The Annals of Probability
1973, Vol. 1, No. 6, 926–946

SELF ANNIHILATING BRANCHING PROCESSES[1]

BY K. BRUCE ERICKSON

Stanford University

In this paper criticality-limit theorems are proved for a two-type branching process in which cancellation of particles of opposite type occurs after reproduction.

1 Introduction. In this paper we derive the criticality-limit behavior of a stochastic growth process whose probabilistic description is the same as for an ordinary two-type Galton-Watson process with this difference: immediately after reproduction a one-to-one annihilation occurs between offspring of opposite type. Thus if $Z_n(i)$ is the number of particles of type i, $i = 1, 2$, active in the nth generation, then, since only those particles remaining after annihilation are counted as active, we have $\min\{Z_n(1), Z_n(2)\} = 0$ for all n. Given that $Z_n(i) > 0$ and $Z_n(i') = 0$, each of these $Z_n(i)$ type i particles independently of each other produces particles of both types according to a fixed distribution F_i. The original $Z_n(i)$ particles then die. Assume that a total of $\xi_n(1)$, $\xi_n(2)$ particles of each type have been produced; then annihilation takes place between as many pairs as possible, viz. $\min\{\xi_n(1), \xi_n(2)\}$, and thus

$$Z_{n+1}(l) = \xi_n(l) - \min\{\xi_n(1), \xi_n(2)\}, \qquad l = 1, 2. \tag{1.1}$$

The data for the process are the distributions F_1 and F_2.

This process and similar others originated with Professor Peter Ney as a model for the antigenic behavior of Lymphoma cell populations and it should be applicable to a variety of antibody reactions. Many other writers have considered branching type processes which allow particle interactions in various ways; for a general survey with an extensive bibliography see Kesten [3]. See also Karlin and Kaplan [2]. (We have followed Kesten in the description of the self-annihilating process just given, see [3] page 500.) Unlike these other processes the interactions we are considering here are of a negative character. For example, increasing the production of a given type can result in a decrease in over-all growth rate (see Section 3).

I wish to express my thanks to Professor Kesten for a very instructive exchange of correspondence on this and other problems; The proof in Section 8 of sure extinction for critical, symmetric processes is due to him.

2. Redefinition of the process. The symmetric case. Let $\{Z_n\}_{n\geq 0}$ be defined

Received February 14, 1972; revised March 26, 1973.

[1] Research supported in part by National Science Foundation Grant GP 28876, Stanford University.

AMS 1970 *subject classifications.* Primary 6067, 6069.

Key words and phrases. Two type branching process, annihilation of particles, criticality, Malthusian parameter, growth rate limit theorems, submartingale, random walks with mean 0.

as in Section 1. Define a sequence $\{W_n\}$ of integer valued random variables by

$$W_n = Z_n(1) - Z_n(2) .$$

Then, since $\min\{Z_n(1), Z_n(2)\} = 0$,

$$W_n{}^+ = Z_n(1) \qquad \text{and} \qquad W_n{}^- = Z_n(2)$$

where $W^\pm = \frac{1}{2}(|W| \pm W)$. Thus $Z_n \mapsto W_n$ is one-to-one on the state space of Z_n and we have:

PROPOSITION. *The sequence $\{W_n\}$ is a Markov chain on the integers with*

$$(2.1) \qquad \begin{aligned} p_{ij} = P(W_n = j \mid W_{n-1} = i) &= g_j{}^{|i|*} , && \textit{when } i < 0 ; \\ &= \delta_{0j} , && \textit{when } i = 0 \ (0 \textit{ is absorbing}) , \\ &= f_j{}^{i*} , && \textit{when } i > 0; j = 0, \pm 1, \cdots \end{aligned}$$

where $f = \{f_j\}_{j=-\infty}^{\infty}$ and $g = \{g_j\}_{j=-\infty}^{\infty}$ are the probability distributions

$$\begin{aligned} f_j &= \textstyle\sum_k P\{\textit{a single type } 1 \textit{ particle produces } j + k \textit{ type } 1, \ k \textit{ type } 2 \textit{ particles}\} \\ &= \textstyle\sum_k F_1(j + k, k) , \\ g_j &= \textstyle\sum_k P\{\textit{a single type } 2 \textit{ particle produces } j + k \textit{ type } 1, \ k \textit{ type } 2 \textit{ particles}\} \\ &= \textstyle\sum_k F_2(j + k, k) \end{aligned}$$

and f^{r*}, g^{r*} denote their r-fold convolutions.

Given the distributions f and g we can easily construct a chain $\{W_n\}$ satisfying (2.1). Here is one such construction. (We give another in Section 5.) Let $\{X_i^{(n)}\}$ and $\{Y_i^{(n)}\}$, $n, i \geqq 0$, be two doubly indexed sequences of independent random variables with

$$P\{X_i^{(n)} = j\} = f_j , \qquad P\{Y_i^{(n)} = j\} = g_j , \qquad j = 0, \pm 1, \pm 2, \cdots$$

and put $S_0^{(n)} = T_0^{(n)} \equiv 0$,

$$S_r^{(n)} = X_1^{(n)} + \cdots + X_r^{(n)} , \qquad T_r^{(n)} = Y_1^{(n)} + \cdots + Y_r^{(n)} , \qquad r \geqq 1 .$$

Let W_0 be any integer (random variable) and define $\{W_n\}_{n \geqq 0}$ inductively from

$$(2.2) \qquad W_n = S_{W_{n-1}^+}^{(n)} + T_{W_{n-1}^-}^{(n)} , \qquad n = 1, 2, \cdots .$$

One easily sees that $\{W_n\}$ is a Markov chain with transition probabilities (2.1).[2] Note that at most one of the two partial sums in (2.2) is nonzero. In terms of the $\{Z_n\}$ process of Section 1, the X's (Y's) denote the excess number of type 1 particles over type 2 produced by a single type 1 (type 2); a negative excess indicates more type 2.

[2] To be perfectly accurate one should interpret (2.2) in a distributional sense: if $\{\hat{W}_n\}_{n=1}^{\infty}$ is any chain with transition probabilities (2.1) and if $\{W_n\}_{n=0}^{\infty}$ is the chain defined by (2.2) and if $P\{W_0 = k\} = P\{\hat{W}_0 = k\}$ for all i then $P\{A\} = P\{\hat{A}\}$ for any event A defined on $\{W_n\}_{n=0}^{\infty}$ and $\hat{A}$ the corresponding event of $\{W_n\}_{n=0}^{\infty}$. For example $P\{W_n = 0 \text{ eventually}\} = P\{\hat{W}_n = 0 \text{ eventually}\}$, $P\{W_n/\rho^n \text{ converges}\} = P\{\hat{W}_n/\rho^n \text{ converges}\}$, etc.

Symmetric self-annihilating process. This is the process which arises when the distributions f and g are the same: $f_j = g_j$ for all j. Then (2.1) becomes

$$(2.3)\qquad P(W_n = j \mid W_{n-1} = i) = f_j^{|i|*}, \qquad i \neq 0$$
$$= \delta_{0j}, \qquad i = 0;\ j = 0, \pm 1, \cdots$$

and (2.2) simplifies to

$$(2.4)\qquad W_0 - \text{arbitrary integer (rv)}$$
$$W_n = S^{(n)}_{|W_{n-1}|}, \qquad n = 1, 2, \cdots .$$

Observe that (2.3)/(2.4) reduces exactly to the definition of ordinary Galton–Watson process whenever $f_j = 0$ for $j < 0$, see [1]. Our seemingly slight generalization, allowing negative as well as positive values of numbers of offspring, immediately forces one to abandon the use of generating functions and functional iteration so prominent in the classical set up. Aside from curiosity, principal justification for treating the symmetric process separately is Theorem 3 below; the criticality problem is essentially solved for (2.3)/(2.4) but not for (2.1)/(2.2). (Also the results in the symmetric case will be needed in the general case [see 1° in Section 6 and 5° in Section 7].)

3. Statements of results. We define "extinction" to be the event $\{W_n = 0$ ultimately$\}$, "non-extinction" $= \{W_n \neq 0$ all $n\}$. Throughout this paper we assume

(A) For every i, $P\{W_n = 0 \mid W_0 = i\} > 0$ for some n.

LEMMA 1. *If* (A), *then for every* i

$$P\{\textit{extinction} \mid W_0 = i\} = P\{\liminf_{n\to\infty} |W_n| < \infty \mid W_0 = i\}$$
$$= \lim_{n\to\infty} P\{W_n = 0 \mid W_0 = i\} > 0$$

and $\sum_{n=0}^{\infty} P\{1 \leqq |W_n| \leqq k \mid W_0 = i\} < \infty$ *for every finite* $k > 0$, $i \neq 0$.

PROOF. $\{W_n\}$ is a Markov chain with 0 an absorbing state which by (A) can be reached from every state i. Hence every $i \neq 0$ is transient.

REMARK. To ensure (A) it suffices to assume $f_0 = \sum_0^\infty F_1(k, k) > 0$ and $g_0 = \sum_0^\infty F_2(k, k) > 0$, or, more generally, that $\min(\liminf f_0^{n*}, \liminf g_0^{n*}) > 0$.

CONVENTION. In this paper we interpret statements such as $E|W_n| \to 0$, $P\{A\} = 1$, $P\{A\} \neq 0$, etc., to mean $E(|W_n| \mid W_0 = i) \to 0$, $P\{A \mid W_0 = i\} = 1$ $P\{A \mid W_0 = i\} \neq 0$, etc. for *all* initial $W_0 = i \neq 0$.

Criticality limit theorems for the symmetric self-annihilating processes (2.3)/(2.4). The Malthusian parameter for (2.3)/(2.4) is $|\mu|$ where

$$\mu = E(W_1 \mid W_0 = \pm 1) = EX_i^{(n)} = \textstyle\sum_{-\infty}^{\infty} jf_j .$$

THEOREM 1. *If* $|\mu| < 1$, *then* $P\{\textit{extinction}\} = 1$ *and* $\sum_0^\infty E|W_n| < \infty$, *while if* $|\mu| > 1$ (*including possibly* $\mu = \pm\infty$), *then* $P\{\textit{non-extinction}\} > 0$ *and*

$$(3.1)\qquad \lim_{n\to\infty} W_n/(|W_0| + |W_1| + \cdots + |W_{n-1}|) = (|\mu| - 1)\,\mathrm{sign}\,(\mu),$$

a.s. *on* $\{\textit{non-extinction}\}$.

COROLLARY. *If* $|\mu| > 1$ *then* $\lim(\log|W_n|)/n = \log|\mu|$ a.s. *on* {*non-extinction*}.

THEOREM 2. *If* $|\mu| > 1$ *and* $E(|W_1|^\gamma \mid W_0 = \pm 1) < \infty$ *for some* $\gamma > 1$, *then* $W_n/|\mu|^n$ *converges* a.s. *and in mean of order* γ *to a non-degenerate random variable* V.

THEOREM 3. *If* $|\mu| = 1$ *and* $E(|W_1|^3 \mid W_0 = \pm 1) < \infty$ *then* $P\{extinction\} = 1$. (*The assumption* $E|W_1|^3 < \infty$ *can be weakened slightly*; *see Section* 8.)

Criticality limit theorems for the general self-annihilating process (2.1)/(2.2). Let us put

$$\mu = E(W_1 \mid W_0 = 1) = EX_i^{(n)} = \sum_{-\infty}^{\infty} jf_j$$
$$\lambda = E(W_1 \mid W_0 = -1) = EY_i^{(n)} = \sum_{-\infty}^{\infty} jg_j$$

or in terms of the $Z_n(1)$, $Z_n(2)$ process,

$$\mu = m_{11} - m_{12} \qquad \text{and} \qquad \lambda = m_{21} - m_{22}$$

where $m_{ij} = E\{\#\text{ type } j \text{ particles produced by a single type } i\}$. The Malthusian parameter ρ turns out to be the largest nonnegative eigenvalue of the matrix

$$\begin{pmatrix} \mu^+ & \mu^- \\ \lambda^+ & \lambda^- \end{pmatrix}$$

where $Q^\pm = \frac{1}{2}(|Q| \pm Q)$ for any quantity Q. Considering cases easily gives

$$\rho = \max\{\mu^+, \lambda^-, (\mu^-\lambda^+)^{\frac{1}{2}}\}.$$

The gist of the next three theorems is that $\rho < 1$ implies sure extinction while $\rho > 1$, implies $P\{\text{non-extinction}\} > 0$ and $W_n = O(\rho^n)$ a.s.

For Theorem 5 (second part) and Theorem 6 (but not Theorem 4), we need the quantity

$$\sigma^2 = \max\{\text{Var}(W_1 \mid W_0 = 1), \text{Var}(W_1 \mid W_0 = -1)\} = \max\{\sigma_f^2, \sigma_g^2\}.$$

For Theorem 5 only we will need to assume

(B) $\quad P\{W_1 < 0 \mid W_0 = 1\} > 0 \quad$ and $\quad P\{W_1 > 0 \mid W_0 = -1\} > 0$.

Assumption (A) remains in force throughout.

THEOREM 4. *If* $\rho < 1$, *then* $P\{extinction\} = 1$ *and* $\sum_{n=1}^{\infty} E|W_n| < \infty$ (*hence* $E|W_n| \to 0$ as $n \to \infty$).

THEOREM 5. *Assume* (B). *If* $\mu > 0$ *or* $\lambda < 0$ (*or both*) *but* $\rho = \max\{\mu^+, \lambda^-\} > 1$, *then* $P\{non\text{-}extinction\} > 0$ *and* a.s. *on* {*non-extinction*} $\lim W_n = +\infty$ *or* $\lim W_n = -\infty$, *or, what amounts to the same thing,*

(3.2) $$P\{W_n < 0 \text{ and } W_{n+1} > 0 \text{ infinitely often}\} = 0.$$

Suppose also $\sigma^2 < \infty$. *Then* $\rho^{-n}W_n$ *converges* a.s. *and in mean square to a non-degenerate random variable*; *if both* $\mu > 1$ *and* $\lambda < -1$ *then both* W_n^+/μ^n *and* $W_n^-/|\lambda|^n$ *converges* a.s. *and in mean of order* 1 *to non-degenerate limits* (*of course* $P\{\lim W_n^+/\mu^n \neq 0$ *and* $\lim W_n^-/|\lambda|^n \neq 0\} = 0$).

THEOREM 6. *Assume* $\sigma^2 < \infty$. *If* $\mu < 0$ *and* $\lambda > 0$ *but* $\rho^2 = |\mu\lambda| > 1$, *then* $P\{$*non-extinction*$\} > 0$, $\rho^{-n}(|\mu|^{\frac{1}{2}} W_n^+ + (\lambda)^{\frac{1}{2}} W_n^-)$ *converges* a.s. *and in mean square to a non-degenerate limit* V, *and on the event* $\{V \neq 0\}$ W_n *alternates in sign as* $n \to \infty$; i.e.,

$$P\{W_n W_{n+1} < 0 \text{ for all } n \text{ sufficiently large} \mid V \neq 0\} = 1 . \tag{3.3}$$

Notes.

(i) When $\rho = 1$ it is probably true that $P\{\text{extinction}\} = 1$. This is true in special cases, see Section 9, but a complete theorem is lacking.

(ii) Assumption (B) is not necessary and in possibly biologically interesting cases undesirable. For example, if type 1 particles can produce only type 1 particles then $P(W_n < 0 \mid W_0 > 0) = 0$; once W_n becomes positive, type 2 particles will never reappear and W_n is then an ordinary Galton–Watson process. Here if $\lambda < -1$ and $\mu < 1$ we will have $P\{\text{extinction} \mid W_0 > 0\} = 1$ but $P\{W_n \to -\infty \mid W_0 < 0\} > 0$. Theorem 5 requires only trivial, but verbose, modifications when (B) fails.

(iii) One can still get convergence of $V_n = (aW_n^+ + bW_n^-)/\rho^n$ in Theorems 5 and 6 when $\sigma^2 < \infty$ is weakened as in Theorem 2. Whether or not probability 1 convergence is true without any finite moments (other than the first) and the corresponding question of limit degeneracy (the $EZ \log Z < \infty$ criterion in the classical case) are open problems and likely quite difficult. (For an appreciation of the difficulties in the classical case see [1] or [5]. In Theorem 6 one should be able to prove $P\{\text{non-extinction}\} > 0$ without assuming $\sigma^2 < \infty$.

(iv) The matrix $\binom{\mu^+ \ \mu^-}{\lambda^+ \ \lambda^-}$ arises in a natural way. Put $V_n = aW_n^+ + bW_n^-$. By looking for solutions ρ, a and b, to $\lim_{|r|\to\infty} E(V_n \mid W_{n-1} = r)/V_{n-1} = \rho$, one immediately gets

$$\mu^+ a + \mu^- b = \rho a$$
$$\lambda^+ a + \lambda^- b = \rho b .$$

Clearly if the largest ρ is less than 1 one expects $P\{\text{extinction}\} = 1$, and if $\rho > 1$, $P\{\text{non-extinction}\} \neq 0$.

(v) Independently of this author and by different methods, S. Karlin and N. Kaplan [2] have also obtained the criticality criteria under a finite second moment assumption, for (2.1)/(2.2). Their method does not apply to the critical case $\rho = 1$; and they do not obtain the limit theorems of this paper. H. Kesten in [4] (announced in [3]) has proved some very technical a.s. limit theorems applicable to a wide variety of stochastic population growth models. As in [2] his results also depend on more than a finite first moment and do not give mean convergence, his method also does not appear to apply in critical cases.

4. Proof of Theorem 4.

Let us assume for the moment the truth of the statement: "For every $\varepsilon > 0$ there are numbers $\delta_n = \delta_n(\varepsilon) \geqq 0$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and

$$E|W_n| \leqq \delta_n + (\gamma_1 + \varepsilon)E|W_{n-1}| + (\gamma_2 + \varepsilon)E|W_{n-2}| , \qquad n \geqq 2 , \tag{4.1}$$

where $\gamma_1 = \max\{\mu^+, \lambda^-\}$ and $\gamma_2 = \mu^- \cdot \lambda^+$."

The partial sums $Q_n = E|W_1| + \cdots + E|W_n|$ must then satisfy

$$Q_n \leqq \Delta + (\gamma_1 + \varepsilon)Q_{n-1} + (\gamma_2 + \varepsilon)Q_{n-2}, \qquad n \geqq 3$$

where

$$\Delta = \Delta(\varepsilon) = (\gamma_2 + \varepsilon)E|W_0| + E|W_1| + \sum_{k=2}^{\infty} \delta_k < \infty, \qquad \varepsilon > 0.$$

Because $\min\{\gamma_1, \gamma_2\} = 0$, $\max\{\gamma_1, \gamma_2\} = \max\{\rho, \rho^2\} = \rho < 1$, we may pre-choose $\varepsilon > 0$ so that

$$\theta = 1 - \gamma_1 - \gamma_2 - 2\varepsilon > 0.$$

Now note that if $M = \max\{\Delta/\theta, Q_1, Q_2\}$, then $Q_3 \leqq \Delta + (\gamma_1 + \varepsilon)Q_2 + (\gamma_2 + \varepsilon)Q_1 \leqq \theta M + (1 - \theta)M = M$ and by induction $Q_n \leqq M$ for all n. Consequently,

$$\sum_{n=1}^{\infty} E|W_n| = \lim_{n\to\infty} Q_n \leqq M < \infty. \tag{4.2}$$

From (4.2) $E(\liminf |W_n|) \leqq \lim E|W_n| = 0$ which, according to Lemma 1, is more than enough to conclude $P\{\text{extinction}\} = 1$. Therefore, except for (4.1), the proof of Theorem 4 is done.

PROOF OF (4.1). In calculations of conditional expectations, the superscript n in (2.2) may be safely omitted. Thus

$$\begin{aligned} EW_n^{\pm} &= \sum_{r\neq 0} E(W_n^{\pm} \mid W_{n-1} = r)P\{W_{n-1} = r\} \\ &= \sum_{r>0} (ES_r^{\pm})P\{W_{n-1} = r\} + \sum_{r>0} (ET_r^{\pm})P\{W_{n-1} = -r\} \\ &\leqq kaP\{1 \leqq |W_{n-1}| \leqq k\} \\ &\quad + \sum_{r>k} ((ES_r^{\pm})P\{W_{n-1}^{+} = r\} + (ET_r^{\pm})P\{W_{n-1}^{-} = r\}) \end{aligned} \tag{4.3}$$

where $S_r = X_1 + \cdots + X_r$, $T_r = Y_1 + \cdots + Y_r$, $a = \max\{E|X_1|, E|Y_1|\}$. By a slightly strengthened law of large numbers, S_r/r and T_r/r converge in mean (of order 1), as well as a.s., to μ and λ, hence

$$ES_r^{\pm} \leqq (\mu^{\pm} + \varepsilon_k)r \quad \text{and} \quad ET_r^{\pm} \leqq (\lambda^{\pm} + \varepsilon_k)r \quad \text{for } r \geqq k \tag{4.4}$$

where

$$\begin{aligned} \varepsilon_k &= \sup_{r\geqq k} \max\left\{\left|\frac{E|S_r|}{r} - |\mu|\right|, \left|\frac{E|T_r|}{r} - |\lambda|\right|\right\} \\ &\leqq \sup_{r\geqq k} \max\left\{E\left|\frac{S_r}{r} - \mu\right|, E\left|\frac{T_r}{r} - \lambda\right|\right\} \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

(Recall that $Q^+ + Q^- = |Q|$, $Q^+ - Q^- = Q$.) Applying (4.4) to (4.3) and writing $\delta_{n,k} = kaP\{1 \leqq |W_{n-1}| \leqq k\}$, we get

$$\begin{aligned} EW_n^{+} &\leqq \delta_{n,k} + \mu^+ EW_{n-1}^{+} + \lambda^+ EW_{n-1}^{-} + \varepsilon_k E|W_{n-1}| \equiv z_{n-1}^{+} \\ EW_n^{-} &\leqq \delta_{n,k} + \lambda^- EW_{n-1}^{-} + \mu^- EW_{n-1}^{+} + \varepsilon_k E|W_{n-1}| \equiv z_{n-1}^{-}. \end{aligned} \tag{4.5}$$

In (4.5) replace the term $\lambda^+ EW_{n-1}^{-}$ by $\lambda^+ z_{n-2}^{-}$ and $\mu^- EW_{n-1}^{+}$ by $\mu^- z_{n-2}^{+}$ where $z_{n-2}^{\pm}$ are the right hand sides of (4.5) when n is changed to $n - 1$. Add the two new inequalities and simplify using $Q^+ + Q^- = |Q|$ and

$$\mu^+\mu^- = \lambda^+\lambda^- = 0, \qquad \max\{\mu^+, \lambda^-\} = \gamma_1, \qquad \mu^-\lambda^+ = \gamma_2.$$

The result is

$$E|W_n| \leqq \delta_n + (\gamma_1 + \varepsilon')E|W_{n-1}| + (\gamma_2 + \varepsilon'')E|W_{n-2}|$$

where

$$\begin{aligned}\delta_n &= 2\delta_{n,k} + \lambda^+\delta_{n-1,k} + \mu^-\delta_{n-1,k} \\ &\leqq \text{const} \times k \times [P\{1 \leqq |W_{n-1}| \leqq k\} + P\{1 \leqq |W_{n-2}| \leqq k\}] \\ &\qquad \varepsilon' = 2\varepsilon_k , \qquad \varepsilon'' = (\lambda^+ + \mu^-)\varepsilon_k .\end{aligned}$$

On the one hand ε', ε'' can be made as small as we please by taking k sufficiently large; on the other, the series $\sum_{n=2}^{\infty} \delta_n$ converges for each k by Lemma 1 in Section 3. This completes the proof.

REMARK. If $a = \max\{E|X_1|, E|Y_1|\} < 1$, Theorem 4 is almost trivial, for, as one can easily check,

$$E|W_n| \leqq aE|W_{n-1}| \leqq \cdots \leqq a^{n-1}E|W_1| .$$

We are saved from triviality because we can have $\rho < 1$ while $\min\{E|X_1|, E|Y_1|\} > 1$.

5. Proof of Theorem 1 and the corollary. That $|\mu| < 1$ implies $P\{\text{extinction}\} = 1$ and $\sum_0^\infty E|W_n| < \infty$ follows from Theorem 4 since $\rho = |\mu|$ in the symmetric case.

To prove the remaining assertions of the theorem when $|\mu| > 1$ we first give a more useful representation of the symmetric process than (2.4). Let $\{X_i\}$, $i \geqq 1$ be independent random variables with common distribution

$$f_j = P(X = j) , \qquad j = 0, \pm 1, \cdots$$

and let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Define two sequences $\{W_n\}$, $n \geqq 0$, and $\{\tau_n\}$, $n \geqq 1$, of random variables as follows: $W_0 \neq 0$, constant, arbitrary integer; $\tau_0 = 0$, and for $n \geqq 1$

$$\begin{aligned}\tau_n &= |W_0| + |W_1| + \cdots + |W_{n-1}| \\ W_n &= S_{\tau_n} - S_{\tau_{n-1}} .\end{aligned} \tag{5.1}$$

Clearly (5.1) defines a Markov chain with transition probabilities (2.3). Now put $X_i' = X_i - 1$ and let $\{S_n'\}$, $n \geqq 0$, be the random walk $S_0' = W_0$,

$$S_k' = S_0' + X_1' + \cdots + X_k' = W_0 + S_k - k , \qquad k \geqq 1 . \tag{5.2}$$

Then

$$W_n = S'_{\tau_n} + \sum_{j=0}^{n-1}(|W_j| - W_j) , \qquad n \geqq 1 , \tag{5.3}$$

and since $|W_j| - W_j \geqq 0$

$$W_n \geqq S'_{\tau_n} , \qquad \text{for every } n \geqq 1 . \tag{5.4}$$

(Equality holds in (5.4) for some n if and only if $W_k \geqq 0$ for all $0 \leqq k \leqq n - 1$. This fact will be fully exploited in Section 8.)

We can now finish proving Theorem 1. Let us assume

$$E(W_n \mid W_{n-1} = \pm 1) = EX_i = \mu > 1 .$$

(If $\mu < -1$, replace W_n by $\hat{W}_n = -W_n$; $\{\hat{W}_n\}$ is a symmetric self-annihilating process with $\hat{\mu} = -\mu > 1$.) Because the transition probabilities (2.3) are symmetric in i, we have

$$P\{B \mid W_0 = i\} = P\{B \mid W_0 = -i\}$$

for any event B defined on $\{W_n, n \geqq 1\}$, so we may also assume

$$W_0 = i > 0 .$$

Now the steps $X_j' = X_j - 1$ of the random walk $\{S_n'\}$ have mean

$$EX_j' = \mu - 1 > 0 ,$$

so S_n' drifts to $+\infty$ and with positive probability S_n' will lie to the right of S_0' for every $n \geqq 1$. Noting (5.4) we conclude

$$\text{(5.5)} \qquad \begin{aligned} P\{\text{non-extinction}\} &\geqq P\{W_n > 0 \quad \text{for all} \quad n \geqq 1\} \\ &\geqq P\{S_k' > 0 \quad \text{for every} \quad k \geqq 1 \mid S_0' = i\} > 0 . \end{aligned}$$

It remains to prove (3.1). Put $B = \{\text{non-extinction}\}$. On B, $\tau_n = |W_0| + \cdots + |W_{n-1}| \geqq n \to \infty$, so since B has positive probability

$$\lim \frac{S'_{\tau_n}}{\tau_n} = \mu - 1 \qquad \text{a.s. on } B$$

by the strong law of large numbers. From (5.4) and $S_n' \to +\infty$, we also have a.s. on B that $W_n > 0$ for all n sufficiently large. Hence $\sum_{j=0}^{\infty} (|W_j| - W_j) < \infty$ or

$$\lim \frac{1}{\tau_n} \sum_{j=0}^{n} (|W_j| - W_j) = 0 \qquad \text{a.s. on } B .$$

The limit (3.1) follows on dividing (5.3) by τ_n and letting $n \to \infty$.

Proof of the Corollary. Let $\omega = \{W_n\}_{n=0}^{\infty}$ be a sample point at which (3.1) occurs. Then there is an $M = M(\omega, \varepsilon)$ so that

$$c_1 u_{n-1} < u_n - u_{n-1} < c_2 u_{n-1} , \qquad n \geqq M$$

where $u_n = |W_0| + \cdots + |W_n|$ and $c_1 = (|\mu| - 1)(1 - \varepsilon)$, $c_2 = (|\mu| - 1)(1 + \varepsilon)$. Consequently $(c_1 + 1)u_{n-1} < u_n < (c_2 + 1)u_{n-1}$ and thus

$$(c_1 + 1)^{n-M} u_M < u_n < (c_2 + 1)^{n-M} u_M , \qquad n \geqq M .$$

Taking logarithms, dividing by n and letting $n \to \infty$ and then $\varepsilon \to 0$ we get

$$\lim_{n\to\infty} \frac{\log u_n}{n} = \log |\mu| .$$

Hence

$$\lim_{n\to\infty} \frac{\log |W_n|}{n} = \lim_{n\to\infty} \left(\frac{\log (|W_n|/u_{n-1})}{n} + \frac{\log u_{n-1}}{n} \right) = \log |\mu| .$$

6. Proof of Theorems 2 and 5. We first prove Theorem 5. In (2.1)/(2.2) we are assuming that either $\mu = EX_i \equiv E(W_1/W_0 = 1) \equiv m_{11} - m_{12} > 0$, or $\lambda = EY_i \equiv E(W_1/W_0 = -1) \equiv m_{21} - m_{22} < 0$, (or both) and that $\rho = \max\cdot(\mu^+, \lambda^-) > 1$. By replacing, if necessary, W_n by $\hat{W}_n = -W_n$ we can assume $\lambda^- \leqq \mu^+ = \rho$ or equivalently

(6.1) $$1 < \mu = \rho \leqq \infty \quad \text{and} \quad -\mu \leqq \lambda \leqq \infty .$$

1°. $P\{\text{non-extinction}\} > 0$.

PROOF. On account of assumption (B) $P\{W_n > 0 \text{ for some } n \geqq 1 \mid W_0 < 0\} > 0$ so clearly we may assume

$$W_0 = i > 0 .$$

Now as long as W_n remains positive, W_n coincides with the symmetric self-annihilating process

$$\tilde{W}_{n+1} = S^{(n)}_{|\tilde{W}_n|}$$

for which $E(\tilde{W}_1 \mid \tilde{W}_0 = \pm 1) = \mu > 1$. So by (5.5)

(6.2) $$P\{W_n \neq 0 \text{ for all } n\} \geqq P\{W_n > 0 \text{ for all } n\} = P\{\tilde{W}_n > 0 \text{ for all } n\} > 0 .$$

2°. Let us prove (3.2). Again we may assume $W_0 > 0$. Define two sequences $\{\alpha_j\}_{j\geqq 0}$, and $\{\beta_j\}_{j\geqq 1}$ of possibly defective random variables as follows $\alpha_0 = 0$, $\beta_1 = \min\{n: n > \alpha_0, W_n < 0\}$ and for $j \geqq 1$

$$\alpha_j = \min\{n: n > \beta_j, W_n > 0\}$$
$$\beta_{j+1} = \min\{n: n \geqq \alpha_j, W_n < 0\} .$$

From (6.2) and the random walk representation (5.2)—(5.4) for symmetric processes we have for $t > 0$

$$P\{\beta_{j+1} < \infty \mid W_{\alpha_j} = t, \alpha_j < \infty\} = P\{\beta_1 < \infty \mid W_0 = t\}$$
$$= P\{\tilde{W}_n < 0 \text{ for some } n \geqq 1 \mid \tilde{W}_0 = t > 0\}$$
$$\leqq P\{S_m' < 0 \text{ for some } m \geqq 1 \mid S_0' = t\} .$$

Since the walk $\{S_n'\}$ has step mean $\mu - 1 > 0$ the last written probabilities are bounded away from 1 and tend to 0 as $t \to \infty$. Put

$$\theta = \sup_{t\geqq 1} P\{\beta_{j+1} < \infty \mid W_{\alpha_j} = t, \alpha_j < \infty\}$$

(independent of j). Then $\theta < 1$ and we have

$$P\{\beta_{j+1} < \infty\} = \textstyle\sum_{t\geqq 1} P\{\beta_{j+1} < \infty \mid W_{\alpha_j} = t, \alpha_j < \infty\} P\{W_{\alpha_j} = t, \alpha_j < \infty\}$$
$$\leqq \theta P\{\alpha_j < \infty\} \leqq \theta P\{\beta_j < \infty\} .$$

Therefore, $P\{\beta_j < \infty\} \leqq \theta^{j-1} P\{\beta_1 < \infty\} \to 0$, $j \to \infty$, and consequently

$$P\{W_n \text{ changes sign i.o.}\} = P\{\beta_j < \infty \text{ for all } j\}$$
$$= \lim_{j\to\infty} P\{\beta_j < \infty\} = 0 .$$

3°. Put $V_n = W_n/\rho^n = W_n/\mu^n$, $U_n = W_n^-/|\lambda|^n$. Then $\{V_n\}$ and $\{V_n^+\}$ are submartingales; $\{U_n\}$ is a submartingale provided $\lambda < 0$ and in the special case $\lambda^- = \mu^+$ the V_n form a martingale.

PROOF. If $r < 0$ then $E(W_n \mid W_{n-1} = r) = ET_{|r|} = \lambda|r| \geqq \mu r = \mu W_{n-1}$ by (6.1) (equality holds if $-\lambda = \mu > 0$) and $E(W_n^+ \mid W_{n-1} = r) \geqq 0 = r^+ = \mu W_{n-1}^+$; if $r > 0$ then $E(W_n \mid W_{n-1} = r) = ES_r = \mu W_{n-1}$ and $E(W_n^+ \mid W_{n-1} = r) = ES_r^+ \geqq ES_r = \mu W_{n-1}^+$; divide all inequalities by μ^n. Similarly prove $\{U_n\}$ is a submartingale if $\lambda < 0$.

4°. The series $\sum_{n=1}^\infty s^n E|V_n|$ converges for every $|s| < 1$. From (6.1) and (4.1): given $\varepsilon > 0$ there is a $c < \infty$, independent of n, such that

$$E|W_n| \leqq c + (\mu + \varepsilon)E|W_{n-1}| + \varepsilon E|W_{n-2}| \,,$$

or

$$E|V_n| \leqq c\mu^{-n} + (1 + \varepsilon')E|V_{n-1}| + \varepsilon' E|V_{n-2}|$$

where $\varepsilon' = \max\{\varepsilon/\mu, \varepsilon/\mu^2\}$. Clearly $E|V_{n-1}|$ is also dominated by the right hand side of this last inequality, so if we put

$$a_n = \max\{E|V_n|, E|V_{n-1}|\} \,,$$

then $a_n \leqq c\mu^{-n} + (1 + 2\varepsilon')a_{n-1}$. Iterating and setting $\theta = 1 + 2\varepsilon'$ we get

$$a_n \leqq c\mu^{-n}\left\{\frac{(\theta\mu)^{n-1} - 1}{\theta\mu - 1}\right\} + \theta^{n-1}a_1 \leqq c_1\mu^{-n} + c_2\theta^n$$

for some numbers c_1, c_2 independent of n. For a fixed positive $s < 1$, $\theta s < 1$ for $\varepsilon > 0$ sufficiently small, hence $\sum_1^\infty a_n s^n$ and *a fortiori* $\sum_1^\infty E|V_n|s^n$ converges.

From now on we assume

$$\sigma^2 = \max\{\mathrm{Var}(W_1 \mid W_0 = 1), \mathrm{Var}(W_1 \mid W_0 = -1)\} < \infty \,.$$

5°. $\sup EV_n^2 < \infty$. In what follows c's denote finite positive constants independent of n and r.

Case 1. $\lambda \geqq 0$. From the estimates $E(S_r^-)^2 = O(r)$, $E(T_r^-)^2 = O(r)$ (see the lemma at the end of this section) we obtain

$$\begin{aligned} E\{(W_n^-)^2 \mid W_{n-1} = r\} &= E(S_{r^+}^- + T_{r^-}^-)^2 = E(S_{r^+}^-)^2 + E(T_{r^-}^-)^2 \\ &\leqq c_1(r^+ + r^-) = c_1|r| = c_1|W_{n-1}| \end{aligned}$$

and hence

(6.3) $$E(W_n^-)^2 \leqq c_1 E|W_{n-1}| \,.$$

We also have

$$\begin{aligned} E(W_{n+1}^2 \mid W_n = r) &= ES_{r^+}^2 + ET_{r^-}^2 \\ &= \sigma_f^2 r^+ + \mu^2(r^+)^2 + \sigma_g^2 r^- + \lambda^2(r^-)^2 \\ &\leqq \sigma^2|r| + \mu^2 r^2 + \lambda^2(r^-)^2 \\ &= \sigma^2|W_n| + \mu^2 W_n^2 + \lambda^2(W_n^-)^2 \,. \end{aligned}$$

Taking expectations and applying (6.3) we get

$$EW_{n+1}^2 \leqq \sigma^2 E|W_n| + \mu^2 EW_n^2 + \lambda^2 c_1 E|W_{n-1}|$$
$$\leqq c_2 \mu^n a_n + \mu^2 EW_n^2$$

($a_n = \max\{E|V_n|, E|V_{n-1}|\}$), or since $W_n = \mu^n V_n$

$$EV_{n+1}^2 - EV_n^2 \leqq \text{const} \times \mu^{-n} a_n .$$

Summing these inequalities immediately gives us $\sup EV_n^2 < \infty$, for $1/\mu < 1$ implies $\sum \mu^{-n} a_n < \infty$ by 4°.

Case 2. $\lambda < 0$, $|\lambda| \leqq \mu$. Here $ET_r^2 \leqq \sigma^2 r + \mu^2 r^2$, so

$$E(W_n^2 \mid W_{n-1} = r) = ES_{r^+}^2 + ET_{r^-}^2$$
$$\leqq \sigma^2|r| + \mu^2[(r^+)^2 + (r^-)^2] = \sigma^2|r| + \mu^2 r^2 .$$

Hence $EW_n^2 \leqq \sigma^2 E|W_{n-1}| + \mu^2 EW_{n-1}^2$ and we finish up as in Case 1.

6°. If $\lambda^- < \mu$ then $\lim E(V_n^-)^2 = 0$.

PROOF. For suppose $\lambda \geqq 0$. Then (6.3) implies $E(V_n^-)^2$ is dominated by the nth term of a convergent series (namely $\sum_0^\infty \mu^{-n} E|V_n|$), hence it must tend to 0. If $\lambda < 0$, $|\lambda| < \mu$, then $E(S_r^-)^2 = O(r)$, $E(T_r^-)^2 \leqq cr + \lambda^2 r^2$ and we have

$$E\{(W_n^-)^2 \mid W_{n-1} = r\} \leqq \sigma^2|r| + \lambda^2(r^-)^2 ;$$

therefore,

$$E(V_n^-)^2 \leqq \frac{c}{\mu^n} E|V_{n-1}| + \frac{\lambda^2}{\mu^2} E(V_{n-1}^-)^2 = o(1) + \frac{\lambda^2}{\mu^2} E(V_{n-1}^-)^2 .$$

But $L = \limsup E(V_n^-)^2 < \infty$ by 5°, hence $L \leqq (\lambda/\mu)^2 L$ which is impossible unless $L = 0$.

7°. V_n and V_n^+ converge a.s. and in mean square. If $\lambda^- = \mu$ then $\{V_n\}$ is a martingale by 3° with $\sup EV_n^2 < \infty$ by 5°. This implies the assertion by standard martingale theorems. Suppose $\lambda^- < \mu$. Then V_n forms a submartingale with $\sup EV_n^2 < \infty$ so V_n converges a.s. (and in mean of order $\gamma < 2$) to a nondegenerate random variable V. But V_n^+ being a nonnegative submartingale with $\sup E(V_n^+)^2 \leqq \sup EV_n^2 < \infty$ must also converge in mean square to V^+; V_n^- must also converge in mean square by Fatou and 6° to $V^- \equiv 0$. It follows that $V_n = V_n^+ - V_n^-$ must also converge in mean square. (Note that we have proved slightly more than Theorem 5 asserts, viz. $\lambda^- < \mu^+$ implies $\lim \rho^{-n} W_n = V \geqq 0$ a.s., $\lambda^- > \mu^+$ implies $\lim \rho^{-n} W_n = V \leqq 0$ a.s.)

8°. If $\lambda < -1$ (but $|\lambda| \leqq \mu$) then $U_n = W_n^-/|\lambda|^n$ converges a.s. and in mean.

PROOF. We may suppose $|\lambda| < \mu$. (If $\lambda < -1$, $|\lambda| = \mu$ then $W_n^-/|\lambda|^n = V_n^-$ converges a.s. and in mean square by the first part of 7°.) It suffices to prove

$$\sum_{n=1}^\infty E|U_{n+1} - U_n| < \infty . \tag{6.4}$$

From Lemma 2 below there exists a constant $c_1 < \infty$ such that $ES_r^- \leqq c_1$ and

$ET_r^- \leqq c_1 + |\lambda| r$ for all r. Hence

$$EW_n^- = \sum_{r>0} (ES_r^-) P\{W_{n-1} = r\} + \sum_{r>0} (ET_r^-) P\{W_{n-1} = -r\}$$
$$= c_1 + |\lambda| \sum_{r>0} rP(W_{n-1} = -r) = c_1 + |\lambda| EW_{n-1}^-,$$

or, $EU_n - EU_{n-1} \leqq c_1 |\lambda|^{-n}$. Since $|\lambda| > 1$, $\sum_0^\infty |\lambda|^{-n} < \infty$ and it follows that

$$\sup EU_n = c_2 < \infty .$$

(At this point we could of course conclude a.s. convergence, since $\{U_n\}$ is a submartingale.) From the obvious inequality $|T_r^- - r|\lambda|| \leqq |T_r - r\lambda|$ we get for $r > 0$

$$E\{|U_n - U_{n-1}| \mid W_{n-1} = -r\} = |\lambda|^{-n} E|T_r^- - |\lambda| r| \leqq \sigma |\lambda|^{-n} (r)^{\frac{1}{2}} = c_3 |\lambda|^{-n/2} (U_{n-1})^{\frac{1}{2}} .$$

Also

$$E\{|U_n - U_{n-1}| \mid W_{n-1} = r > 0\} = |\lambda|^{-n} ES_r^- \leqq c_1 |\lambda|^{-n} .$$

Hence

$$E|U_n - U_{n-1}| \leqq c_3 |\lambda|^{-n/2} E(U_{n-1})^{\frac{1}{2}} + c_1 |\lambda|^{-n}$$
$$\leqq c_3 |\lambda|^{-n/2} (EU_{n-1})^{\frac{1}{2}} + c_1 |\lambda|^{-n} = O(|\lambda|^{-n/2})$$

and (6.4) follows since $|\lambda| > 1$. This completes the proof of Theorem 5.

PROOF OF THEOREM 2. For the process (2.3)/(2.4) we assume

$$EX_i = E(W_1 \mid W_0 = \pm 1) = \mu > 1$$
$$E|X_i|^\gamma = E(|W_1|^\gamma \mid W_0 = \pm 1) < \infty .$$

If $\gamma = 2$, Theorem 2 follows from the second part of Theorem 5. For any $\gamma > 1$ all but one (Step 3 below) of the key steps are similar, though simpler, to the latter steps in the proof of Theorem 5; so, we omit some details.

Put $S_r = X_1 + \cdots + X_r$ and $V_n = W_n/\mu^n$.

Step 1. With respect to $\{\mathscr{B}(W_0, \cdots, W_n)\}_{n \geqq 0}$, $\{V_n\}_{n \geqq 0}$ is a submartingale and $\{|V_n|^\theta\}_{n \geqq 0}$ for $1 \leqq \theta \leqq \gamma$ is a nonnegative submartingale. Also, $E|W_n|^\theta \geqq \mu^{n\theta} E|W_0| \to \infty$ as $n \to \infty$. $W_0 \neq 0$.

PROOF. All assertions follow from the inequalities

$$E(|V_n| \mid W_{n-1} = r) \geqq E(V_n \mid W_{n-1} = r) = (ES_{|r|})/\mu^n$$
$$= |r|\mu/\mu^n = |V_{n-1}| \geqq V_{n-1} ,$$

and the fact that $\Phi(t) = |t|^\theta$ is convex, $\theta \geqq 1$.

Step 2.

$$\lim_{n\to\infty} \frac{E|V_n|^\theta}{E|V_{n-1}|^\theta} = 1 .$$

To see this note that because $E|X_i|^\gamma < \infty$ we have as $r \to \infty$, $E|S_r|^\theta/r^\theta \to \mu^\theta$ by the Laws of large numbers. So for $|r| \geqq K = K(\varepsilon)$, $E(|W_n|^\theta \mid W_{n-1} = r) =$

$E|S_{|r|}|^\theta \leqq |r\mu|^\theta(1+\varepsilon)$. This and Step 1 give us

$$E|V_{n-1}|^\theta \leqq E|V_n|^\theta \leqq \delta_n + (1+\varepsilon)E|V_{n-1}|^\theta$$

where $\delta_n = \mu^{-n\theta}KP\{1 \leqq |W_{n-1}| \leqq K\} \to 0$ as $n \to \infty$. This does Step 2.

Step 3. $\sup_{n\geqq 1} E|V_n|^\gamma < \infty$.

PROOF. An inequality due to Marcenkiewicz and Zygmund (cf. [7] page 87) gives

$$E|S_r - r\mu|^\gamma \leqq c_1 E\{\textstyle\sum_{i=1}^r (X_i - \mu)^2\}^{\gamma/2} \tag{6.5}$$

where c_1 is a constant independent of r. For step 3 we consider two cases.

Case I. $1 < \gamma < 2$. In this case $\gamma/2 < 1$ so $\{\sum_{i=1}^r (X_i - \mu)^2\}^{\gamma/2} \leqq \sum_{i=1}^r |X_i - \mu|^\gamma$. From this, (6.5), and Minkowskie's inequality we obtain

$$\begin{aligned} E|S_r|^\gamma &\leqq \{(E|S_r - r\mu|^\gamma)^{1/\gamma} + r\mu\}^\gamma \\ &\leqq (r^{1/\gamma}c_1^{1/\gamma}\{E|X_1 - \mu|^\gamma\}^{1/\gamma} + r\mu)^\gamma \leqq c_2 r^\theta + (r\mu)^\gamma , \end{aligned}$$

where $\theta = \mu + \gamma^{-1} - 1 < \gamma$. Since $E(|V_n|^\gamma \,|\, |W_{n-1}| = r) = E|S_r|^\gamma/\mu^{n\gamma}$ we have therefore

$$E|V_n|^\gamma \leqq E|V_{n-1}|^\gamma + c_3 s^{n-1} E|V_{n-1}|^\theta$$

where $s = \mu^{\theta-\gamma} < 1$, c_3 is a constant independent of n. By Step 2 the series $\sum_0^\infty s^n E|V_n|^\theta$ is finite and hence

$$\begin{aligned} \sup_{n\geqq 1} E|V_n|^\gamma &= E|V_0|^\gamma + \lim_{n\to\infty} \textstyle\sum_{k=1}^n (E|V_k|^\gamma - E|V_{k-1}|^\gamma) \\ &\leqq c_3 \textstyle\sum_{n=0}^\infty s^n E|V_n|^\theta < \infty . \end{aligned}$$

Case II. $\gamma > 2$. Here we take the $\gamma/2$ root of both sides of (6.5) and apply Minkowski's inequality to obtain

$$E|S_r|^\gamma \leqq (c_2 r^{\frac{1}{2}} + r\mu)^\gamma \leqq c_3 r^\theta + (r\mu)^\gamma$$

where $\theta = \gamma - \frac{1}{2}$. Proceed as before.

Step 4. We now complete the proof of Theorem 2. By Steps 1 and 3, $|V_n|$ converges with probability 1 and in mean or order γ to a non-degenerate real value V_∞. But $V_n \geqq 0$ for all n sufficiently large with probability 1 by Theorem 1. Thus we have $V_n \to V_\infty$ a.s. and $E||V_n| - V_\infty|^\gamma \to 0$. This implies $|V_n|^\gamma$ is uniformly integrable and hence $E|V_n - V_\infty|^\gamma \to 0$.

REMARK. It is worth noting the equality

$$EV_\infty^2 = EW_0^2 + \left(\frac{\sigma}{\mu}\right)^2 \textstyle\sum_0^\infty \mu^{-2n} E|W_n|$$

which occurs in the case $\gamma = 2$ of Theorem 2. This follows from

$$E(W_n^2 \,|\, W_{n-1} = r) = ES_{|r|}^2 = |r|\sigma^2 + r^2\mu^2$$

which replaces the inequalities in Cases I and II.

We have used the following lemma at various places in the proof of Theorem 5.

LEMMA 2. *Let $\{\xi_i\}$ be a sequence of independent random variables with $E\xi_i = m$ and $E(\xi_i - m)^2 = \sigma^2$ for all $i(\sigma^2 < \infty)$. Put $\eta_r = \xi_1 + \xi_2 + \cdots + \xi_r$. If $m > 0$ then for all r*

$$rm \leqq E\eta_r^+ \leqq rm + \sigma^2/4m\,, \qquad E\eta_r^- \leqq \sigma^2/4m$$

and $E(\eta_r^-)^2 \leqq r\sigma^2$. If $m < 0$ interchange $+$ and $-$; if $m = 0$ then $E(\eta_r^\pm)^2 \leqq r\sigma^2$.

PROOF. All inequalities follow from

$$E\eta_r^+ + E\eta_r^- = E|\eta_r| \leqq (E\eta_r^2)^{\frac{1}{2}} = (r\sigma^2 + r^2m^2)^{\frac{1}{2}} \leqq rm + \frac{1}{2}\frac{\sigma^2}{m}$$

$$E\eta_r^+ - E\eta_r^- = E\eta_r = rm \qquad \text{(now add)}\,,$$

and

$$E(\eta_r^-)^2 \leqq E[(\eta_r - rm)^-]^2 \leqq r\sigma^2\,,$$

$$E\eta_r^+ \geqq E\eta_r = rm\,, \qquad E\eta_r^2 = E(\eta_r^+)^2 + E(\eta_r^-)^2\,.$$

7. Proof of Theorem 6. We are assuming in (2.1)/(2.2), that $\sigma^2 < \infty$ and

$$\mu < 0\,, \qquad \lambda > 0\,, \qquad \rho^2 = |\mu\lambda| > 1\,.$$

Put

$$V_n = \rho^{-n}(|\mu|^{\frac{1}{2}}W_n^+ + \lambda^{\frac{1}{2}}W_n^-)\,.$$

1°. $\{V_n\}$ is a nonnegative submartingale (with respect to $\mathscr{B}_n = \mathscr{B}(W_0, \cdots, W_n)$ as usual). To see this let $\Delta_n = \rho^n V_n = |\mu|^{\frac{1}{2}}W_n^+ + \lambda^{\frac{1}{2}}W_n^-$. Then

$$\begin{aligned} E(\Delta_n \mid W_{n-1} = r > 0) &= |\mu|^{\frac{1}{2}}ES_r^+ + \lambda^{\frac{1}{2}}ES_r^- \geqq \lambda^{\frac{1}{2}}ES_r^- \\ &\geqq \lambda^{\frac{1}{2}}E(-S_r) = \lambda^{\frac{1}{2}}(-\mu)r \\ &= \rho\Delta_{n-1}\,. \end{aligned}$$

(The last equality is because $\Delta_{n-1} = |\mu|^{\frac{1}{2}}r^+ + \lambda^{\frac{1}{2}}r^- = |\mu|^{\frac{1}{2}}r$ when $W_{n-1} = r > 0$.) Similarly, $E(\Delta_n \mid W_{n-1} = r < 0) \geqq \rho\Delta_{n-1}$.

2°. $\lim E\Delta_n/E\Delta_{n-1} = \rho$, hence $\sum_1^\infty s^n EV_n < \infty$ for all $|s| < 1$.

PROOF. We have

$$\frac{E(\Delta_n \mid W_{n-1} = r)}{\Delta_{n-1}} = \frac{|\mu|^{\frac{1}{2}}ES_{r^+}^+ + \lambda^{\frac{1}{2}}ES_{r^+}^- + |\mu|^{\frac{1}{2}}ET_{r^-}^+ + \lambda^{\frac{1}{2}}ET_{r^-}^-}{|\mu|^{\frac{1}{2}}r^+ + \lambda^{\frac{1}{2}}r^-}$$

$$\to \rho \qquad \text{as} \quad r \to \pm\infty\,.$$

(Recall $ES_n^\pm/n \to \mu^\pm$, $ET_n^\pm/ \to \lambda^\pm$ as $n \to \infty$, see Section 4, and $\mu^+ = \lambda^- = 0$.) Hence $E(\Delta_n \mid W_{n-1} = r) \leqq (\rho + \varepsilon)\Delta_{n-1}$ for all $|r| \geqq k(\varepsilon)$ and consequently

$$\rho E\Delta_{n-1} \leqq E\Delta_n \leqq c + (\rho + \varepsilon)E\Delta_{n-1}$$

where c is independent of n. This gives us what we want since $E\Delta_n \geqq \rho^n E\Delta_0 \to \infty$.

3°. V_n converges a.s. and in mean square to a non-degenerate random variable (so $P\{\text{non-extinction}\} > 0$). By 1° we need only show

$$\sup EV_n^2 < \infty\,. \tag{7.1}$$

Using Lemma 2 at the end of the last section we get

$$\begin{aligned} E(\Delta_n^2 \mid W_{n-1} = r > 0) &= |\mu| E(S_r^+)^2 + \lambda E(S_r^-)^2 \\ &\leqq |\mu|\sigma^2 r + \lambda(\mu^2 r^2 + \sigma^2 r) \\ &\leqq c_1 \Delta_{n-1} + \rho^2 \Delta_{n-1}^2 , \end{aligned}$$

and similarly $E(\Delta_n^2 \mid W_{n-1} = r < 0) \leqq c_2 \Delta_{n-1} + \rho^2 \Delta_{n-1}^2$. Taking expectations, dividing by ρ^{2n} we get for some constant c

$$EV_n^2 \leqq c\rho^{-n} EV_{n-1} + EV_{n-1}^2 ,$$

and therefore, $\sup EV_n^2 < \text{const} \times \sum \rho^{-n} EV_n < \infty$ by 2°.

4°. Each of the four processes $\{\rho^{-2n} W_{2n}^+\}$, $\{\rho^{-2n} W_{2n}^-\}$, $\{\rho^{-2n} W_{2n+1}^+\}$ and $\{\rho^{-2n} W_{2n+1}^-\}$ is a nonnegative submartingale and hence converges a.s. (and in mean square) by (7.1) and $E(\rho^{-m} W_m^{\pm})^2 \leqq \text{const} \times EV_m^2$. This assertion follows from inequalities of the form $E(W_{m+2}^- \mid W_m = r < 0) \geqq \rho^2 |r|$, $E(W_{m+2}^+ \mid W_m = r > 0) \geqq \rho^2 r$. We prove only the first one. Let $r < 0$ then

$$\begin{aligned} E(W_{m+2}^- \mid W_m = r) &= \textstyle\sum_{k=-\infty}^{\infty} E(W_{m+2}^- \mid W_{m+1} = k) P\{W_{m+1} = k \mid W_m = r\} \\ &\geqq \textstyle\sum_{k>0} ES_k^- P\{W_{m+1} = k \mid W_m = r\} \\ &\geqq \textstyle\sum_{k>0} (-k\mu) P\{W_{m+1} = k \mid W_m = r\} \\ &= |\mu| E(W_{m+1}^+ \mid W_m = r < 0) = |\mu| ET_{|r|}^+ \geqq |\mu\lambda r| = \rho^2 r . \end{aligned}$$

(The 2nd and 3rd inequalities are from Lemma 2 of Section 6.)

5°. To finish up we must prove (3.3). Let A^+, A^-, B^+, B^- be respectively the events on which $\rho^{-2n} W_{2n}^+$, $\rho^{-2n} W_{2n}^-$, $\rho^{-2n} W_{2n+1}^+$, $\rho^{-2n} W_{2n+1}^-$ have *nonzero* limits and let $V = \lim V_n$, then $A^+A^- = B^+B^- = \phi$ and

$$\{V \neq 0\} = A^+ \cup A^- \cup B^+ \cup B^- .$$

To prove (3.3) it is necessary and sufficient to show

$$P\{A^+B^+ \cup A^-B^-\} = 0 . \tag{7.2}$$

But starting from any $W_{n_0} = i > 0$ the process $\{W_{n_0+n}\}$, $n \geqq 0$ coincides, as long as it is nonnegative with a symmetric self-annihilating process $\{\tilde{W}_n\}$ $n \geqq 0$, $\tilde{W}_0 = i$, which has mean $\mu = E(\tilde{W}_k \mid \tilde{W}_{k-1} = 1) < 0$. Hence, by Theorem 1, $W_{n_0+n_1} = \tilde{W}_{n_0+n_1} \leqq 0$ for some n_1 with probability 1. Similarly, if $W_{n_0} = i < 0$ then $\lambda > 0$ implies $W_{n_0+n_1} \geqq 0$ with probability 1. From these remarks we may conclude

$$P\{W_n < 0,\ W_{n+1} > 0 \ \text{ i.o. or } \ \{W_n\} \ \text{ becomes extinct}\} = 1$$

which clearly implies (7.2).

8. Proof of Theorem 3. The following elegant proof was suggested by Professor Harry Kesten (private communication).

We may suppose

$$\mu = EX_i = E(W_n \mid W_{n-1} = \pm 1) = 1 .$$

We use the representation (5.1) and the identity (5.3). Note that because the step mean is $EX_i' = \mu - 1 = 0$, the random walk $S_k' = S_0' + X_1' + \cdots + X_k' = W_0 + S_k - k$ is recurrent. Define

$$\alpha = \min\{m : S_m' \leqq 0\}$$
$$\beta = \min\{n : \tau_n \geqq \alpha\}$$

$\tau_n = |W_0| + |W_1| + \cdots + |W_{n-1}|$ recall. Both α and β are well defined with probability 1. Roughly the idea is to show $|W_\beta|$ is "small" relative to $|W_0|$ with "high" probability by comparing $W_\beta \equiv S'_{\tau_\beta}$ with S_α'. This leads to $P\{\liminf |W_n| < \infty\} = 1$. We get estimates for $|S_\alpha'|$ and $\tau_\beta - \alpha$ from fluctuation theory for random walks with mean 0.

Let $S_0' = W_0 = k > 0$. By definition of α, $S_j' > 0$ for $j < \alpha$, so by (5.3) and the definition of β we will have $W_0 = S'_{\tau_0} > 0$, $W_1 = S'_{\tau_1} > 0, \cdots, W_{\beta-1} = S'_{\tau_{\beta-1}} > 0$ and then

$$\tau_\beta = S_0' + S'_{\tau_1} + \cdots + S'_{\tau_{\beta-1}}$$
$$W_\beta = S'_{\tau_\beta}\,.$$

It follows that the event $\{\alpha = m, \beta = n, \tau_\beta = m + j\}$ is completely determined from the values of $S_0', S_1', \cdots, S_m'$. Hence the events $\{\alpha = m, \tau_\beta = m + j\}$ and $\{X'_{m+1} = x_1, \cdots, X'_{m+j} = x_j\}$ are independent for each $j \geqq 1, m, x_1, \cdots, x_j$; consequently

$$P\left\{|W_\beta - S_\alpha'| > \frac{d}{2}\,\middle|\, \alpha = m, \tau_\beta = m + j\right\} = P\left\{|S'_{m+j} - S_m'| > \frac{d}{2}\right\}$$
$$= P\left\{|X_1' + \cdots + X_j'| > \frac{d}{2}\right\}.$$

Keeping in mind $P\{\cdot\} = P\{\cdot \mid S_0' = k\}$ and noting $EX_1' = 0$, $E(X_i')^2 = \sigma^2 < \infty$ we get for every $R \geqq 1$

$$P\left\{|W_\beta - S_\alpha'| > \frac{d}{2}\right\} = \textstyle\sum_{j\geqq 1} P\left\{|X_1' + \cdots + X_j'| > \frac{d}{2}\right\} P\{\tau_\beta - \alpha = j\}$$
$$\leqq \textstyle\sum_{1\leqq j \leqq R} P\left\{|X_1' + \cdots + X_j'| > \frac{d}{2}\right\} + P\{\tau_\beta - \alpha > R\}$$
$$\leqq \frac{4\sigma^2 R}{d^2} + P\{\tau_\beta - \alpha > R\}\,.$$

But $P\{|W_\beta| > d\} \leqq P\{|W_\beta - S_\alpha'| > d/2\} + P\{|S_\alpha'| > d/2\}$ hence

$$(8.1) \qquad P\{|W_\beta| > d\} \leqq P\{\tau_\beta - \alpha > R\} + P\left\{|S_\alpha'| > \frac{d}{2}\right\} + \frac{4\sigma^2 R}{d^2}\,.$$

We want to estimate $P\{\tau_\beta - \alpha > R\}$ for $R > k = S_0'$. Let A be the event {the random walk $\{S_n'\}$ starting at $S_0' = k$ exits to the right from the interval $[1, R]$}. From ([8] pages 252–254) we obtain

$$P\{A\} \leqq c_1 \frac{k}{R}$$

for some constant c_1 independent of k and R (provided $1 \leqq k \leqq R$). Now α is the first hitting time of $(-\infty, 0]$ so clearly A occurs if and only if

$$\max\{S_1', S_1', \cdots, S'_{\alpha-1}\} > R .$$

But

$$\tau_\beta - \alpha < \tau_\beta - \tau_{\beta-1} = S'_{\tau_{\beta-1}} \leqq \max\{S_0', S_1', \cdots, S'_{\alpha-1}\} ,$$

hence

$$P\{\tau_\beta - \alpha > R\} \leqq P\{A\} \leqq c_1 \frac{k}{R} . \tag{8.2}$$

Using the methods of proof on page 211 of [8] (see also Exercise 6, page 232) one can show that the assumption $E|X_i'|^3 < \infty$ implies

$$\sup_{k \geq 0} E(|S_\alpha'| \mid S'_0 = k) < \infty .$$

Hence

$$P\left\{|S_\alpha'| > \frac{d}{2}\right\} \leqq \frac{2E|S_\alpha'|}{d} \leqq \frac{c_2}{d} \tag{8.3}$$

with c_2 independent of k. (Note: This is the only place we need $E(|W_1|^3 \mid W_0 = 1) < \infty$. If only $E(|W_1|^{2+\varepsilon} \mid W_0 = 1) < \infty$ for some $\varepsilon > 0$ then the right hand side of (8.3) is $c_2 k^{1-\varepsilon}/d$. This is still enough to prove Theorem 3.)

Let us now put the bounds (8.2) and (8.3) in (8.1) and also set

$$R = k^{\frac{3}{2}} , \qquad d = \tfrac{1}{2}k .$$

Then we obtain

$$P\{|W_\beta| > \tfrac{1}{2}k \mid W_0 = k\} \leqq \frac{c}{k^{\frac{1}{2}}}$$

for some constant $c < \infty$. Hence

$$P\{|W_n| \leqq \tfrac{1}{2}k \text{ for some } n \geqq 1 \mid W_0 = k\} \geqq P\{|W_\beta| \leqq \tfrac{1}{2}k\} \geqq 1 - \frac{c}{k^{\frac{1}{2}}}$$

and for reasons of symmetry

$$P\{|W_n| \leqq \tfrac{1}{2}k \text{ for some } n \geqq 1 \mid W_0 = -k\} \geqq 1 - \frac{c}{k^{\frac{1}{2}}} .$$

From Lemma 3 it follows that $\liminf |W_n| < \infty$ a.s. and therefore by Lemma 1 in Section 3.

$$P\{\text{extinction}\} = 1 .$$

LEMMA 3. *Let $\{W_n\}$ be a Markov chain and suppose there are numbers θ, $0 < \theta < 1$, $\delta > 0$, $c < \infty$ such that for all $|k|$ sufficiently large*

$$P\{|W_n| \leqq \theta|k| \text{ for some } n \geqq 1 \mid W_0 = k\} \geqq 1 - \frac{c}{|k|^\delta} . \tag{8.4}$$

Then

$$P\{\liminf |W_n| < \infty\} = 1 .$$

PROOF. Suppose we could show

$$P\{|W_n| \leqq r \text{ for some } n \geqq 0 \mid W_0 = k\} \geqq 1 - \frac{b}{r^\delta} \tag{8.5}$$

for all $|k| \geqq r > r_0 > 0$ where $b < \infty$ is independent of k and r. Then for $r \geqq r_0$ and with

$$B_m = \{|W_n| \leqq r \text{ for some } n \geqq m\}$$

we have $P\{B_m \mid W_m = k\} = 1$ if $|k| \leqq r$ and $P\{B_m \mid W_m = k\} = P\{B_0 \mid W_0 = k\} \geqq 1 - b/r^\delta$ by stationarity and (8.5). Hence

$$\begin{aligned} P\{B_m\} &= (\textstyle\sum_{|k|\leqq r} + \sum_{|k|>r})P\{B_m \mid W_m = k\}P\{W_m = k\} \\ &\geqq P\{|W_m| \leqq r\} + \left(1 - \frac{b}{r^\delta}\right) P\{|W_m| > r\} \geqq 1 - \frac{b}{r^\delta} \end{aligned}$$

and therefore

$$\begin{aligned} P\{\liminf |W_m| < \infty\} &= \lim_{r\to\infty} P\{|W_n| < r \text{ i.o.}\} \\ &= \lim_{r\to\infty} \lim_{m\to\infty} P\{|W_n| \leqq r \text{ for some } n \geqq m\} \\ &= \lim_{r\to\infty}\left(1 - \frac{b}{r^\delta}\right) = 1 . \end{aligned}$$

Assume (8.4) holds for $|k| \geqq k_0 > 0$. To establish (8.5) keep $r > k_0$ and let

$$A_r = \{|W_n| > r \text{ for every } n \geqq 0\}$$

and define a sequence of times $\{\nu_j\}$ by $\nu_0 = 0$ and for $j \geqq 1$, $\nu_{j-1} < \infty$,

$$\nu_j = \min\{n : n > \nu_{j-1}, |W_n| \leqq \theta |W_{\nu_{j-1}}|\} .$$

If $\nu_{j-1} = \infty$ or if $|W_n| > \theta|W_{\nu_{j-1}}|$ for all $n > \nu_{j-1}$, then we define $\nu_j = W_{\nu_j} = \infty$. Also put

$$u = \sup\{j : \nu_j < \infty\} .$$

Then

$$P\{A_r\} = P\{A_r, u = \infty\} + \textstyle\sum_{j=0}^\infty P\{A_r, u = j\}$$

where $P\{\cdot\} \equiv P\{\cdot \mid W_0 = k\}$.

Now since $\theta < 1$, we have on $[u = \infty]$

$$|W_{\nu_j}| \leqq \theta^j |W_0| = \theta^j k \leqq r \tag{8.6}$$

for all j sufficiently large (provided $r > 1$). So

$$P\{A_r, u = \infty\} = 0 .$$

(Note that it is possible for $P\{u = \infty\} > 0$ since we are not assuming 0 is absorbing here.) Next

$$\begin{aligned} P\{A_r, u = j\} &\leqq P\{\nu_{j+1} = \infty, |W_{\nu_j}| > r, \nu_j < \infty\} \\ &= E\{P(\nu_{j+1} = \infty \mid W_{\nu_j}, \nu_j); |W_{\nu_j}| > r, \nu_j < \infty\} . \end{aligned}$$

But from (8.4), if $|t| > k_0$, then

$$P\{\nu_{j+1} = \infty \mid W_{\nu_j} = t, \nu_j = m\} = P\{|W_n| > \theta|t| \text{ for all } n > m \mid W_m = t\}$$
$$\leqq \frac{c}{|t|^\delta} = \frac{c}{|W_{\nu_j}|^\delta}.$$

Hence since $r > k_0$

$$P\{A_r, u = j\} \leqq E\left\{\frac{c}{|W_{\nu_j}|^\delta} I_{[|W_{\nu_j}|>r,\, \nu_j<\infty]}\right\}.$$

Now $|W_{\nu_0}| > |W_{\nu_1}| > \cdots$ so if we put

$$s = \max\{j\colon |W_{\nu_j}| > r, \nu_j < \infty\}$$

then $\{|W_{\nu_j}| > r, \nu_j < \infty\} = \{s \geqq j\}$ and by (8.6) $s < \infty$. Consequently,

$$P\{A_r\} \leqq \sum_{j=0}^\infty E\left\{\frac{c}{|W_{\nu_j}|^\delta} I_{[s\geqq j]}\right\} = E\left\{\sum_{j=0}^\infty \frac{c}{|W_{\nu_j}|^\delta} I_{[s\geqq j]}\right\}$$
$$= E\left\{\sum_{j=0}^s \frac{c}{|W_{\nu_j}|^\delta}\right\}.$$

Put

$$Q = \sum_{j=0}^s \frac{c}{|W_{\nu_j}|^\delta}.$$

Since $r < |W_{\nu_j}| \leqq \theta|W_{\nu_{j-1}}|$, $1 \leqq j \leqq s$ it follows that

$$|W_{\nu_{j-1}}|^{-\delta} \leqq \theta^\delta |W_{\nu_j}|^{-\delta}.$$

Hence

$$Q - \frac{c}{r^\delta} < Q - \frac{c}{|W_{\nu_s}|^\delta} \leqq \theta^\delta\left(Q - \frac{c}{|W_0|^\delta}\right) < \theta^\delta Q,$$

or

$$Q < \left(\frac{c}{1-\theta^\delta}\right)\frac{1}{r^\delta} = \frac{b}{r^\delta}.$$

and finally

$$P\{|W_n| \leqq r \text{ for some } n \geqq 0 \mid W_0 = k\} = 1 - P\{A_r\} \geqq 1 - EQ$$
$$\geqq 1 - \frac{b}{r^\delta}$$

for all $|k| > r \geqq r_0 = k_0$.

9. Discussion of criticality in the general case. Suppose $\{W_n\}$ is a self-annihilating process (2.1)/(2.2) with first moment parameters μ and λ as defined in Section 3. If

$$\rho = \max\{\mu^+, \lambda^-, (\mu^-\lambda^+)^{\frac{1}{2}}\} = 1,$$

then one of the following must hold

(i) $\rho = \max\{\mu^+, \lambda^-\} = 1$, or
(ii) $\mu < 0$ and $\lambda > 0$ but $|\mu\lambda| = \rho^2 = 1$.

In Case (i) we can show $P\{\text{extinction}\} = 1$ by essentially the same methods used in Section 8. In Case (ii) the process tends to "flip-flop" instantaneously, see Theorem 6, and the random walk trick appears to be useless.

Here is a sketch of a proof for sure extinction in Case (i). Without loss of generality we may assume

$$\mu = 1 \qquad \text{and} \qquad -1 \leqq \lambda < \infty .$$

We also suppose both $E(|W_1|^3 \mid W_0 = 1)$ and $E(|W_1|^3 \mid W_0 = -1)$ are finite where necessary and that assumption (B) is in effect.

If $W_0 = k > 0$, then until and including the first time $W_n \leqq 0$ we can imbed W_n in a random walk $\{S_m'\}$ with mean $\mu - 1 = 0$. Using the estimates (8.1)—(8.3) we have for every $k > 0$ and $b > 0$

$$(9.1) \qquad P\{|W_n| \leqq bk \ \text{ for some } \ n \geqq 1 \mid W_0 = k\} \geqq 1 - \frac{B_1}{k^{\frac{1}{2}}}$$

where B_1 depends on b but not k. To get a similar estimate when $W_0 = -k < 0$ we consider three possibilities:

(a) $\lambda = -1$. If $W_0 = -k < 0$, we imbed $\hat{W}_n = -W_n$ in a random walk with mean $\lambda + 1 = 0$, and we may conclude

$$P\{|W_n| \leqq \tfrac{1}{2}k \ \text{ for some } \ n \geqq 1 \mid W_0 = -k\} \geqq 1 - \frac{B_2}{k^{\frac{1}{2}}}$$

for every k. This and $b = \frac{1}{2}$ in (9.1) imply $P\{\text{extinction}\} = 1$ by Lemmas 3 and 1.

(b) $-1 < \lambda \leqq 0$. If $W_0 = -k < 0$, then $W_1 = T_k = Y_1 + \cdots + Y_k$ where $EY_i = \lambda = -|\lambda|$, so by Chebyshev's inequality

$$P\{-\theta k \leqq W_1 \leqq \theta k \mid W_0 = -k\} \geqq P\{|T_k - k\lambda| \leqq (\theta - |\lambda|)k\}$$
$$\geqq 1 - \frac{\sigma^2}{(\theta - |\lambda|)^2 k} = 1 - \frac{B_2}{k}$$

for any θ such that $|\lambda| < \theta < 1$. Hence

$$P\{|W_n| \leqq \theta k \ \text{ for some } \ n \geqq 1 \mid W_0 = -k\}$$
$$\geqq P\{|W_1| \leqq \theta k \mid W_0 = -k\} \geqq 1 - \frac{B_2}{k} .$$

This combined with (9.1) again gives $P\{\text{extinction}\} = 1$ by Lemma 3.

(c) $0 < \lambda < \infty$. From Chebyshev's inequality we have

$$P\{0 \leqq W_1 \leqq Ak \mid W_0 = -k\} \geqq 1 - \frac{\sigma^2}{\lambda^2 k}$$

where $A = 2\lambda + 1$. For $\frac{1}{2}k < j \leqq Ak$ and $b = 1/2A$ we have from (9.1)

$$P\{|W_n| \leqq \tfrac{1}{2}k \ \text{ for some } \ n \geqq 1 \mid W_1 = j\}$$
$$\geqq P\{|W_n| \leqq bj \ \text{ for some } \ n \geqq 2 \mid W_1 = j\}$$
$$\geqq 1 - \frac{B_1}{j^{\frac{1}{2}}} \geqq 1 - \frac{B_1'}{k^{\frac{1}{2}}}$$

for some $B_1' = 2^{\frac{1}{2}}B_1$ independent of k. Therefore

$$\begin{aligned} P\{|W_n| \leqq \tfrac{1}{2}k \text{ for some } n \geqq 1 \mid W_0 = -k\} & \\ \geqq \textstyle\sum_{0 \leqq j \leqq Ak} P\{|W_n| \leqq \tfrac{1}{2}k \text{ for some } n \geqq 1, W_1 = j \mid W_0 = -k\} & \\ \geqq P\{0 \leqq W_1 \leqq \tfrac{1}{2}k\} + \textstyle\sum_{\frac{1}{2}k < j \leqq Ak} (1 - B_1' k^{-\frac{1}{2}}) P\{W_1 = j \mid W_0 = -k\} & \\ \geqq P\{0 \leqq W_1 \leqq \tfrac{1}{2}k\} + \left(1 - \frac{B_1'}{k^{\frac{1}{2}}}\right) P\{\tfrac{1}{2}k < W_1 \leqq Ak\} & \\ \geqq \left(1 - \frac{B_1'}{k^{\frac{1}{2}}}\right)\left(1 - \frac{\sigma^2}{\lambda^2 k}\right) \geqq 1 - \frac{B_2}{k^{\frac{1}{2}}} & \end{aligned}$$

for some B_2 independent of k. Once again it follows that (8.4) is valid and hence $P\{\text{extinction}\} = 1$.

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DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305