The Annals of Probability
1975, Vol. 3, No. 4, 732–736

ENTROPY ZERO × BERNOULLI PROCESSES ARE CLOSED IN THE $\bar{d}$-METRIC

BY PAUL SHIELDS[1] AND J.-P. THOUVENOT[2]

Stanford University and Université de Paris

An entropy zero × Bernoulli process is a stationary finite state process whose shift transformation is the direct product of an entropy zero transformation and a Bernoulli shift. We show that the class of such transformations which are ergodic is closed in the $\bar{d}$-metric. The $\bar{d}$-metric measures how closely two processes can be joined to form a third stationary process.

An entropy zero × Bernoulli process is a stationary finite state process whose shift transformation is the direct product of an entropy zero transformation and a Bernoulli shift. We show that the class of ergodic entropy zero × Bernoulli processes is closed in the $\bar{d}$-metric. The second author has shown [7] that this class is closed under the taking of factors; and thus, for example, the class of rotations × Bernoulli shifts is a regular class as defined by the first author [4].

This paper is divided into two sections. In the first section we give a representation for a sequence of processes. This representation (Theorem 1) is of independent interest and will enable us to prove the main theorem (Theorem 2) in the second section.

1. Our measure spaces (X, B, μ) will be assumed to be isomorphic to the unit interval with Lebesgue sets and Lebesgue measure. All sets and functions will be assumed (or must be shown) to be measurable and such qualifying phrases as "almost everywhere" will be omitted. Partitions of X will be assumed to be finite and ordered, the join of P and Q is defined as $P \vee Q = \{P_i \cap Q_j\}$ with lexicographic ordering, and the distribution of $P = \{P_1, P_2, \cdots, P_k\}$ is the vector $d(P) = d_\mu(P) = (\mu(P_1), \mu(P_2), \cdots, \mu(P_k))$. The cardinality of P is denoted by $|P|$. If $\{P^i\}_{i=1}^n$ and $\{\bar{P}^i\}_{i=1}^n$ are two sequences of partitions such that $|P^i| = |\bar{P}^i| = k$ for $1 \leqq i \leqq n$, then

$$\bar{d}_n(\{P^i\}_{i=1}^n, \{\bar{P}^i\}_{i=1}^n) = \inf \frac{1}{n} \sum_{i=1}^n |Q^i - \bar{Q}^i|$$

where this infimum is taken over all sequences of partitions $\{Q^i\}_{i=1}^n$ and $\{\bar{Q}^i\}_{i=1}^n$ of a fixed non-atomic probability space (Y, ν) for which

$$d_\nu(\textstyle\bigvee_1^n Q^i) = d(\bigvee_1^n P^i)\,; \qquad d_\nu(\bigvee_1^n \bar{Q}^i) = d(\bigvee_1^n \bar{P}^i)$$

and $|Q - \bar{Q}| = |Q - \bar{Q}|_\nu$ denotes $\frac{1}{2} \sum' \nu(Q_j - \bar{Q}_j) + \frac{1}{2} \sum \nu(\bar{Q}_j - Q_j)$.

Received October 3, 1974; revised December 30, 1974.

[1] This research was supported in part by the National Science Foundation Grant GP-33581X. Author now at University of Toledo.

[2] Equipe de Recherche n° 1 "Processus stochastiques et applications" dépendant de la Section n° 1 "Mathématiques Informatique" associée au C.N.R.S.

AMS 1970 *subject classifications*. Primary 28A65; Secondary 60G10.

Key words and phrases. Bernoulli shift, entropy zero, $\bar{d}$-metric.

An alternative description can be given on a sequence space where the partitions are fixed and the measure is varied. To formulate this put $S = \{1, 2, \cdots, k\}$, $S^n = S^{n-1} \times S$ and let $\{R^i\}_{i=1}^n$ and $\{\bar{R}^i\}_{i=1}^n$ be partitions of $S^n \times S^n$ defined by

$$R_j^{\,i} = \{x, \bar{x}) \mid x_i = j\} \qquad \bar{R}_j^{\,i} = \{(x, \bar{x}) \mid \bar{x}_i = j\} .$$

Let M_n be the set of all probability measures ν on the discrete space $S^n \times S^n$ such that

$$d_\nu(\textstyle\bigvee_1^n R^i) = d_\mu(\bigvee_1^n P^i) ; \qquad d_\nu(\bigvee_1^n \bar{R}^i) = d_\mu(\bigvee_1^n \bar{P}^i) .$$

It is then easy to see that

$$\bar{d}_n(\{P^i\}_{i=1}^n, \{\bar{P}^i\}_{i=1}^n) = \min_{\nu \in M_n} \frac{1}{n} \sum |R^i - \bar{R}^i|_\nu .$$

Note this infimum is indeed a minimum for $n^{-1} \sum |R^i - \bar{R}^i|_\nu$ is continuous on the compact set M_n.

An invertible measure preserving transformation is called an automorphism. If T is an automorphism and P is a partition, the pair T, P is called a process (see [2]) with $|P|$ states. If T, P and $\bar{T}$, $\bar{P}$ are k state processes then

$$\bar{d}((T, P)), (\bar{T}, \bar{P})) = \lim_n \bar{d}_n(\{T^i P\}_{i=1}^n, \{\bar{T}^i \bar{P}\}_{i=1}^n)$$

a limit which can easily be shown to exist. We say that $U, R \sim T, P$ if for all n, $d(\bigvee_1^n U^i R) = d(\bigvee_1^n T^i P)$. Let $\mathscr{P}$ be the class of all processes $U, R \vee \bar{R}$ such that $U, R \sim T, P$ and $U, \bar{R} \sim \bar{T}, \bar{P}$. Then (see [2])

$$\bar{d}((T, P)(\bar{T}, \bar{P})) = \inf_{U, R \vee \bar{R} \in \mathscr{P}} |R - \bar{R}| . \tag{1}$$

This can be described in sequence space as follows. Let U be the shift on $(S \times S)^Z$ and let R and $\bar{R}$ be the partitions defined by

$$R_j = \{(x, \bar{x}) \mid x_0 = j\} ; \qquad \bar{R}_j = \{(x, \bar{x}) \mid \bar{x}_0 = j\} .$$

Let M denote the class of all U-invariant Borel probability measures ν such that $(U, R)_\nu \sim T, P$ and $(U, \bar{R})_\nu \sim \bar{T}, \bar{P}$, where $(U, R)_\nu$ means the process defined by the measure ν. The class M is convex and weakly compact and $|R - \bar{R}|_\nu$ is continuous, hence

$$\bar{d}((T, P)(\bar{T}, \bar{P})) = \min_{\nu \in M} |R - \bar{R}|_\nu . \tag{2}$$

The fact that if T and $\bar{T}$ are ergodic one need only consider ergodic automorphisms U in (1) was established in [2] by an ergodic decomposition argument. We note in passing that this is a simple consequence of (2), for if ν is an extreme point of M and $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ where ν_1 and ν_2 are U-invariant then (if T and $\bar{T}$ are ergodic) we must have

$$(U, R)_{\nu_i} \sim T, P , \qquad (U, \bar{R})_{\nu_i} \sim \bar{T}, \bar{P} , \qquad i = 1, 2$$

so that $\nu_1, \nu_2 \in M$ and hence $\nu_1 = \nu_2 = \nu$. Thus ν must be an extreme point in the class of U-invariant measures, hence U-ergodic.

Our main theorem in this section is

THEOREM 1. *If T_n, P^n is a sequence of k state processes then there is an automorphism U and partitions R^n such that $U, R^n \sim T_n, P^n$ and*

$$|R^n - R^{n-1}| = \bar{d}((T_n, P^n), (T_{n-1}, P^{n-1})) . \tag{3}$$

PROOF. Let V be the shift on $X = S^Z$ and let U be the infinite direct product of V with itself, that is, U is defined on $Y = \prod^\infty X_i$, where $X_i = X$ by the formula

$$(Ux)_n = V(x_n) , \qquad n = 1, 2, \cdots$$

where $x_n \in S^Z$. R^n is the partition defined by

$$R_i^n = \{x \mid x_n(0) = i\} ,$$

that is R^n is the time 0 partition of the nth coordinate.

Define X^n, V_n by the relations $X^1 = X$, $V_1 = V$ and $X^n = X^{n-1} \times S^Z$, $V_n = V_{n-1} \times V$ for $n > 1$. In an abuse of notation R^n will denote the partition of X^m for $m \geqq n$ defined by the relation

$$R_i^n = \{(x_1, x_2, \cdots, x_m) \mid x_n(0) = i\} .$$

We shall also write $(T, P)_\mu \sim (\bar{T}, \bar{P})_\nu$ if

$$d_\mu(\bigvee_1^m T^i P) = d_\nu(\bigvee_1^m \bar{T}^i \bar{P}) , \qquad m = 1, 2, \cdots$$

omitting reference to the measures when understood.

We shall now construct a sequence $\{\mu_n\}$ of U-invariant measures such that

$$(U, R^j)_{\mu_n} \sim (T_j, P^j) \qquad \text{for } 1 \leqq j \leqq n \tag{4}$$

$$|R^{j+1} - R^j|_{\mu_n} = \bar{d}((T_{j+1}, P^{j+1}), (T_j, P^j)) \qquad 1 \leqq j \leqq n - 1 . \tag{5}$$

Any weak limit of the sequence $\{\mu_n\}$ will give a measure for which (3) holds. The sequence $\{\mu_n\}$ is constructed by induction. First construct the V_1-invariant measure ν_1 such that $(V_1, R^1)_\nu \sim (T, P^1)$ and let $\mu_1 = \prod_1^\infty \varphi_i$ where $\varphi_i \equiv \nu_1$. Then μ_1 satisfies condition (4). Assume $\mu_1, \mu_2, \cdots, \mu_n$ have been constructed satisfying (4) and (5).

Construct a V_n-invariant measure ν_n such that $(V_n, R^1 \vee \cdots \vee R^n)_{\nu_n} \sim (U, R^1 \vee \cdots \vee R^n)_{\mu_n}$ and then use (2) to construct a V_2 invariant measure ν on X^2 such that

$$(V_2, R^1)_\nu \sim (T_n, P^n) , \qquad (V_2, R^2)_\nu \sim (T_{n+1}, P^{n+1})$$

and $|R^1 - R^2|_\nu = \bar{d}((T_{n+1}, P^{n+1}), (T_n, P^n))$. Now apply Furstenberg's Theorem [1, Lemma 1.2] to construct the V_{n+1}-invariant measure $\bar{\mu}_{n+1}$ such that

$$(V_{n+1}, R^1 \vee \cdots \vee R^n)_{\bar{\mu}_{n+1}} \sim (V_n, R^1 \vee \cdots \vee R^n)_{\nu_n}$$

and $(V_{n+1}, R^n \vee R^{n+1})_{\bar{\mu}_{n+1}} \sim (V_2, R^1 \vee R^2)_\nu$. Let μ_{n+1} be the product measure on Y induced by the measure $\bar{\mu}_{n+1}$ on X^{n+1}, where we identify Y with $\prod^\infty Y_i$, where $Y_i = X^n$ for all i. The measure μ_{n+1} the satisfies the conditions (4) and (5). This complete the proof of Theorem 1.

COROLLARY. *If* T_n, P_n *coverges in* $\bar{d}$ *to* T, P *and if each* T_n, P_n *is ergodic (respectively weak mixing, mixing) then* T, P *is ergodic (weak mixing, mixing).*

PROOF. We can assume that $\bar{d}((T_n, P^n), (T_{n+1}, P^{n+1})) < 2^{-n}$ and apply the theorem to construct U, R^n so that (3) holds. Then $|R^n - R^{n+1}| < 2^{-n}$ so $\lim R^n = R$ exists. If each U, R^n is ergodic, each satisfies the ergodic theorem and hence so does U, R. A similar proof establishes the other parts of the corollary.

2. In this section we make free use of the concepts of entropy, Bernoulli shifts and partitions discussed in [5]. If T is an automorphism and $\mathscr{A}$ is a sub-σ-algebra invariant under T and T^{-1} we let $T_{\mathscr{A}}$ denote the factor automorphism. If the collection $\bigcup_{i \in Z} \{T^i P\}$ generates $\mathscr{A}$ we shall also write $\mathscr{A} = (P)_T$ and $T_P = T_{\mathscr{A}}$. The Pinsker algebra of T is the smallest-σ-algebra which contains all P for which $H(T, P) = 0$. If $\mathscr{A}$ is the Pinsker algebra of T then (see [3]) $H(T_{\mathscr{A}}) = 0$ and

(6) $\mathscr{A}$ and $\mathscr{B}$ are independent if $T_{\mathscr{B}}$ is a Bernoulli shift.

An entropy zero × Bernoulli process is a pair T, P where T is isomorphic to the direct product $T_0 \times T_1$, where $H(T_0) = 0$ and T_1 is a Bernoulli shift. Note that $T_0 = T_{\mathscr{A}}$ where $\mathscr{A}$ is the Pinsker algebra of T.

If T is an ergodic automorphism and P and Q are partitions then P is Q-conditionally finitely determined if given $\varepsilon > 0$, $\exists \delta > 0$ and $\bar{n} \geqq 0$ such that if $\bar{T}$ is any ergodic automorphism and $\bar{P}$ and $\bar{Q}$ any partitions such that

(a) $d(\bigvee_0^m \bar{T}^i \bar{Q}) = d(\bigvee_0^m T^i Q)$ for all $m \geqq 0$
(b) $|d(\bigvee_0^{\bar{n}} \bar{T}^i(\bar{Q} \vee \bar{P})) - d(\bigvee_0^{\bar{n}} T^i(Q \vee P))| \delta$
(c) $|H(\bar{T}, \bar{Q} \vee \bar{P}) - H(T, Q \vee P)| < \delta$

then given any space Z and any integer n there are sequences of partitions of Z, $Q^{(i)}$, $P^{(i)}$, $\bar{P}^{(i)}$, $0 \leqq i \leqq n$ such that

(d) $d(\bigvee_0^n \bar{T}^i(\bar{Q} \vee \bar{P})) = d(\bigvee_0^n Q^{(i)} \vee \bar{P}^{(i)})$
$d(\bigvee_0^n T^i(Q \vee P)) = d(\bigvee_0^n Q^{(i)} \vee P^{(i)})$
(e) $(n+1)^{-1} \sum_{i=0}^n |P^{(i)} - \bar{P}^{(i)}| < \varepsilon$.

We make use of the following results about this concept (see [6, 7]):

(7) If $B \vee Q$ is a generator for an ergodic automorphism T, if T_B is a Bernoulli shift and $(Q)_T$ is the Pinsker algebra of T, then any partition P is Q-conditionally finitely determined.

(8) If P is Q-conditionally finitely determined for an ergodic automorphism T then there is a partition B such that T_B is isomorphic to a Bernoulli shift and $(B \vee Q)_T = (P \vee Q)_T$.

(9) If $\{P_n\}$ is a sequence of Q-conditionally finitely determined partitions for an ergodic automorphism T which converges to a partition P, then P is Q-conditionally finitely determined.

THEOREM 2. *If T_n, P^n is sequence of ergodic entropy zero $\times$ Bernoulli processes which converges in $\bar{d}$ to T, P, then T, P is an ergodic entropy zero $\times$ Bernoulli process.*

PROOF. We can assume that $\bar{d}((T_n, P^n), (T_{n+1}, P^{n+1})) < 2^{-n}$ and apply Theorem 1 to construct U and a sequence of partitions R^n such that for each n, U, $R^n \sim T_n$, P^n and $|R^n - R^{n+1}| < 2^{-n}$. Let $R = \lim_n R^n$. As we noted in Section 1, U, R is an ergodic process.

Let Q be a generator for the Pinsker algebra of U and let Q^n be a generator for the Pinsker algebra of U_{R^n}. The result (7) implies that for each n, R^n is Q^n conditionally finitely determined so we can find $\bar{Q}^n$, B^n such that $(\bar{Q}^n)_U \subseteqq (Q^n)_U$, U_{B^n} is Bernoulli and $(B^n \vee \bar{Q}^n)_U = (R^n)_U$. Thus (6) implies that $(B^n)_U$ and $(Q)_U$ are independent, so another application of (7) implies that R^n is Q-conditionally finitely determined. The result (9) then implies that R is Q-conditionally finitely determined. Thus we can apply (8) and (6) to conclude that U, R is indeed an entropy zero $\times$ Bernoulli process.

REFERENCES

[1] FURSTENBERG, H. (1967). Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation. *Math. Systems Theory* **1** 1–50.

[2] ORNSTEIN, D. S. (1973). An application of ergodic theory to probability theory. *Ann. Probability* **1** 43–58.

[3] ROHLIN, V. A. and SINAI, J. G. (1961). Construction and properties of invariant measurable partitions. *Dokl. Akad. Nauk SSSR* **141** 1038–1041.

[4] SHIELDS, P. (1973). Discussion on Professor Ornstein's paper. *Ann. Probability* **1** 59.

[5] SHIELDS, P. (1973). The Theory of Bernoulli Shifts. Univ. of Chicago Press.

[6] THOUVENOT, J.-P. (1975). Quelques propriétés des systèmes dynamiques qui se décomposent en un produit de deux systèmes dont l'un est un schéma de Bernoulli. To appear in *Israel J. Math.*

[7] THOUVENOT, J.-P. (1975). Une classe de systèmes pour lesquels la conjecture de Pinsker est vraie. To appear in *Israel J. Math.*

LABORATOIRE DE PROBABILITÉ
UNIVERSITÉ PARIS VI
9 QUAI SAINT BERNARD
75230 PARIS CEDEX 05
FRANCE

MATHEMATICS DEPARTMENT
UNIVERSITY OF TOLEDO
TOLEDO, OHIO 43606