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LIMIT THEOREMS FOR DISCONTINUOUS RANDOM EVOLUTIONS WITH APPLICATIONS TO INITIAL VALUE PROBLEMS AND TO MARKOV PROCESSES ON N LINES

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Let $X(t)$; $t \geqq 0$ be a stationary continuous-time Markov chain with state space $\{1, 2, \cdots, N\}$ and jump times $t_1, t_2, \cdots$. Let $T_\alpha(t)$; $t \geqq 0$, $1 \leqq \alpha \leqq N$, be semi-groups and $\prod_{jk}(u)$; $u \geqq 0$, $1 \leqq j \neq k \leqq N$, operators defined on Banach space B. Under suitable conditions on these operators, including commutativity, and an appropriate time change in $\varepsilon > 0$ on $X(t)$, we give limiting behavior for the discontinuous random evolutions $T_{X(0)}(t_1^\varepsilon) \prod_{X(0)X(t_1)}(\varepsilon) T_{X(t_1)}(t_2^\varepsilon - t_1^\varepsilon) \cdots T_{X(t_\nu)}(t - t_\nu^\varepsilon)$ as $\varepsilon \to 0$. By considering the 'expectation semi-group' of the discontinuous random evolutions, we prove a type of singular perturbation theorem and give formulas for the asymptotic solution. These results rely on a limit theorem for the joint distribution of the occupation-time and number-of-jump random variables of the chain $X(\cdot)$. We prove this theorem and with 'random evolution' techniques use it to give new proofs of limit theorems for Markov processes on N lines. Analogous results are obtained when the controlling process is a discrete-time finite-state Markov chain.

0. Introduction. Research on the 'random evolution' of a set of semi-groups, with switching among semi-groups controlled by a finite-state, continuous-time Markov chain, has been surveyed by Griego–Hersh [6], and more recently by Pinsky [23]. Pinsky has given representation theorems for multiplicative operator functionals (MOF) in the case of a finite state Markov chain in terms of 'random evolutions' of semi-groups $T_j(t)$, $1 \leqq j \leqq N$, $t \geqq 0$, and jump operators $\prod_{jk}$, $1 \leqq j, k \leqq N$ [22]. Limit theorems for the MOF's without introduction of the jump operators, i.e., in the continuous case, have been proved by Griego–Hersh [6] and by Hersh–Pinsky [8] under the assumption of commutativity of the operators. In this paper we extend these results to MOF's with jump operators in their representation, i.e., in the more general discontinuous case. Our method of proof is based on a limit theorem for the joint distribution of the 'occupation-time' and 'number-of-jumps' random variables for a continuous-time, finite-state Markov chain.

Griego–Hersh have shown that perturbation theorems for systems of partial differential equations follow from the limit theorems for the MOF's when the analytic semi-groups associated with the continuous MOF's are considered [6]. In Section 2 we obtain from our limit theorems for discontinuous MOF's new

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perturbation theorems for various initial value problems. One of the perturbation theorems is stated as follows.

We consider the initial value problem

$$(1)\qquad \begin{aligned} \frac{\partial u_\alpha^\varepsilon}{\partial t} &= \varepsilon^{-2}\psi_{\alpha\alpha}(\varepsilon)u_\alpha^\varepsilon + \varepsilon^{-2}\textstyle\sum_{\beta\neq\alpha} q_{\alpha\beta}(P_{\alpha\beta}(\varepsilon)u_\beta^\varepsilon - u_\alpha^\varepsilon) \\ u_\alpha^\varepsilon(0) &= f_\alpha \in B \qquad\qquad 1 \leqq \alpha \leqq N,\ \varepsilon > 0\,. \end{aligned}$$

Here B is a Banach space. For each $1 \leqq j,\ k \leqq N$, Ψ_{jk}, and Φ_{jk} and $\Psi_{jk}(u)$, are respectively generators of a strongly continuous group and semi-groups of bounded, linear operators on B which are mutually commutative; these operators satisfy $\Psi_{jk}(\varepsilon) = \varepsilon\Psi_{jk} + \varepsilon^2\Phi_{jk} + o(\varepsilon^2)$ as $\varepsilon \to 0$. For $1 \leqq j \neq k \leqq N$, $P_{jk}(u)$ is defined by $P_{jk}(u) = \exp(\Psi_{jk}(u))$, $u \geqq 0$. The numbers $q_{\alpha\beta}$, $1 \leqq \alpha,\ \beta \leqq N$, are real constants with $q_{\alpha\beta} \geqq 0$ for $1 \leqq \alpha \neq \beta \leqq N$, $\sum_{\beta=1}^N q_{\alpha\beta} = 0$, and with p_α, $1 \leqq \alpha \leqq N$, the numbers satisfying $\sum_{\alpha=1}^N p_\alpha q_{\alpha\beta} = 0$, $\sum_{\alpha=1}^N p_\alpha = 1$. If we assume also that $\prod_{j=1}^N \exp(tp_j\Psi_{jj}) \prod_{1\leq j\neq k\leq N} \exp(tp_j q_{jk}\Psi_{jk}) = I$, the identity operator, then it follows that $u(t) = \lim_{\varepsilon\to 0} u_\alpha^\varepsilon(t)$ exists, is independent of α, and is the unique B-valued solution of

$$(2)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= \{\textstyle\sum_{j=1}^N p_j\Phi_{jj} + \sum_{1\leq j\neq k\leq N} p_j q_{jk}\Phi_{jk}\}u \\ &\quad + (\tfrac{1}{2})\{\textstyle\sum_{1\leq j,k,m,n\leq N} d_{jk,mn}\Psi_{jk}\Psi_{mn}\}u \\ u(0) &= \textstyle\sum_{1\leq\alpha\leq N} p_\alpha f_\alpha \end{aligned}$$

where $(d_{jk,mn})$ is a certain $N^2 \times N^2$ nonnegative definite matrix.

Our Markov chain limit theorem, together with 'random evolution' techniques, is also used in Section 3 to give new proofs of a limit theorem of Fukushima–Hitsuda for a Markov process on N-lines [5]. Limit theorems for discrete-time, finite state Markov chains and MOF's being 'controlled' by these chains are given in Section 4. In this context the 'random evolution' approach is used to give a new interpretation to a limit theorem of Keilson–Wishart [12], [13] in Section 5.

In other papers the author proves limit theorems for discontinuous MOF's without the assumption of commutativity of the operators [16] [17]. Applications are given there to a model for the approximation, by certain jump Markov processes, to Brownian motion with inertia and to a model in storage theory. Discontinuous MOF's controlled by more general processes are also studied there. Limit theorems and applications for continuous MOF's with possibly non-commutative operators and more general controlling processes have appeared in several places [7], [14], [18], [20].

1. Throughout Sections 1–3 of this paper we use the following notation and assumptions. $Q = (q_{\alpha\beta})$, $1 \leqq \alpha,\ \beta \leqq N$, is a matrix satisfying the conditions that $q_{\alpha\beta} \geqq 0$ for $1 \leqq \alpha \neq \beta \leqq N$, and $\sum_{\beta=1}^N q_{\alpha\beta} = 0$. We assume Q has zero as a simple eigenvalue. Hence Q has a unique left eigenvector $p = (p_\alpha)$, $1 \leqq \alpha \leqq N$, with $pQ = 0$ and $\sum_{\alpha=1}^{N} p_\alpha = 1$. $\{X(t);\, t \geqq 0\}$ is a stationary Markov chain taking values in $\{1, 2, \cdots, N\}$, having generator Q, and having transition probabilities $\{p_{\alpha\beta}(t);\, t \geqq 0\}$, $1 \leqq \alpha, \beta \leqq N$. For $1 \leqq \alpha \neq \beta \leqq N$ and $n \geqq 0$, we define $N_{\alpha\beta}(u) =$ number of transitions from state α to state β during $(0, u]$ by the Markov chain $\{X(t);\, t \geqq 0\}$; $N_{\alpha\beta}(u) = \sum_{0 \leqq s \leqq u} I_{\{X(s-)=\alpha;\, X(s)=\beta\}}$. For each sample path $X(\cdot)$, we let $t_j^* =$ epoch of the jth transition, and $N(t) =$ number of transitions until time $t = \sum_{1 \leqq j \neq k \leqq N} N_{jk}(t)$. For $1 \leqq \alpha \leqq N$ and $u \geqq 0$, we define $\gamma_\alpha(u) =$ amount of time before time u during which the Markov chain $\{X(t);\, t \geqq 0\}$ is in state α; $\gamma_\alpha(u) =$ Lebesgue measure of $\{s;\, X(s) = \alpha$ and $0 \leqq s \leqq u\}$. $P_j\{\cdot\}$ denotes the probability measure defined on the sample paths $X(\cdot)$, given that $X(0) = j$; $E_j\{\cdot\}$ denotes integration with respect to $P_j\{\cdot\}$.

The numbers $\{\theta_{jk}\}$, $\{\rho_{jk}\}$, $1 \leqq j \neq k \leqq N$, and $\{v_j\}$, $\{w_j\}$, $1 \leqq j \leqq N$, λ_1, λ_2, and μ are complex numbers with λ_1 and λ_2 constant. $Q(\mu) = Q\{\mu;\, \lambda_1, \lambda_2;\, (v_j), (w_j), (\theta_{jk}), (\rho_{jk})\} = (q_{jk}(\mu))$, $1 \leqq j, k \leqq N$, denotes the $N \times N$ matrix which has entries given by

$$\begin{aligned} q_{jk}(\mu) &= q_{jj} + \lambda_1 v_j \mu + \lambda_2 w_j \mu^2 && \text{for } 1 \leqq j = k \leqq N \\ &= q_{jk} \exp(\lambda_1 \theta_{jk} \mu + \lambda_2 \rho_{jk} \mu^2) && \text{for } 1 \leqq j \neq k \leqq N. \end{aligned}$$

We define numbers b_{jk}, $1 \leqq j, k < \infty$, d_1 and d_2 by $A(\gamma, \mu) = \det(Q(\mu) - \gamma) = \sum_{j,k=1}^{\infty} b_{jk} \gamma^j \mu^k$, $d_1 = b_{01}/(-b_{10})$, and $d_2 = (b_{02} + b_{11} d_1 + b_{20} d_1^2)/(-b_{10})$. The numbers $\tau_j(\mu)$, $1 \leqq j \leqq N$, denote the eigenvalues of $Q(\mu)$. In the appendix we give algebraic formulas for the coefficients b_{10}, b_{20}, b_{01}, b_{02}, b_{11} and d_1 in terms of (p_j), (q_{jk}), (v_j), (w_j), (θ_{jk}), (ρ_{jk}), λ_1 and λ_2; we use these to identify parameters in the limit and perturbation theorems. In the appendix we also give results on the behavior of the eigenvalues $\tau_j(\mu)$, $1 \leqq j \leqq N$, for the matrix $Q(\mu)$, which we require in the proof of Theorem 2. We also use Theorem 1 due to Pinsky (see [23] Section 1.3). To state this theorem we introduce the following notation. We let $\mathbf{f} = (f_j)_{1 \leqq j \leqq N} \in \mathbf{B} = B_1 x \cdots x B_N$, where $B_j = B$, $1 \leqq j \leqq N$, is a Banach space with any norm such that $\|\mathbf{f}_\alpha\| \to 0$ as $\|f_{j,\alpha}\| \to 0$ for every $1 \leqq j \leqq N$. $\{T_j(t);\, t \geqq 0\}$, $1 \leqq j \leqq N$, are strongly continuous semi-groups of bounded, linear operators on B with generators A_j, $1 \leqq j \leqq N$; $\{S_{jk}\}$, $1 \leqq j \neq k \leqq N$, are bounded, linear operators defined on B. We define the 'random evolution' $\{M(t);\, t \geqq 0\}$ by

$$M(t) = T_{X(0)}(t_1^*) S_{X(0)X(t_1^*)} T_{X(t_1^*)}(t_2^* - t_1^*) \cdots S_{X(t^*_{N(t)-1})X(t^*_{N(t)})} T_{X(t^*_{N(t)})}(t - t^*_{N(t)})$$

and define the 'expectation semi-group' $\{\mathbf{T}(t);\, t \geqq 0\}$ associated with $M(t)$, on $\mathbf{B}$, by $(\mathbf{T}(t)\mathbf{f})_j = E_j\{M(t) f_{X(t)}\}$, $t \geqq 0$.

THEOREM 1. (i) $\{\mathbf{T}(t);\, t \geqq 0\}$ *is a strongly continuous semi-group of bounded, linear operators on* $\mathbf{B}$, *and*

(ii) $\mathbf{u}(t) = \mathbf{T}(t)\mathbf{f}$, $t \geqq 0$, *solves the Cauchy problem*

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t} &= A_\alpha u_\alpha + \sum_{\beta \neq \alpha} q_{\alpha\beta}(S_{\alpha\beta} u_\beta - u_\alpha) \\ \mathbf{u}(0) &= \mathbf{f} \in \textstyle\bigtimes_{\alpha=1}^{N} \mathscr{D}_{A_\alpha}. \end{aligned} \tag{3}$$

2. We give a direct proof of a limit theorem for the joint distribution of occupation-time and number-of-jumps random variables of the Markov chain, together with the Markov chain itself. We consider two scalings concurrently. To prove convergence results for expectation semi-groups of random evolutions, we use this theorem with $\lambda_1 = 0$, $\lambda_2 = i\lambda$ as a weak-of-law-of-large-numbers result (Theorem 3) and with $\lambda_1 = \lambda_2 = i\lambda$ as a type of central limit theorem (Theorem 4). In Theorem 5 we use this result with $\lambda_1 = i\lambda$, $\lambda_2 = -\lambda^2/2$ to prove the central limit theorem for Markov processes on N lines. This theorem can be proved in more general settings by using techniques of Hitsuda and Shimizu [10], or Pyke and Schaufele [24]. Of special interest in the direct approach are explicit formulas for limiting parameters and the application made of Theorem 1.

THEOREM 2.

$$\begin{aligned}
&\lim_{\varepsilon\to 0} E_\alpha\{\exp[\lambda_1(\textstyle\sum_{j=1}^N v_j \varepsilon \gamma_j'(t/\varepsilon^2) + \sum_{1\leqq j\neq k\leqq N} \theta_{jk}\varepsilon N'_{jk}(t/\varepsilon^2)) \\
&\qquad + \lambda_2(\textstyle\sum_{j=1}^N w_j \varepsilon^2\gamma_j(t/\varepsilon^2) + \sum_{1\leqq j\neq k\leqq N} \rho_{jk}\varepsilon^2 N_{jk}(t/\varepsilon^2))]; X(t/\varepsilon^2) = \beta\} \\
&\qquad = p_\beta \exp(\lambda_2 mt + (\sigma^2\lambda_1^2 t/2))
\end{aligned} \tag{4}$$

for $1 \leqq \alpha \leqq N$, *where*

$$\gamma_j'(t) = \gamma_j(t) - p_j t$$
$$N'_{jk}(t) = N_{jk}(t) - p_j q_{jk} t\,.$$

$$m = \textstyle\sum_{j=1}^N p_j w_j + \sum_{1\leqq j\neq k\leqq N} p_j q_{jk}\rho_{jk}\,. \tag{5}$$

$$\sigma^2 = \textstyle\sum_{1\leqq j,k,m,n\leqq N} d_{jk,mn}\eta_{jk}\eta_{mn} \tag{6}$$

with $\eta_{jk} = \theta_{jk}$ *for* $j \neq k$ *and* $\eta_{jk} = v_j$ *for* $j = k$ *and* $(d_{jk,mn})$, *an* $N^2 \times N^2$ *real-valued, nonnegative definite matrix.*

PROOF. From Theorem 1 we have the representation

$$\begin{aligned}
&E_\alpha\{\exp[\lambda_1(\textstyle\sum_{j=1}^N v_j \varepsilon \gamma_j'(t/\varepsilon^2) + \sum_{1\leqq j\neq k\leqq N} \theta_{jk}\varepsilon N'_{jk}(t/\varepsilon^2)) \\
&\qquad + \lambda_2(\textstyle\sum_{j=1}^N w_j \varepsilon^2\gamma_j(t/\varepsilon^2) + \sum_{1\leqq j\neq k\leqq N} \rho_{jk}\varepsilon^2 N_{jk}(t/\varepsilon^2))]; X(t/\varepsilon^2) = \beta\} \\
&\qquad = [\exp(tQ(\varepsilon)/\varepsilon^2)]_{\alpha\beta}\exp(-\lambda_1\omega t/\varepsilon)
\end{aligned}$$

where $\omega = \sum_{j=1}^N p_j v_j + \sum_{1\leqq j\neq k\leqq N} p_j q_{jk}\theta_{jk}$. We can write $Q(\varepsilon)/\varepsilon^2 = (P^\varepsilon)(J^\varepsilon)(P^\varepsilon)^{-1}$ where J^ε is the Jordan form of the matrix $Q(\varepsilon)/\varepsilon^2$, and let $\gamma_j(\varepsilon) =$ eigenvalue of $Q(\varepsilon)/\varepsilon^2$ found in the jth row of J^ε. It is known that if γ_j, $1 \leqq j \leqq N$, denote the eigenvalues of Q, then $\gamma_1 = 0$ and $\mathrm{Re}\,(\gamma_j) < 0$ for all $j \neq 1$ (see [21] page 104). From this fact, together with Lemma A.4, we have that $\lim_{\varepsilon\to 0}\mathrm{Re}\,(\gamma_j(\varepsilon)) = -\infty$ for $j \neq 1$. From Lemma A.5, we obtain $\varepsilon^2\gamma_1(\varepsilon) = d_1\varepsilon + d_2\varepsilon^2 + O(\varepsilon^3)$ as $\varepsilon \to 0$; we can write $\gamma_1(\varepsilon) = (\lambda_1\omega/\varepsilon) + (\lambda_2 m + (\sigma^2\lambda_1^2/2)) + O(\varepsilon)$ as $\varepsilon \to 0$, where $d_1 = \lambda_1\omega$ from Lemma A.2 and σ^2, defined by $\sigma^2 = -2(d_2 - \lambda_2 m)/(-\lambda_1)^2$, is independent of λ_1 and λ_2 and is non-negative.

Since J^ε is in Jordan form and $\gamma_1(\varepsilon)$ is a simple eigenvalue of $Q(\varepsilon)/\varepsilon^2$, we have $\{\exp(tJ^\varepsilon)\}_{1,1} = \exp(t\gamma_1(\varepsilon))$ and for $j \neq 1$ or $k \neq 1$, either $\{\exp(tJ^\varepsilon)\}_{j,k} = 0$ or $\{\exp(tJ^\varepsilon)\}_{j,k} = \{\exp(t\gamma_n(\varepsilon))\}\cdot\{\text{polynomial in } \gamma_n(\varepsilon)\}$. Thus $\lim_{\varepsilon\to 0}\{\exp(tJ^\varepsilon)\}_{j,k} \times \exp(-\lambda_1\omega t/\varepsilon) = \exp\{\lambda_2 mt + (\sigma^2\lambda_1^2 t/2)\}$ if $j = k = 1$ and $= 0$ if $j \neq 1$ or $k \neq 1$.

The first column of P^ε is a right eigenvector associated with the eigenvalue $\gamma_1(\varepsilon)$, and has limit as $\varepsilon \to 0$ given by a right null vector of Q, $q \cdot (1, \cdots, 1)$, for some scalar q. The first row of $(P^\varepsilon)^{-1}$ is a left eigenvector associated with the eigenvalue $\gamma_1(\varepsilon)$, and has limit as $\varepsilon \to 0$ given by a left null vector of Q, $r \cdot p$, for some scalar r and p satisfying $\sum_{\alpha=1}^N p_\alpha = 1$. It follows that $\lim_{\varepsilon\to 0} (P^\varepsilon)_{\alpha,1}(P^\varepsilon)^{-1}_{1,\beta} = qrp_\beta = p_\beta$.

Thus we have $\lim_{\varepsilon\to 0} \exp\{tQ(\varepsilon)/\varepsilon^2\}_{\alpha,\beta} \exp(-\lambda_1 \omega t/\varepsilon) = \lim_{\varepsilon\to 0} \sum_{1\le m,n\le N} (P^\varepsilon)_{\alpha,m} \times (\exp(tJ^\varepsilon))_{m,n}(P^\varepsilon)^{-1}_{n,\beta} \exp(-\lambda_1\omega t/\varepsilon) = p_\beta \exp\{\lambda_2 mt + (\sigma^2\lambda_1^2 t/2)\}$ and (4) is proved.

By setting $\lambda_1 = i\lambda$, with λ possibly complex valued, and $\lambda_2 = 0$ in (4), we have $\{\sum_{1\le j\ne k\le N} \theta_{jk} N'_{jk}(t) + \sum_{j=1}^N v_j \gamma_j'(t)\} t^{-\frac{1}{2}}$ converges in distribution as $t \to \infty$ to a Gaussian distributed random variable with mean zero and variance σ^2. The moment generating functions and all moments converge (see [1] page 164) and thus for any $\{\theta_{jk}\}_{1\le j\ne k\le N}$, $\{v_j\}_{1\le j\le N}$, it follows that

$$\begin{aligned}\sigma^2 = \lim_{t\to\infty} (1/t)\{&\textstyle\sum_{1\le j\ne k, m\ne n\le N} \theta_{jk}\theta_{mn} \operatorname{Cov}(N_{jk}(t), N_{mn}(t)) \\ &+ \textstyle\sum_{1\le j,m\le n} v_j v_m \operatorname{Cov}(\gamma_j(t), \gamma_m(t)) + \sum_{1\le j, m\ne n\le N} v_j \theta_{mn} \operatorname{Cov}(\gamma_j(t), N_{mn}(t)) \\ &+ \textstyle\sum_{1\le j\ne k, m\le N} \theta_{jk} v_m \operatorname{Cov}(N_{jk}(t), \gamma_m(t))\}\,.\end{aligned}$$

By correct choice of the parameters $\{\theta_{jk}\}$ and $\{v_j\}$, the following limits are seen to exist: $\lim_{t\to\infty}\{\operatorname{Cov}(N_{jk}(t), N_{mn}(t))\}/t$, $\lim_{t\to\infty}\{\operatorname{Cov}(\gamma_j(t), \gamma_m(t))\}/t$, and $\lim_{t\to\infty}\{\operatorname{Cov}(\gamma_j(t), N_{mn}(t))\}/t$. Thus (6) holds with

$$\begin{aligned} d_{jk,mn} &= \lim_{t\to\infty}\{\operatorname{Cov}(N_{jk}(t), N_{mn}(t))\}/t && \text{if } j\ne k\,, && m\ne n \\ &= \lim_{t\to\infty}\{\operatorname{Cov}(\gamma_j(t), \gamma_m(t))\}/t && \text{if } j = k\,, && m = n \\ &= \lim_{t\to\infty}\{\operatorname{Cov}(N_{jk}(t), \gamma_m(t))\}/t && \text{if } j\ne k\,, && m = n \\ &= \lim_{t\to\infty}\{\operatorname{Cov}(\gamma_j(t), N_{mn}(t))\}/t && \text{if } j = k\,, && m\ne n\,. \end{aligned} \tag{7}$$

$(d_{jk,mn})_{1\le j,k,m,n\le N} = (s_{\alpha,\beta})_{1\le\alpha,\beta\le N^2}$ is a limiting covariance matrix and thus is nonnegative definite. □

REMARK 1. From the proof of Theorem 2 and Lemma A.5, we have $\sigma^2 = -2r$. Hence from Lemmas A.1 and A.2, we see that the coefficients $(d_{jk,mn})_{1\le j,k,m,n\le N}$ of σ^2 are functions of the entries of the matrix Q.

One can check by computation that we can write σ^2 in the following form

$$\begin{aligned}\sigma^2 = (2/\textstyle\sum_{j=1}^N q'_{j,j})\{&\textstyle\sum_{j<k} q'_{jj,kk} v_j' v_k' + \sum_{j\ne k, k\ne m, m\ne n} q_{jk} q'_{jk,mm}(-1)^\nu v_m'\theta_{jk} \\ &+ \textstyle\sum' [q_{hj} q_{mk}(-1)^{h+j+\mu} q'_{hj,mk}]\theta_{hj}\theta_{mk} + (\frac{1}{2})\sum_{j\ne k} q_{jk} q'_{j,k}\theta_{jk}^2\}\end{aligned} \tag{8}$$

where $v_j' = v_j - \omega$, ω as in Theorem 2; and other notation is given in Lemma A.1.

REMARK 2. The integrals $\nu_{jk} = \int_0^\infty (p_{jk}(t) - p_k)\,dt$, $1 \le j, k \le N$, converge by exponential convergence of the integrands $p_{jk}(t) - p_k$ to zero (see [2] page 236). In [16] the author shows that the quantities ν_{jk}, $1 \le j, k \le N$, and p_α, $1 \le \alpha \le N$, together determine the Markov chain. With the aid of Lemmas A.6 and A.7 we have $\sigma^2 = 2p \cdot (w \cdot h)$ where $p = (p_j)_{1\le j\le N}$ and $w = (w_{jk})_{1\le j,k\le N}$

and $h = (h_j)_{1 \leq j \leq N}$ are defined respectively by $w_{jk} = v_j$ for $j = k$ and $= q_{jk}\theta_{jk}$ for $j \neq k$ and $h_j = \sum_{k=1}^N \nu_{jk} v_k + \sum_{1 \leq k \neq m \leq N} \nu_{jk} q_{km} \theta_{km}$.

The following two theorems are extensions of perturbation theorems of Hersh–Pinsky [8]. In the proofs we represent the solutions of the Cauchy problems as expectation semi-groups of discontinuous random evolutions and then prove limit relations from Theorem 1 for these expectations.

THEOREM 3. *We let B denote a Banach space and assume that the following hold:*

(i) *For each $1 \leqq j, k \leqq N$, $u \geqq 0$, $\Psi_{jk}(u)$ and Ψ_{jk} are operators respectively with domain $\mathscr{D}_{jk,u}$ and $\mathscr{D}_{jk}$ in B satisfying $\Psi_{jk}(\varepsilon)f = \varepsilon \Psi_{jk} f + O(\varepsilon)$ as $\varepsilon \to 0$ for f in $\mathscr{D} \subset \bigcap (\mathscr{D}_{jk,\varepsilon} \cap \mathscr{D}_{jk})$, where $\mathscr{D}$ is a dense subset of B. $\Psi_{jk}(u)$ and Ψ_{jk} generate respectively the strongly-continuous semi-groups $\{\exp(t\Psi_{jk}(u));\ t \geqq 0\}$ and $\{\exp(t\Psi_{jk});\ t \geqq 0\}$ of bounded, linear operators on B.*

(ii) *The operators $T_j^\varepsilon(t) = \exp(t\varepsilon^{-1}\Psi_{jj}(\varepsilon))$, $\varepsilon, t \geqq 0$, and $P_{jk}(u) = \exp(\Psi_{jk}(u))$, $u \geqq 0$, are all mutually commutative, and the families of operators $\exp(t\Psi_{jk})$ $t \geqq 0$, $1 \leqq j, k \leqq N$, are mutually commutative.*

(iii) *$Q = (q_{\alpha\beta})$, $1 \leqq \alpha, \beta \leqq N$, is an $N \times N$ matrix with $q_{\alpha\beta} \geqq 0$ for $\alpha \neq \beta$, with $\sum_{\beta=1}^N q_{\alpha\beta} = 0$, and with zero as a simple eigenvalue. $p = (p_\alpha)$, $1 \leqq \alpha \leqq N$, is the unique left eigenvector of Q satisfying $\sum_{\alpha=1}^N p_\alpha = 1$, $p_\alpha \geqq 0$.*

Suppose $(w_j^\varepsilon(t))$, $1 \leqq j \leqq N$, in B^N satisfies

$$\frac{\partial w_j^\varepsilon}{\partial t} = \varepsilon^{-1}\Psi_{jj}(\varepsilon) w_j^\varepsilon + \varepsilon^{-1} \textstyle\sum_{k;k\neq j} q_{jk}(P_{jk}(\varepsilon) w_k^\varepsilon - w_j^\varepsilon) \tag{9}$$

$$w_j^\varepsilon(0) = f_j \qquad\qquad 1 \leqq j \leqq N,\ \varepsilon > 0,\ t \geqq 0, f_j \text{ in } B.$$

Then $w^0(t) = \lim_{\varepsilon\to0} w_j^\varepsilon(t)$ exists, is independent of j and is the unique B-valued solution of

$$\frac{\partial w}{\partial t} = (\textstyle\sum_{j=1}^N p_j \Psi_{jj} + \sum_{1\leq j\neq k\leq N} p_j q_{jk} \Psi_{jk}) w \tag{10}$$

$$w(0) = \textstyle\sum_{j=1}^N p_j f_j$$

PROOF. From Theorem 1, together with assumptions (i) and (ii), we obtain that the solution $(w_\alpha^\varepsilon(t))$, $1 \leqq \alpha \leqq N$, $t \geqq 0$, of (9) has the form

$$\begin{aligned} w_\alpha^\varepsilon(t) &= E_\alpha\{T^\varepsilon_{X(0)}(\varepsilon t_1^*) P_{X(0)X(t_1^*)}(\varepsilon) T^\varepsilon_{X(t_1^*)}(\varepsilon(t_2^* - t_1^*)) \cdots \\ &\qquad T^\varepsilon_{X(t^*_{N(t/\varepsilon)})}(t - \varepsilon t^*_{N(t/\varepsilon)}) f_{X(t/\varepsilon)}\} \\ &= E_\alpha\{T_1^\varepsilon(\varepsilon\gamma_1(t/\varepsilon)) \cdots T_N^\varepsilon(\varepsilon\gamma_N(t/\varepsilon))(P_{1,2}(\varepsilon))^{N_{1,2}(t/\varepsilon)} \cdots \\ &\qquad (P_{N,N-1}(\varepsilon))^{N_{N,N-1}(t/\varepsilon)} f_{X(t/\varepsilon)}\} \\ &= E_\alpha[\textstyle\prod_{1\leq j\leq N} \exp\{\varepsilon\gamma_j(t/\varepsilon)\Psi_{jj} + \gamma_j(t/\varepsilon) o(\varepsilon)\} \\ &\qquad \times \textstyle\prod_{1\leq j\neq k\leq N} \exp\{\varepsilon N_{jk}(t/\varepsilon)\Psi_{jk} + N_{jk}(t/\varepsilon) o(\varepsilon)\} f_{X(t/\varepsilon)}]\,. \end{aligned}$$

We apply Lemma A.8 and Theorem 2 with $\lambda_1 = 0$ and $\lambda_2 = i\gamma$ to obtain

$$\begin{aligned} w^0(t) &= \lim_{\varepsilon\to0} \textstyle\sum_{\beta=1}^N \iint_{R^{N^2}} \prod_{1\leq j,k\leq N} \exp\{x_{jk}\Psi_{jk}\} f_\beta \, dH_\alpha^\varepsilon(t, (x_{mn})) \\ &= \textstyle\prod_{1\leq j\leq N} \exp\{tp_j\Psi_{jj}\} \prod_{1\leq j\neq k\leq N} \exp\{tp_j q_{jk}\Psi_{jk}\} \sum_{\beta=1}^N p_\beta f_\beta\,. \end{aligned}$$

Hence, by using that the $\exp(t\Psi_{jk})$ are commuting semi-groups, we obtain that $w^0(t)$ satisfies (10) (see Theorem 1 of [28]). By evaluation at $t = 0$ the initial conditions are satisfied; uniqueness follows by use of resolvents as in Theorem 1.3 of [3]. □

THEOREM 4. *We let B denote a Banach space and assume that the following hold:*

(i) *For each $1 \leqq j, k \leqq N$, $u \geqq 0$, $\Psi_{jk}(u)$, Ψ_{jk}, and Φ_{jk} are operators respectively with domain $\mathscr{D}_{jk,u}$, $\mathscr{D}^{(1)}_{jk}$, and $\mathscr{D}^{(2)}_{jk}$ in B satisfying $\Psi_{jk}(\varepsilon)f = \varepsilon\Psi_{jk}f + \varepsilon^2\Phi_{jk}f + o(\varepsilon^2)$ as $\varepsilon \to 0$ for f in $\mathscr{D} \subset \bigcap (\mathscr{D}^{(i)}_{jk} \cap \mathscr{D}_{jk,\varepsilon})$ where $\mathscr{D}$ is a dense subset of B. Ψ_{jk}, and Φ_{jk} and $\Psi_{jk}(u)$, are respectively generators of a strongly continuous group $\{\exp(t\Psi_{jk}\}$; $-\infty < t < \infty\}$ and semi-groups $\exp(t\Phi_{jk})$ and $\exp(t\Psi_{jk}(u))$, $t \geqq 0$ of bounded linear operators on B.*

(ii) *The operators $T_j^\varepsilon(t) = \exp(t\varepsilon^{-2}\Psi_{jj}(\varepsilon))$, $\varepsilon, t \geqq 0$, and $P_{jk}(u) = \exp(\Psi_{jk}(u))$, $u \geqq 0$, are all mutually commutative, and the families of operators $\exp(t\Psi_{jk})$ and $\exp(t\Phi_{jk})$, $t \geqq 0$, $1 \leqq j, k \leqq N$, are mutually commutative.*

(iii) *$Q = (q_{\alpha\beta})$ and $p = (p_\alpha)$ are given as in Theorem 3.*

(iv) *$\prod_{j=1}^N \exp(tp_j\Psi_{jj}) \prod_{1\leq j\neq k\leq N} \exp(tp_jq_{jk}\Psi_{jk}) = I$, the identity operator for $t \geqq 0$.*

Suppose $(u_j^\varepsilon(t))$, $1 \leqq j \leqq N$, in B^N satisfies

$$\text{(a)} \qquad \frac{\partial u_j^\varepsilon}{\partial t} = \varepsilon^{-2}\Psi_{jj}(\varepsilon)u_j^\varepsilon + \varepsilon^{-2} \textstyle\sum_{k\neq j} q_{jk}(P_{jk}(\varepsilon)u_k^\varepsilon - u_j^\varepsilon)$$

$$u_j^\varepsilon(0) = f_j \qquad\qquad 1 \leqq j \leqq N,\ \varepsilon > 0, f_j \text{ in } B\,.$$

Then $u^0(t) = \lim_{\varepsilon\to 0} u_j^\varepsilon(t)$ exists, is independent of j, and is the unique B-valued solution of

$$\text{(b)} \qquad \frac{\partial u}{\partial t} = V^{(2)}u + V^{(1)}u$$

$$u(0) = \textstyle\sum_{j=1}^N p_jf_j$$

where

$$V^{(2)} \equiv \textstyle\sum_{j=1}^N p_j\Phi_{jj} + \sum_{1\leq j\neq k\leq N} p_jq_{jk}\Phi_{jk} \tag{11}$$

$$V^{(1)} \equiv (\tfrac{1}{2}) \textstyle\sum_{1\leq j,k,m,n\leq N} d_{jk,mn}\Psi_{jk}\Psi_{mn} \tag{12}$$

with $(d_{jk,mn})$ the coefficients defined in (7).

REMARK. By assumptions (i) and (ii), the operator $V^{(2)} + V^{(1)}$ generates on B a strongly continuous semi-group, $\exp(t(V^{(2)} + V^{(1)}))$, $t \geqq 0$.

PROOF. From Theorem 1 and assumptions (i), (ii), and (iv), we have that the solution $(u_\alpha^\varepsilon(t))$, $1 \leqq \alpha \leqq N$, $t \geqq 0$, of (1) has the form

$$\begin{aligned} u_\alpha^\varepsilon(t) &= E_\alpha[T^\varepsilon_{X(0)}(\varepsilon^2t_1{}^*)P_{X(0)X(t_1{}^*)}(\varepsilon)\cdots T_{X(t^*_{N(t/\varepsilon^2)})}(t - \varepsilon^2t^*_{N(t/\varepsilon^2)})f_{X(t/\varepsilon^2)}] \\ &= E_\alpha[\textstyle\prod_{1\leq j\leq N} \exp\{\varepsilon\gamma_j{}'(t/\varepsilon^2)\Psi_{jj}\}\exp\{\varepsilon^2\gamma_j(t/\varepsilon^2)\Phi_{jj}\}\exp\{\gamma_j(t/\varepsilon^2)o(\varepsilon^2)\} \\ &\quad \times \textstyle\prod_{1\leq j\neq k\leq N} \exp\{\varepsilon N'_{jk}(t/\varepsilon^2)\Psi_{jk}\}\exp\{\varepsilon^2N_{jk}(t/\varepsilon^2)\Phi_{jk}\} \\ &\quad \times \exp\{N_{jk}(t/\varepsilon^2)o(\varepsilon^2)\}f_{X(t/\varepsilon^2)}]\,. \end{aligned}$$

From Lemma A.8 and Theorem 2 with $\lambda_1 = \lambda_2 = i\lambda$ we obtain

$$\begin{aligned} u^0(t) &= \lim_{\varepsilon\to 0} \textstyle\sum_{\beta=1}^N \int \cdots \int_{R^{2N^2}} \prod_{1\leq j,k\leq N} \exp\{x_{jk}\Phi_{jk}\} \\ &\quad \times \textstyle\prod_{1\leq j,k\leq N} \exp\{y_{jk}\Psi_{jk}\} f_\beta \, dH_\alpha^\varepsilon(t, \cdots, x_{jk}, \cdots, y_{jk}, \cdots, \beta) \\ &= \textstyle\prod_{1\leq j\leq N} \exp\{p_j t\Phi_{jj}\} \prod_{1\leq j\neq k\leq N} \exp\{p_j q_{jk} t\Phi_{jk}\} \\ &\quad \times \textstyle\int \cdots \int_{R^{N^2}} \prod_{1\leq j,k\leq N} \exp\{y_{jk}\Psi_{jk}\} \sum_{\beta=1}^N p_\beta f_\beta \, dF(t, (y_{jk})) . \end{aligned}$$

From this representation of $u^0(t)$ we have that $u^0(t)$ is independent of α. Here $F(t, \mathbf{z})$ is the distribution function of a Gaussian random variable with mean zero and covariance matrix $t(d_{jk,mn})$, $1 \leqq j, k, m, n \leqq N$. $F(t, \mathbf{z})$ is the fundamental solution with pole at $z = 0$ of the Cauchy problem for the parabolic equation

$$\frac{\partial F}{\partial t}(t, \mathbf{z}) = (\tfrac{1}{2}) \textstyle\sum_{1\leq j,k,m,n\leq N} d_{jk,mn} \dfrac{\partial^2 F}{\partial z_{jk}\partial z_{mn}}(t, \mathbf{z}) .$$

We use this and integration by parts as in Theorem 5 of [6], together with Theorem 1 of [28] to obtain that $u^0(t)$ satisfies $\partial u/\partial t = (V^{(2)} + V^{(1)})u$ where $V^{(2)}$ and $V^{(1)}$ are the two operators in (11) and (12) respectively.

The initial condition $u^0(0) = \sum_{j=1}^N p_j f_j$ is satisfied, since the Gaussian kernel at $t = 0$ is just the δ-function at the mean zero. Uniqueness follows by use of resolvents as in Theorem 1.3 of [3]. □

REMARK. Suppose we have $\{\prod_{jk}(u); u \geqq 0\}$, $1 \leqq j \neq k \leqq N$, a family of bounded, linear operators on B and $\prod_{jk}^{(1)}$ and $\prod_{jk}^{(2)}$ operators satisfying

$$\textstyle\prod_{jk}(\varepsilon)f = f + \varepsilon \prod_{jk}^{(1)} f + \varepsilon^2 \prod_{jk}^{(2)} f + o(\varepsilon^2)$$

as $\varepsilon \to 0$ for $f \in \mathscr{D} \subset \bigcap (\mathscr{D}(\prod_{jk}^{(1)}) \cap \mathscr{D}(\prod_{jk}(\varepsilon)))$ with $\mathscr{D}$ dense in B. Assume also that $\ln(\prod_{jk}(\varepsilon))$ can be defined (e.g., if $\|\prod_{jk}(\varepsilon) - I\| < 1$ as $\varepsilon \to 0$, this holds) and generates a semi-group on B. Then $\Psi_{jk} = \prod_{jk}^{(1)}$ and $\Phi_{jk} = \prod_{jk}^{(2)} - (\frac{1}{2})(\prod_{jk}^{(1)})^2$ and $P_{jk}(u) = \exp\{\ln(\prod_{jk}(u)\} = \exp\{\varepsilon \prod_{jk}^{(1)} + \varepsilon^2[\prod_{jk}^{(2)} - (\frac{1}{2})(\prod_{jk}^{(1)})^2] + o(\varepsilon^2)\}$. Theorem 4 holds with $V^{(2)} = \sum_{j=1}^N p_j\Phi_j + \sum_{1\leq j\neq k\leq N} p_j q_{jk}\{\prod_{jk}^{(2)} - (\frac{1}{2})(\prod_{jk}^{(1)})^2\}$.

3. In this section we apply Theorem 2 and some of the techniques used in the proof of Theorems 3 and 4 to give a new proof of a limit theorem for the Markov processes on N-lines studied by Fukushima and Hitsuda and others ([5], [14]). This theorem and these processes also arise as a special case of work of Schäl on Markov renewal processes with auxiliary paths [27]. For the facts used in this section pertaining to infinitely divisible processes, see Itô [11].

We let $\{X(t); t \geqq 0\}$, $\{t_n^*\}$, $n \leqq 1$, and $N(t)$ be respectively the Markov chain, jump times for this Markov Chain, and number of jumps up to time t for the Markov chain, as given in Section 1. B_1 denotes the space of complex-valued functions on the real numbers R which are bounded and measurable with respect to the Borel σ-algebra on R. We define $\{T_\alpha(t); t \geqq 0\}$, $1 \leqq \alpha \leqq N$, on B_1 by

$$T_\alpha(t)f(x) = \textstyle\int_R f(x + u)p_\alpha(t, du) \tag{13}$$

where $\{p_\alpha(t); t \geqq 0\}$ is the transition function of an infinitely divisible process with characteristic function

$$\int_R e^{i\lambda u} p_\alpha(t, du) = \exp(t\Psi_\alpha(\lambda))$$

where Ψ_α is a Lévy exponent:

$$\Psi_\alpha(\lambda) = im_\alpha \lambda - (\tfrac{1}{2})\sigma_\alpha{}^2\lambda^2 + \int_R \{e^{i\lambda x} - 1 - (i\lambda x/(1 + x^2))\} M_\alpha(dx) .$$

We define $\{\prod_{\alpha\beta}\}$, $1 \leqq \alpha \neq \beta \leqq N$, by

$$\prod_{\alpha\beta} f(x) = \int_R f(x + u) p_{\alpha\beta}(du) \qquad f \in B_1 \tag{14}$$

where $\{p_{\alpha\beta}(du)\}$, $1 \leqq \alpha \neq \beta \leqq N$, are probability measures on R. The random evolution $M(t)$, $t \geqq 0$, is given by

$$M(t) = T_{X(0)}(t_1{}^*) \prod_{X(0)X(t_1{}^*)} T_{X(t_1{}^*)}(t_2{}^* - t_1{}^*) \cdots T_{X(t^*_{N(t)})}(t - t^*_{N(t)}) . \tag{15}$$

Let $\{Y(t); t \geqq 0\}$ be the stochastic process with state space R satisfying

$$E\{e^{i\lambda Y(t)} | X(s), 0 \leqq s \leqq t\} = \{M(t)e^{i\lambda y}\}|_{y=0} \tag{16}$$

$\{(X(t), Y(t)); t \geqq 0\}$ is then a Markov process. $\{Y(t); t \geqq 0\}$ evolves in the following way:

(i) between jump times of $X(s)$, the process $Y(s)$ evolves like an infinitely divisible process with Lévy exponent $\Psi_{X(s)}$ on R; and

(ii) at jump times $t_n{}^*$ of $X(s)$, the $Y(s)$ process stops evolving like an infinitely divisible process with exponent $\Psi_{X(t_n{}^*-)}$; then it 'jumps' to a new position $y_n \in R$ according to the probability distribution $\operatorname{Prob}\{y_n \in A \mid X(t); t \leqq t_n{}^*\} = p_{X(t^*_{n-1}), X(t_n{}^*)}(Y(t_n{}^*-), A)$; and then starting at y_n, the $Y(s)$ process evolves like an infinitely divisible process with exponent $\Psi_{X(t_n{}^*)}$ for a length of time up to the next jump time t^*_{n+1} of the $X(s)$ process, and so on. For the Fukushima–Hitsuda formulation of this process, see [5].

We assume $\Psi_j(\lambda)$ and $p_{jk}^\wedge(\lambda) = \int_R e^{i\lambda u} p_{jk}(du)$ are both in C^3. Let $\{Z_j(t); t \geqq 0\}$ be an infinitely divisible process on R with Lévy exponent $\Psi_j(\lambda)$. Let $X_{\alpha,\beta}$ be a random variable on R with probability distribution $p_{\alpha\beta}(du)$ for $1 \leqq \alpha \neq \beta \leqq N$.

THEOREM 5. *As $t \to \infty$, $(Y(t) - mt)t^{-1/2}$ converges in distribution to a random variable which has a Gaussian distribution with mean zero and variance $\sigma^2 + \tau^2$, $N(0, \sigma^2 + \tau^2)$. Here the parameter $m = \sum_{j=1}^N p_j m_j + \sum_{1\leqq j\neq k\leqq N} p_j q_{jk} m_{jk}$ with $m_j = E\{Z_j(1)\}$ and $m_{jk} = E\{X_{j,k}\}$; and the parameters σ^2 and τ^2 have the representations*:

$$\sigma^2 = \sum_{1\leqq j,k,m,n\leqq N} d_{jk,mn} \eta_{jk} \eta_{mn} \tag{17}$$

with $\eta_{jk} = m_j$ if $j = k$ and $\eta_{jk} = m_{jk}$ if $j \neq k$; and

$$\tau^2 = \sum_{j=1}^N p_j v_j + \sum_{1\leqq j\neq k\leqq N} p_j q_{jk} v_{jk} \tag{18}$$

with $v_j = V(Z_j(1)) =$ variance of $Z_j(1)$ and $v_{jk} = V(X_{jk})$. The coefficients $(d_{jk,mn})$, $1 \leqq j, k, m, n \leqq N$, are those in the representations of σ^2 given in (6), (8), and Remark 2 after Theorem 2.

PROOF. We have from (13)–(16) that

$$
\begin{aligned}
&E\{\exp i\lambda[(Y(t) - mt)t^{-\frac{1}{2}}]\} \\
&\quad = E\{M(t)\exp(i\lambda t^{-\frac{1}{2}}y))|_{y=0}\exp(-i\lambda m t^{\frac{1}{2}})\} \\
&\quad = E\{\exp[\textstyle\sum_{j=1}^{N} \Psi_j(\lambda t^{-\frac{1}{2}})\gamma_j(t) + \sum_{1\leq j\neq k\leq N}(\ln p_{jk}^{\wedge}(\lambda t^{-\frac{1}{2}}))N_{jk}(t)]\exp(-i\lambda m t^{\frac{1}{2}})\}\,.
\end{aligned}
$$

We use $\Psi_j'(0) = im_j$, $\Psi_j''(0) = -v_j$, $p_{jk}^{\wedge\prime}(0) = im_{jk}$, and $p_{jk}^{\wedge\prime\prime}(0) - (p_{jk}^{\wedge\prime}(0))^2 = -v_{jk}$ in the expansions for $\Psi_j(\lambda)$ and $p_{jk}^{\wedge}(\lambda)$ to obtain that as $t \to \infty$

$$
\begin{aligned}
\Psi_j(\lambda t^{-\frac{1}{2}}) &= (i\lambda m_j t^{-\frac{1}{2}}) - (\lambda^2 v_j/2t) + O((\lambda t^{-\frac{1}{2}})^3) \\
\ln(p_{jk}^{\wedge}(\lambda t^{-\frac{1}{2}})) &= (i\lambda m_{jk} t^{-\frac{1}{2}}) - (\lambda^2 v_{jk}/2t) + O((\lambda t^{-\frac{1}{2}})^3)\,.
\end{aligned}
$$

By an application of Theorem 2 with $\lambda_1 = i\lambda$ and $\lambda_2 = -\lambda^2/2$, it follows that

$$
\begin{aligned}
&\lim_{t\to\infty} E[\exp i\lambda\{(Y(t) - mt)t^{-\frac{1}{2}}\}] \\
&\quad = \lim_{t\to\infty} E[\exp i\lambda\{\textstyle\sum_{j=1}^{N} m_j(\gamma_j'(t)t^{-\frac{1}{2}}) + \sum_{1\leq j\neq k\leq N} m_{jk}(N_{jk}'(t)t^{-\frac{1}{2}})\} \\
&\qquad \times \exp-(\lambda^2/2)\{\textstyle\sum_{j=1}^{N} v_j(\gamma_j(t)/t) + \sum_{1\leq j\neq k\leq N} v_{jk}(N_{jk}(t)/t)\}] \\
&\quad = \exp(-\lambda^2(\sigma^2 + \tau^2)/2)
\end{aligned}
$$

where σ^2 and τ^2 are given in (17) and (18) respectively. □

REMARK. These techniques can also be used to show that as $t \to \infty$, $Y(t)/t$ converges in distribution to the constant random variable $= m$.

4. In Sections 4 and 5 we use the following notation and assumptions. $P = (p_{\alpha\beta})$, $1 \leqq \alpha, \beta \leqq N$, is a matrix satisfying the conditions that $p_{\alpha\beta} \geqq 0$ for $1 \leqq \alpha, \beta \leqq N$ and $\sum_{\beta=1}^{n} p_{\alpha\beta} = 1$. We assume P has one as a simple eigenvalue. Hence P has a unique left eigenvector $p = (p_\alpha)$, $1 \leqq \alpha \leqq N$, with $p \cdot P = p$ and $\sum_{\alpha=1}^{n} p_\alpha = 1$. Let $Z^+ = \{1, 2, \cdots\}$. $\{X(n); n \in Z^+ \cup \{0\}\}$ is a stationary Markov chain taking values in $\{1, 2, \cdots, N\}$ and having transition matrix P. For $1 \leqq \alpha, \beta \leqq N$ and $u \in Z^+$, we define $N_{\alpha\beta}(u)$ = number of transitions from state α to state β before and including time u in $\{X(n); n \in Z^+ \cup \{0\}\}$; $N_{\alpha\beta}(u) = \sum_{n=0}^{u-1} I_{\{X(n)=\alpha, X(n+1)=\beta\}}$. For $1 \leqq \alpha \leqq N$ and $u \in Z^+$, we define $\gamma_\alpha(u)$ = amount of time before and including time u during which $\{X(n); n \in Z^+\}$ is in state α;

$$\gamma_\alpha(u) = \textstyle\sum_{n=1}^{u} I_{\{X(n)=\alpha\}} = \sum_{a=1}^{N} N_{a\alpha}(u)\,.$$

We suppose B and $\mathbf{B}$ are Banach spaces as given in Section 1 and let $\{S_{jk}\}$, $1 \leqq j, k \leqq N$, be bounded, linear operators defined on B. We define the 'random evolution' $\{M(n); n \in Z^+\}$ by

$$M(n) = S_{X(0)X(1)} S_{X(1)X(2)} \cdots S_{X(n-1)X(n)} \tag{19}$$

and the 'expectation semi-group' $\{\mathbf{T}(n); n \in Z^+\}$, associated with $M(n)$, on $\mathbf{B}$ by $(\mathbf{T}(n)\mathbf{f})_j = E_j\{M(n)f_{X(n)}\}$, $n \in Z^+$. With this notation the following theorem becomes immediate from material in [23]. (We let $M(0) = I$ and $\mathbf{T}(0) = \mathbf{I}$).

THEOREM 6. (i) *$\{\mathbf{T}(n); n \in Z^+ \cup \{0\}\}$ is a discrete semi-group of bounded, linear operators on* $\mathbf{B}$, *and*

(ii) $\mathbf{u}(n) = \mathbf{T}(n)\mathbf{f}$, $n \in Z^+ \cup \{0\}$ *solves the initial value problem*

$$u_j(n+1) = \sum_{k=1}^{N} p_{jk} S_{jk} u_k(n) \qquad (20)$$
$$\mathbf{u}(0) = \mathbf{f} \in \mathbf{B} .$$

We use the following theorem in proving limit results for the discrete-time expectation semi-groups in Theorems 8, 9, and 10.

THEOREM 7.

$$\lim_{k\to\infty} E_a\{\exp[\lambda_1 \textstyle\sum_{1\leq\alpha,\beta\leq N} \theta_{\alpha\beta}(N'_{\alpha\beta}(k)/k^{\frac{1}{2}}) + \lambda_2 \sum_{1\leq\alpha,\beta\leq N} v_{\alpha\beta}(N_{\alpha\beta}(k)/k)]X(k) = b\}$$
$$= \exp\{(\lambda_1^2/2)(\sigma^2 - m_\theta^2) + \lambda_2 m_v\} p_b$$

where

$$N'_{\alpha\beta}(k) = N_{\alpha\beta}(k) - p_\alpha p_{\alpha\beta} k \qquad (21)$$

$$m_\theta = \textstyle\sum_{\alpha,\beta} p_{\alpha\beta}\theta_{\alpha\beta} \qquad (22)$$

$$m_v = \textstyle\sum_{\alpha,\beta} p_\alpha p_{\alpha\beta} v_{\alpha\beta} \qquad (23)$$

$$\sigma^2 - m_\theta^2 = \textstyle\sum_{1\leq\alpha,\beta,\gamma,\delta\leq N} c_{\alpha\beta,\gamma\delta}\theta_{\alpha\beta}\theta_{\gamma\delta} \qquad (24)$$

with $(c_{\alpha\beta,\gamma\delta})$ *an* $N^2 \times N^2$ *real-valued, nonnegative definite matrix.*

PROOF. The proof follows along the lines of Theorem 2. Using Theorem 6, we see that it suffices to show

$$\lim_{k\to\infty} \exp(-\lambda_1 m_\theta k^{\frac{1}{2}})(p^{(k)}(k^{-\frac{1}{2}}))_{a,b} = \exp((\lambda_1^2/2)(\sigma^2 - m_\theta^2) + \lambda_2 m_v) p_b$$

where for $1 \leqq \alpha, \beta \leqq N$,

$$(p(\mu))_{\alpha,\beta} = p_{\alpha\beta}\exp\{\lambda_1\mu\theta_{\alpha\beta} + \lambda_2\mu^2 v_{\alpha\beta}\} .$$

We write $p(\mu) = T^{-1}(\mu)J(\mu)T(\mu)$ where $J(\mu)$ is the Jordan form of $p(\mu)$ and let $(\gamma_j(\mu))_{1\leq j\leq N}$ be the eigenvalues of $p(\mu)$. Using eigenvalue expansion arguments similar to those used in Theorem 2, we obtain

$$\begin{aligned}\lim_{k\to\infty} \exp(-\lambda_1 m_\theta k^{\frac{1}{2}})J^{(k)}(k^{-\frac{1}{2}}) &= \lim_{k\to\infty} \exp(-\lambda_1 m_\theta k^{\frac{1}{2}})J_{1,1}^{(k)}(k^{-\frac{1}{2}}) \\ &= \lim_{k\to\infty} \exp(-\lambda_1 m_\theta k^{\frac{1}{2}})\gamma_1^{k}(k^{-\frac{1}{2}}) \\ &= \exp((\gamma_1''(0) - (\gamma_1'(0))^2)/2) .\end{aligned}$$

Finally we use that $\sigma^2 \equiv -2[(\gamma_1''(0)/2) - \lambda_2 m_v]/-\lambda_1^2$ is non-positive and real-valued, that $\gamma_1'(0)k = \lambda_1 m_\theta$, and that $\lim_{k\to\infty} T_{a,1}^{-1}(k^{-\frac{1}{2}})T_{1,b}(k^{-\frac{1}{2}}) = p_b$ to obtain

$$\lim_{k\to\infty} \exp(-\lambda_1 m_\theta k^{\frac{1}{2}})(p^{(k)}(k^{-\frac{1}{2}}))_{a,b} = \exp\{(\lambda_1^2/2)(\sigma^2 - m_\theta^2) + \lambda_2 m_v\} p_b .$$

Since this convergence holds for λ_1 complex, we can represent $\sigma^2 - m_\theta^2$ as in (24) with $c_{\alpha\beta,\gamma\delta}$, $1 \leqq \alpha, \beta, \gamma, \delta \leqq N$, defined by

$$c_{\alpha\beta,\gamma\delta} = \lim_{k\to\infty} \{(1/k)\operatorname{Cov}(N_{\alpha\beta}(k), N_{\gamma\delta}(k))\} \qquad (25)$$

and obtain that $\sigma^2 - m_\theta^2$ is an $N^2 \times N^2$ nonnegative definite matrix. □

REMARK 3. From results analogous to Lemmas A.1 and A.2, we can conclude that the coefficients $c_{\alpha\beta,\gamma\delta}$ are functions of the entries of matrix P and that σ^2 has the representation

$$(26)\qquad \sigma^2 = (2/\textstyle\sum_{j=1}^N p'_{j,j})\{\frac{1}{2}(\sum_{j,k} p_{jk}p'_{j,k}\theta_{jk}^2 + \sum_{h\neq m,j\neq k}(-1)^{h+j+\mu}p_{hj}p_{mk}\theta_{hj}\theta_{mk}) - m_\theta(\sum_{j\neq m,k\neq m} p_{jk}p'_{jk,mm}(-1)^\nu\theta_{jk}) + \frac{1}{2}m_\theta^2(\sum_{j\neq k} p'_{jj,kk})\}$$

where notation is analogous to that of Lemma A.1, with matrix Q replaced by matrix P.

REMARK 4. The parameters $(c_{\alpha\beta,\gamma\delta})$ and $\sigma^2 - m_\theta^2$ have been characterized in terms of (p_α), $(p_{\alpha\beta})$, and $(z_{\alpha\beta})$, the fundamental matrix of the Markov chain, by Kemeney–Snell (see [15] page 145). Another useful representation of $\sigma^2 - m_\theta^2$ in terms of (p_α), $(p_{\alpha\beta})$, and $s_{\alpha\beta} = \sum_{n=1}^\infty (p_{\alpha\beta}^{(n)} - p_\beta)$ is given in Romanovsky (see [26] page 194). (See also Frechet [4], and Kielson and Wishart [12], [13]).

The following two theorems are discrete-time analogues to perturbation Theorems 3 and 4. The techniques of proof here are analogous to those used in the previous theorems. From Theorem 6 the solutions have 'expectation semi-group' representations. Convergence of the expectations follows from Theorem 7 and Lemma A.8.

THEOREM 8. *We let B denote a Banach space and assume that the following hold:*

(i) *For each $1 \leqq j, k \leqq N$, $u \geqq 0$, $\Psi_{jk}(u)$ and Ψ_{jk} are operators respectively with domain $\mathscr{D}_{jk,u}$ and $\mathscr{D}_{jk}$ in B satisfying $\Psi_{jk}(\varepsilon)f = \varepsilon\Psi_{jk}f + o(\varepsilon)$ as $\varepsilon \to 0$ for $f \in \mathscr{D} \subset \bigcap(\mathscr{D}_{jk} \cap \mathscr{D}_{jk,\varepsilon})$, where $\mathscr{D}$ is a dense subset of B. Ψ_{jk} and $\Psi_{jk}(u)$ are respectively generators of strongly continuous semi-group* $\exp(t\Psi_{jk})$ *and* $\exp(t\Psi_{jk}(u))$, $t \geqq 0$, *of bounded, linear operators on B.*

(ii) *For $1 \leqq j, k \leqq N$, $P_{jk}(u) = \exp(\Psi_{jk}(u))$ are mutually commutative operators on B; the operators $\exp(t\Psi_{jk})$, $1 \leqq j, k \leqq N$, are also mutually commutative.*

(iii) *$P = (p_{\alpha\beta})$, $1 \leqq \alpha, \beta \leqq N$, is an $N \times N$ matrix with $p_{\alpha\beta} \geqq 0$, $\sum_{\beta=1}^N p_{\alpha\beta} = 1$, and with one as a simple eigenvalue. $p = (p_\alpha)$, $1 \leqq \alpha \leqq N$, is the unique left eigenvector of P satisfying $\sum_{\alpha=1}^N p_\alpha = 1$, $p_\alpha \geqq 0$.*

Suppose $\mathbf{w}(n) = (w_j^t(n))$, $1 \leqq j \leqq N$, $n \in Z^+$, $t \geqq 0$, in B^N satisfies

$$(27)\qquad \begin{aligned} w_j(n+1) &= \textstyle\sum_{k=1}^N p_{jk}P_{jk}(t/n)w_k(n) \\ w_j(0) &= f_j \qquad\qquad 1 \leqq j \leqq N, f_j \in B\,. \end{aligned}$$

Then $w(t) = \lim_{n\to\infty} w_j^t(n)$ exists, is independent of j, and is the unique B-valued solution of

$$(28)\qquad \begin{aligned} \frac{\partial w}{\partial t} &= (\textstyle\sum_{1\leq j,k\leq N} p_j p_{jk}\Psi_{jk})w \\ w(0) &= \textstyle\sum_{j=1}^N p_j f_j\,. \end{aligned}$$

THEOREM 9. *We let B denote a Banach space and assume that the following hold:*

(i) *For each $1 \leqq j, k \leqq N$, $u \geqq 0$, $\Psi_{jk}(u)$, Ψ_{jk}, and Φ_{jk} are operators respectively with domain $\mathscr{D}_{jk,u}$, $\mathscr{D}_{jk}^{(1)}$, and $\mathscr{D}_{jk}^{(2)}$ in B satisfying $\Psi_{jk}(\varepsilon)f = \varepsilon\Psi_{jk}f + \varepsilon^2\Phi_{jk}f + o(\varepsilon^2)$ as $\varepsilon \to 0$ for $f \in \mathscr{D} \subset \bigcap(\mathscr{D}_{jk}^{(i)} \cap \mathscr{D}_{jk,\varepsilon})$, where $\mathscr{D}$ is a dense subset* *of* B. Ψ_{jk}, *and* Φ_{jk} *and* $\Psi_{jk}(u)$, *are respectively generators of a strongly continuous group* $\{\exp(t\Psi_{jk}); -\infty < t < \infty\}$ *and semi-groups* $\exp(t\Phi_{jk})$ *and* $\exp(t\Psi_{jk}(u))$, $t \geqq 0$ *of bounded, linear operators on* B.

(ii) *For* $1 \leqq j, k \leqq N$, $P_{jk}(u) = \exp(\Psi_{jk}(u))$ *are mutually commutative operators on* B; *the operators* $\exp(t\Psi_{jk})$ *and* $\exp(t\Phi_{jk})$ *are also mutually commutative.*

(iii) $P = (p_{\alpha\beta})$, $1 \leqq \alpha, \beta \leqq N$, *is an* $N \times N$ *matrix with* $p_{\alpha\beta} \geqq 0$, $\sum_{\beta=1}^N p_{\alpha\beta} = 1$, *and with one as a simple eigenvalue.* $p = (p_\alpha)$, $1 \leqq \alpha \leqq N$, *is the unique left eigenvector of* P *satisfying* $\sum_{\alpha=1}^N p_\alpha = 1$, $p_\alpha \geqq 0$. $(c_{jk,mn})$, $1 \leqq j, k, m, n \leqq N$, *is given in* (25).

(iv) $\prod_{1 \leqq j,k \leqq N} \exp(tp_j p_{jk}\Psi_{jk}) = I$, *the identity operator, for* $t \geqq 0$.

Suppose $\mathbf{u}(n) = (u_j^t(n))$, $1 \leqq j \leqq N$, $n \in Z^+$, $t \geqq 0$, *in* B^N *satisfies*

$$u_j(n+1) = \sum_{k=1}^N p_{jk} P_{jk}((t/n)^{\frac{1}{2}}) u_k(n) \qquad \tag{29}$$
$$u_j(0) = f_j \qquad 1 \leqq j \leqq N, f_j \in B\,.$$

Then $u(t) = \lim_{n\to\infty} u_j^t(n)$ *exists, is independent of* j, *and is the unique* B*-valued solution of*

$$\frac{\partial u}{\partial t} = V^{(2)}u + V^{(1)}u \tag{30}$$
$$u(0) = \sum_{j=1}^N p_j f_j$$

where

$$V^{(2)} \equiv \sum_{1\leqq j,k\leqq N} p_j p_{jk} \Phi_{jk}$$
$$V^{(1)} \equiv \tfrac{1}{2} \sum_{1\leqq j,k,m,m\leqq N} c_{jk,mn} \Psi_{jk} \Psi_{mn}\,.$$

5. We use Theorem 7 and discrete-time 'random evolution' techniques as found in the proof of Theorems 8 and 9 to obtain a new proof of the Central Limit Theorem of Keilson–Wishart [12] for a discrete time Markov process defined on N-lines. We deal with the temporally homogeneous Markov process $\{(X(n), Y(n)), n \in Z^+ \cup \{0\}\}$ with state space $E \times R$, where $E = \{1, 2, \cdots, N\}$ and $R =$ real numbers. The vector of probability measures $\mathbf{F}(y, n) = (F_k(y, n))_{1\leqq k\leqq N}$ has entries $F_k(y, n) = \operatorname{Prob}\{X(n) = k, Y(n) \leqq y\}$ which satisfy $F_k(y, n+1) = \sum_{j=1}^N \int_{-\infty}^{\infty} F_j(y - z, n)\, dB_{jk}(z)$. Here $\mathbf{B} = (B_{jk}(z))$, $1 \leqq j, k \leqq N$, is the matrix increment distribution whose elements $B_{jk}(z)$ $1 \leqq j, k \leqq N$ are nonnegative, monotonically increasing functions satisfying $\sum_{k=1}^N B_{jk}(+\infty) = 1$, $1 \leqq j \leqq N$; $\int_{-\infty}^{\infty} z^2\, dB_{jk}(z) < \infty$, $1 \leqq j, k \leqq N$; and $\mathbf{B}(+\infty)$ is irreducible and aperiodic. For $1 \leqq \alpha, \beta \leqq N$, we denote $p_{\alpha\beta} = B_{\alpha\beta}(+\infty)$ and $b_{\alpha\beta}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda z}\, dB_{\alpha\beta}(z)$ and let $\xi_{\alpha\beta}$ be the random variable giving the increment change in going from state α to state β. The relation of $\xi_{\alpha\beta}$ to $B_{\alpha\beta}(z)$ is given by $B_{\alpha\beta}(dy) = \Pr(\alpha \to \beta; \xi_{\alpha\beta} \in dy)$ and $\Psi_{\alpha\beta}(\lambda) \equiv E[\exp(i\lambda\xi_{\alpha\beta})] = b_{\alpha\beta}(\lambda)/p_{\alpha\beta}$.

We define operators $\{\prod_{\alpha\beta}\}$ $1 \leqq \alpha, \beta \leqq N$ on $B_1 =$ space of complex-valued, bounded, measurable functions on R by $\prod_{\alpha\beta} f(x) = \int_R f(x + z) P(\xi_{\alpha\beta} \in dz)$ and define a 'random evolution' $\{M(n), n \in Z^+\}$ by

$$M(n) = \prod\nolimits_{X(0)X(1)} \prod\nolimits_{X(1)X(2)} \cdots \prod\nolimits_{X(n-1)X(n)}\,.$$

Then the process $\{Y(n)\}$ satisfies

$$E\{\exp(i\lambda Y(n)) \mid X(m), 0 \leqq m \leqq n\} = \{M(n)e^{i\lambda y}\}|_{y=0} .$$

THEOREM 10. *$(Y(k) - mk)/k^{\frac{1}{2}}$ converges in distribution to $N(0, s_1^2 + s_2^2)$ as $k \to \infty$. Here $N(0, s_1^2 + s_2^2)$ denotes a Gaussian-distributed random variable with mean zero and variance $s_1^2 + s_2^2$,*

$$\begin{aligned} m &= \textstyle\sum_{1\leqq\alpha,\beta\leqq N} p_\alpha p_{\alpha\beta} E[\xi_{\alpha\beta}] \\ s_1^2 &= \textstyle\sum_{\alpha,\beta} p_\alpha p_{\alpha\beta} \operatorname{Var}(\xi_{\alpha\beta}) \\ s_2^2 &= \textstyle\sum_{\alpha,\beta,\gamma,\delta} c_{\alpha\beta,\gamma\delta} E(\xi_{\alpha\beta}) E(\xi_{\gamma\delta}) \end{aligned}$$

where $c_{\alpha\beta,\gamma\delta}$, $1 \leqq \alpha, \beta, \gamma, \delta \leqq N$, are the parameters defined in (25).

PROOF. Denoting $m_{\alpha\beta}(1) = E[\xi_{\alpha\beta}]$ and $m_{\alpha\beta}(2) = E[\xi^2_{\alpha\beta}]$, we have

$$\begin{aligned} &\lim_{k\to\infty} E[\exp i\lambda\{Y(k) - mk)/k^{\frac{1}{2}}\}] \\ &\quad = \lim_{k\to\infty} E[\textstyle\prod_{1\leqq\alpha,\beta\leqq N} (\Psi_{\alpha b}(\lambda k^{-\frac{1}{2}}))^{N_{\alpha\beta}(k)}] \exp(-i\lambda m k^{\frac{1}{2}}) \\ &\quad = \lim_{k\to\infty} E[\exp \textstyle\sum_{1\leqq\alpha,\beta\leqq N} \ln (\Psi_{\alpha\beta}(\lambda k^{-\frac{1}{2}})) N_{\alpha\beta}(k)] \exp(-i\lambda m k^{\frac{1}{2}}) \\ &\quad = \lim_{k\to\infty} E\{\exp \textstyle\sum_{1\leqq\alpha,\beta\leqq N} \{i\lambda k^{-\frac{1}{2}} m_{\alpha\beta}(1) - \frac{1}{2}\lambda^2 k^{-1}(m_{\alpha\beta}(2) - (m_{\alpha\beta}(1))^2) \\ &\qquad\qquad + O(|\lambda k^{-\frac{1}{2}}|^3)\} N_{\alpha\beta}(k)\} \\ &\quad = \lim_{k\to\infty} E\{\exp i\lambda[\textstyle\sum_{\alpha,\beta} m_{\alpha\beta}(1)(N'_{\alpha\beta}(k)/k^{\frac{1}{2}})] \\ &\qquad\qquad \times \exp -\frac{1}{2}\lambda^2[\textstyle\sum_{\alpha,\beta} (m_{\alpha\beta}(2) - (m_{\alpha\beta}(1))^2)(N_{\alpha\beta}(k)/k)]\} \\ &\quad = \exp(-\tfrac{1}{2}(s_1^2 + s_2^2)) . \end{aligned}$$

In the last equality we have used Theorem 7 with $\lambda_1 = i\lambda$, $\lambda_2 = -\frac{1}{2}\lambda^2$, $\theta_{\alpha\beta} = m_{\alpha\beta}(1)$, and $v_{\alpha\beta} = m_{\alpha\beta}(2) - (m_{\alpha\beta}(1))^2$. □

REMARK. By using the representation for $s_2^2 = \sigma^2 - m_\theta^2$ referred to in Remark 4 and $p_{\alpha\beta} m_{\alpha\beta}(1) = \int z\, dB_{\alpha\beta}(z)$ and $p_{\alpha\beta} m_{\alpha\beta}(2) = \int z^2\, dB_{\alpha\beta}(z)$, we obtain the form of $s_1^2 + s_2^2$ given in Keilson and Wishart [13].

APPENDIX

We use the notation given at the beginning of Section 1.

LEMMA A.1. *We have the following representations for the coefficients b_{10}, b_{20}, b_{01}, b_{02} and b_{11} in $A(\gamma, \mu)$:*

$$\begin{aligned} b_{10} &= (-1) \textstyle\sum_{j=1}^N q'_{j,j} \\ b_{20} &= \textstyle\sum_{1\leqq j<k\leqq N} q'_{jj,kk} \\ b_{11} &= (-\lambda_1)\{\textstyle\sum_{1\leqq j\neq k\leqq N} q'_{jj,kk} v_j + \sum_{1\leqq j\neq k, k\neq m, m\neq j\leqq N} (-1)^\nu q'_{jk,mm} q_{jk} \theta_{jk}\} \\ b_{01} &= \lambda_1\{\textstyle\sum_{j=1}^N q'_{j,j} v_j + \sum_{1\leqq j\neq k\leqq N} q'_{j,k} q_{jk} \theta_{jk}\} \\ b_{02} &= \lambda_2\{\textstyle\sum_{j=1}^N q'_{j,j} w_j + \sum_{1\leqq j\neq k\leqq N} q'_{j,k} q_{jk} \rho_{jk}\} \\ &\quad + \lambda_1^2\{(\tfrac{1}{2}) \textstyle\sum_{1\leqq j\neq k\leqq N} q'_{j,k} q_{jk} \theta^2_{jk} + \sum_{1\leqq j<k\leqq N} q'_{jj,kk} v_j v_k \\ &\quad + \textstyle\sum_{1\leqq j\neq k, k\neq m, m\neq j\leqq N} (-1)^\nu q'_{jk,mm} q_{jk} v_m \theta_{jk} \\ &\quad + \textstyle\sum' (-1)^{h+j+\mu} q'_{hj,mk} q_{hj} q_{mk} \theta_{hj} \theta_{mk}\} \end{aligned}$$

where

$\sum' =$ *summation over* $1 \leqq h, m, j, k \leqq N$ *satisfying* $h \neq j$, $k \neq m$, $h \neq m$, *and* $j \neq k$, $h + j < m + k$, *and* $h + j = m + k$ *is taken just once*;

$q'_{j,k} =$ *determinant of the matrix obtained from deleting the* jth *row and the* kth *column from the matrix* Q;

$q'_{\alpha\beta,\gamma\delta} =$ *determinant of the matrix obtained from deleting the rows numbered* α *and* γ *and the columns numbered* β *and* δ *from the matrix* Q;

$\nu = j + k$ *if either* $m < j$ *and* $m < k$ *or* $m > j$ *and* $m > k$
$= j + k - 1$ *if either* $m < j$ *and* $m > k$ *or* $m > j$ *and* $m < k$;

$\mu = k + m$ *if either* $h < m$ *and* $j < k$ *or* $h > m$ *and* $j > k$
$= k + m - 1$ *if either* $h < m$ *and* $j > k$ *or* $h > m$ *and* $j < k$.

PROOF. We use partial differentiation of $A(\gamma, \mu)$. For example,

$$b_{02} = \text{coefficient of } \mu^2 \text{ in } A(\gamma, \mu) = \left(\frac{1}{2}\right) \frac{\partial^2}{\partial \mu^2} A(\gamma, \mu)\Big|_{\gamma=0,\mu=0} . \qquad \square$$

LEMMA A.2. *The following representation for the coefficient* d_1 *holds*:

$$d_1 = \lambda_1\{\textstyle\sum_{j=1}^N p_j v_j + \sum_{1\leqq j\neq k\leqq N} p_j q_{jk} \theta_{jk}\} . \tag{A.1}$$

PROOF. It is known that $p_\alpha = q'_{\alpha,\beta}/\sum_{j=1}^N q'_{j,j}$, independent of β, for $1 \leqq \alpha \leqq N$ (see [7]). Then by Lemma A.1, it follows that $d_1 = b_{01}/(-b_{10}) = \lambda_1\{\sum_{j=1}^N q'_{j,j} v_j + \sum_{1\leqq j\neq k\leqq N} q'_{j,k} q_{jk} \theta_{jk}\}/\sum_{j=1}^N q'_{j,j}$ and that (A.1) holds. $\square$

LEMMA A.3. *Given* $\{f_n\}_{n\geqq 1}$ *analytic functions. Let* $\varepsilon > 0$. *Suppose* $f_n \to f$ *uniformly on* $\{z; |a - z| < r\}$ *and that* f *has* μ *zeros in* $\{z; |a - z| < r - \varepsilon\}$ *and no zeros on* $|a - z| = r - \varepsilon$. *Then there exists* N *such that for each* $n > N$, f_n *has* μ *zeros in* $\{z; |a - z| < r - \varepsilon\}$.

PROOF. Since $f_n \to f$ uniformly on $\{z; |a - z| < r\}$, we have that $f_n' \to f'$ uniformly on $\{z; |a - z| \leqq r - \varepsilon\}$, and thus $f_n'/f \to f'/f$ uniformly on $\{z; |a - z| = r - \varepsilon\}$. It follows that

$$(1/2\pi i) \int_{\{z;|a-z|=r-\varepsilon\}} f_n'/f_n \, dz \to (1/2\pi i) \int_{\{z;|a-z|=r-\varepsilon\}} f'/f \, dz ,$$

and by [9] page 252, we conclude that there exists N such that the number of zeros of f_n in $\{z; |a - z| < r - \varepsilon\}$ equals the number of zeros of f in $\{z; |a - z| < r - \varepsilon\}$, if $n > N$. $\square$

LEMMA A.4. *Denote the distinct eigenvalues of* Q *by* $\gamma_1 = 0, \gamma_2, \cdots, \gamma_m$ *with* γ_j *of algebraic multiplicity* c_j. *Denote the eigenvalues of* $Q(\mu)$ *by* $\tau_k(\mu)$, $1 \leqq k \leqq N$. *Let* N_j *be a neighborhood containing* γ_j, $1 \leqq j \leqq m$, *with* $N_j \cap N_i = \emptyset$ *for* $j \neq i$.
Then there is an $\varepsilon > 0$ *such that if* $|\mu| < \varepsilon$, *then* c_j *of the eigenvalues* $\{\tau_k(\mu)\}$, $1 \leqq k \leqq N$, *lie in the neighborhood* N_j, *for* $1 \leqq j \leqq m$.

PROOF. We apply Lemma A.3 and use that for each μ complex, the function $\gamma \to A(\gamma, \mu) = \det(Q(\mu) - \gamma)$ is analytic, and that $A(\gamma, \mu) \to A(\gamma, 0)$ uniformly in γ on a large disc as $\mu \to 0$.

In particular, denote by $\tau_1(\mu)$ that eigenvalue of $Q(\mu)$ which is near $\gamma = 0$ as $\mu \to 0$.

LEMMA A.5. *We have the representation* $\tau_1(\mu) = \gamma(\mu) = \sum_{n=1}^{\infty} d_n \mu^n$ *as an analytic function, in a neighborhood* N *of* $\mu = 0$, *with the following properties*:

(i) $A(\gamma(\mu), \mu) = 0$ in the neighborhood N;

(ii) $\gamma(0) = 0$;

(iii) $d_1 = b_{01}/(-b_{10}) = \gamma'(0)$ is purely imaginary if and only if λ_1 is purely imaginary; and

(iv) $d_2 = (b_{02} + b_{11}d_1 + b_{20}d_1^2)/(-b_{10}) = \gamma''(0)/2$ and the number r, defined by

$$r \equiv (d_2 - \lambda_2 m)/(-\lambda_1^2) \tag{A.2}$$

is real-valued, independent of λ_1 and λ_2, and non-positive with m given by (5).

PROOF. Consider $A(\gamma, \mu) = \det(Q(\mu) - \gamma) = \sum_{j,k} b_{jk}\gamma^j\mu^k$. Now, $b_{00} = \det Q(0) = 0$ since zero is an eigenvalue of Q, and $b_{10} \neq 0$ since zero is a simple eigenvalue of Q. From Theorem 9.4.4 of [9] page 269, there exist the unique function $\gamma(\mu) = \sum_{n=1}^{\infty} d_n\mu^n$ analytic in a neighborhood N of $\mu = 0$ such that (i) and (ii) hold, and $d_1 = b_{01}/(-b_{10}) = \gamma'(0)$ and $d_2 = \{b_{02} + b_{11}d_1 + b_{20}d_1^2\}/(-b_{10}) = \gamma''(0)/2$.

Lemmas A.1 and A.2 give that d_1 is purely imaginary if and only if λ_1 is purely imaginary and that $r = (d_2 - \lambda_2 m)/(-\lambda_1^2)$ is real-valued and independent of λ_1 and λ_2.

We show that r is non-positive. We note that $r = \gamma_*''(0)/2$, where $\gamma_*(\mu)$ is the analytic function obtained as above for the special case of $Q(\mu)$ in which $\lambda_2 = 0$, $\lambda_1 = i$. It suffices to show that $\operatorname{Re}(\gamma_*(\mu_r)) \leqq 0$ for each real number μ_r; since then, if $\mu = \mu_r + i\mu_{im}$ with μ in the neighborhood N_* of zero,

$$\begin{aligned}\gamma_*''(0)/2 = \operatorname{Re}(\gamma_*''(0))/2 &= (\tfrac{1}{2})\operatorname{Re}\left\{\frac{d^2\gamma_*(\mu_r)}{d\mu_r^2}\right\}\bigg|_{\mu=0} \\ &= \left(\frac{1}{2}\right)\frac{d^2\{\operatorname{Re}(\gamma_*(\mu_r))\}}{d\mu_r^2}\bigg|_{\mu_r=0} \leqq 0\,.\end{aligned}$$

In the inequality we have used that $\operatorname{Re}(\gamma_*(\mu_r)) \leqq 0$ and that $\operatorname{Re}(\gamma_*(\mu_r))$ achieves its maximum at $\mu_r = 0$.

Assume μ is real-valued. Let $\mathbf{x} = (x_\beta)$ be a right eigenvector for $\gamma_*(\mu) = \gamma_*$, i.e.,

$$\sum_{\beta\neq\alpha} q_{\alpha\beta}\exp(i\mu\theta_{\alpha\beta})x_\beta + q_{\alpha\alpha}x_\alpha + i\mu v_\alpha x_\alpha - \gamma_* x_\alpha = 0 \qquad \text{for each } 1 \leqq \alpha \leqq N.$$

Then we have

$$x_\alpha(q_{\alpha\alpha} + i\mu v_\alpha - \gamma_*) = -\sum_{\beta\neq\alpha} q_{\alpha\beta}\exp(i\mu\theta_{\alpha\beta})x_\beta\,.$$

Thus, $|x_\alpha|\,|q_{\alpha\alpha} + i\mu v_\alpha - \gamma_*| \leqq |q_{\alpha\alpha}| \max_n |x_n|$. If k is chosen so that $\max_n |x_n| = |x_k|$, then $|q_{kk} + i\mu v_k - \gamma_*| \leqq |q_{kk}|$. Hence $\gamma_*(\mu)$ lies within a circle with center $q_{kk} + i\mu v_k$ and radius $|q_{kk}|$. This implies that $\operatorname{Re}(\gamma_*(\mu)) < 0$ or $\gamma_*(\mu) = i\mu v_k$.

The techniques in this portion of the proof are found in Gersgorin's theorem (see [19] page 226). □
Let I_A denote the indicator random variable for the set A.

LEMMA A.6. *The following representations hold*:

$$E\{\int_0^T \int_0^r \{I_{(X(\rho)=\alpha)} I_{(X(r)=\beta)}\, d\rho\, dr\} = \int_0^T \int_0^r p_\alpha p_{\alpha\beta}(r-\rho)\, d\rho\, dr\,; \tag{A.3}$$

$$E\{\int_0^T \sum_{\{t_m^*;0<t_m^*\leqq r\}} I_{(X(t_{m-1}^*)=j,X(t_m^*)=k)} I_{(X(r)=\alpha)}\, dr\} = \int_0^T \int_0^r p_j q_{jk} p_{k\alpha}(r-\rho)\, d\rho\, dr\,; \tag{A.4}$$

$$E\{\sum_{\{t_m^*;0<t_m^*\leqq T\}} \int_0^{t_m^*} I_{(X(\rho)=\alpha)} I_{(X(t_{m-1}^*)=j,X(t_m^*)=k)}\, d\rho\} = \int_0^T \int_0^r p_\alpha p_{\alpha j}(r-\rho) q_{jk}\, d\rho\, dr\,; \tag{A.5}$$

$$E\{\sum_{\{t_s^*;0<t_s^*\leqq T\}} \sum_{\{t_v^*;0<t_v^*\leqq t_s^*\}} I_{(X(t_{v-1}^*)=j,X(t_v^*)=k)} I_{(X(t_{s-1}^*)=y,X(t_s^*)=z)}\} = \int_0^T \int_0^r p_j q_{jk} p_{ky}(r-\rho) q_{yz}\, d\rho\, dr\,. \tag{A.6}$$

PROOF. We give the proof of (A.5). The proofs of (A.3), (A.4), and (A.6) are completed similarly.

$$\begin{aligned}
&E\{\textstyle\sum_{\{t_m^*;0<t_m^*\leqq T\}} \int_0^{t_m^*} I_{(X(\rho)=\alpha)} I_{(X(t_{m-1}^*)=j;X(t_m^*)=k)}\, d\rho\} \\
&\quad = \lim_{n\to\infty} E\{\textstyle\sum_{\{b;0<b/2^n\leqq T\}} \int_0^{(b-1)/2^n} I_{(X(\rho)=\alpha)} I_{(X((b-1)/2^n)=j;X(b/2^n)=k)}\, d\rho\} \\
&\quad = \lim_{n\to\infty} \textstyle\sum_{\{b;0<b/2^n\leqq T\}} \int_0^{(b-1)/2^n} P\{X(\rho)=\alpha,\ X((b-1)/2^n)=j, \\
&\qquad\qquad X(b/2^n)=k\}\, d\rho \\
&\quad = \lim_{n\to\infty} \textstyle\sum_{\{b;0<b/2^n\leqq T\}} \int_0^{(b-1)/2^n} P\{X(\rho)=\alpha\} p_{\alpha j}\{((b-1)/2^n)-\rho\} \\
&\qquad\quad \times \{q_{jk}+o(1)\}(1/2^n)\, d\rho \\
&\quad = \int_0^T \int_0^r p_\alpha p_{\alpha j}(r-\rho) q_{jk}\, d\rho\, dr\,. \qquad \square
\end{aligned}$$

LEMMA A.7. *We have the following representation for the limits given in* (7):

$$\lim_{T\to\infty} \{\mathrm{Cov}\,(\gamma_j(T), \gamma_m(T))\}/T = 2p_j \nu_{jm} \tag{A.7}$$

$$\lim_{T\to\infty} \{\mathrm{Cov}\,(\gamma_j(T), N_{mn}(T))\}/T = p_j \nu_{jm} q_{mn} + p_m q_{mn} \nu_{nj} \tag{A.8}$$

$$\lim_{T\to\infty} \{\mathrm{Cov}\,(N_{jk}(T), N_{mn}(T))\}/T = p_j q_{jk} \nu_{km} q_{mn} + p_m q_{mn} \nu_{nj} q_{jk} \tag{A.9}$$

where the constants ν_{jk}, $1 \leqq j,\ k \leqq N$, *are defined in Remark* 2.

PROOF. We give the proof of (A.8). The representations (A.7) and (A.9) are proved similarly.

$$\begin{aligned}
&\lim_{T\to\infty} \{\mathrm{Cov}\,(\gamma_j(T), N_{mn}(T))\}/T \\
&\quad = \lim_{T\to\infty} (1/T)[E\{\textstyle\int_0^T \sum_{\{t_k^*;0<t_k^*\leqq T\}} I_{(X(\rho)=j)} I_{(X(t_{k-1}^*)=m,X(t_k^*)=n)}\, d\rho\} \\
&\qquad\quad - p_j p_m q_{mn} T^2] \\
&\quad = \lim_{T\to\infty} (1/T) \textstyle\int_0^T \int_0^r p_m q_{mn}(p_{nj}(r-\rho) - p_j)\, d\rho\, dr \\
&\qquad + \lim_{T\to\infty} (1/T) \textstyle\int_0^T \int_r^T p_j(p_{jm}(\rho-r) - p_m) q_{mn}\, d\rho\, dr \\
&\quad = p_m q_{mn} \nu_{nj} + p_j \nu_{jm} q_{mn}\,.
\end{aligned}$$

Here we have used Lemma A.6 and results proved using the techniques of proof of Lemma A.6. We have also used convergence of $p_{\alpha\beta}(u) - p_\beta$ to zero at an exponential rate to obtain finiteness of the integrals $\int_0^\infty (p_{\alpha\beta}(u) - p_\beta)\, du$ for $1 \leqq \alpha, \beta \leqq N$. $\square$

Hersh–Pinsky proved the following lemma in [23], page 39. We have used it in the proof of Theorems 3, 4, 8, and 9.

LEMMA A.8. *Let $\{P_n\}$, $n \geqq 1$, be a sequence of probability measures on R^N which converge weakly to a limit P. Assume that, for some $k_1 > 0$,*

$$\limsup_{n\to\infty} \int e^{k_1|z|}\, dP_n(z) < \infty .$$

Then for any strongly continuous B-valued function $f(z)$ with $||f(z)|| \leqq Me^{k|z|}$, $k < k_1$, the Bochner integral $\int f(z)\, dP_n(z)$ converges strongly to $\int f(z)\, dP(z)$.

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