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PASSAGES AND MAXIMA FOR A PARTICULAR GAUSSIAN PROCESS[1]

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Expressions for first and last passage probabilities, conditioned on both an initial and a subsequent value, for the Gaussian process with triangular covariance and mean zero, are derived. We use these to bound the passage probabilities of arbitrary functions, to derive formulas for the expectation of passage and excursion times, to prove the uniqueness of the maximum of the process in the interval [0, 1], and to find a formula for the joint probability density function of the maximum and its instant of occurrence.

1. Introduction. Let $x(t) = x_t$, $0 \leqq t \leqq s \leqq 1$ be the Gaussian random process with mean zero and covariance

$$\begin{aligned} Ex(t)x(t+\tau) &= 1 - |\tau|\,, && |\tau| \leqq 1 \\ &= 0\,, && |\tau| > 1\,. \end{aligned}$$

It was found by Slepian (1961) that $x(t)$ has the following peculiar Markov-like property (MLP): "let $0 \leqq t_1 \leqq t_2 \leqq 1$ be two instants in the unit interval. Denote the open interval (t_1, t_2) by A and the set $(0, t_1) \cup (t_2, 1)$ by B. Then, given the values of $x(t_1)$ and $x(t_2)$, events defined on A are statistically independent of events defined on B." Jamison (1970) called it the reciprocal property and showed that it was shared by a (rather small) class of other processes.

The "first passage" conditional probability $Q_a(T\,|\,x_0)\,dT$ that, for $t > 0$, $x(t)$ first assumes the value a in the interval $T \leqq t \leqq T + dT$ given that $x(0) = x_0$ has been found by Slepian:

$$Q_a(T\,|\,x_0) = \frac{|x_0 - a|}{T[2\pi T(2-T)]^{\frac{1}{2}}} \exp\left\{-\frac{[x_0(1-T)-a]^2}{2T(1-T)}\right\}; \qquad x_0 \neq a,\ 0 \leqq T \leqq 1\,. \tag{1}$$

This problem was also studied by Mehr and McFadden (1965) and Shepp (1966), who derived directly the integral of (1) over $[0, T]$. Shepp (1971) found a formula valid for arbitrary T.

We consider the questions of passage of arbitrary functions and that of location of the maximum of the process in the interval [0, 1].

In Section 2 we present several variants of (1) concerning first, last, or both passage time probability density functions (pdf's), conditioned on the initial,

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final, or both values of $x(t)$. Iterated use of a doubly conditioned version of (1) is instrumental in writing out bounds on the probability of passing a given function in [0, 1]. We conclude Section 2 with formulas for the expected time to and from the passage of a level and of the excursion above it. All results are in doubly conditioned form. Either conditioning can, of course, be removed by weighted integration along the respective variable.

As a first attempt to locate the maximum of this non-differentiable process we derive in Section 3 the (doubly conditioned) probability of crossing a given level only within a given subinterval of length ε. We show that as $\varepsilon \to 0$, this probability is proportional to $\varepsilon^{\frac{3}{2}}$ so that a corresponding density function cannot be defined. On the other hand, using the reciprocal property we prove that the maximum of $x(t)$ in a closed interval within [0, 1] is unique. This enables the derivation of the joint probability density of the maximum and its instant of occurrence.

Among previous work on the location of the maximum is Heyde's (1969) in which the problem of the distribution of the maxima of processes with independent increments is treated and the limiting distribution of the *first* maximum is derived.

2. Passages. We denote by $p(x_1, \cdots, x_n)$ the joint pdf of $x_i = x(t_i)$, $i = 1, 2, \cdots, n$, $0 \leqq t_1 < t_2 < \cdots < t_n \leqq 1$, which is given by (Slepian, 1961)

$$(2) \quad p(x_1, x_2, \cdots x_n) = 2(2\pi)^{-n/2}[2(2 - t_n + t_1)]^{-\frac{1}{2}} \prod_{j=2}^{n} [2(t_j - t_{j-1})]^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2}\left\{\frac{(x_1 + x_n)^2}{2(2 - t_n + t_1)} + \sum_{j=2}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right\}\right].$$

We also use the notation $p(u_1, u_2, \cdots, u_j \mid u_{j+1}, \cdots, u_n)$ for the joint conditional pdf, given the values u_i, $i = j + 1, \cdots, n$,

$$(3) \quad p(u_1, u_2, \cdots, u_j \mid u_{j+1}, \cdots, u_n) = \frac{p(u_1, u_2, \cdots, u_n)}{p(u_{j+1}, \cdots, u_n)}$$

where u_i denote possibly permuted values of x_i.

Let θ and η denote, respectively, the first and the last passage times of the level a within the time interval [0, 1]. In the sequel, passage pdf's are indexed by either θ, η or both, and are conditioned on either one, two, or more values of $x(t)$ at different instants. Thus, $p_\theta(\cdot \mid x_s)$ and $p_\eta(\cdot \mid x_s)$ denote, respectively, the pdf of θ and η given $x(s) = x_s$. Similarly, $p_\theta(\cdot \mid x_r, x_s)$ and $p_\eta(\cdot \mid x_r, x_s)$ denote, respectively, the pdf of θ and η, given $x(r) = x_r$ and $x(s) = x_s$. Also $p_{\theta\eta}(\cdot, \cdot \mid x_r, x_s)$ denotes the joint pdf of θ and η given r_r and x_s. Additional conditionings should be self-explanatory.

We present the following preliminary results, valid for $0 \leqq \theta < \eta \leqq s \leqq 1$:

$$(4) \quad p_\theta(\theta \mid x_0) = \left|\frac{a - x_0}{\theta}\right| p(x_\theta = a \mid x_0); \qquad a \neq x_0$$

$$(5) \quad p_\eta(\eta \mid x_s) = \left|\frac{a - x_s}{s - \eta}\right| p(x_\eta = a \mid x_s); \qquad a \neq x_s$$

$$(6)\qquad p_\theta(\theta \mid x_0, x_s) = \left|\frac{a-x_0}{\theta}\right| p(x_\theta = a \mid x_0, x_s)\,; \qquad a \neq x_0$$

$$(7)\qquad p_\eta(\eta \mid x_0, x_s) = \left|\frac{a-x_s}{s-\eta}\right| p(x_\eta = a \mid x_0, x_s)\,; \qquad a \neq x_s$$

$$(8)\qquad p_{\theta\eta}(\theta, \eta \mid x_0, x_s) = \left|\frac{(a-x_0)(a-x_s)}{\theta(s-\eta)}\right| p(x_\theta = x_\eta = a \mid x_0, x_s)\,; \qquad a \neq x_0, x_s\,.$$

Equation (4) is just (1) rewritten in our notation; (5) is a consequence of the stationarity of the process and is obtained by inverting the direction of time. To derive (6) we note that, due to the MLP, the conditional pdf of x_s given $x(0) = x_0$, $x(\theta) = x_\theta = a$ and that $x(t) < a$, $0 \leqq t \leqq \theta$, namely $p(x_s \mid x_0, x_\theta = a, x(t) < a, 0 \leqq t \leqq \theta)$ is equal to $p(x_s \mid x_0, x_\theta = a)$. Then by Bayes' rule $p_\theta(\theta \mid x_0, x_s) = p(x_s \mid x_0, x_\theta = a)p_\theta(\theta \mid x_0)/p(x_s \mid x_0)$ from which (6) follows, using (4) and (2). The derivation of (7) is similar. Now choose an instant β, $0 \leqq \theta < \beta < \eta \leqq s$. Then by the MLP,

$$p_\theta(\theta \mid x_0, x_\beta) = p_\theta(\theta \mid x_0, x_\beta, x_s)$$
$$p_\eta(\eta \mid x_\beta, x_s) = p_\eta(\eta \mid x_0, x_\beta, x_s)$$

and using (6) and (7) and again the MLP,

$$p_{\theta\eta}(\theta, \eta \mid x_0, x_\beta, x_s) = \left|\frac{a-x_0}{\theta}\right|\left|\frac{a-x_s}{s-\eta}\right| p(x_\theta = a \mid x_0, x_\beta)\cdot p(x_\eta = a \mid x_\beta, x_s)\,.$$

Taking the expected value over x_β and using again the MLP we obtain (8).

Of interest is also the event $C_{rs}(a)$: The level a has been crossed in the interval $[r, s]$, $0 \leqq r < s \leqq 1$. Shepp (1966) found the formula for $P_a{}^+(s \mid x_0) = 1 - \Pr[C_{0s} \mid x_0]$. We derive, for further reference, the doubly conditioned probability

$$(9)\qquad \Pr[C_{0s}(a) \mid x_0, x_s] = \exp\left(-\frac{(a-x_0)(a-x_s)}{s}\right), \qquad (a-x_0)(a-x_s) > 0$$
$$= 1\,, \qquad (a-x_0)(a-x_s) \leqq 0\,.$$

To obtain (9), integrate over $[0, s]$ either one of (6), (7) or (8). Integration is eased by the observation that the integrals are convolutions and that Laplace transforms can be used. The expectation of (9) restricted to $x_s \leqq a$ gives, of course, Shepp's result. When not needed for clarity, we shall drop the argument of C_{qr}.

An application of (9) is to the passage of arbitrary functions. Let $\hat{P}_{0s}(A \mid x_0, x_s)$ be the probability that $x(t) < A(t)$, $0 \leqq t \leqq s$, given $x(0) < A(0)$ and $x(s) < A(s)$, where $A(t)$ is a simple function: $A(t) = a_i$, $0 \leqq t_i \leqq t \leqq t_{i+1} \leqq s \leqq 1$, $i = 0, 1, \cdots, N-1$, $t_0 = 0$, $t_N = s$. Let $\hat{P}_{qr}(a \mid x_q, x_r) = 1 - P_r(C_{qr}(a) \mid x_q, x_r)$. From (9) due to the stationarity of $x(t)$ and assuming that $x_q, x_r < a$, $\hat{P}_{qr}(a \mid x_q, x_r) = 1 - \exp[-(a-x_q)(a-x_r)/(r-q)]$. Using repeatedly the MLP one can write by inspection

$$(10)\qquad \hat{P}_{0s}(A \mid x_0, x_s) = \int_{-\infty}^{a'_{N-1}} \cdots \int_{-\infty}^{a_2'} \hat{P}_{0t_1}(a_0 \mid x_0, x_1) \prod_{i=1}^{N-1} \hat{P}_{t_i t_{i+1}}(a_i \mid x_i, x_{i+1}) \times p(x_i \mid x_0, x_{i+1})\, dx_1\, dx_2, \cdots, dx_{N-1}\,,$$

where $a_i' = \min(a_{i-1}, a_i)$. Let $\hat{P}_{0s}(f \mid x_0, x_s)$ be the probability that $x(t)$ does not pass an arbitrary function $f(t)$ in $[0, s]$. $\hat{P}_{0s}(t)$ can be bounded from above, and from below by $\hat{P}_{0s}(A)$ where $A(t)$ is a simple function, respectively, $\geqq f(t)$, $\leqq f(t)$.

The derivation of the following expectations is also eased by use of Laplace transforms. They are, respectively, the mean time to first passage, the mean time since the last passage, both conditioned, of course, on passage having occurred, i.e. on the event C_{0s}.

$$(11)\qquad E[\theta \mid x_0, x_s, C_{0s}] = \tfrac{1}{2}|a - x_0|(\pi s)^{\frac{1}{2}} \exp\left(\frac{(x_s - x_0)^2}{4s}\right) \operatorname{erfc} \frac{|a - x_0| + |a - x_s|}{2s^{\frac{1}{2}}} \times [\Pr(C_{0s} \mid x_0, x_s)]^{-1}$$

where $\Pr(C_{0s} \mid x_0, x_s)$ is as above. Also $E[s - \eta \mid x_0, x_s, C_{0s}]$ is given by the same expression with x_0 and x_s interchanged. Use was made in the derivation of the fact that $p(\theta, C_{0s} \mid x_0, x_s)$ equals $p(\theta \mid x_0, x_s)$, as a consequence of the MLP.

The above can be used to derive expected excursion times. Denote by T_{qr} the time spent by $x(t)$ above a given level, say a, $0 \leqq q \leqq t \leqq r \leqq s \leqq 1$. We derive a formula for $E[T_{0s} \mid x_0, x_s]$.

Let C'_{0s} be the complementary event to C_{0s}. Then

$$(12)\qquad E[T_{0s} \mid x_0, x_s] = E[T_{0s} \mid x_0, x_s, C_{0s}] \Pr(C_{0s} \mid x_0, x_s) + E[T_{0s} \mid x_0, x_s, C'_{0s}](1 - \Pr(C_{0s} \mid x_0, x_s))\,.$$

We have

$$(13)\qquad \begin{aligned} E[T_{0s} \mid x_0, x_s, C'_{0s}] &= 0\,, & a &> x_0, x_s \\ &= s\,, & a &< x_0, x_s\,. \end{aligned}$$

Also, adding and then removing additional conditionings,

$$(14)\qquad E[T_{0s} \mid x_0, x_s, C_{0s}] = \int_0^s \int_0^\eta E[T_{0s} \mid x_0, x_s, C_{0s}, \theta, \eta] \cdot p_{\theta\eta}(\theta, \eta \mid x_0, x_s, C_{0s})\, d\theta\, d\eta\,.$$

To evaluate $E[T_{0s} \mid x_0, x_s, C_{0s}, \theta, \eta]$ we identify four possibilities and make use of the MLP, recalling that the conditions θ and η imply that $x_\theta = x_\eta = a$ and that crossings have occurred neither in $(0, \theta)$ nor in (η, s). Thus

$$\begin{aligned} E[T_{0s} \mid x_0, x_s, C_{0s}, \theta, \eta] &= \gamma & a &\geqq x_0, x_s \\ &= \theta + \gamma & x_0 &\geqq a \geqq x_s \\ &= s - \eta + \gamma & x_0 &\leqq a \leqq x_s \\ &= s - \eta + \theta + \gamma & a &\leqq x_0, x_s \end{aligned}$$

where γ is given by

$$\gamma = E[T_{\theta\eta} \mid x_\theta = a, x_\eta = a] = \tfrac{1}{2}(\eta - \theta)\,,$$

the last step being due to symmetry.

Then $E[T_{0s} \mid x_0, x_s, C_{0s}]$ is equal to

$$(15)\qquad \begin{aligned} e_1 &= \tfrac{1}{2}E[\eta \mid x_0, x_s] - \tfrac{1}{2}E[\theta \mid x_0, x_s]\,, && a \geqq x_0, x_s \\ e_2 &= \tfrac{1}{2}E[\eta \mid x_0, x_s) + \tfrac{1}{2}E[\theta \mid x_0, x_s]\,, && x_s \leqq a \leqq x_0 \\ & s - e_2\,, && x_s \geqq a \geqq x_0 \\ & s - e_1\,, && a \leqq x_0, x_s\,. \end{aligned}$$

Using (11) in (15) and substituting, along with (13), in (12), yields for $a > x_0, x_s$, the formula

$$(16)\qquad E[T_{0s} \mid x_0, x_s] = \frac{s}{2}\left\{\Pr[C_{0s} \mid x_0, x_s] - \pi^{\frac{1}{2}} \frac{|a - x_0| + |a - x_s|}{2s^{\frac{1}{2}}} \times \exp\left(\frac{x_s - x_0}{2s^{\frac{1}{2}}}\right)^2 \operatorname{erfc}\left(\frac{|a - x_0| + |a - x_s|}{2s^{\frac{1}{2}}}\right)\right\}.$$

Similar formulae valid for the other relations between a, x_0 and x_s can easily be written out.

3. Maxima. Since $x(t)$ has no derivative, one possible approach to the problem of its maximum value, in a closed interval, is to investigate the crossing of a given level within only a subinterval i.e. within a "horizontal window," and then let the window shrink to infinitesimal size. Let the *(q, r)-crossing at a*, $D_{qr}(a)$, be the event: $x(t)$, $0 \leqq t \leqq s$, has crossed the level a only within the subinterval (q, r), $0 \leqq q \leqq r \leqq s \leqq 1$. Then an ε-maximum *at q and at a* is the event $D_{q,q+\varepsilon}(a)$, considered only for $(a - x_0)(a - x_s) > 0$. We show that, as $\varepsilon \to 0$,

$$(17)\qquad \Pr[D_{q,q+\varepsilon}(a) \mid x_0, x_s] \to \frac{2}{3\pi^{\frac{1}{2}}} \frac{|(a - x_0)(a - x_s)|}{q(s - q)} p(x_q = a \mid x_0, x_s)\varepsilon^{\frac{3}{2}}\,.$$

By definition,

$$(18)\qquad \operatorname{Prob}[D_{q,r}(a) \mid x_0, x_s] = \textstyle\int_q^r \int_q^\eta p_{\theta\eta}(\theta, \eta \mid x_0, x_s)\, d\theta\, d\eta\,.$$

Using the MLP, (2) and (6), we can perform one integration to obtain for (18)

$$\int_q^r p_\eta(\eta \mid x_0, x_s) \operatorname{erf}\left(\frac{|a - x_0|}{2}\left(\frac{1}{q} - \frac{1}{\eta}\right)^{\frac{1}{2}}\right) d\eta\,,$$

from which (17) follows, as $r - q = \varepsilon \to 0$. Since the probability of the ε-maximum is proportional to $\varepsilon^{\frac{3}{2}}$, an associated pdf cannot be defined. This behavior is, however, explicable since if $x(t)$ is confined to within ε in time, at a level a, its excursion should also be limited about a by implication. By (17) the corresponding limitation is of the order of $\varepsilon^{\frac{1}{2}}$, which is reasonable considering that $x(t)$ is the difference of two Wiener processes.

The above discussion motivates the investigation of the joint pdf of the value m and of the instant of occurrence τ of the maximum of $x(t)$ within $[0, s]$. We first note that the pdf of m, conditioned on x_0 and x_s, exists and is given by minus the derivative of (9). However, due to the non-differentiability of $x(t)$, the instant of occurrence τ might not be well defined; we recall that with such processes there is an infinite number of expected level-crossings in any finite interval (Rice (1944)). The following theorem, on additivity of measure in function space, excludes the possibility of fuzziness about τ: except for a set of functions of zero measure, τ is either to the left or to the right of any t_0, $0 \leqq t_0 \leqq 1$.

THEOREM. *Let* $0 \leqq t_1 \leqq t_2 \leqq s \leqq 1$, *let* $0 \leqq t \leqq s$, $a \geqq x_s, x_0$ *and let*

$$G_a(t_1, t_2) = \{\omega: x(0, \omega) = x_0,\, x(s, \omega) = x_s,\, x_\omega(t) < a,\; t \notin (t_1, t_2),\, a \leqq x(t, \omega) < a + da,\, t \in (t_1, t_2)\}\,. \tag{19}$$

Then, as $da \to 0$,

$$\Pr[G_a(0, t)] + \Pr[G_a(t, s)] = \Pr[G_a(0, s)]\,.$$

To prove the theorem we use again the notation

$$\hat{P}_{qr}(a \mid x_q, x_r) = 1 - \Pr[C_{qr}(a) \mid x_q, x_r] \tag{20}$$

so that

$$\Pr[G_a(0, s)] = d\hat{P}_{0s}(a \mid x_0, x_s)\,. \tag{21}$$

We now impose the additional condition $x(t) = x_0$ and note that, as $da \to 0$, by the MLP,

$$\Pr[G_a(0, t) \mid x_t] = d\hat{P}_{0t}(a \mid x_0, x_t)\hat{P}_{ts}(a \mid x_t, x_s)\,, \tag{22}$$

$$\Pr[G_a(t, s) \mid x_t] = \hat{P}_{0t}(a \mid x_0, x_s)\, d\hat{P}_{ts}(a \mid x_t, x_s)\,. \tag{23}$$

Furthermore, using again the MLP

$$\begin{aligned}\Pr[G_a(0, s) \mid x_t] &= d\hat{P}_{0s}(a \mid x_0, x_s, x_t) = d\{\hat{P}_{0t}(a \mid x_0, x_s, x_t)\hat{P}_{ts}(a \mid x_0, x_s, x_t)\} \\ &= d\hat{P}_{0t}(a \mid x_0, x_t)\hat{P}_{ts}(a \mid x_s, x_t) + \hat{P}_{0t}(a \mid x_0, x_t)\, d\hat{P}_{ts}(a \mid x_s, x_t) \\ &= \Pr[G_a(0, t) \mid x_t] + \Pr[G_a(t, s) \mid x_t]\,.\end{aligned}$$

The theorem follows by eliminating the conditioning on x_t.

We proceed to derive the joint pdf of the maximum of $x(t)$, m and its unique instant of occurrence τ in $[0, s]$, $s \leqq 1$, conditioned on the initial and final values, which we denote by $\hat{p}(m, \tau \mid x_0, x_s)$. We show that

$$\hat{p}(m, \tau \mid x_0, x_s) = \frac{(m - x_0)(m - x_s)}{\tau(s - \tau)}\, p(x_\tau = m \mid x_0, x_s)\,, \qquad m > x_0, x_s,\ 0 \leqq \tau \leqq s\,, \tag{24}$$

and note that $\hat{p}(m, \tau \mid x_0, x_s)\, dm\, d\tau$ is proportional to (17), if indeed dm is proportional to $(d\tau)^{\frac{1}{2}}$ by implication, as discussed in the context of (17).

Since τ can be taken to be the first maximum we obtain its joint pdf with m, given x_0 and x_s, from the time derivative of (19), namely

$$\hat{p}(m, \tau \mid x_0, x_s) = \frac{d}{dt}\Pr[G_m(0, t)]\Big|_{t=\tau}\,. \tag{25}$$

Using (22) we have

$$\Pr[G_m(0, t)] = \int_{-\infty}^{m} \hat{P}_{ts}(m \mid x_t, x_s)\, d\hat{P}_{0t}(m \mid x_0, x_t) p(x_t \mid x_0, x_t)\, dx_t\,,$$

which can be calculated, using (20), (9), (3) and (2).

$$\Pr[G_m(0, t)] = \frac{1}{s}\exp\left\{-\frac{1}{2}\frac{(m-x_0)^2}{2t} + \frac{(m-x_s)^2}{2(s-t)} - \frac{(x_s-x_0)^2}{2s}\right\}$$

$$\times\left\{\left[\frac{m-x_0}{2} + \frac{m-x_s}{2}\right]G(u-v)\right. \tag{26}$$

$$\left. - \left[\frac{m-x_0}{2} - \frac{m-x_s}{2}\right]G(u+v)\right\},$$

where $G(\omega) = \exp(\omega^2)\operatorname{erfc}(\omega)$, $u = (m - x_0)/2(t^{-1} - s^{-1})^{\frac{1}{2}}$ and $v = (m - x_s)/2 \times ((s - t)^{-1} - s^{-1})^{\frac{1}{2}}$. Formula (24) follows from (25) and (26).

The case of $x_s = x_0 = b$ is of interest. Integrating (24) we obtain for the distribution of the maximum

$$F_b(\alpha, \theta) = \operatorname{Prob}[m \leqq \alpha, \tau \leqq \theta \mid x_0 = x_s = b], \qquad b < \alpha,\ 0 \leqq \theta \leqq s \leqq 1, \tag{27}$$

the following particular values: For any $\alpha > b$,

$$\frac{F_b(\alpha, s/2)}{F_b(\alpha, s)} = \tfrac{1}{2},$$

which agrees with the temporal symmetry of the statistics of $x(t)$. For large $(\alpha - b)$

$$\frac{F_b(\alpha, \theta)}{F_b(\alpha, s)} \to \frac{\theta}{s}, \qquad \alpha - b \to \infty,$$

showing that the maxima are uniformly distributed between the two fixed values. As the last observation, as $\alpha \to b$

$$\frac{F_b(\alpha, \theta)}{F_b(\alpha, s)} \to \tfrac{1}{2}, \qquad 0 < \theta < s,$$

indicating that if $x_0 = x_s = \alpha$, then at no other point in $(0, s)$ will $x(t)$ reach the level α, in agreement with the theorem.

As another application of (24), an upper bound to $\hat{P}_{0s}(f(t) \mid x_0, x_s)$, a problem that has been considered in Section 2, is given by

$$\textstyle\int_0^s d\tau \int_{-\infty}^{f(\tau)} \hat{p}(m, \tau \mid x_0, x_s)\, dm .$$

This is the probability that the maximum of $x(t)$ be below $f(t)$; it is only a bound because smaller peaks may protrude above it if $f(t)$ is not constant.

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