

## SOME REMARKS ON SUFFICIENCY, INVARIANCE AND CONDITIONAL INDEPENDENCE

BY A. G. NOGALES AND J. A. OYOLA

*Universidad de Extremadura*

In this paper results and counterexamples are given to study the relationship between some conditions which appear in the literature on sufficiency, invariance and conditional independence.

Let  $(\Omega, \mathcal{A}, \mathcal{P})$  be a statistical experiment [i.e.,  $\mathcal{P}$  is a family of probability measures on the measurable space  $(\Omega, \mathcal{A})$ ] and  $G$  is a group of bijective and bimeasurable maps of  $(\Omega, \mathcal{A})$  onto itself leaving the family  $\mathcal{P}$  invariant, that is,  $gP \in \mathcal{P}, \forall P \in \mathcal{P}, \forall g \in G$ , where  $gP$  is the probability measure on  $\mathcal{A}$  defined by  $gP(A) = P(g^{-1}A), A \in \mathcal{A}$ . If  $P \in \mathcal{P}$ , two events  $B, C \in \mathcal{A}$  are said to be  $P$ -equivalent (and we shall write  $B \sim_P C$ ) if  $P(B \Delta C) = 0$ ; these events are said to be equivalent (we write  $B \sim C$ ) if they are  $P$ -equivalent for all  $P \in \mathcal{P}$ . Let  $\mathcal{A}_I = \{A \in \mathcal{A}: gA = A, \forall g \in G\}$  be the  $\sigma$ -field of  $G$ -invariant sets and let  $\mathcal{A}_A = \{A \in \mathcal{A}: gA \sim A, \forall g \in G\}$  be the  $\sigma$ -field of  $\mathcal{P}$ -almost- $G$ -invariant sets.  $\mathcal{A}_S$  will always be a sufficient sub- $\sigma$ -field of  $\mathcal{A}$ . Let  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  be three sub- $\sigma$ -fields of  $\mathcal{A}$ . For  $P \in \mathcal{P}$ , the  $\sigma$ -fields  $\mathcal{B}$  and  $\mathcal{C}$  are said to be  $P$ -conditionally independent given  $\mathcal{D}$ , and we shall write  $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}$ , if

$$E_P(I_{B \cap C} | \mathcal{D}) \sim_P E_P(I_B | \mathcal{D}) \cdot E_P(I_C | \mathcal{D})$$

for every  $B \in \mathcal{B}$  and  $C \in \mathcal{C}$ . It is well known that  $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}$  if and only if

$$E_P(I_C | \mathcal{B} \vee \mathcal{D}) \sim_P E_P(I_C | \mathcal{D}), \quad \forall C \in \mathcal{C},$$

where  $\mathcal{B} \vee \mathcal{D}$  is the smallest  $\sigma$ -field containing  $\mathcal{B}$  and  $\mathcal{D}$ . The  $\sigma$ -fields  $\mathcal{B}$  and  $\mathcal{C}$  are said to be conditionally independent given  $\mathcal{D}$ , and we shall write  $\mathcal{B} \perp \mathcal{C} | \mathcal{D}$  if  $\mathcal{B} \perp_P \mathcal{C} | \mathcal{D}, \forall P \in \mathcal{P}$ . Other concepts not defined here can be found, for example, in Lehmann (1986).

This paper is concerned with four propositions:

P1. For every  $A \in \mathcal{A}_I$ , there exists an  $\mathcal{A}_S \cap \mathcal{A}_I$ -measurable function  $P$ -equivalent to  $E(I_A | \mathcal{A}_S)$  for every  $P \in \mathcal{P}$ .

P2.  $\mathcal{A}_S \perp \mathcal{A}_I | \mathcal{A}_S \cap \mathcal{A}_I$ .

P3.  $\mathcal{A}_S \cap \mathcal{A}_I$  is sufficient for  $\mathcal{P}_{|\mathcal{A}_I}$ .

P4. For every  $A \in \mathcal{A}_I, E(I_A | \mathcal{A}_S)$  is almost invariant.

Received June 1994; revised June 1995.

AMS 1991 subject classifications. Primary 62B05; secondary 62A05.

Key words and phrases. Sufficiency, invariance, conditional independence.

These propositions have been considered in relation with a theorem of C. Stein on sufficiency and invariance. In Hall, Wijsman and Ghosh (1965), Stein's theorem is stated as follows: "Under conditions A.i)  $g\mathcal{A}_S = \mathcal{A}_S, \forall g \in G$  and A.ii)  $\mathcal{A}_S \cap \mathcal{A}_I \sim \mathcal{A}_S \cap \mathcal{A}_A$ , we have that  $\mathcal{A}_S \cap \mathcal{A}_I$  is sufficient for  $\mathcal{P}_{|\mathcal{A}_I}$ ." They pose the question if condition A.ii) can be dropped in this result. Example 1 of Landers and Rogge (1973) solved this problem in the negative. This example will be frequently referred to in this paper, where results and counterexamples are given to clarify the relationship between the propositions above.

The following theorem states positive results between the propositions.

**THEOREM 1.** (a) P1  $\Leftrightarrow$  P2 + P3.  
(b) P2  $\Rightarrow$  P4.

**PROOF.** Part (a) is a restatement of Lemma 3.3 of Hall, Wijsman and Ghosh (1965); just a little change in their proof is necessary to prove it. To prove part (b), let  $A \in \mathcal{A}_I$ ,  $h$  be a version of  $E(I_A | \mathcal{A}_S)$  and  $g \in G$ . For  $P \in \mathcal{P}$ , let  $\varphi_P$  be a version of  $E_P(I_A | \mathcal{A}_S \cap \mathcal{A}_I)$ . The conditional independence implies

$$P(h \circ g \neq \varphi_{gP} \circ g) = gP(h \neq \varphi_{gP}) = 0.$$

In (2.1) of Hall, Wijsman and Ghosh (1965), take  $f = I_A$ ,  $\mathcal{A}_0 = \mathcal{A}_S \cap \mathcal{A}_I$  and use the invariance of  $f$ ,  $\varphi_P$  and  $\mathcal{A}_0$  to obtain  $\varphi_{gP} \sim_{gP} \varphi_P$ , for every  $P, g$ . Then a simple substitution yields  $\varphi_{gP} \sim_P \varphi_P$ . On the other hand,  $\varphi_{gP} \circ g = \varphi_{gP}$  by invariance.

Since

$$P(h \circ g \neq h) \leq P(h \circ g \neq \varphi_{gP} \circ g) + P(\varphi_{gP} \circ g \neq h),$$

we have that  $h \circ g \sim h$ , as desired.  $\square$

**REMARK 1.** (i) Lemma 3.3 of Hall, Wijsman and Ghosh (1965) states that P1 and P2 are equivalent, but only the implication P1  $\Rightarrow$  P2 is true: Example 1 of Landers and Rogge (1973) shows that P2  $\not\Rightarrow$  P1. Nevertheless, P1 is implied by the stronger condition that  $\mathcal{A}_S \perp_Q \mathcal{A}_I | \mathcal{A}_S \cap \mathcal{A}_I$  for a privileged dominating probability  $Q$  on  $\mathcal{A}$ .

(ii) The reader can also find in Hall, Wijsman and Ghosh [(1965), pages 595 and 602] the erroneous assertion that P2  $\Rightarrow$  P3: Example 1 of Landers and Rogge (1973) is a counterexample. See also Remark 2(i) below.

(iii) An equivalent formulation of P1 is that  $E(I_A | \mathcal{A}_S)$  has an invariant version if  $A \in \mathcal{A}_I$ . To obtain Stein's theorem, Hall, Wijsman and Ghosh [(1965), Lemma 3.2] show that A.i) + A.ii)  $\Rightarrow$  P1. Their proof is also valid to prove the strictly stronger statement that  $E(I_A | \mathcal{A}_S)$  has an invariant version for all  $A \in \mathcal{A}_A$ . The last statement is, in fact, equivalent to A.ii) in the presence of A.i).

(iv) Lemma 3.1 of Hall, Wijsman and Ghosh (1965) shows that P4 is true under the hypothesis  $g\mathcal{A}_S = \mathcal{A}_S, \forall g \in G$ .

(v) The proof of the equivalence of (i) and (ii) in Lemma 3 of Berk (1972) sends the reader to Lemma 3.3 of Hall, Wijsman and Ghosh (1965), which is

false as is pointed out in Remark 1(i) above. Nevertheless, an analogous argument to that used in the proof of the part (b) of the theorem also shows that Lemma 3 of Berk (1972) remains true.

The next example shows that neither  $P3 \Rightarrow P2$  nor  $P4 \Rightarrow P2$ .

EXAMPLE 1. Let us consider the statistical experiment

$$(\mathbb{R}^3, \mathcal{R}^3, \{P_\mu/\mu \in \mathbb{R}\}),$$

where  $\mathcal{R}^3$  is the Borel  $\sigma$ -field on  $\mathbb{R}^3$  and  $P_\mu$  is the trivariate normal distribution with mean  $(\mu, 2\mu, 0)$  and covariance matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If  $X, Y$  and  $Z$  denote the coordinates on  $\mathbb{R}^3$ , the  $\sigma$ -field  $\mathcal{A}_S$  induced by the statistics  $(X, Y)$  is sufficient for this experiment. Let  $G$  be the group of all bijective maps of  $\mathbb{R}^3$  onto itself moving at most a finite set of  $\mathbb{R}^3$  and leaving  $(Y, Z)$  invariant. The  $\sigma$ -field  $\mathcal{A}_S \cap \mathcal{A}_I = Y^{-1}(\mathcal{R})$  is sufficient for  $\mathcal{P}|_{\mathcal{A}_I}$ . Furthermore,  $P4$  is satisfied because  $\mathcal{A}_A = \mathcal{R}^3$ . Nevertheless,  $\mathcal{A}_S$  and  $\mathcal{A}_I$  are not conditionally independent given  $\mathcal{A}_S \cap \mathcal{A}_I$ , since  $Z$  is invariant and  $E(Z|\mathcal{A}_I) = 0$  is not equivalent to  $E(Z|\mathcal{A}_S \cap \mathcal{A}_I) = (2X - Y)/3$ .  $\square$

REMARK 2. (i) In this example, condition A.i) is not satisfied, but  $g\mathcal{A}_S \sim \mathcal{A}_S, \forall g \in G$ . However, implication A.i) +  $P3 + P4 \Rightarrow P2$  is not true either. Replacing  $\mathcal{A}_S$  by  $\mathcal{A}_S \vee \mathcal{N}$  ( $\mathcal{N}$  being the family of null Borel sets), a counterexample is obtained. Note that the sufficient statistic  $(X, Y)$  is not complete. In fact, it is pointed out in Hall, Wijsman and Ghosh [(1965), page 602] that  $P3 +$  completeness of  $\mathcal{A}_S$  implies  $P2$ .

(ii) The above-mentioned example of Landers and Rogge (1973) shows that  $P3$  is not implied by  $P4$ .

The next example shows that implication  $P3 \Rightarrow P4$  is also false.

EXAMPLE 2. Let  $\Omega = \{-2, -1, 1, 2\}$ ,  $\mathcal{A}$  be the family of all subsets of  $\Omega$  and  $\mathcal{P} = \{\varepsilon_+, \varepsilon_-\}$ , where  $\varepsilon_+ = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$  and  $\varepsilon_- = \frac{1}{2}(\varepsilon_{-1} + \varepsilon_{-2})$ ,  $\varepsilon_i$  being the probability measure concentrated at point  $i$ . The smallest  $\sigma$ -field  $\mathcal{A}_S$  containing the subsets  $\{-1\}$  and  $\{-2\}$  is sufficient for this experiment. The family  $\mathcal{P}$  remains invariant under the action of the group  $G = \{I, Z\}$  where  $I$  is the identity map on  $\Omega$  and  $Zi = -i, i \in \Omega$ , and  $\mathcal{A}_A = \mathcal{A}_I$  is the smallest  $\sigma$ -field containing  $\{-1, 1\}$ . The  $\sigma$ -field  $\mathcal{A}_S \cap \mathcal{A}_I$  is sufficient for  $\mathcal{A}_I$  since the restrictions of  $\varepsilon_+$  and  $\varepsilon_-$  to  $\mathcal{A}_I$  coincide. On the other hand, the event  $\{-1, 1, 2\}$  is not almost-invariant and its indicator function is a common version of  $\varepsilon_\pm(\{-1, 1\}|\mathcal{A}_S)$ .

**Acknowledgments.** We wish to thank an Associate Editor, the referee and Professor J. K. Ghosh and Professor R. A. Wijsman for their helpful comments on an earlier draft of the paper.

#### REFERENCES

- BERK, R. H. (1972). A note on sufficiency and invariance. *Ann. Math. Statist.* **43** 647–650.  
HALL, W. J., WIJSMAN, R. A. and GHOSH, J. K. (1965). The relationship between sufficiency and invariance with applications in sequential analysis. *Ann. Math. Statist.* **36** 575–614.  
LANDERS, D. and ROGGE, L. (1973). On sufficiency and invariance. *Ann. Statist.* **1** 543–544.  
LEHMANN, E. L. (1986). *Testing Statistical Hypotheses*, 2nd ed. Wiley, New York.

DEPARTAMENTO DE MATEMÁTICAS  
FACULTAD DE CIENCIAS  
UNIVERSIDAD DE EXTREMADURA  
AVDA. ELVAS S / N  
06071-BADAJOS  
SPAIN