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LIMIT THEOREMS FOR THE MAXIMUM TERM OF A STATIONARY PROCESS[1]

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This paper contains necessary and sufficient conditions, which sometimes coincide, for the limiting distribution of a uniformly (or strongly) mixing stationary process to be the same as for the independent process with the same marginal distributions. Examples with different limits are given. Let H be any distribution function and let $c_n(\xi) = \inf\{x \in R: H(x) \geqq 1 - \xi/n\}$. The limiting behavior of $H^n(c_n(\xi))$ is determined.

1. Introduction. Throughout this paper we let $\{X_n, n = 1, 2, \cdots\}$ be a strictly stationary process with marginal distribution function H. We assume H is continuous at $x_0 = \sup\{x \in R \mid H(x) < 1\}$. Otherwise the problems considered here are trivial. We also assume $\{X_n\}$ is *uniformly* (or strongly) *mixing*, that is, there is a real-valued function g on the positive integers such that $g(k) \to 0$ as $k \to \infty$ and, if $A \in \mathscr{B}(X_1, \cdots, X_m)$ and $B \in \mathscr{B}(X_{m+k}, X_{m+k+1}, \cdots)$ for some m, then $|P(A \cap B) - P(A)P(B)| \leqq g(k)$. (Here $\mathscr{B}(X)$ is the Borel field generated by X.) We call g a *mixing function* for $\{X_n\}$.

Let $Z_n = \max\{X_1, X_2, \cdots, X_n\}$. We extend some of Gnedenko's results (1943) about the limiting behavior of Z_n in the independent identically distributed (i.i.d.) case. Loynes (1965) obtained some results in this direction. We use similar methods to obtain additional results.

Let $\{Y_n\}$ be an i.i.d. process with marginal distribution H. Then $P[\max(Y_1, \cdots, Y_n) \leqq c] = H^n(c)$. Gnedenko showed that there are only three possible nondegenerate limit law types for $H^n(a_n x + b_n)$. Loynes extended that result to the present circumstances.

A sequence $\{c_n\}$ of real numbers with $c_n < x_0$ is said to satisfy condition R if there exist sequences $\{p = p_m\}$, $\{q = q_m\}$, and $\{r = r_m\}$ of positive integers such that, if $m \to \infty$, then $r \to \infty$, $rg(q) \to 0$, $p^{-1}q \to 0$ and (writing $t = t(m) = r(p+q)$)

$$(1) \qquad p^{-1} \sum_{i=1}^{p-1} (p - i) P[X_{i+1} > c_t \mid X_1 > c_t] \to 0 .$$

The sequence $\{c_n\}$ satisfies R_1 if $\{c_n\}$ satisfies R and, in addition, $r_{m+1}p_{m+1}(r_m p_m)^{-1} \to 1$.

Gnedenko also obtained conditions on H for convergence to each of the types mentioned above. Feller ((1966) page 270) and Marcus and Pinsky (1969) and de Haan (1970) have obtained further results of this nature. Loynes showed that if $\{a_n x + b_n\}$ satisfies R_1 for all x such that $0 < \Phi(x) < 1$, then $H^n(a_n x + b_n) \to \Phi(x)$ (where convergence is in distribution) implies $P[Z_n \leqq a_n x + b_n] \to \Phi(x)$.

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Our Theorem 4 shows that if $\{a_n x + b_n\}$ satisfies R_1 then $H^n(a_n x + b_n) \to \Phi(x)$ if and only if $P[Z_n \leqq a_n x + b_n] \to \Phi(x)$.

For ξ between 0 and ∞, define $c_n(\xi) = \inf\{x \in R : H(x) \geqq 1 - n^{-1}\xi\}$ for $n > \xi$. If H is continuous, $H^n(c_n(\xi)) \to e^{-\xi}$. Theorem 2 determines more completely the limiting behavior of $H^n(c_n(\xi))$. Loynes showed that if $\{c_n(\xi)\}$ satisfies R_1 for some $\xi > 0$ and if $H^n(c_n(\xi)) \to e^{-\xi}$, then $P[Z_n \leqq c_n(\xi)] \to e^{-\xi}$. We give as Theorem 3 the more general result that under R_1, $H^n(c_n(\xi)) - P[Z_n \leqq c_n(\xi)] \to 0$.

Loynes showed that the only possible non-degenerate limits of $P[Z_n \leqq c_n(\xi)]$ are the functions $e^{-\alpha\xi}$, $0 < \alpha \leqq 1$. We give examples to show each such function is possible. In O'Brien and Denzel (1975), we give further examples. In one, $P[Z_n \leqq c_n(\xi)] \to 1$ for $\xi > 0$ (that is, $\alpha = 0$).

2. General results. In this section we obtain some results for general sequences $\{c_n\}$. Theorem 1 gives necessary conditions for $H^n(c_n) - P[Z_n \leqq c_n] \to 0$, while Lemma 3 gives sufficient conditions.

LEMMA 1. *Let $\{c_n\}$ be a sequence of real numbers. Assume there are sequences $\{q = q_m\}$, $\{r = r_m\}$ and $\{t = t_m\}$ of positive integers such that $rg(q) \to 0$, $r \to \infty$, $rq(t)^{-1} \to 0$ and $P[Z_t \leqq c_t] \to a > 0$ as $m \to \infty$. Then $P[Z_p \leqq c_t]^r \to a$, where $p = p_m = \operatorname{int}(r^{-1}t) - q$ (where* int *stands for the greatest integer function).*

PROOF. We have $r(p + q) \leqq t \leqq r(p + q + 1)$. Let $W_1, V_1, \cdots, W_r, V_r$ be the maxima of $2r$ successive sets of the X_i's, alternately of sizes p and q or $q + 1$, the latter choices being made so that the total number of random variables is t.

We note that we may assume without loss of generality that g is a decreasing function. If not, we may replace $g(m)$ by $g'(m) = \min\{g(1), g(2), \cdots, g(m)\}$.

We therefore have

$$\begin{aligned}|a - P[Z_p \leqq c_t]^r| &\leqq |a - P[Z_t \leqq c_t]| + |P[Z_t \leqq c_t] - P[\textstyle\bigcap_{i=1}^r (W_i \leqq c_t)]| \\ &\quad + |P[\textstyle\bigcap_{i=1}^r (W_i \leqq c_t)] - P[W_1 \leqq c_t]^r| \\ &\leqq |a - P[Z_t \leqq c_t]| + P[\textstyle\bigcup_{i=1}^r (V_i > c_t)] + rg(q) \\ &\leqq |a - P[Z_t \leqq c_t]| + 1 - P[V_1 \leqq c_t]^r + rg(p) + rg(q)\,.\end{aligned}$$

The first and two last terms go to zero by hypothesis. We show the third term goes to one.

It is enough to show $\liminf P[V_1 \leqq c_t]^{\alpha_m} > 0$ for a sequence α_m such that $r_m = o(\alpha_m)$. If $g(k)$ is eventually zero, let $\alpha_m = \operatorname{int}(rp(2q)^{-1})$; otherwise, let $\alpha_m = \min\{\operatorname{int}[-r\log(rg(q))], \operatorname{int}(rp(2q)^{-1})\}$. Then $\{\alpha_m\}$ is a sequence of positive integers such that $r = o(\alpha_m)$ (since $pq^{-1} \to \infty$ and $-\log(rg(q)) \to \infty$), $\alpha_m g(q) \to 0$ and $2q\alpha_m \leqq rp$. Divide the first $2q\alpha_m$ random variables into $2\alpha_m$ successive sets of q each, with maxima $U_1, U_2, \cdots, U_{2\alpha_m}$. Then

$$\begin{aligned}P[V_1 \leqq c_t]^{\alpha_m} &\geqq P[U_1 \leqq c_t, U_3 \leqq c_t, \cdots, U_{2\alpha_m - 1} \leqq c_t] - \alpha_m g(q) \\ &\geqq P[Z_t \leqq c_t] - \alpha_m g(q) \\ &\to a > 0\,,\end{aligned}$$

as required. Thus we conclude that $P[W_1 \leqq c_t]^r \to a$. This completes the proof.

THEOREM 1. *Let $\{c_n\}$ be a sequence of real numbers and let $\{t = t_m\}$ be a sequence of positive integers which goes to infinity. Suppose $H^t(c_t) \to a$ and $P[Z_t \leqq c_t] \to a$ where $0 < a < 1$. Then $P[X_{i+1} > c_t \mid X_1 > c_t] \to 0$ for $i = 1, 2, \cdots$.*

PROOF. We have $g(m) \to 0$. Therefore $u_m g(m) \to 0$ for some sequence of positive integers $\{u_m\}$ for which $u_m \to \infty$ (for example, if $e^{-1} > g(m) > 0$ for all m, let $u_m = \text{int}(-\log g(m))$, where $\text{int}(x)$ is the greatest integer not greater than x). There is a sequence $\{q = q_m\}$ of positive integers such that $q \to \infty$; $u_q g(q) \to 0$ and $qu_q = o(t)$. Define $r_m = u_q$. By Lemma 1, $P(Z_p \leqq c_t)^r \to a$. Since $H^t(c_t) \to a$, $H(c_t) = 1 + t^{-1}\log a + o(t^{-1})$. Thus $(1 - p(1 - H(c_t)))^r \to a$. Fix i. Place the random variables $X_1, \cdots, X_{ki}$ in pairs as follows $(X_1, X_{i+1}), (X_2, X_{i+2}), \cdots, (X_i, X_{2i})$, $(X_{2i+1}, X_{3i+1}), \cdots, (X_{3i}, X_{4i}), \cdots, (X_{(k-1)i}, X_{ki})$, where k is even and $(k)i \leqq p < (k+2)i$. Let $U_1, U_2, \cdots, U_{ki/2}$ be the maxima of these pairs, respectively. The number of pairs $(\text{int}(p/2i))i = ki/2 > p/3$ for m sufficiently large. By elementary inequalities,

$$\begin{aligned} P[Z_p > c_t] &\leqq P[\textstyle\bigcup_{j=1}^{\frac{1}{2}ki}(U_j > c_t)] + P[\bigcup_{j=ki}^{p}(X_j > c_t)] \\ &\leqq p(1 - H(c_t)) - \tfrac{1}{3}pP[X_1 > c_t, X_{i+1} > c_t] \\ &\leqq p(1 - H(c_t)) . \end{aligned}$$

Subtracting each expression from one and taking the rth power of each, the outside terms both go to a. Hence

$$rp(1 - H(c_t)) - \tfrac{1}{3}rpP[X_1 > c_t, X_{i+1} > c_t]) \to -\log a .$$

But the first term goes to $-\log a$. Therefore the second goes to 0. Hence the quotient goes to 0, which is the required result.

LEMMA 2. *Suppose $\{c_n\}$ satisfies condition R. Then $P[Z_t \in c_t] - H^t(c_t) \to 0$.*

PROOF. It is sufficient to prove the result on any subsequence of m's on which both $P[Z_t \leqq c_t]$ and $H^t(c_t)$ converge. The result now follows (using Lemma 1) as in the proof of Lemma 1 of Loynes (1965).

LEMMA 3. *Let $\{c_n\}$ be a sequence satisfying R_1 such that $a = \liminf_{m\to\infty} H^t(c_t) > 0$, $b = \limsup (P(X_1 < c_t))^t < 1$ and, for all $\varepsilon > 0$, there exists an m_0 such that if $m > m_0$ and $m < n(1-\varepsilon)$, then $c_m < c_n$. Then $P[Z_n \leqq c_n] - H^n(c_n) \to 0$.*

PROOF. Let $s(n) = \max_m \{t_m : c_{t_m} \leqq c_n\}$. If $s(n) = t_m$, let $t(n) = t_{m+1}$. Then $c_{s(n)} \leqq c_n < c_{t(n)}$, $s(n)/n \to 1$ and $t(n)/n \to 1$. For any subsequence of $\{n\}$, pick a sub-subsequence $\{k\}$ such that $s(k)$ is strictly increasing. Since $\limsup_k P(X_1 > c_{s(k)})/P(X_1 > c_k) \leqq \limsup P(X_1 > c_{s(k)})/P(X_1 \geqq c_{t(k)}) \leqq \log a/\log b < \infty$, $P(X_1 > c_{s(k)})/P(X_1 > c_k)$ is bounded for large k, say by $M \geqq 1$. Define $d_n = c_k$ if $n = s(k)$ and c_n otherwise. Then $d_n \geqq c_n$ for all n. Therefore

$$P[X_1 > d_t, X_{i+1} > d_t]/P[X_1 > d_t] \leqq MP[X_1 > c_t, X_{i+1} > c_t]/P[X_1 > c_t] .$$

Thus R_1 holds for $\{d_n\}$. $P[Z_k \leqq c_k] - H^k(c_k) = P[Z_k \leqq d_{s(k)}] - H^k(d_{s(k)}) \to 0$ by Lemma 2.

3. Limit theorems for $P[Z_n \leqq c_n(\xi)]$. We first examine the limiting behavior in the independent case.

THEOREM 2. *The set of limit points of $H^n(c_n(\xi))$, for any $\xi > 0$, is the interval $[e^{-\xi}, e^{-\xi/\delta}]$ where*

$$\delta = \limsup_{x \to x_0 -} (1 - H(x-))/(1 - H(x)) \leqq \infty . \tag{2}$$

PROOF. First suppose $\delta > 1$. Fix $\xi > 0$. Let a and b be such that $1 < a < b < \delta$. There is a sequence $\{x_k\}$ such that, for each k, $(1 - H(x_k-))/(1 - H(x_k)) > b$, $x_k \to x_0$ and $\xi/(1 - H(x_k)) > (ab)/(b - a)$. There is an integer n in the interval $[(1 - H(x_k))^{-1}\xi b^{-1}, (1 - H(x_k))^{-1}\xi a^{-1}]$. Then

$$(1 - H(x_k))b > \xi n^{-1} > (1 - H(x_k))a . \tag{3}$$

Therefore $1 - H(x_k-) > \xi n^{-1} > 1 - H(x_k)$, so that $c_n(\xi) = x_k$. By (3) also we obtain

$$(1 - \xi/(na))^n < H^n(c_n(\xi)) < (1 - \xi/(nb))^n .$$

The left term converges to $c^{-\xi/a}$; the right goes to $e^{-\xi/b}$. Thus $H^n(c_n(\xi))$ has a limit point in $[e^{-\xi/a}, e^{-\xi/b}]$. Therefore the closed set of limit points contains $[e^{-\xi}, e^{-\xi/\delta}]$.

We note that $H^n(c_n(\xi)) \geqq (1 - \xi n^{-1})^n \to e^{-\xi}$. We show $\limsup H^n(c_n(\xi)) \leqq e^{-\xi/\delta}$. If $\delta = \infty$, this is obvious. We may assume $\delta < \infty$. Suppose conversely that there is a subsequence $\{n'\}$ such that $H^{n'}(c_{n'}(\xi)) \to e^{-\zeta}$ where $0 \leqq \zeta < \xi/\delta$. Then

$$n'(1 - H(c_{n'}(\xi))) = \zeta + o(1) .$$

Since H is continuous at x_0 and $n(1 - H(c_n(\xi)-)) \geqq \xi > 0$, it must be that $1 - H(c_{n'}(\xi)) > 0$. If $\zeta > 0$ (the case $\zeta = 0$ is similar),

$$\begin{aligned} n'(1 - H(c_{n'}(\xi)-)) &= n'(1 - H(c_{n'}(\xi)))\left(\frac{1 - H(c_{n'}(\xi)-)}{1 - H(c_{n'}(\xi))}\right) \\ &\leqq \zeta(\delta + \xi/\zeta)/2 \end{aligned}$$

for n sufficiently large, since $\delta < \xi/\zeta$. Therefore $\limsup H^{n'}(c_{n'}(\xi)-) \geqq e^{-\zeta(\delta+\xi/\zeta)/2} > e^{-\xi}$. But this contradicts $H^n(c_n(\xi)-) \leqq (1 - \xi n^{-1})^n \to e^{-\xi}$. This completes the proof.

THEOREM 3. *Assume $\{c_n(\xi)\}$ satisfies R_1 for some $\xi > 0$. Then $P[Z_n \leqq c_n(\xi)] - H^n(c_n(\xi)) \to 0$.*

This follows immediately from Lemma 3 by the definition of $c_n(\xi)$. Thus, the result of Theorem 2 holds also for strongly mixing processes if R_1 holds. This generalizes Loyne's Lemma 1 which was for the case when $\delta = 1$. It includes the case studied by Watson (1954), when g is eventually 0. Another theorem of this general type has been obtained by Galambos (1972), who does not assume stationarity. The result of Berman (1964) for sequences of the form $\{a_n x + b_n\}$

is valid also for $\{c_n(\xi)\}$. We note that the random variable Y in his result is identically 1 in our circumstances.

4. Convergence of $P[Z_n \leqq d_n]$ for general $\{d_n\}$. Of particular interest in this section are the sequences $\{d_n = a_n x + b_n\}$.

First we note the following fact in the i.i.d case. If $\lim_{n\to\infty} H^n(d_n) = a \in (0, 1)$, then $\delta = 1$. Suppose $\delta > 1$ and $\liminf H^n(d_n) = a \in (0, 1)$. Let $\xi > 0$ satisfy $e^{-\xi} < a < e^{-\xi/\delta}$. Then $\limsup H^n(c_n(\xi)-) \leqq e^{-\xi} < a$; thus $d_n \geqq c_n(\xi)$ for all large n. Therefore, $\limsup H^n(d_n) \geqq \limsup H^n(c_n(\xi)) = e^{-\xi/\delta} > a$, by Theorem 2. This proves the result, which Gnedenko (1943) showed under the hypothesis $H^n(a_n x + b_n) \to \Phi(x)$, for some extreme value distribution Φ.

THEOREM 4. *Assume $\{d_n\}$ satisfies R_1. Then $H^n(d_n) \to a \in (0, 1)$ if and only if $P[Z_n \leqq d_n] \to a$.*

PROOF. Assume $P[Z_n \leqq d_n] \to a \in (0, 1)$. Let $\varepsilon > 0$ and suppose $m < (1-\varepsilon)n$. By an argument like that of Lemma 1, $P(Z_n \leqq d_m) < P(Z_n \leqq d_n)$ for large m and n. Thus, $d_m < d_n$. By Lemma 2, $H^t(d_t) \to a$. Lemma 3 now gives the required result.

As a corollary, we note that if $P[Z_n \leqq d_n] \to a \in (0, 1)$ and $\{d_n\}$ satisfies R_1, then $\delta = 1$.

5. Examples. We show each function $e^{-\alpha\xi}$, $0 < \alpha \leqq 1$ is a possible limit of $P[Z_n \leqq c_n(\xi)]$.

First let $\{J_n\}$ and $\{Y_n\}$ be two sequences of random variables, all independent, with $P[J_n = 1] = \alpha = 1 - P[J_n = 0]$ and $P[Y_n \leqq y] = H(y)$ for some distribution function H for which $\delta = 1$. Let $X_1 = Y_1$. For $n > 1$, let $X_n = Y_n$ if $J_n = 1$ and $X_n = X_{n-1}$ otherwise. Let $A \in \mathscr{B}(X_1, \cdots, X_m)$ and $B \in \mathscr{B}(X_{m+k}, X_{m+k+1}, \cdots)$. Let C be the event that $J_l = 1$ for some l with $m < l \leqq m + k$. Then $P(ABC) = P(B \mid AC)P(C \mid A)P(A) = P(B)P(C)P(A) = P(BC)P(A)$. Thus, $|P(AB) - P(A)P(B)| \leqq P(C^c) + |P(ABC) - P(A)P(BC)| = P(C^c) = (1-\alpha)^k \to 0$. $\{X_n\}$ is uniformly mixing with $g(k) = (1-\alpha)^k$. Up to time n, there are approximately αn independent X_i's. By the weak law of large numbers, $P[Z_n \leqq c_n(\xi)] = H^{\alpha n}(c_n(\xi)) + o(1) \to e^{-\alpha\xi}$. Note that $H^n(a_n x + b_n) \to \Phi(x)$ if and only if $P[Z_n \leqq a_n x + b_n] \to (\Phi(x))^\alpha$. In fact, for fixed x, pick ξ so that $e^{-\xi} < \Phi(x)$. Then $H^n(c_n(\xi)) \to e^{-\xi}$. Thus $c_n(\xi) < a_n x + b_n$ for large n. Thus $\liminf P[Z_n \leqq a_n x + b_n] \geqq \lim P[Z_n \leqq c_n(\xi)] = e^{-\alpha\xi}$ which implies $\liminf P[Z_n \leqq a_n x + b_n] \geqq (\Phi(x))^\alpha$. The rest is similar.

We now give a similar example for which $g(2) = 0$. Fix $\alpha \in [\frac{1}{2}, 1]$ and let $\{J_n\}$ be a Markov chain on $\{0, 1\}$ with transition matrix (P_{ij}) given by $P_{00} = 0$, $P_{01} = 1$, $P_{10} = \alpha^{-1} - 1$ and $P_{11} = 2 - \alpha^{-1}$. Assume $\{J_n\}$ is started with its stationary probability measure (π_i) given by $\pi_0 = 1 - \alpha$ and $\pi_1 = \alpha$. Define $\{X_n\}$ as above. Since $P_{00} = 0$, $g(2) = 0$. As before, one can achieve the limit $e^{-\alpha\xi}$. In general, if $g(k) = 0$, $e^{-\alpha\xi}$ is a possible limit for any $\alpha \in [k^{-1}, 1]$.

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