The Annals of Statistics
1999, Vol. 27, No. 6, 1948–1973

THEORY OF OPTIMAL BLOCKING OF 2^{n-m} DESIGNS

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In this paper, we define the blocking wordlength pattern of a blocked fractional factorial design by combining the wordlength patterns of treatment-defining words and block-defining words. The concept of minimum aberration can be defined in terms of the blocking wordlength pattern and provides a good measure of the estimation capacity of a blocked fractional factorial design. By blending techniques of coding theory and finite projective geometry, we obtain combinatorial identities that govern the relationship between the blocking wordlength pattern of a blocked 2^{n-m} design and the split wordlength pattern of its blocked residual design. Based on these identities, we establish general rules for identifying minimum aberration blocked 2^{n-m} designs in terms of their blocked residual designs. Using these rules, we study the structures of some blocked 2^{n-m} designs with minimum aberration.

1. Introduction. Fractional factorial designs with factors at two levels have a long history of successful use in factor screening experiments and many scientific investigations. A 2^{-m}th fraction of a 2^n factorial design consisting of 2^{n-m} distinct combinations will be referred to as a 2^{n-m} fractional factorial design. Such a design is called regular if it can be constructed by using a defining relation. This is discussed in many textbooks on experimental design; see, for example, Hinkelmann and Kempthorne (1994). How to choose a good 2^{n-m} fractional factorial design is an important issue. Resolution [Box and Hunter (1961)] and its refinement, minimum aberration [Fries and Hunter (1980)], are commonly used criteria for selecting regular 2^{n-m} designs. In general, these criteria give good measures of the estimation capacity of a fractional factorial design [Cheng, Steinberg and Sun (1999)].

Blocking is an effective method for improving the efficiency of an experiment by eliminating systematic variations due to inhomogeneities of experimental units. However, how to block a 2^{n-m} design in an optimal way, a problem of practical importance, has not been addressed until the recent work of Bisgaard (1994), Sun, Wu and Chen (1997) and Sitter, Chen and Feder (1997). For unblocked 2^{n-m} designs, the criteria of resolution and minimum aberration are based on the *wordlength pattern* of the defining

Received August 1997; revised October 1999.
[1]Supported in part by NSF Grant DMS-97-04548.
AMS 1991 *subject classifications*. Primary 62K15; secondary 62K05.
Key words and phrases. Fractional factorial design, linear code, MacWilliams identities, resolution, projective geometry, weight distribution, wordlength pattern.

relation. In the blocked case, the defining relation contains two kinds of words, those involving only treatment factors and those containing at least one blocking factor. These two kinds of defining words play very different roles, leading to two different wordlength patterns and causing some difficulties. In attempts to alleviate the problem, Bisgaard (1994) and Sitter, Chen and Feder (1997) presented different proposals to modify the definition of the length of a defining word that contains at least one blocking factor. Sun, Wu and Chen (1997) treated the two wordlength patterns separately and introduced a notion of admissibility. Because of subtle differences and intricate relations between the two wordlength patterns, these approaches do not completely resolve the problem of optimal blocking of fractional factorial designs (see Section 2).

The objective of this article is to present a proper extension of the concept of minimum aberration to blocked fractional factorial designs. In Section 3, we introduce the blocking wordlength pattern of a blocked fractional factorial design, which combines the two wordlength patterns mentioned above, and use it to define blocking resolution and minimum aberration. Advantages of our formulation are discussed by comparing it with the optimal blocking schemes obtained by other approaches. The rest of the paper develops tools for the construction of minimum aberration blocked designs under our criterion. Chen and Hedayat (1996), Tang and Wu (1996) and Suen, Chen and Wu (1997) studied the characterization of unblocked regular 2^{n-m} designs with minimum aberration in terms of their complementary designs. This approach is very powerful for identifying minimum aberration designs when their complementary designs are small. It turns out that in the blocked case, minimum aberration designs can also be identified by classifying the split wordlength patterns of their blocked residual designs, which are defined in Section 4. This section also contains some preliminary material in coding theory including the MacWilliams identities. Section 5 derives combinatorial identities relating the split wordlength pattern of a blocked fractional factorial design to that of its blocked residual design. Based on these identities, we establish general rules for identifying minimum aberration designs in terms of their blocked residual designs. These rules are then used to characterize several families of blocked fractional factorial designs with minimum aberration in Section 6.

In the spirit in which the concept of minimum aberration was originally introduced, we shall impose the hierarchical assumption that (1) lower order effects are more important than higher order effects and (2) effects of the same order are equally important. Our approach can be extended to the case where some effects are more important than other effects of the same order (see Section 3). Throughout this paper, we shall only consider blocked designs in which no treatment main effect is aliased with other treatment main effects or block effects. We also note that, for simplicity, we focus entirely on two-level designs, even though the results can easily be generalized to the case where the number of levels is a prime power.

2. Existing approaches. Before studying the problem of optimal blocking of fractional factorial designs, we briefly describe some of the basic concepts associated with regular fractional factorial designs.

Each factor is represented by one of the numbers $1, 2, \dots, n$, which are called letters. An interaction can then be represented by a product (juxtaposition) of a subset of these letters, and is called a word. The number of letters in a word is called its length. Associated with every regular 2^{n-m} fractional factorial design is a set of m independent defining words $\mathbf{w}_1, \dots, \mathbf{w}_m$. The set of distinct words formed by all possible products involving $\mathbf{w}_1, \dots, \mathbf{w}_m$ gives the defining relation of the fraction. For each regular 2^{n-m} fractional factorial design $D(2^{n-m})$, let $A_i(D)$ be the number of words of length i in its defining relation and $W(D)$ be the vector

$$W(D) = (A_3(D), A_4(D), \dots, A_n(D)). \tag{1}$$

Here without loss of generality, we may assume that $A_1(D) = A_2(D) = 0$. Then $W(D)$ is referred to as the wordlength pattern of $D(2^{n-m})$. With this notation, the resolution of a $D(2^{n-m})$ is the smallest i with positive $A_i(D)$. Designs with the same resolution can be further discriminated by the following criterion. For two fractions $D_1(2^{n-m})$ and $D_2(2^{n-m})$ with wordlength patterns $W(D_1)$ and $W(D_2)$, respectively, D_1 is said to have less aberration than D_2 if $A_s(D_1) < A_s(D_2)$ where s is the smallest integer such that $A_s(D_1) \neq A_s(D_2)$. A 2^{n-m} design has minimum aberration if no other 2^{n-m} design has less aberration. Simply put, the criterion of minimum aberration sequentially minimizes $A_3(D), A_4(D), \dots$, etc.

Blocking of a 2^{n-m} design can be treated as a special case of fractionation [Lorenzen and Wincek (1992)]. Let $D(2^{n-m} : 2^r)$ be a 2^{n-m} design in 2^r blocks of size 2^{n-m-r} $(r < n - m)$. Then $D(2^{n-m} : 2^r)$ can be viewed as a $2^{(n+r)-(m+r)}$ design where the factors are divided into two different types: n treatment factors $1, \dots, n$, and r blocking factors $b_1, \dots, b_r$. The 2^r combinations of the blocking factors are used to divide the 2^{n-m} treatment combinations into 2^r blocks. In this $2^{(n+r)-(m+r)}$ design, there are two different kinds of defining words: those containing the treatment factors only (called *treatment defining words*) and those containing at least one blocking factor (called *block defining words*). This can be illustrated by the following example from Bisgaard (1994).

EXAMPLE 1. Consider the 2^{6-2} design with four blocks (i.e., a $2^{(6+2)-(2+2)}$ design where factors $1, \dots, 6$ are treatment factors and b_1, b_2 are blocking factors) defined by

$$\begin{aligned} I &= 1235 = 2346 = 1456 \\ &= 46b_1b_2 = 23b_1b_2 = 15b_1b_2 \\ &= 134b_1 = 245b_1 = 356b_1 = 126b_1 = 345b_2 = 136b_2 = 256b_2 = 124b_2 \\ &= 123456b_1b_2. \end{aligned}$$

In the above defining relation, both $23b_1b_2$ and $134b_1$ have four letters. However, the former confounds a two-factor interaction with a block effect, while the latter confounds a less important three-factor interaction. Bisgaard (1994) rightly argued that it is no longer appropriate to define the length of a block-defining word as the number of letters in it. In general, he proposed to define the length of a block-defining word as the number of treatment letters it contains plus 1. According to this definition, both 1456 and $356b_1$ in the defining relation have length 4. However, 1456 causes some two-factor interactions to be aliased with other two-factor interactions, while $356b_1$ confounds a less important three-factor interaction with block effects. This shows that Bisgaard's definition is still not adequate. Another problem with Bisgaard's (1994) definition is that it cannot distinguish between designs of the same resolution. Clearly, the different roles played by the treatment and block-defining words cannot simply be captured by naive wordlengths.

Recognizing the inadequacy in Bisgaard's (1994) definition, Sun, Wu and Chen (1997) proposed to consider the treatment and block defining words separately. For a blocked design $D(2^{n-m}:2^r)$, let $A_{i,0}(D)$ be the number of treatment-defining words containing i treatment letters, and let $A_{i,1}(D)$ be the number of block-defining words also containing i treatment letters. Since we will only consider designs in which none of the treatment main effects is aliased with other main effects or confounded with blocks, we may assume that $A_{1,0} = A_{0,1} = A_{2,0} = A_{1,1} = 0$. The two vectors $W_t = (A_{3,0}(D), A_{4,0}(D), \dots)$ and $W_{bt} = (A_{2,1}(D), A_{3,1}(D), \dots)$ together are called the *split wordlength pattern* of D. The criterion of minimum aberration can be applied to W_t and W_{bt} separately. However, minimum aberration designs with respect to W_t may not have minimum aberration with respect to W_{bt} and vice versa. Facing such multiple criteria, Sun, Wu and Chen (1997) adopted a concept of admissible designs. One difficulty with this approach is that often there are too many admissible designs. For example, only one of the six nonisomorphic 2^{8-4} designs with 4 blocks is inadmissible under Sun, Wu and Chen's (1997) criterion.

As another device to separate treatment-defining words from block-defining words, Sitter, Chen and Feder (1997) proposed to modify the length of a block-defining word as the number of treatment letters it contains plus 1.5. Effectively, this combines W_t and W_{bt} in a simple manner,

$$W_{SCF}(D) = \big(A_{3,0}(D), A_{2,1}(D), A_{4,0}(D), A_{3,1}(D), \dots, \\ A_{n,0}(D), A_{n-1,1}(D), A_{n,1}(D)\big),$$

where the components of $W_{SCF}(D)$ are successively the numbers of words of lengths 3, 3.5, 4, 4.5, ..., etc., according to Sitter, Chen and Feder's (1997) definition of wordlength. The criterion of minimum aberration applied to this combined wordlength pattern then amounts to sequentially minimizing $A_{3,0}, A_{2,1}, A_{4,0}, A_{3,1}, A_{5,0}, A_{4,1}, A_{6,0}, \dots$, etc. This approach, say, deems $A_{4,1}$ less desirable than $A_{6,0}$. However, if $A_{6,0}$ is nonzero, then some three-factor interactions are aliased with other three-factor interactions; on the other hand, if $A_{4,1}$ is nonzero, then the design confounds certain less important four-factor interactions with the blocks. Therefore, $A_{6,0}$ should be less desirable than $A_{4,1}$, and their order should be reversed. From this point on, the ordering of the components in W_{SCF} is no longer in accordance with the hierarchical assumption. The following example shows that this may cause Sitter, Chen and Feder (1997) to misclassify the resolution of a blocked fractional factorial design.

EXAMPLE 2. Consider the following two blocked designs $D(2^{8-1}:2^2)$:

$$D_1: I = 12345678 = 1234b_1 = 1256b_2,$$
$$D_2: I = 123458 = 1236b_1 = 3467b_2.$$

Their wordlength patterns under Sitter, Chen and Feder's (1997) definition are

$$W_{SCF}(D_1) = (0,0,0,0,0,6,0,0,0,0,1,0,0),$$
$$W_{SCF}(D_2) = (0,0,0,0,0,5,1,0,0,1,0,0,0).$$

These two designs have the same resolution according to Sitter, Chen and Feder (1997), and as they indicated, D_2 has minimum aberration. Since $A_{6,0}(D_2) = 1$, D_2 cannot estimate all three-factor interactions. However, the "more aberration" design D_1 [with $A_{4,1}(D_1) = 6$ and $A_{6,0}(D_1) = 0$] can estimate all three-factor interactions when the four-factor and higher-order interactions are negligible and, in our opinion, is a better design.

3. Blocking wordlength pattern. A proper extension of the concept of minimum aberration to blocked designs needs to somehow combine the two wordlength patterns W_t and W_{bt}. We have pointed out that Sitter, Chen and Feder's (1997) approach is not entirely adequate. In view of the hierarchical assumption, we propose the following ordering:

$$\underbrace{A_{3,0}, A_{2,1}}, A_{4,0}, \underbrace{A_{5,0}, A_{3,1}}, A_{6,0}, \underbrace{A_{7,0}, A_{4,1}}, A_{8,0}, \dots . \tag{2}$$

When $A_{3,0}$ is nonzero, certain two-factor interactions are aliased with main effects, while if $A_{2,1}$ is positive, then some two-factor interactions are confounded with blocks. Under the hierarchical assumption, both will cause certain two-factor interactions to be unestimable. On the other hand, they are less desirable than $A_{4,0}$ since when $A_{4,0}$ is positive, even though some two-factor interactions will be aliased with other two-factor interactions, their information is not completely lost.

One could then define a criterion of minimum aberration for blocked factorials by sequentially minimizing $A_{3,0}$, $A_{2,1}$, $A_{4,0}$, $A_{5,0}$, $A_{3,1}$, $A_{6,0}$, $A_{7,0}$, $A_{4,1}$, $A_{8,0}, \dots$. However, as will be seen later, this is still not satisfactory from the estimation capacity point of view. Instead, we propose the following definition of blocking wordlength pattern.

For each j, let

$$(3)\quad A_j^b(D) = \begin{cases} A_{j,0}, & \text{for even } j \le n, \\ \binom{j}{(j+1)/2} A_{j,0}(D) + A_{(j+1)/2,1}(D), & \text{for odd } j \le n, \\ A_{(j-[n/2]),1}(D), & \text{for } n+1 \le j \le n+[n/2], \end{cases}$$

where $[x]$ is the largest integer less than or equal to x. Notice that the length j can be greater than the number of treatment factors n. The vector

$$W_b(D) = \big(A_3^b(D), A_4^b(D), \dots, A_{n+[n/2]}^b(D)\big),$$

is called the *blocking wordlength pattern* of a blocked design $D(2^{n-m}:2^r)$.

The criterion of minimum aberration can be applied to this wordlength pattern in the usual way. Thus our criterion of minimum aberration for blocked factorials sequentially minimizes A_3^b, A_4^b, A_5^b, $A_6^b, \dots$, that is, $3A_{3,0} + A_{2,1}$, $A_{4,0}, 10A_{5,0} + A_{3,1}$, $A_{6,0}, \dots$, etc. This follows the order proposed in (2), with the modification that pairs which are connected by a parenthesis in (2) are dealt with together via a linear combination. This is motivated by the following consideration based on estimation capacity.

Cheng, Steinberg and Sun (1999) showed that in the unblocked case, minimum aberration is a good surrogate for some model-robustness criteria. For simplicity, assume that the main effects are of primary interest and their estimates are required. Furthermore, all the three-factor and higher-order interactions are negligible. We say that a model can be estimated by a 2^{n-m} design D if all the effects in the model are jointly estimable under D.

For any $1 \le k \le \binom{n}{2}$, define $E_k(D)$ as the number of models containing all the main effects and k two-factor interactions which can be estimated by D. With equal weights for the two-factor interactions (a kind of noninformative prior representing the experimenter's ignorance), roughly, $E_k(D)/\binom{n(n-1)/2}{k}$ can be thought of as the conditional probability that the true model can be estimated by D given that it contains all the main effects and k two-factor interactions. It is desirable to have $E_k(D)$ as large as possible. A design D_1 is said to dominate D_2 if $E_k(D_1) \ge E_k(D_2)$ for all k, with strict inequality for at least one k. A design is said to have maximum estimation capacity if it maximizes $E_k(D)$ for all k.

In a 2^{n-m} design D of resolution III or higher, let f be the number of alias sets not containing main effects, $m_1(D), \dots, m_f(D)$ be the number of two-factor interactions in these alias sets and let $\mathbf{m}(D)$ be the vector $(m_1(D), \dots, m_f(D))$. It turns out that for all k, $E_k(D)$ is a Schur-concave function of $\mathbf{m}(D)$ and is nondecreasing in each component of $\mathbf{m}(D)$ [Cheng, Steinberg and Sun (1999)]. Therefore a design has large estimation capacity if $\sum_{i=1}^f m_i(D)$ is as large as possible and the $m_i(D)$'s are as equal as possible. Since $\sum_{i=1}^f m_i(D)$ is equal to the number of two-factor interactions that are not aliased with main effects, a design has large estimation capacity if it (1) maximizes the number of two-factor interactions that are not aliased with main effects, and (2) these interactions are distributed among the alias sets as uniformly as possible. It is clear that (1) is equivalent to the minimization of $A_3(D)$ in equation (1). On the other hand, Cheng, Steinberg and Sun (1999) showed that (2) is related to the minimization of $A_4(D)$.

The concept of estimation capacity can be extended to the blocked case. Again define $E_k^b(D)$ as the number of models containing all the main effects and k two-factor interactions which can be estimated by a blocked 2^{n-m} design D. Now let f be the number of alias sets which neither contain main effects nor are confounded with blocks, and $m_1^b(D), \dots, m_f^b(D)$ the number of two-factor interactions in these alias sets. Then a design has large estimation capacity if (1) $\sum_{i=1}^f m_i^b(D)$ (= the number of two-factor interactions which are neither aliased with main effects nor confounded with blocks $= \binom{n}{2} - 3A_{3,0} - A_{2,1}$) is maximized, and (2) the $m_i^b(D)$'s are as equal as possible. This makes good sense when one does not know which two-factor interactions will be active. In this case, maximizing $\sum_{i=1}^f m_i^b(D)$ is equivalent to minimizing $3A_{3,0} + A_{2,1}$, and (2) is related to the minimization of $A_{4,0}$ [Cheng and Mukerjee (1997)]. Thus it is desirable to minimize $3A_{3,0} + A_{2,1}$ and $A_{4,0}$. For resolution III or IV designs, it is rare that one would ever need to consider longer words, whereas if both $3A_{3,0} + A_{2,1}$ and $A_{4,0}$ are zero, then no two-factor interaction is aliased with main effects, other two-factor interactions or block effects. In this case, the number of three-factor interactions which are aliased with some lower-order effects or confounded with blocks is equal to $\binom{5}{3}A_{5,0} + A_{3,1} = 10A_{5,0} + A_{3,1}$, the next component of the blocking wordlength pattern in equation (3). The subsequent components can be justified in a similar fashion.

We believe that this is a reasonable and natural extension of minimum aberration to blocked designs. It is important to maximize $3A_{3,0} + A_{2,1}$ instead of $A_{3,0}$ alone. The following is an example where sequential minimization of $A_{3,0}$, $A_{2,1}$ and $A_{4,0}$ leads to an inferior design.

EXAMPLE 3. Consider a 2^{13-8} design with 8 blocks defined by

$$\begin{aligned} D_3\colon I &= 1236 = 1247 = 1348 = 2340 = 125t_{10} = 135t_{11} = 235t_{12} = 145t_{13} \\ &= 13b_1 = 14b_2 = 15b_3, \end{aligned}$$

where $t_{10}, \dots, t_{13}$ are factors $10, \dots, 13$. This design has minimum aberration under Sitter, Chen and Feder (1997)'s definition, with wordlength pattern

$$W_{SCF}(D_3) = (0, 36, 55, 0, 0, 310, 96, \dots).$$

It also has minimum aberration with respect to the ordering in (2).

Now consider another blocked design D_4 which is generated by

$$\begin{aligned} I &= 126 = 137 = 148 = 2349 = 1234t_{10} = 235t_{11} = 245t_{12} = 345t_{13} \\ &= 23b_1 = 24b_2 = 15b_3. \end{aligned}$$

This design has

$$W_{SCF}(D_4) = (4, 22, 39, 76, 32, 124, 48, \dots).$$

Therefore D_4 has "more aberration" than D_3 under Sitter, Chen and Feder's (1997) criterion [and also under the minimum aberration criterion with respect to the ordering in (2)].

However, we have

$$\mathbf{m}(D_3) = (0, 0, 0, 5, 5, 5, 5, 5, 5, 6, 6),$$
$$\mathbf{m}(D_4) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4).$$

Not only does D_4 yield more two-factor interactions that are neither aliased with main effects nor confounded with blocks, but also they are distributed among the alias sets more uniformly. Clearly, D_4 dominates D_3 with respect to estimation capacity. In fact, since D_4 maximizes $\sum_{i=1}^{f} m_i^b(D)$ and all the $m_i^b(D_4)$'s are equal, it has maximum estimation capacity among all 2^{13-8} designs in eight blocks. It also has minimum aberration under our criterion, with blocking wordlength pattern

$$W_b(D_4) = (34, 39, \dots),$$

while D_3 has the following (inferior) blocking wordlength pattern

$$W_b(D_3) = (36, 55, \dots).$$

Our criterion correctly shows that D_4 is a better design from the viewpoint of estimation capacity. We also point out that D_3 cannot entertain models which contain more than eight two-factor interactions, but D_4 can handle certain models with up to eleven two-factor interactions.

Our definition of blocking wordlength pattern can be extended to the case where certain effects are more important than other effects of the same order. For example, suppose the experimenter has prior knowledge that certain two-factor interactions are negligible, then for each such two-factor interaction, $3A_{3,0} + A_{2,1}$ should be reduced by the number of block-defining words and three-letter treatment-defining words in which it appears. The wordlength $A_{4,0}$ can be modified similarly; the details are omitted. We also point out that the technical tools we develop in Sections 4 and 5 can be used even if other minimum aberration criteria are adopted. For example, if one sticks to the minimum aberration criterion with respect to the ordering in (2), our results can be adapted to handle it.

EXAMPLE 2 (Revisited). According to our definition, the blocking wordlength patterns of the two designs D_1 and D_2 in Example 2 are

$$W_b(D_1) = (0, 0, 0, 0, 6, 0, \dots) \quad \text{and}$$
$$W_b(D_2) = (0, 0, 0, 1, 5, 0, \dots),$$

respectively. Our criterion correctly shows that D_1 is a better design.

We conclude this section by commenting on the concept of resolution.

Unlike minimum aberration, it serves more the purpose of describing certain properties of a design than as a criterion for design selection. Defined as the length of the shortest word in the defining relation, for unblocked regular fractional factorial designs, it can be interpreted in more than one way, leading to different possible extensions to the blocked case. For example, for odd j with $3 \le j \le n$, an unblocked fractional factorial design is of resolution j if none of the $(j-1)/2$-factor effects is aliased with any other treatment effect involving at most $(j-1)/2$ factors, but some $(j+1)/2$-factor effects are aliased with $(j-1)/2$-factor effects. For even j, it has resolution j if none of the $j/2$-factor effects is aliased with any lower-order effects, but some of them are aliased with other treatment effects of the same order. If we define the *blocking resolution* as the smallest integer j such that A_j^b is positive, then for odd j with $j \le n$, a blocked fractional factorial design is of resolution j if none of the $(j-1)/2$-factor effects is aliased with any other treatment effect involving at most $(j-1)/2$ factors or confounded with block effects, but some $(j+1)/2$-factor effects are aliased with $(j-1)/2$-factor effects or confounded with block effects. For even j, it has resolution j if none of the $j/2$-factor effects is aliased with any lower-order effects or confounded with block effects, but some of them are aliased with other treatment effects of the same order. This extends the concept of resolution to blocked fractional factorial designs in a natural way.

From (3), a blocked design can have blocking resolution j larger than the number of treatment factors n. In this case, none of the treatment effects is aliased with any other treatment effect, none of the treatment effects involving less than $j-[n/2]$ factors is confounded with block effects, but there are treatment effects involving $j-[n/2]$ factors which are confounded with block effects.

On the other hand, it is also known that an unblocked regular fractional factorial design has resolution at least j, where $j \le n$, if and only if all the main effects and interactions involving at most $[(j-1)/2]$ factors are estimable, under the assumption that all the interactions involving more than $[j/2]$ factors are negligible. This interpretation can be carried over to the blocked case by defining the resolution as the smallest integer j such that B_j^b is positive, where for each j,

$$(4) \quad B_j^b(D) = \begin{cases} A_{j,0}(D) + A_{j/2,1}(D), & \text{for even } j \le n, \\ A_{j,0}(D), & \text{for odd } j \le n, \\ A_{j-[(n+1)/2],1}(D), & \text{for } n+1 \le j \le n + [(n+1)/2]. \end{cases}$$

These two definitions of resolution for blocked factorials do not always agree. As mentioned earlier, we shall only consider designs under which none of the main effects is aliased with other main effects or confounded with block effects. Such designs have resolution III or higher under both definitions. Other than referring to such designs as having resolution III or higher, we do not need the concept of resolution in this paper.

4. Blocked residual designs and some technical tools. The rest of the paper is devoted to the development of some tools for constructing minimum aberration blocked designs. Let V_n be the n-dimensional vector space over the finite field $GF(2)$ consisting of all the $1 \times n$ vectors $\mathbf{x}$ with elements from $GF(2)$. As mentioned earlier, each blocked design $D(2^{n-m}:2^r)$ can be viewed as a $2^{(n+r)-(m+r)}$ design. Then each defining word can also be considered as a vector in $V_{n+r} = V_n \times V_r$, where V_n corresponds to the treatment factors, and V_r corresponds to the blocking factors. The length of a defining word is nothing but the Hamming weight, that is, the number of nonzero components, in the corresponding vector. For each defining word $\mathbf{u}$ of $D(2^{n-m}:2^r)$, write $\mathbf{u}$ as $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2$, where $\mathbf{u}_1 \in V_n$, and $\mathbf{u}_2 \in V_r$. Then $A_{i_1,0}(D)$ is the number of vectors $\mathbf{u}$ in the defining relation with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = 0$, and $A_{i_1,1}(D)$ denotes the number of vectors $\mathbf{u}$ with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) \neq 0$, where $wt(\mathbf{u}_i)$ is the Hamming weight of $\mathbf{u}_i$. Then the split wordlength pattern of D can be represented as the $(n+1) \times 2$ array,

$$A = \left\{A_{i_1,b}(D): 0 \le i_1 \le n, \text{ and } b = 0, 1\right\}. \tag{5}$$

Following the notation and concepts in Chen and Hedayat (1996), the combinations in $D(2^{(n+r)-(m+r)})$ can be represented as row vectors as follows:

$$D(2^{(n+r)-(m+r)}) = \{\mathbf{x} : \mathbf{x} = \mathbf{v}K_{n+r}, \mathbf{v} \in V_{n-m}\}, \tag{6}$$

where K_{n+r} is an $(n-m) \times (n+r)$ matrix of rank $n-m$ over the finite field $GF(2)$. The matrix K_{n+r} is called the *factor representation* of the fractional factorial design $D(2^{(n+r)-(m+r)})$. One such matrix K_{n+r} can be obtained by writing down the coordinates of $n+r$ points of $PG(n-m-1, 2)$ as columns, where $PG(n-m-1, 2)$ is the projective geometry of dimension $n-m-1$ over $GF(2)$. So a regular fractional factorial design as in (6) is determined by a set of $n+r$ points of $PG(n-m-1, 2)$. Since the blocked design under consideration has resolution III or higher, the corresponding $2^{(n+r)-(m+r)}$ design must be of resolution III or higher.

Let $k = n - m$ and a $2^{(n+r)-(n+r-k)}$ design of resolution III or higher be determined by a subset $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+r}\}$ of $n+r$ distinct points of $PG(k-1, 2)$. The subset $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+r}\}$ can be obtained by deleting $2^k - 1 - n - r$ points from $PG(k-1, 2)$. Without loss of generality, we can represent all the points of $PG(k-1, 2)$ as

$$\mathbf{a}_1, \dots, \mathbf{a}_{n+r}, \mathbf{a}_{n+r+1}, \dots, \mathbf{a}_{2^k-1}. \tag{7}$$

Among $\mathbf{a}_1, \dots, \mathbf{a}_{n+r}$, n points, say $\mathbf{a}_1, \dots, \mathbf{a}_n$, are used to construct the unblocked 2^{n-m} design. The remaining r points, $\mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+r}$, are used for blocking. Let $T = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$, and let F be the set of all the nonnull linear combinations of $\mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+r}$. Then F is an $(r-1)$-flat in $PG(k-1, 2)$. If a blocked design $D(2^{n-(n-k)}:2^r)$ has resolution III or higher, then F and T must be disjoint. Therefore $PG(k-1, 2)$ can be partitioned into three parts,

$$PG(k-1, 2) = \Big\{\underbrace{\mathbf{a}_1, \dots, \mathbf{a}_n}_{T}, \underbrace{\mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+2^r-1}}_{F}, \underbrace{\mathbf{a}_{n+2^r}, \dots, \mathbf{a}_{2^k-1}}_{C}\Big\}, \tag{8}$$

where T corresponds to an unblocked $2^{n-(n-k)}$ design, F is an $(r-1)$-flat and C contains the remaining points. All the vectors of the k-dimensional subspace generated by the rows of (8) can be displayed as a $2^k \times (2^k - 1)$ matrix

$$\left\{\underbrace{\mathbf{c}_1, \dots, \mathbf{c}_n}_{D_T}, \underbrace{\mathbf{c}_{n+1}, \dots, \mathbf{c}_{n+2^r-1}}_{D_F} \underbrace{\mathbf{c}_{n+2^r}, \dots, \mathbf{c}_{2^k-1}}_{D_C}\right\}. \tag{9}$$

The blocked design $D(2^{n-(n-k)} : 2^r)$, denoted by D, can be viewed as $\{D_T, D_F\}$. The set $\{D_C, D_F\}$ corresponds to a blocked design D_R, that is, the design D_C blocked in 2^r blocks. The blocked design D_R is called the *blocked residual design* of D. In the unblocked case, Tang and Wu (1996) and Suen, Chen and Wu (1997) obtained combinatorial identities that govern the relationship between the wordlength pattern of a regular fractional factorial design and that of its complementary design. Analogously we need to find the relationship between the split wordlength pattern of the blocked design $D(2^{n-m} : 2^r)$ and that of its blocked residual design D_R. Then using this relationship, we can establish general rules for identifying minimum aberration blocked designs in terms of their blocked residual designs.

First we briefly describe some connections between fractional factorial designs and algebraic coding theory. See Bose (1961) and Suen, Chen and Wu (1997) for basic concepts, notation and detailed discussions. Let the defining words of a fractional factorial design be represented by binary row vectors. A regular 2^{n-m} fractional factorial design can be considered as an $[n, n-m]$ linear code E which is the null space of an $m \times n$ matrix G, whose rows are the m independent defining words of the 2^{n-m} design, that is, $\mathbf{w}_1, \dots, \mathbf{w}_m$. The dual code $E^{\perp}$ of E, generated by $\mathbf{w}_1, \dots, \mathbf{w}_m$, has the same structure as the defining relation of the 2^{n-m} design. Therefore, the wordlength pattern of a 2^{n-m} design is the same as the weight distribution of the corresponding dual code. MacWilliams identities in coding theory give a fundamental relationship between the weight distributions of a code and its dual code. There are a number of versions of MacWilliams identities, one of which is given below.

LEMMA 1 (MacWilliams). *If E is a linear $[n, n-m]$ code over $GF(2)$ with dual code $E^{\perp}$, then $\{B_i(E)\}$, the weight distribution of E and $\{B_i'(E^{\perp})\}$, the weight distribution of $E^{\perp}$, are related by*

$$B_i'(E^{\perp}) = 2^{m-n} \sum_{j=0}^{n} P_i(j; n) B_j(E),$$

$$B_i(E) = 2^{-m} \sum_{j=0}^{n} P_i(j; n) B_j'(E^{\perp}),$$

for $i = 0, \dots, n$, where $P_i(j; n) = \sum_{s=0}^{i} (-1)^s \binom{j}{s}\binom{n-j}{i-s}$ is a Krawtchouk polynomial.

Throughout this paper, we extend the definition of $\binom{n}{s}$ to allow n and s to be any integers

$$\binom{n}{s} = \begin{cases} \dfrac{n(n-1)\cdots(n-s+1)}{s(s-1)\cdots 1}, & \text{for positive } s, \\ 1, & \text{for } s = 0, \\ 0, & \text{for negative } s. \end{cases}$$

In the following we use *MacWilliams transform* [see MacWilliams and Sloane (1977), Chapter 5] to link the various weight distributions and split word-length patterns.

The row vectors in (9) constitute a subspace, say H_k, of the $(2^k - 1)$-dimensional linear vector space V_{2^k-1}, where V_{2^k-1} can be considered as $V_n \times V_{2^r-1} \times V_{2^k-2^r-n}$. A typical $\mathbf{u}$ vector in H_k can be written as

$$\mathbf{u} = \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3,$$

where $\mathbf{u}_1 \in V_n$, $\mathbf{u}_2 \in V_{2^r-1}$ and $\mathbf{u}_3 \in V_{2^k-2^r-n}$. We let $B_{i_1,i_2,i_3}(H_k)$ denote the number of row vectors $\mathbf{u}$ in H_k with $wt(\mathbf{u}_1) = i_1$, $wt(\mathbf{u}_2) = i_2$ and $wt(\mathbf{u}_3) = i_3$. The triple weight distribution of H_k partitioned into $V_n \times V_{2^r-1} \times V_{2^k-2^r-n}$ is defined as the following $(n+1) \times 2^r \times (2^k - 2^r - n + 1)$ array,

$$\left\{B_{i_1,i_2,i_3}(H_k): 0 \le i_1 \le n,\ 0 \le i_2 \le 2^r - 1 \text{ and } 0 \le i_3 \le 2^k - 2^r - n\right\}.$$

We also define the triple weight distribution of $H_k^{\perp}$ (the dual code of H_k) partitioned into $V_n \times V_{2^r-1} \times V_{2^k-2^r-n}$ as

$$\left\{B'_{i_1,i_2,i_3}\left(H_k^{\perp}\right): 0 \le i_1 \le n,\ 0 \le i_2 \le 2^r - 1 \text{ and } 0 \le i_3 \le 2^k - 2^r - n\right\},$$

where $B'_{i_1,i_2,i_3}(H_k^{\perp})$ is the number of vectors $\mathbf{u}$ in $H_k^{\perp}$ with $wt(\mathbf{u}_1) = i_1$, $wt(\mathbf{u}_2) = i_2$ and $wt(\mathbf{u}_3) = i_3$.

From Sloane and Stufken (1996), the relationship between the numbers $\{B_{i_1,i_2,i_3}\}$ and $\{B'_{i_1,i_2,i_3}\}$ may be expressed in terms of Krawtchouk polynomials,

$$B'_{i_1,i_2,i_3} = \frac{1}{2^k} \sum_{j_1=0}^{n} \sum_{j_2=0}^{2^r-1} \sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(j_1; n) P_{i_2}(j_2; 2^r - 1) \times P_{i_3}\left(j_3; 2^k - 2^r - n\right) B_{j_1,j_2,j_3}, \tag{10}$$

for $0 \le i_1 \le n$, $0 \le i_2 \le 2^r - 1$ and $0 \le i_3 \le 2^k - 2^r - n$.

The blocked design $D(2^{n-(n-k)}: 2^r)$, that is, $\{D_T, D_F\}$ can be viewed as the projection of H_k onto $V_n \times V_{2^r-1}$. Each vector $\mathbf{u}$ in $\{D_T, D_F\}$ can be written as

$$\mathbf{u} = \mathbf{u}_1\mathbf{u}_2,$$

where $\mathbf{u} \in V_n$, and $\mathbf{u}_2 \in V_{2^r-1}$. Let $B_{i_1,i_2}(D)$ denote the number of vectors $\mathbf{u}$ in $\{D_T, D_F\}$ with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = i_2$. The double weight distribution of $\{D_T, D_F\}$ partitioned into $V_n \times V_{2^r-1}$ is the $(n+1) \times 2^r$ array

$$\left\{B_{i_1,i_2}(D): 0 \le i_1 \le n \text{ and } 0 \le i_2 \le 2^r - 1\right\}.$$

The double weight distribution of $D^{\perp}$ (the dual code of $\{D_T, D_F\}$) partitioned into $V_n \times V_{2^r-1}$ is defined as

$$\left\{B'_{i_1,i_2}(D^{\perp}): 0 \le i_1 \le n \text{ and } 0 \le i_2 \le 2^r - 1\right\},$$

where $B'_{i_1,i_2}(D^{\perp})$ is the number of vectors $\mathbf{u}$ in $D^{\perp}$ with $wt(\mathbf{u}_1) = i_1$, and $wt(\mathbf{u}_2) = i_2$.

Following the same argument in Sloane and Stufken (1996), the relationship between the numbers $\{B_{i_1,i_2}(D)\}$ and $\{B'_{i_1,i_2}(D^{\perp})\}$ can also be expressed in terms of Krawtchouk polynomials,

$$(11)\qquad B'_{i_1,i_2}(D^{\perp}) = \frac{1}{2^k}\sum_{j_1=0}^{n}\sum_{j_2=0}^{2^r-1} P_{i_1}(j_1;n)P_{i_2}(j_2;2^r-1)B_{j_1,j_2}(D),$$

for $0 \le i_1 \le n$, and $0 \le i_2 \le 2^r - 1$.

From (7), a defining word of a $2^{(n+r)-(n+r-k)}$ design with length i corresponds to an i-subset of $\{\mathbf{a}_1, \dots, \mathbf{a}_{n+r}\}$, say $\{\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_i}\}$, such that $\mathbf{a}_{j_1} + \cdots + \mathbf{a}_{j_i} = \mathbf{0}$. Therefore the number of defining words of length i is equal to the number of such i-subsets with zero sum. For a blocked design viewed as (8), a treatment-defining word of length i corresponds to an i-subset of $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ with zero sum, that is, a vector $\mathbf{u}$ in $D^{\perp}$ (the dual code of $\{D_T, D_F\}$) with $wt(\mathbf{u}_1) = i$, and $wt(\mathbf{u}_2) = 0$ or $2^r - 1$ (since the sum of all the $2^r - 1$ points in F is equal to $\mathbf{0}$). Meanwhile a block-defining word of length i is represented by i points of T in which the sum of the i points is equal to a point in F, which correspond to a vector $\mathbf{u}$ in $D^{\perp}$ with $wt(\mathbf{u}_1) = i$, and $wt(\mathbf{u}_2) = 1$. Therefore the split wordlength pattern $\{A_{i_1,b}(D)\}$ of a blocked design $D(2^{n-(n-k)}:2^r)$ has the following relation to the double weight distribution of $D^{\perp}$:

$$(12)\quad A_{i_1,0}(D) = B'_{i_1,0}(D^{\perp}) = B'_{i_1,2^r-1}(D^{\perp}) \text{ and } A_{i_1,1}(D) = B'_{i_1,1}(D^{\perp}).$$

Similarly the blocked residual design D_R, that is, $\{D_C, D_F\}$, may be viewed as the projection of H_k onto $V_{2^k-2^r-n} \times V_{2^r-1}$. We let $\{B_{i_1,i_2}(D_R)\}$ be the double weight distribution of $\{D_C, D_F\}$ partitioned into $V_{2^k-2^r-n} \times V_{2^r-1}$, and denote the double weight distribution of $D_R^{\perp}$ (the dual code of $\{D_C, D_F\}$) partitioned into $V_{2^k-2^r-n} \times V_{2^r-1}$ as $\{B'_{i_1,i_2}(D_R^{\perp})\}$. The weight distributions $\{B_{i_1,i_2}(D_R)\}$ and $\{B'_{i_1,i_2}(D_R^{\perp})\}$ have a similar relation to (11). The split wordlength pattern $\{A_{i_1,b}(D_R)\}$ of the blocked residual design D_R is

$$(13)\quad A_{i_1,0}(D_R) = B'_{i_1,0}\left(D_R^{\perp}\right) = B'_{i_1,2^r-1}\left(D_R^{\perp}\right) \text{ and } A_{i_1,1}(D_R) = B'_{i_1,1}\left(D_R^{\perp}\right).$$

Write the $2^r - 1$ vectors in a $PG(r-1,2)$ as the columns of an $r \times (2^r - 1)$ matrix, and let H_r be the subspace generated by the row vectors of this matrix. Then H_r is the dual code of the Hamming code, with $2^r - 1$ vectors of weight 2^{r-1} and one vector of weight zero [Peterson and Weldon (1972), page 75], that is, $B_0(H_r) = 1$ and $B_{2^{r-1}}(H_r) = 2^r - 1$. For $3 \le i \le 2^r - 1$, let $\gamma_r(i)$ be the number of i-subsets of $PG(r-1,2)$ with zero sum, and let $\gamma_r(i) = 0$ for $i = 0, 1, 2$. As in the previous discussion, $\gamma_r(i)$ is equal to the number of vectors in $H_r^{\perp}$ with length i. Then by Lemma 1, we have

$$\begin{aligned}\gamma_r(i) = B_i\left(H_r^{\perp}\right) &= \frac{1}{2^r}\sum_{j=0}^{2^r-1} P_i(j; 2^r-1) B_j(H_r)\\ &= \frac{1}{2^r}\left[\binom{2^r-1}{i} + P_i(2^{r-1}; 2^r-1)(2^r-1)\right],\end{aligned}$$

for $i = 0, \dots, 2^r - 1$.

The double weight distribution $\{B_{i_1,i_2}(H_r)\}$ of H_r partitioned into $V_1 \times V_{2^r-2}$ has

$$\begin{gathered}B_{0,0}(H_r) = 1,\ B_{0,2^{r-1}}(H_r) = 2^{r-1} - 1, \qquad B_{1,2^{r-1}-1}(H_r) = 2^{r-1}\\ \text{and} \quad B_{i_1,i_2}(H_r) = 0 \quad \text{otherwise}.\end{gathered}$$

Therefore, the double weight distribution $\{B'_{i_1,i_2}(H_r^{\perp})\}$ of $H_r^{\perp}$ (the Hamming code) partitioned into $V_1 \times V_{2^r-2}$ is

$$\begin{aligned}B'_{i_1,i_2}\left(H_r^{\perp}\right) &= \frac{1}{2^r}\sum_{j_1=0}^{1}\sum_{j_2=0}^{2^r-2} P_{i_1}(j_1;1) P_{i_2}(j_2; 2^r-2) B_{j_1,j_2}(H_r)\\ &= \frac{1}{2^r}\left[\binom{2^r-2}{i_2} + P_{i_2}(2^{r-1}; 2^r-2)(2^{r-1}-1)\right.\\ &\qquad\qquad \left. + P_{i_1}(1;1) P_{i_2}(2^{r-1}-1; 2^r-2) 2^{r-1}\right].\end{aligned}$$

For $i_1 = 1$ and $i_2 = 2, \dots, 2^r - 2$, $B'_{1,i_2}(H_r^{\perp})$ represents the number of i_2-subsets of $PG(r-1,2)$ such that the sum of the i_2 points is equal to the same point in $PG(r-1,2)$. Let

$$\alpha_r(s) = \begin{cases} B'_{1,s}\left(H_r^{\perp}\right), & \text{for } s = 2, \dots, 2^r - 2,\\ 1, & \text{for } s = 1.\end{cases}$$

Then we have the following lemma.

LEMMA 2. *Let* $\{B'_{i_1,i_2}(D^{\perp})\}$ *and* $\{B'_{i_1,i_2}(D_R^{\perp})\}$ *be the double weight distributions of* $D^{\perp}$ *and* $D_R^{\perp}$, *respectively. Then*:

(i) $B'_{i_1,i_2}(D^{\perp}) = \alpha_r(i_2) B'_{i_1,1}(D^{\perp}) + \gamma_r(i_2) B'_{i_1,0}(D^{\perp})$ *and*
(ii) $B'_{i_1,i_2}(D_R^{\perp}) = \alpha_r(i_2) B'_{i_1,1}(D_R^{\perp}) + \gamma_r(i_2) B'_{i_1,0}(D_R^{\perp})$,

where $i_2 = 1, \dots, 2^r - 2$.

PROOF. We only prove the equation in part (i). The equation in part (ii) follows from the same argument. Each vector $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2 \in D^\perp$ with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = i_2$ corresponds to i_1 points of T and i_2 points of F in (8) such that the sum of these $i_1 + i_2$ points is zero. There are two types of such vectors. One consists of vectors with $\mathbf{u}_1 \in D_T^\perp$ and $\mathbf{u}_2 \in D_F^\perp$, that is, the sum of the i_1 corresponding points in T is zero, and so is the sum of the i_2 corresponding points in F. We have $B'_{i_1,0}(D^\perp)$ vectors in $D^\perp$ with $\mathbf{u}_1 \in D_T^\perp$, $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = 0$. Each of these vectors combined with $\gamma_r(i_2)$ vectors of weight i_2 in $D_F^\perp$ forms $\gamma_r(i_2)$ vectors in $D^\perp$ with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = i_2$. The total number of such vectors is $\gamma_r(i_2)B'_{i_1,0}(D^\perp)$. The other type consists of all vectors whose $\mathbf{u}_1$ is not in $D_T^\perp$, that is, the sum of the i_1 corresponding points in T is not zero, but is equal to the sum of the i_2 corresponding points in F. Since F is an $(r-1)$-flat, the sum of any i_2 points in F belongs to F. A vector in $D^\perp$ with $wt(\mathbf{u}_1) = i_1$ and $wt(\mathbf{u}_2) = 1$ can be formed by combining the i_1 corresponding points in T with their sum in F. For each of the $B'_{i_1,1}(D^\perp)$ vectors in $D^\perp$ so obtained, there are $\alpha_r(i_2)$ i_2-subsets of F such that the sum of these i_2 points is equal to the sum of the i_1 points in T. Therefore the total number of the second type of vectors is equal to $\alpha_r(i_2)B'_{i_1,1}(D^\perp)$. This completes the proof. □

From (12), (13) and Lemma 2, we can see that the split wordlength patterns $\{A_{i_1,b}(D)\}$ and $\{A_{i_1,b}(D_R)\}$ have the following relation to the double weight distributions of $D^\perp$ and $D_R^\perp$:

$$\sum_{t=0}^{2^r-1} B'_{s,t}(D^\perp) = 2^{2^r-1-r}A_{s,0}(D) + 2^{2^r-2-r}A_{s,1}(D), \tag{14}$$

$$\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^\perp) = 2^{2^r-1-r}A_{s,0}(D_R) + 2^{2^r-2-r}A_{s,1}(D_R). \tag{15}$$

Equations (12), (13), (14) and (15) indicate that the split wordlength patterns of a blocked design and its blocked residual design are uniquely determined by the double weight distributions of their duals. Hence it suffices to study the relationship between the double weight distributions of $D^\perp$ and $D_R^\perp$.

To link $\{B'_{i_1,i_2}(D^\perp)\}$ and $\{B'_{i_1,i_2}(D_R^\perp)\}$, we shall study their relations to H_k. The space H_k has $2^k - 1$ vectors of weight 2^{k-1} and one vector of weight zero. On the other hand, since F is an $(r-1)$-flat, the weight distribution of D_F is 2^{k-r} copies of that of the dual code of Hamming code, that is, there are 2^{k-r} vectors of weight zero and $2^k - 2^{k-r}$ vectors of weight 2^{r-1}. Since H_k is split into $\{D_T, D_F, D_C\}$, the weight of a vector in H_k is the sum of the weights of the corresponding vectors in D_T, D_C and D_F. From (11), we have the following lemma.

LEMMA 3.

(i) $$B_{2^{k-1}-i_1,0,i_1} = B_{i_1,0}(D_R) = \frac{1}{2^{2^k-1-n-k}} \times \left[\sum_{s=0}^{2^k-2^r-n} P_{i_1}(s; 2^k - 2^r - n) \left(\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^{\perp}) \right) \right],$$

(ii) $$B_{2^{k-1}-2^{r-1}-i_1,2^{r-1},i_1} = B_{i_1,2^{r-1}}(D_R) = \frac{1}{2^{2^k-1-n-k}} \left[\sum_{s=0}^{2^k-2^r-n} P_{i_1}(s; 2^k - 2^r - n) \times \sum_{t=0}^{2^r-1} P_{2^{r-1}}(t; 2^r - 1) B'_{s,t}(D_R^{\perp}) \right].$$

PROOF. Let $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$ be a nonzero vector $\mathbf{u} \in H_k$ where $\mathbf{u}_1 \in D_T$, $\mathbf{u}_2 \in D_F$ and $\mathbf{u}_3 \in D_C$. Then the weight of $\mathbf{u}$ is 2^{k-1}. Since D_F is 2^{k-r} copies of H_r, the weight of $\mathbf{u}_2$ is either 0 or 2^{r-1}. If $wt(\mathbf{u}_2) = 0$, and $wt(\mathbf{u}_1) = i_1$, then $wt(\mathbf{u}_3) = 2^{k-1} - i_1$. Therefore the number of vectors $\mathbf{u}$ in H_k with $wt(\mathbf{u}_1) = i_1$, $wt(\mathbf{u}_2) = 0$ and $wt(\mathbf{u}_3) = 2^{k-1} - i_1$ equals the number of vectors $\mathbf{u}_R = \mathbf{u}_3\mathbf{u}_2$ in D_R with $wt(\mathbf{u}_3) = 2^{k-1} - i_1$ and $wt(\mathbf{u}_2) = 0$. The equation in (i) results from (11). Following the same argument, we have the equation in (ii). □

Lemma 3 provides the crucial connection between various weight distributions. This connection is essential for establishing the relationship between the split wordlength pattern of a blocked design and that of its blocked residual design in the next section.

5. General rules for identifying minimum aberration blocked designs. For a regular $2^{n-(n-k)}$ fractional factorial design D_T in (9), its wordlength pattern corresponds to $W_t = (A_{3,0}(D), A_{4,0}(D), \dots)$. From Corollary 2 of Suen, Chen and Wu (1997), $A_{i,0}(D)$ can be expressed in terms of the wordlength pattern of the complementary design of D_T. The complementary design $\overline{D_T}$ of D_T is $\{D_C, D_F\}$ which can be viewed as the blocked residual design of $D(2^{n-(n-k)}: 2^r)$. The following theorem shows that $A_{i,0}(D)$ can be written in terms of the split wordlength pattern of its blocked residual design.

THEOREM 1. *Let $\{A_{i_1,b}(D)\}$ and $\{A_{i_1,b}(D_R)\}$ be the split wordlength pat-*

terns of a blocked design $D(2^{n-(n-k)}:2^r)$ and its blocked residual design D_R, respectively. Then

$$
\begin{aligned}
A_{i,0}(D) = C_i + C_{i0} &+ \sum_{s=2}^{i-1}\Bigg(\sum_{t=1}^{i-1-s} C_{i,t+s} I_{[t\le 2^r-2]}\alpha_r(t) \\
&\qquad + (-1)^i I_{[i-s\le 2^r-2]}\alpha_r(i-s)\Bigg) A_{s,1}(D_R) \\
&+ \sum_{s=2}^{i-1}\Bigg(\sum_{t=1}^{i-1-s} C_{i,t+s} I_{[t\le 2^r-2]}\gamma_r(t) \\
&\qquad + (-1)^i I_{[i-s\le 2^r-2]}\gamma_r(i-s)\Bigg) A_{s,0}(D_R) \\
&+ \sum_{s=3}^{i} C_{i,s}\big(A_{s,0}(D_R) + I_{[s>2^r]}A_{s-2^r+1,0}(D_R)\big),
\end{aligned}
\tag{16}
$$

for $i = 3,\ldots,n$, where

$$
C_i = 2^{-k}\big[P_i(0;n) - P_i(2^{k-1};n)\big], \quad C_{ij} = (-1)^{i-[(i-j)/2]}\binom{n-2^{k-1}}{[(i-j)/2]},
$$

$[x]$ is the largest integer less than or equal to x and $I_{[\cdot]}$ is the indicator function which takes the value 1 or 0 depending on whether condition $[\cdot]$ is true or not.

PROOF. Since $\overline{D_T} = \{D_C, D_F\}$, the double weight distribution $\{B'_{i_1,i_2}(D_R^{\perp})\}$ has the following relation to $A_j(\overline{D_T})$:

$$
\begin{aligned}
A_j(\overline{D_T}) &= \sum_{s+t=j} B'_{s,t}(D_R^{\perp}) = \sum_{s=2}^{j} B'_{s,j-s}(D_R^{\perp}) = \sum_{s=2}^{j-1} B'_{s,j-s}(D_R^{\perp}) + B'_{j,0}(D_R^{\perp}) \\
&= \sum_{s=2}^{j-1} I_{[j-s\le 2^r-2]}\big(\alpha_r(j-s)B'_{s,1}(D_R^{\perp}) + \gamma_r(j-s)B'_{s,0}(D_R^{\perp})\big) \\
&\quad + B'_{j,0}(D_R^{\perp}) + I_{[j>2^r]}B'_{j-2^r+1,0}(D_R^{\perp}).
\end{aligned}
$$

By (13), we have

$$
\begin{aligned}
A_j(\overline{D_T}) = &\sum_{s=2}^{j-1} I_{[j-s\le 2^r-2]}\big(\alpha_r(j-s)A_{s,1}(D_R) + \gamma_r(j-s)A_{s,0}(D_R)\big) \\
&+ A_{j,0}(D_R) + I_{[j>2^r]}A_{j-2^r+1,0}(D_R).
\end{aligned}
\tag{17}
$$

Then (16) results from replacing $A_j(\overline{D_T})$ with (17) in the equation of Corollary 2 of Suen, Chen and Wu (1997). $\square$

In the following we can show that $A_{i,1}(D)$, the number of block-defining words containing i treatment letters can also be expressed in terms of the split wordlength pattern $\{A_{i,b}(D_R)\}$. First we need the following two lemmas which can be obtained from the proof of Theorem 2 in Suen, Chen and Wu (1997) and MacWilliams and Sloane [(1977), page 153], respectively.

LEMMA 4. *Let* $\beta_{ij} = 2^{-(2^k-2^r-n)}\sum_{s=0}^{2^k-2^r-n} P_i(2^{k-1}-s;n)P_s(j;2^k-2^r-n)$, *for* $i=0,\dots,n$, *and* $j=0,\dots,2^k-2^r-n$. *Then*:

(i) $\beta_{ij}=0$, *when* $j>i$,

(ii) $\beta_{ij}=\sum_{u,v\geq 0,\,u+v=i-j}\binom{2^{k-1}-n}{v}\binom{2^{k-1}-2^r-n}{u}(-1)^{v+j}$ *when* $j\leq i$,

(iii) $\beta_{i,i-1}=(-1)^i 2^r$ *and*

(iv) $\beta_{ii}=(-1)^i$.

LEMMA 5. *For Krawtchouk polynomials* $P_s(j;n)$, *we have*:

(i) $P_k(0;n)=\binom{n}{k}$,

(ii) $P_0(x;n)=1$, *for any* x,

(iii) $\sum_{i=0}^{n}P_i(s;n)=2^n\delta_{s,0}$, *where* $\delta_{s,r}=1$ *if* $s=r$ *and* $\delta_{s,r}=0$ *if* $s\neq r$.

By Lemmas 3–5 and MacWilliams transform, we have the following theorem which clearly indicates that $A_{i,1}(D)$ can indeed be expressed in terms of the split wordlength pattern of the blocked residual design of D.

THEOREM 2. *Let* $\{A_{i_1,b}(D)\}$ *and* $\{A_{i_1,b}(D_R)\}$ *be the split wordlength patterns of a blocked design* $D(2^{n-(n-k)}:2^r))$ *and its blocked residual design* D_R, *respectively. Then we have*

$$2A_{i_1,0}(D)+A_{i_1,1}(D)=\sum_{s=0}^{i_1}\beta_{i_1,s}(2A_{s,0}(D_R)+A_{s,1}(D_R)), \tag{18}$$

where $\beta_{i_1,s}$ *is defined in Lemma* 4.

PROOF. Let $\{B'_{i_1,i_2}(D^\perp)\}$ and $\{B'_{i_1,i_2}(D_R^\perp)\}$ be the double weight distributions of $D^\perp$ and $D_R^\perp$, respectively. From (10) and Lemma 5, we have

$$\begin{aligned}
B'_{i_1,i_2}(D^\perp) &= B'_{i_1,i_2,0}\\
&=\frac{1}{2^k}\sum_{j_1=0}^{n}\sum_{j_2=0}^{2^r-1}P_{i_1}(j_1;n)P_{i_2}(j_2;2^r-1)\\
&\qquad\times\sum_{j_3=0}^{2^k-2^r-n}P_0\big(j_3;2^k-2^r-n\big)B_{j_1,j_2,j_3}\\
&=\frac{1}{2^k}\sum_{j_1=0}^{n}\sum_{j_2=0}^{2^r-1}P_{i_1}(j_1;n)P_{i_2}(j_2;2^r-1)\sum_{j_3=0}^{2^k-2^r-n}B_{j_1,j_2,j_3}\\
&=\frac{1}{2^k}\sum_{j_1=0}^{n}\sum_{j_3=0}^{2^k-2^r-n}P_{i_1}(j_1;n)\sum_{j_2=0}^{2^r-1}P_{i_2}(j_2;2^r-1)B_{j_1,j_2,j_3}.
\end{aligned}$$

Since D_F in (9) is 2^{k-r} replications of H_r, $B_{j_1,j_2,j_3}=0$, except $j_2=0$ and 2^{r-1}. Therefore,

$$\begin{aligned}
B'_{i_1,i_2,0} &= \frac{1}{2^k}\sum_{j_1=0}^{n}\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(j_1;n)\big[P_{i_2}(0;2^r-1)B_{j_1,0,j_3} \\
&\qquad\qquad + P_{i_2}(2^{r-1};2^r-1)B_{j_1,2^{r-1},j_3}\big] \\
&= \frac{1}{2^k}\left[\binom{2^r-1}{i_2}\left(\sum_{j_1+j_3=2^{k-1}} P_{i_1}(j_1;n)B_{j_1,0,j_3}\right) + P_{i_2}(2^{r-1};2^r-1)\right. \\
&\qquad\left.\times\left(\sum_{j_1+j_3=2^{k-1}-2^{r-1}} P_{i_1}(j_1;n)B_{j_1,2^{r-1},j_3}\right)\right] \\
&= \frac{1}{2^k}\left[\binom{2^r-1}{i_2}\left(\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-j_3;n)B_{2^{k-1}-j_3,0,j_3}\right)\right. \\
&\qquad + P_{i_2}(2^{r-1};2^r-1) \\
&\qquad\left.\times\left(\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-2^{r-1}-j_3;n)B_{2^{k-1}-2^{r-1}-j_3,2^{r-1},j_3}\right)\right].
\end{aligned}$$

By Lemma 3, we have

$$\begin{aligned}
B'_{i_1,i_2,0} &= 2^{n+1-2^k}\left[\binom{2^r-1}{i_2}\left(\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-j_3;n)\right.\right. \\
&\qquad\left.\times\sum_{s=0}^{2^k-2^r-n} P_{j_3}(s;2^k-2^r-n)\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^{\perp})\right) \\
&\qquad + \left(P_{i_2}(2^{r-1};2^r-1)\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-2^{r-1}-j_3;n)\right. \\
&\qquad\left.\times\sum_{s=0}^{2^k-2^r-n} P_{j_3}(s;2^k-2^r-n)\right) \\
&\qquad\left.\times\left(\sum_{t=0}^{2^r-1} P_{2^{r-1}}(t;2^r-1)B'_{s,t}(D_R^{\perp})\right)\right].
\end{aligned}$$

Since $\sum_{i_2=0}^{2^r-1} B'_{i_1,i_2}(D^{\perp}) = \sum_{i_2=0}^{2^r-1} B'_{i_1,i_2,0}$,

$$\begin{aligned}
\sum_{i_2=0}^{2^r-1} B'_{i_1,i_2}(D^{\perp}) &= \sum_{i_2=0}^{2^r-1} B'_{i_1,i_2,0} \\
&= 2^{n+1-2^k}\left[\sum_{i_2=0}^{2^r-1}\binom{2^r-1}{i_2}\right]
\end{aligned}$$

$$\times\left[\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-j_3;n)\sum_{s=0}^{2^k-2^r-n} P_{j_3}(s;2^k-2^r-n) \times\left(\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^{\perp})\right)\right]$$

$$= 2^{2^r+n-2^k}\left[\sum_{j_3=0}^{2^k-2^r-n} P_{i_1}(2^{k-1}-j_3;n) \times \sum_{s=0}^{2^k-2^r-n} P_{j_3}(s;2^k-2^r-n)\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^{\perp})\right].$$

By Lemma 4, it can be simplified as

$$\sum_{i_2=0}^{2^r-1} B'_{i_1,i_2}(D^{\perp}) = \sum_{s=0}^{i_1}\beta_{i_1,s}\left(\sum_{t=0}^{2^r-1} B'_{s,t}(D_R^{\perp})\right).$$

From (14) and (15), we have (18). □

The identities in Theorems 1 and 2 provide an important relationship between the split wordlength patterns of a blocked design $D(2^{n-m}:2^r)$ and its blocked residual design. From (16) and (18), we have the following identities relating the blocking wordlength pattern of a blocked design $D(2^{n-(n-k)}:2^r)$ to the split wordlength pattern of its blocked residual design D_R:

$$\begin{aligned} A_3^b(D) &= 3A_{3,0}(D) + A_{2,1}(D) \\ &= 3C_3 + 3C_{3,0} + 2\beta_{2,0} - (3A_{3,0}(D_R) + 2A_{2,1}(D_R)), \\ A_4^b(D) &= A_{4,0}(D) \\ &= C_4 + C_{4,0} + A_{3,0}(D_R) \\ &\quad + (1+\alpha_r(2))A_{2,1}(D_R) + A_{4,0}(D_R) + A_{3,1}(D_R), \end{aligned} \tag{19}$$

where $\alpha_r(2) = 2^{r-1}(2^{r-1}-1)$. Based on the identities in (19), we can establish some general rules for identifying minimum aberration blocked 2^{n-m} designs in terms of their blocked residual designs.

Rule 1. A blocked design $D^*(2^{n-(n-k)}:2^r)$ with D_R^* as its blocked residual design has minimum aberration if:

(i) $3A_{3,0}(D_R^*) + 2A_{2,1}(D_R^*)$ is the maximum among the blocked residual designs of all the $D(2^{n-(n-k)}:2^r)$'s;

(ii) D_R^* is the unique design satisfying (i).

Rule 2. A blocked design $D^*(2^{n-(n-k)}:2^r)$ has minimum aberration if

(i) $3A_{3,0}(D_R^*) + 2A_{2,1}(D_R^*)$ is the maximum among the blocked residual designs of all the $D(2^{n-(n-k)}:2^r)$'s;

(ii) $A_{3,0}(D_R^*) + (1 + 2^{r-1}(2^{r-1} - 1))A_{2,1}(D_R^*) + A_{4,0}(D_R^*) + A_{3,1}(D_R^*)$ is the minimum among all blocked residual designs D_R whose $A_3^b(D_R)$ equals $A_3^b(D_R^*)$;

(iii) D_R^* is the unique design satisfying (ii).

By Theorems 1 and 2, it is not difficult to develop similar rules for words of lengths greater than four.

Now we apply Rule 1 to identify a blocked design $D(2^{9-5}:2^2)$ with minimum aberration. Let $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ be four independent columns of H_4, where H_4 is a 16×15 matrix as defined in (9), and denote $\mathbf{1}^i\mathbf{2}^l\mathbf{3}^j\mathbf{4}^k = i\mathbf{1} + l\mathbf{2} + j\mathbf{3} + k\mathbf{4}$, where i, l, j and k are in $GF(2)$. Let L be the set of the 15 columns of H_4,

$$L = \{\mathbf{1}, \mathbf{2}, \mathbf{12}, \mathbf{3}, \mathbf{13}, \mathbf{23}, \mathbf{123}, \mathbf{4}, \mathbf{14}, \mathbf{24}, \mathbf{34}, \mathbf{124}, \mathbf{134}, \mathbf{234}, \mathbf{1234}\}.$$

Without loss of generality, we choose $D_F = \{\mathbf{1}, \mathbf{2}, \mathbf{12}\}$. Any $D(2^{9-5}:2^2)$ design can be determined by its blocked residual design D_R, that is, any subset of six columns in L which contains D_F. There are two nonisomorphic blocked residual designs, that is, $D_R^1 = \{\mathbf{3}, \mathbf{13}, \mathbf{23}, \mathbf{1}, \mathbf{2}, \mathbf{12}\}$, and $D_R^2 = \{\mathbf{3}, \mathbf{4}, \mathbf{34}, \mathbf{1}, \mathbf{2}, \mathbf{12}\}$, with the following split wordlength patterns:

$$A_{3,0}(D_R^1) = 0 \quad \text{and} \quad A_{2,1}(D_R^1) = 3,$$

$$A_{3,0}(D_R^2) = 1 \quad \text{and} \quad A_{2,1}(D_R^2) = 0.$$

Let D_1 and D_2 be the $D(2^{9-5}:2^2)$ designs with D_R^1 and D_R^2 as their blocked residual designs respectively. Since D_R^1 maximizes $3A_{3,0}(D_R) + 2A_{2,1}(D_R)$, the blocked design $D(2^{9-5}:2^2)$, which consists of the last nine columns of L and D_F, has minimum aberration.

6. Some structures of optimal blocked designs. In this section, we study the structures of some blocked designs $D(2^{n-(n-k)}:2^r)$ with minimum aberration.

Let $W_b(D)$ be the blocking wordlength pattern of a blocked design $D(2^{n-(n-k)}:2^r)$ in (9). If a blocked design $D(2^{n-(n-k)}:2^r)$ has minimum aberration, then it must minimize $A_3^b(D)$, or equivalently its blocked residual design D_R must maximize $3A_{3,0}(D_R) + 2A_{2,1}(D_R)$ (by Rule 1 in Section 5). Let $\bar{n} = 2^k - 2^r - n$. From (8), the blocked residual design D_R is determined by its factor representation,

$$\underbrace{\{\mathbf{a}_1, \ldots, \mathbf{a}_{\bar{n}}\}}_{C} \cup F, \tag{20}$$

where F is an $(r-1)$-flat and C is a subset of $\bar{n}$ distinct points in the complement of F in $PG(k-1, 2)$. As discussed in Chen and Hedayat (1996), a line of $PG(k-1, 2)$ corresponds to a word of length three. The number $A_{3,0}(D_R)$ is equal to the number of lines in C, and $A_{2,1}(D_R)$ is the number of lines containing two points of C and one point of F. Since each pair of points in C determines a line, $A_{2,1}(D_R)$ can not exceed $\binom{\bar{n}}{2} - 3A_{3,0}(D_R)$. Therefore we have the following upper bound for $3A_{3,0}(D_R) + 2A_{2,1}(D_R)$,

$$3A_{3,0}(D_R) + 2A_{2,1}(D_R) \leq \bar{n}(\bar{n}-1) - 3A_{3,0}(D_R) \leq \bar{n}(\bar{n}-1). \tag{21}$$

Using (21), some structures of blocked residual designs D_R with maximum $3A_{3,0}(D_R) + 2A_{2,1}(D_R)$ can be identified. The upper bound in (21) is achieved if and only if $A_{2,1}(D_R) = \binom{\bar{n}}{2}$ and $A_{3,0}(D_R) = 0$, that is, C contains no lines and the third point of the line determined by any two points of C is in F. This implies that if $\bar{n} \leq 2^r$, then the upper bound in (21) is attained if and only if the blocked residual design has a factor representation,

$$\{\mathbf{a}, \mathbf{a} + \mathbf{a}_1, \ldots, \mathbf{a} + \mathbf{a}_{\bar{n}-1}\} \cup F, \tag{22}$$

where F is an $(r-1)$-flat, $\mathbf{a}_1, \ldots, \mathbf{a}_{\bar{n}-1} \in F$, and $\mathbf{a} \notin F$; also see Cheng and Mukerjee (1997). When $\bar{n} = 2^r$, an r-flat containing F is the only subset which has the kind of structure in (22). Therefore we have the following theorem.

THEOREM 3. *For $n = 2^k - 2^{r+1}$, $r < k-1$, a blocked design $D(2^{n-(n-k)}: 2^r)$ has minimum aberration if $F \cup C$ is an r-flat, where F and C are as in* (8).

When $0 \leq \bar{n} \leq 2^r$, any blocked residual design D_R with (22) as its factor representation has $A_{3,0}(D_R) = 0$, $A_{2,1}(D_R) = \binom{\bar{n}}{2}$, $A_{3,1}(D_R) = 0$, $A_{4,0}(D_R) + A_{4,1}(D_R) = \binom{\bar{n}}{4}$, $A_{5,0}(D_R) = 0$, $A_{5,1}(D_R) = 0$, etc. To search for a minimum aberration blocked design, we only need to consider the $\bar{n}$-subsets of $\{\mathbf{a}, \mathbf{a} + \mathbf{a}_1, \ldots, \mathbf{a} + \mathbf{a}_{2^r-1}\}$, where $\{\mathbf{a}_1, \ldots, \mathbf{a}_{2^r-1}\}$ is an $(r-1)$-flat. By Rule 2 in Section 5, minimizing the number of words of length four $A_4^b(D)$ is equivalent to minimizing $A_{4,0}(D_R)$. We can go on and minimize $A_i^b(D)$ for larger i if necessary. Following this rule, we have identified the factor representations of the blocked residual designs of several families of minimum aberration blocked designs.

For $\bar{n} = 1$ or 2, structure (22) as the factor representation of a blocked residual design is unique up to equivalence; hence the corresponding blocked design has minimum aberration. For convenience, a point of $PG(k-1, 2)$ is denoted by $i_1 i_2, \ldots, i_l$ if the i_1th, i_2th, $\ldots$, i_lth coordinates of this point are 1 and all others are zero. The following are the factor representations of the blocked residual designs of some minimum aberration blocked designs.

$$\begin{aligned} r=2, \quad & F = \{1, 2, 12\}, \\ & C_3 = \{3, 13, 23\}, \\ & C_4 = PG(2,2) \setminus F, \end{aligned}$$

$$\begin{aligned} r=3, \quad & F = \{1, 2, 12, 3, 13, 23, 123\}, \\ & C_3 = \{4, 14, 24\}, \\ & C_4 = \{4, 14, 24, 34\}, \\ & C_5 = \{4, 14, 24, 34, 1234\}, \\ & C_6 = \{4, 14, 24, 34, 124, 134\}, \end{aligned}$$

$$C_7 = \{4, 14, 24, 34, 124, 134, 234\},$$
$$C_8 = PG(3,2)\setminus F,$$

$$r=4, \qquad F = \{1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 34, 124, 134, 234, 1234\}$$
$$C_3 = \{5, 15, 25\},$$
$$C_4 = \{5, 15, 25, 35\},$$
$$C_5 = \{5, 15, 25, 35, 45\},$$
$$C_6 = \{5, 15, 25, 35, 45, 12345\},$$
$$C_7 = \{5, 15, 25, 35, 45, 125, 345\},$$
$$C_8 = \{5, 15, 25, 35, 45, 125, 135, 245\},$$
$$C_9 = \{5, 15, 25, 35, 45, 125, 135, 245, 345\},$$
$$C_{10} = \{5, 15, 25, 35, 45, 125, 135, 245, 345, 12345\},$$
$$C_{11} = \{5, 15, 25, 35, 45, 125, 135, 235, 145, 245, 345\},$$
$$C_{12} = \{5, 15, 25, 35, 45, 125, 135, 1235, 235, 145, 245, 345\},$$
$$C_{13} = \{5, 15, 25, 35, 45, 125, 135, 1235, 1245, 235, 145, 245, 345\},$$
$$C_{14} = \{5, 15, 25, 35, 45, 125, 135, 1235, 1245, 1345, 235, 145, 245, 345\},$$
$$C_{15} = \{5, 15, 25, 35, 45, 125, 135, 1235, 1245, 1345, 2345, 235, 145, 245, 345\},$$
$$C_{16} = PG(4,2)\setminus F.$$

Deleting these subsets from a $PG(k-1, 2)$ yields subsets corresponding to blocked designs $D(2^{n-(n-k)}: 2^r)$ with minimum aberration.

Based on the above technique and computer search, we obtain a catalog of 8-, 16- and 32-run blocked 2^{n-m} designs with minimum aberration which are represented in the same fashion as Sun, Wu and Chen (1997). In Table 1, the columns correspond to all the factorial effects for each run size. We use columns to represent both treatment and blocking factors. The independent columns are indicated in boldface; all the other columns are generated by these columns. For example **1**, **2** and **4** are three independent columns for 8 runs, and the other columns are 3 = **12**, 5 = **14**, 6 = **24** and 7 = **124**.

In Tables 2 and 3, minimum aberration designs with 8, 16 and 32 runs are given. The columns of treatment and blocking factors for each design are listed under "Treatment" and "Block," respectively. To save space, in the tables all independent columns are omitted for each design.

Acknowledgments. We thank Jiahua Chen and John Stufken, each of whom kindly sent us early copies of their respective preprints. Our thanks also go to the Associate Editor and two referees for their constructive comments.

TABLE 1
Matrices for 8, 16, 32-*run designs**

1	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

*For 8 runs use the first 3 rows and 7 columns; for 16 runs use the first 4 rows and 15 columns; for 32 runs use the entire matrix. The independent columns are in boldface and numbered 1, 2, 4, 8 and 16.

TABLE 2
All 8- *and* 16-*run minimum aberration* 2^{n-m} *designs**

Design	**Treatment**	**Block**	W_b
	8 Runs		
$2^{(4+1)-(1+1)}$	7	3	2 1 0
$2^{(4+2)-(1+2)}$	7	3 5	6 1 0
$2^{(5+1)-(2+1)}$	3 5	6	8 1 2
$2^{(6+1)-(3+1)}$	3 5 6	7	15 3 4
	16 Runs		
$2^{(5+1)-(1+1)}$	7	11	0 1 2
$2^{(5+2)-(1+2)}$	7	3 13	2 1 4
$2^{(5+3)-(1+3)}$	7	3 5 9	10 1 0
$2^{(6+1)-(2+1)}$	7 11	13	0 3 4
$2^{(6+2)-(2+2)}$	7 11	3 13	3 3 8
$2^{(6+3)-(2+3)}$	7 11	3 5 9	15 3 0
$2^{(7+1)-(3+1)}$	7 11 13	14	0 7 7
$2^{(7+2)-(3+2)}$	7 11 13	3 5	9 7 0
$2^{(7+3)-(3+3)}$	7 11 13	3 5 9	21 7 0
$2^{(8+1)-(4+1)}$	7 11 13 14	3	4 14 0
$2^{(8+2)-(4+2)}$	7 11 13 14	3 5	12 14 0
$2^{(8+3)-(4+3)}$	7 11 13 14	3 5 9	28 14 0
$2^{(9+1)-(5+1)}$	3 5 9 14 15	6	16 14 84
$2^{(9+2)-(5+2)}$	3 5 9 14 15	6 10	24 14 92
$2^{(10+1)-(6+1)}$	3 5 6 9 14 15	10	28 18 96
$2^{(10+2)-(6+2)}$	3 5 6 9 14 15	7 10	37 18 184
$2^{(11+1)-(7+1)}$	3 5 6 9 10 13 14	15	40 25 293
$2^{(11+2)-(7+2)}$	3 5 6 9 10 13 14	7 11	51 26 316
$2^{(12+1)-(8+1)}$	3 5 6 9 10 13 14 15	7	54 39 496
$2^{(12+2)-(8+2)}$	3 5 6 9 10 13 14 15	7 11	66 39 528
$2^{(13+1)-(9+1)}$	3 5 6 7 9 10 11 12 13	14	72 55 742
$2^{(14+1)-(10+1)}$	3 5 6 7 9 10 11 12 13 14	15	91 77 1148

*To save space, the blocking wordlength patterns are truncated.

TABLE 3
Some 32-run minimum aberration 2^{n-m} designs

Design	Treatment	Block	W_b
$2^{(6+1)-(1+1)}$	31	7	0 0 2
$2^{(6+2)-(1+2)}$	7	11 21	0 1 4
$2^{(6+3)-(1+3)}$	31	3 12 21	3 0 8
$2^{(6+4)-(1+4)}$	31	3 5 9 17	15 0 0
$2^{(7+1)-(2+1)}$	7 27	13	0 1 22
$2^{(7+2)-(2+2)}$	7 11	13 19	0 3 7
$2^{(7+3)-(2+3)}$	7 27	5 11 19	5 1 32
$2^{(7+4)-(2+4)}$	7 25	3 5 9 17	21 2 0
$2^{(8+1)-(3+1)}$	7 11 29	19	0 3 43
$2^{(8+2)-(3+2)}$	7 11 29	19 30	1 3 50
$2^{(8+3)-(3+3)}$	7 11 21	3 13 17	7 5 18
$2^{(8+4)-(3+4)}$	7 11 21	3 5 9 17	28 5 0
$2^{(9+1)-(4+1)}$	7 11 19 29	30	0 6 84
$2^{(9+2)-(4+2)}$	7 11 21 25	6 26	2 9 14
$2^{(9+3)-(4+3)}$	7 11 21 25	3 13 17	9 9 27
$2^{(9+4)-(4+4)}$	7 11 21 25	3 5 9 17	36 9 0
$2^{(10+1)-(5+1)}$	7 11 21 25 31	13	0 15 10
$2^{(10+2)-(5+2)}$	7 11 21 25 31	3 13	3 15 20
$2^{(10+3)-(5+3)}$	7 11 13 19 21	3 5 25	12 16 36
$2^{(10+4)-(5+4)}$	7 11 21 25 31	3 5 9 17	45 15 0
$2^{(11+1)-(6+1)}$	7 11 13 19 21 25	14	0 25 13
$2^{(11+2)-(6+2)}$	7 11 13 19 21 25	3 28	4 25 26
$2^{(11+3)-(6+3)}$	7 11 13 14 19 21	3 5 25	15 26 48
$2^{(11+4)-(6+4)}$	7 11 13 19 21 25	3 5 9 17	55 25 0
$2^{(12+1)-(7+1)}$	7 11 13 14 19 21 25	22	0 38 0
$2^{(12+2)-(7+2)}$	7 11 13 14 19 21 25	3 28	5 38 34
$2^{(12+3)-(7+3)}$	7 11 13 14 19 21 22	3 5 25	18 39 64
$2^{(12+4)-(7+4)}$	7 11 13 14 19 21 25	3 5 9 17	66 38 0
$2^{(13+1)-(8+1)}$	7 11 13 14 19 21 22 25	26	0 55 22
$2^{(13+2)-(8+2)}$	7 11 13 14 19 21 22 25	3 28	6 55 44
$2^{(13+3)-(8+3)}$	3 5 9 14 15 22 26 28	6 10 17	34 39 396
$2^{(13+4)-(8+4)}$	7 11 13 14 19 21 22 25	3 5 9 17	78 55 0
$2^{(14+1)-(9+1)}$	7 11 13 14 19 21 22 25 26	28	0 77 28
$2^{(14+2)-(9+2)}$	7 11 13 14 19 21 22 25 26	3 28	7 77 56
$2^{(14+3)-(9+3)}$	7 11 13 14 19 21 22 25 26	5 9 17	0 42 77 0
$2^{(14+4)-(9+4)}$	7 11 13 14 19 21 22 25 26	3 5 9 17	91 77 0
$2^{(15+1)-(10+1)}$	7 11 13 14 19 21 22 25 26 28	31	0 105 0
$2^{(15+2)-(10+2)}$	7 11 13 14 19 21 22 25 26 28	3 5	21 105 0
$2^{(15+3)-(10+3)}$	7 11 13 14 19 21 22 25 26 28	3 5 9	49 150 0
$2^{(15+4)-(10+4)}$	7 11 13 14 19 21 22 25 26 28	3 5 9 17	105 150 0
$2^{(16+1)-(11+1)}$	7 11 13 14 19 21 22 25 26 28 31	3	8 140 0
$2^{(16+2)-(11+2)}$	7 11 13 14 19 21 22 25 26 28 31	3 5	24 140 0
$2^{(16+3)-(11+3)}$	7 11 13 14 19 21 22 25 26 28 31	3 5 9	56 140 0
$2^{(16+4)-(11+4)}$	7 11 13 14 19 21 22 25 26 28 31	3 5 9 17	120 140 0
$2^{(17+1)-(12+1)}$	3 5 9 14 15 17 22 23 26 27 28 29	6	32 140 1128
$2^{(17+2)-(12+2)}$	3 5 9 14 15 17 22 23 26 27 28 29	6 10	48 140 1144
$2^{(17+3)-(12+3)}$	3 5 9 14 15 17 22 23 26 27 28 29	6 10 18	80 140 1176
$2^{(18+1)-(13+1)}$	3 5 6 9 14 15 17 22 23 26 27 28 29	10	56 148 2256
$2^{(18+2)-(13+2)}$	3 5 6 9 14 15 17 22 23 26 27 28 29	10 18	72 148 2288
$2^{(18+3)-(13+3)}$	3 5 6 9 14 15 17 22 23 26 27 28 29	7 10 18	105 148 2252

TABLE 3
(*Continued*)

Design	Treatment	Block	W_b
$2^{(19+1)-(14+1)}$	3 5 6 9 10 14 15 17 22 23 26 27 28 29	18	80 164 3464
$2^{(19+2)-(14+2)}$	3 5 6 9 10 14 15 17 22 23 26 27 28 29	13 18	96 164 3513
$2^{(19+3)-(14+3)}$	3 5 6 9 10 14 15 17 22 23 26 27 28 29	7 11 18	131 164 3608
$2^{(20+1)-(15+1)}$	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29	31	104 188 4832
$2^{(20+2)-(15+2)}$	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29	7 25	121 188 4898
$2^{(20+3)-(15+3)}$	3 5 6 9 10 14 15 17 18 22 23 26 27 28 29	7 11 19	158 188 5024

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