Mesner [1]. The notations used are coincident with those generally used, and especially with those of Kageyama [2]. Since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix are denoted by the same symbol throughout this paper.

2. Kronecker product design of three BIB designs of a certain type. Let N_i be BIB designs with parameters v_i, b_i, r_i, k_i, λ_i and N_i^* be complementary BIB designs with parameters $v_i^* = v_i$, $b_i^* = b_i$, $r_i^* = b_i - r_i$, $k_i^* = v_i - k_i$, $\lambda_i^* = b_i - 2r_i + \lambda_i$ of N_i $(i = 1, 2, 3)$. Consider the Kronecker product in the form $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$, which is different from $N_1 \otimes N_2 \otimes N_3 + N_1 \otimes N_2^* \otimes N_3^* + N_1^* \otimes N_2 \otimes N_3^* + N_1^* \otimes N_2^* \otimes N_3$ constructed by the Sillitto's product of $N_\alpha = N_1 \otimes N_2 + N_1^* \otimes N_2^*$ and N_3, where it is well known (cf. [3]) that N_α is a BIB design provided $b_i = 4(r_i - \lambda_i)$, $i = 1, 2$. Thus the product series considered in this paper is different from the generalization of the Sillitto type of product, which is under investigation.

From the structure of $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$, the parameters of design N are easily given by

$$
\begin{aligned}
v' &= v_1 v_2 v_3\,, \qquad b' = b_1 b_2 b_3\,,\\
r' &= r_1 r_2 r_3 + (b_1 - r_1)(b_2 - r_2)(b_3 - r_3)\,,\\
k' &= k_1 k_2 k_3 + (v_1 - k_1)(v_2 - k_2)(v_3 - k_3)\,,\\
\lambda_1' &= r_1 \lambda_2 r_3 + (b_1 - r_1)(b_2 - 2r_2 + \lambda_2)(b_3 - r_3)\,,\\
\lambda_2' &= \lambda_1 r_2 r_3 + (b_1 - 2r_1 + \lambda_1)(b_2 - r_2)(b_3 - r_3)\,,\\
\lambda_3' &= \lambda_1 \lambda_2 r_3 + (b_1 - 2r_1 + \lambda_1)(b_2 - 2r_2 + \lambda_2)(b_3 - r_3)\,,\\
\lambda_4' &= r_1 r_2 \lambda_3 + (b_1 - r_1)(b_2 - r_2)(b_3 - 2r_3 + \lambda_3)\,,\\
\lambda_5' &= r_1 \lambda_2 \lambda_3 + (b_1 - r_1)(b_2 - 2r_2 + \lambda_2)(b_3 - 2r_3 + \lambda_3)\,,\\
\lambda_6' &= \lambda_1 r_2 \lambda_3 + (b_1 - 2r_1 + \lambda_1)(b_2 - r_2)(b_3 - 2r_3 + \lambda_3)\,,\\
\lambda_7' &= \lambda_1 \lambda_2 \lambda_3 + (b_1 - 2r_1 + \lambda_1)(b_2 - 2r_2 + \lambda_2)(b_3 - 2r_3 + \lambda_3)\\
&\quad + 2(r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3)\,.
\end{aligned}
$$

Further, from the properties of a BIB design we have

$$
\begin{aligned}
N_i N_i' &= (r_i - \lambda_i) I_{v_i} + \lambda_i G_{v_i}\,,\\
N_i^* N_i^{*\prime} &= (r_i - \lambda_i) I_{v_i} + (b_i - 2r_i + \lambda_i) G_{v_i}\,, \qquad (2.1)\\
N_i N_i^{*\prime} &= (\lambda_i - r_i) I_{v_i} + (r_i - \lambda_i) G_{v_i}\,,
\end{aligned}
$$

where I_{v_i} is a $v_i \times v_i$ unit matrix and G_{v_i} is a $v_i \times v_i$ matrix whose elements are all unity $(i = 1, 2, 3)$. Since it is clear that the matrices (2.1) are symmetrical and mutually commutative, there exist orthogonal matrices P_i which make all

N_iN_i', $N_i^*N_i^{*\prime}$, $N_iN_i^{*\prime}$ diagonal simultaneously, such as

$$P_iN_iN_i'P_i' = D_i = \operatorname{diag}\{r_ik_i, r_i - \lambda_i, \cdots, r_i - \lambda_i\}\,,$$
$$(2.2) \qquad P_iN_i^*N_i^{*\prime}P_i' = D_i^* = \operatorname{diag}\{(v_i - k_i)(b_i - r_i), r_i - \lambda_i, \cdots, r_i - \lambda_i\}\,,$$
$$P_iN_iN_i^{*\prime}P_i' = \tilde{D}_i = \operatorname{diag}\{(v_i - 1)(r_i - \lambda_i), \lambda_i - r_i, \cdots, \lambda_i - r_i\}\,,$$

where $\operatorname{diag}\{r_ik_i, r_i - \lambda_i, \cdots, r_i - \lambda_i\}$ denotes a diagonal matrix whose diagonal elements are the elements r_ik_i and $r_i - \lambda_i$, and further, $r_i - \lambda_i$ and $\lambda_i - r_i$ appear $v_i - 1$ times, respectively ($i = 1, 2, 3$). Hence from (2.2) we have

$$(2.3) \qquad (P_1 \otimes P_2 \otimes P_3)NN'(P_1 \otimes P_2 \otimes P_3)' = D_1 \otimes D_2 \otimes D_3 + D_1^* \otimes D_2^* \otimes D_3^* + 2\tilde{D}_1 \otimes \tilde{D}_2 \otimes \tilde{D}_3\,.$$

Then from (2.2) and (2.3) the latent roots of NN' are as follows:

$$\rho_0 = r'k' \qquad \text{with multiplicity } 1\,,$$
$$\rho_1 = (r_3 - \lambda_3)\{r_1k_1r_2k_2 + (b_1 - r_1)(v_1 - k_1)(b_2 - r_2)(v_2 - k_2) - 2(v_1 - 1)(r_1 - \lambda_1)(v_2 - 1)(r_2 - \lambda_2)\} \qquad \text{with multiplicity } v_3 - 1\,,$$
$$\rho_2 = (r_2 - \lambda_2)\{r_1k_1r_3k_3 + (b_1 - r_1)(v_1 - k_1)(b_3 - r_3)(v_3 - k_3) - 2(v_1 - 1)(r_1 - \lambda_1)(v_3 - 1)(r_3 - \lambda_3)\} \qquad \text{with multiplicity } v_2 - 1\,,$$
$$\rho_3 = (r_2 - \lambda_2)(r_3 - \lambda_3)\{r_1k_1 + (b_1 - r_1)(v_1 - k_1) + 2(v_1 - 1)(r_1 - \lambda_1)\} \qquad \text{with multiplicity } (v_2 - 1)(v_3 - 1)\,,$$
$$\rho_4 = (r_1 - \lambda_1)\{r_2k_2r_3k_3 + (b_2 - r_2)(v_2 - k_2)(b_3 - r_3)(v_3 - k_3) - 2(v_2 - 1)(r_2 - \lambda_2)(v_3 - 1)(r_3 - \lambda_3)\} \qquad \text{with multiplicity } v_1 - 1\,,$$
$$\rho_5 = (r_1 - \lambda_1)(r_3 - \lambda_3)\{r_2k_2 + (b_2 - r_2)(v_2 - k_2) + 2(v_2 - 1)(r_2 - \lambda_2)\} \qquad \text{with multiplicity } (v_1 - 1)(v_3 - 1)\,,$$
$$\rho_6 = (r_1 - \lambda_1)(r_2 - \lambda_2)\{r_3k_3 + (b_3 - r_3)(v_3 - k_3) + 2(v_3 - 1)(r_3 - \lambda_3)\} \qquad \text{with multiplicity } (v_1 - 1)(v_2 - 1)\,,$$
$$\rho_7 = 0 \qquad \text{with multiplicity } (v_1 - 1)(v_2 - 1)(v_3 - 1)\,.$$

From the argument similar to that in Sections 4 and 5 in Kageyama [2], among $v_1v_2v_3$ treatments in design N an F_3 type association scheme with seven associate classes can be defined. Thus N is a PBIB design (see also [4]). Here we consider the derivation of necessary and sufficient conditions for a PBIB design $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$ with at most seven associate classes having the F_3 type association scheme to be reducible to a PBIB design with only three distinct associate classes.

It follows from the relations among the parameters of BIB designs that $v_i = v_j$ and $k_i = k_j$ are equivalent to $r_ik_i(r_j - \lambda_j) = r_jk_j(r_i - \lambda_i)$ and $r_i\lambda_j = r_j\lambda_i$ for all i, j $(i \neq j) = 1, 2, 3$. By investigating those among the latent roots ρ_i of NN' and among all the coincidence numbers λ_i' which may be equal to each other, under condition

$$(2.4) \qquad v_1 = v_2 = v_3 \quad \text{and} \quad k_1 = k_2 = k_3\,,$$

the following relations can be obtained:

$$\lambda_1' = \lambda_2' = \lambda_4' , \qquad \lambda_3' = \lambda_5' = \lambda_6' ; \qquad \rho_1 = \rho_2 = \rho_4 , \qquad \rho_3 = \rho_5 = \rho_6 ,$$

if and only if

$$b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1) , \qquad b_2(r_3 - \lambda_3) = b_3(r_2 - \lambda_2) ,$$
$$b_1(r_3 - \lambda_3) = b_3(r_1 - \lambda_1) . \tag{2.5}$$

Now if $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$ is a PBIB design with only three distinct associate classes under (2.4), then from the well-known fact that in general for a PBIB design N with m associate classes, NN' has at most $m + 1$ coincidence numbers λ_i and distinct latent roots ρ_i, condition (2.5) must hold. Conversely, if a PBIB design $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$ with at most seven associate classes satisfies condition (2.5) under (2.4), then it follows from Section 4 in Kageyama [2] that the design N is reducible to a PBIB design with three associate classes having the cubic association scheme.

Thus the following theorem is obtained:

THEOREM 1. *Given the* BIB *designs N_i with parameters v, b_i, r_i, k and λ_i ($i = 1, 2, 3$). Then necessary and sufficient conditions for a* PBIB *design $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$ with at most seven associate classes having the F_3 type association scheme to be reducible to a* PBIB *design with only three distinct associate classes having the cubic association scheme are*

$$b_1(r_2 - \lambda_2) = b_2(r_1 - \lambda_1) , \qquad b_2(r_3 - \lambda_3) = b_3(r_2 - \lambda_2) ,$$
$$b_1(r_3 - \lambda_3) = b_3(r_1 - \lambda_1) .$$

Note that in Theorem 2 of Kageyama [2] condition (2.4) is a necessary and sufficient condition for a PBIB design $N_1 \otimes N_2 \otimes N_3$ with at most seven associate classes having the F_3 type association scheme to be reducible to a PBIB design with only three distinct associate classes having the cubic association scheme and that, however, in Theorem 1, a PBIB design N does not reduce by only condition (2.4). Theorem 2 of Kageyama [2] and Theorem 1 imply that there may be more than one distinct PBIB design based on the same association scheme. Since $N = N_1 \otimes N_2 \otimes N_3 + N_1^* \otimes N_2^* \otimes N_3^*$ does not lead to a BIB design under any of additional conditions, it is conceivable that Theorem 1 is a natural generalization of Theorem 4 of Kageyama [2] in a sense. If the conditions like (2.4) are not entirely assumed, then condition (2.5) becomes more complicated and this fact is uninteresting for us, and hence it is omitted here.

3. Generalization of the type considered here. From the argument of this paper and Section 5 in Kageyama [2], this theorem can be similarly generalized as follows:

THEOREM 2. *Given the* BIB *designs N_i with parameters v, b_i, r_i, k and λ_i ($i = 1, 2, \cdots, m$). Then necessary and sufficient conditions for a* PBIB *design $N = N_1 \otimes N_2 \otimes \cdots \otimes N_m + N_1^* \otimes N_2^* \otimes \cdots \otimes N_m^*$ with at most $2^m - 1$ associate classes* *having the F_m type association scheme to be reducible to a* PBIB *design with the hypercubic association scheme of m associate classes are that*

$$b_i(r_j - \lambda_j) = b_j(r_i - \lambda_i)$$

holds simultaneously for every $i, j\ (i \neq j) = 1, 2, \cdots, m$.

A simple example of Theorem 2 is obtained by letting us take $N_1 = N_2 = \cdots = N_m$ to be a BIB design with the suitable parameters.

From Section 6 in Kageyama [2], the generalization of the other type may be also considered. However, from a point of view of reduction of associate classes, those types are not discussed here.

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REFERENCES

[1] Bose, R. C. and Mesner, D. M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs. *Ann. Math. Statist.* **30** 21–38.

[2] Kageyama, S. (1972). On the reduction of associate classes for certain PBIB designs. *Ann. Math. Statist.* **43** 1528–1540.

[3] Sillitto, G. P. (1957). An extension property of a class of balanced incomplete block designs. *Biometrika* **44** 278–279.

[4] Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs. *Ann. Math. Statist.* **26** 420–438.

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