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A RENEWAL THEOREM FOR CURVED BOUNDARIES AND MOMENTS OF FIRST PASSAGE TIMES[1]

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Let $X_1, X_2, \cdots$ be i.i.d. with a finite positive mean μ and a finite positive variance σ^2 and let $S_n = X_1 + \cdots + X_n$, $n \geqq 1$. Further, let $0 \leqq \alpha < 1$ and t_c be the first $n \geqq 1$ for which $S_n > cn^\alpha$ and let $W_c(a) = \sum_{n=1}^\infty P\{t_c > n, c(n+1)^\alpha - S_n < a\}$. Under some additional conditions on the distribution of X_1 we show that W_c converges weakly to a limit W, where $W'(a) = \beta\mu^{-1}P\{S_k \geqq (k+1)\alpha\mu - a$, for all $k \geqq 0\}$, with $\beta = 1/(1-\alpha)$. We then find the asymptotic distribution of the excess $R_c = S_{t_c} - ct_c^\alpha$ and show that R_c is asymptotically independent of $t_c^* = (t_c - E(t_c))/E(t_c)^{\frac{1}{2}}$, and we compute $E(t_c)$ up to terms which are $o(1)$ as $c \to \infty$.

1. Introduction. Let $X_1, X_2, \cdots$ denote independent and identically distributed random variables which have a finite positive mean μ, let $S_0 = 0$, and let $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. Further, let V be a positive function on $(0, \infty)$ for which $V(x) = o(x)$ as $x \to \infty$, and let

$$t_c = \inf\{n \geqq 1 : S_n > cV(n)\}$$

for $c > 0$, where $\inf \emptyset = \infty$. Observe that t_c is finite w.p. 1. If $V(x) = 1$ for all $x > 0$, $\Pr\{X_1 > 0\} = 1$, and the distribution of X_1 is nonarithmetic, then the renewal theorem (Feller (1966), page 347) asserts that $U(c) = E\{t_c\}$ is finite for every $c > 0$ and that

$$\lim U(c) - U(c-a) = a\mu^{-1} \tag{1.1}$$

as $c \to \infty$ for every $a > 0$.

In this paper we will extend (1.1) to a wider class of boundaries V and to distributions which are not concentrated on $(0, \infty)$, although we will impose some additional conditions on the distribution of X_1.

We will assume throughout this paper that V is a positive, continuous, eventually concave function on $(0, \infty)$ for which $V(x) = x^\alpha L(x)$ for $x > 0$, where $0 \leqq \alpha < 1$, and

$$\frac{L(x+y)}{L(x)} - 1 = o\left(\frac{1}{x}\right) \tag{1.2}$$

as $x \to \infty$ for every $y > 0$. Equation (1.2) is more restrictive than requiring that L vary slowly at ∞, but includes all functions of the form $L(x) = \log^k(1+x)$,

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$x > 0$, with $k \in R$. Let

$$W_c(a) = \sum_{k=0}^{\infty} P\{t_c > k, cV(k+1) - S_k < a\}$$

for $a > 0$ and $c > 0$. We observe that if $V = 1$ and $X_1 > 0$, then $W_c(a) = U(c) - U(c-a)$ for $0 < a < c$. In Section 3, we will show that if X_1 has a finite positive variance, a finite third moment, and a suitably smooth distribution, then

$$\lim W_c(a) = \beta\mu^{-1} \int_0^a \psi(y)\, dy \tag{1.3}$$

for $a > 0$ as $c \to \infty$, where $\beta = 1/(1-\alpha)$ and

$$\psi(y) = P\{S_k \geqq (k+1)\alpha\mu - y, \text{ for all } k \geqq 0\} \tag{1.4}$$

for $0 \leqq y < \infty$. Of course, if $V = 1$ and $X_1 > 0$, then (1.3) reduces to (1.1). If $V = 1$, then (1.3) is closely related to, but distinct from, a result of Spitzer (1960), Theorem 5.1a.

Relation (1.3) has several interesting consequences. Let

$$R_c = S_{t_c} - cV(t_c)\,, \qquad c > 0\,,$$

denote the excess over the boundary. In Section 4 we will show that R_c has a limiting distribution H as $c \to \infty$, where H has density

$$h(y) = \beta\mu^{-1}P\{S_k \geqq k\alpha\mu + y, \text{ for all } k \geqq 1\}$$

for $0 \leqq y < \infty$. Moreover, we will show that R_c is asymptotically independent of $t_c^* = (t_c - E\{t_c\})/E\{t_c\}^{\frac{1}{2}}$, which has an asymptotic normal distribution.

In Section 5 we use the asymptotic distribution of R_c to compute $E\{t_c\}$ up to terms which are $o(1)$ as $c \to \infty$. In fact, we show that if $V(x) = x^\alpha$ for $x > 0$, then

$$E\{t_c\} = c^\beta\mu^{-\beta} + \beta\mu^{-1}\nu - \tfrac{1}{2}\alpha\beta^2\sigma^2\mu^{-2} + o(1)$$

as $c \to \infty$, where σ^2 is the variance of X_1 and ν denotes the expectation of H (and is given explicitly in equation (4.5)).

The asymptotic properties of stopping times of the form t_c have been studied by Chow and Robbins (1963), Chow (1966), Siegmund (1967, 1968, and 1969), and Gut (1972 and 1974) under a variety of assumptions on V and the distribution of $X_1, X_2, \cdots$.

2. Preliminaries. In this section we will suppose that $X_1, X_2, \cdots$ are i.i.d. random variables with a finite positive mean μ and a finite positive variance σ^2. In addition, we will sometimes impose the following condition.

CONDITION C. X_1 has a density f (with respect to Lebesgue measure) which is continuous a.e. (with respect to Lebesgue measure). Moreover, some power of the characteristic function of X_1 is integrable.

To motivate the lemmas of the section, we observe that

$$W_c(b) - W_c(a) = \sum_{k=1}^{\infty} \int_{I_k} P\{t > k \mid S_k^* = z\}\, dG_k^*(z) + I_{[a,b)}(cV(1)) \tag{2.1}$$

for $0 \leqq a < b < \infty$, where

(2.2a) $$S_k^* = \frac{S_k - k\mu}{\sigma k^{\frac{1}{2}}}$$

(2.2b) $$I_k = I_k(a, b) = \left(y_k - \frac{b}{\sigma k^{\frac{1}{2}}}, y_k - \frac{a}{\sigma k^{\frac{1}{2}}}\right]$$

(2.2c) $$y_k = \frac{cV(k+1) - k\mu}{\sigma k^{\frac{1}{2}}},$$

and G_k^* denotes the distribution function of S_k^* for $k \geqq 1$. We observe that for a.e. $z \in I_k[G_k^*]$, $P\{t > k \mid S_k^* = z\} \leqq \psi_{k,c}(b, z)$, where

$$\psi_{k,c}(b, z) = P\{S_{kj} \leqq cV(j) - jk^{-1}[cV(k+1) - b], 1 \leqq j \leqq k \mid S_k^* = z\}$$

with

$$S_{kj} = S_j - jk^{-1}S_k$$

for $j = 1, \cdots, k$. Similarly, $P\{t > k \mid S_k^* = z\} \geqq \psi_{k,c}(a, z)$ for a.e. $z \in I_k[G_k^*]$. In this section we will derive an asymptotic expression for an appropriate version of $\psi_{k,c}$.

Let $X_{ki} = X_i - k^{-1}S_k$, $j = 1, \cdots, k-1$. If Condition C is satisfied, then we may construct a regular conditional distribution for $X_{k1}, \cdots, X_{k(k-1)}$ given S_k^* as follows. By Condition C and the local limit theorem for densities (Feller (1966), pages 489–490), S_k^* has a density g_k^* which is continuous for k sufficiently large and converges to ϕ, the standard normal density, uniformly on R. Let g_0 denote any density on R and let

$$g_k(y_1, \cdots, y_{k-1}, z) = \prod_{i=1}^{k-1} f(y_i + \mu + \sigma k^{-\frac{1}{2}} z)\sigma k^{\frac{1}{2}} f(\mu + \sigma k^{-\frac{1}{2}} z - \sum_{i=1}^{k-1} y_i)$$

and

$$g_k(y_1, \cdots, y_{k-1} \mid z) = g_k(y_1, \cdots, y_{k-1}, z) g_k^*(z)^{-1} I_{(0,\infty)}(g_k^*(z)) + \prod_{i=1}^{k-1} g_0(y_i) I_{\{0\}}(g_k^*(z))$$

for $(y_1, \cdots, y_{k-1}, z) \in R^k$. Then

(2.3) $$Q_k(z, B) = \int_B g_k(y_1, \cdots, y_{k-1} \mid z)\, dy_1 \cdots dy_{k-1},$$

$B \in \mathbf{B}(R^{k-1})$, $z \in R$, defines a regular conditional distribution for $X_{k1}, \cdots, X_{k(k-1)}$ given S_k^*. We will sometimes write $Q_{k,z}$ for $Q_k(z, \cdot)$.

Let $Y_{k1}, \cdots, Y_{k(k-1)}$ denote the coordinate functions on R^{k-1}, $k \geqq 2$, and let $\pi_{kj} = (Y_{k1}, \cdots, Y_{kj})$ denote the projection from R^{k-1} onto R^j, $1 \leqq j < k$. Also, let F_1 denote the distribution of $X_1 - \mu$ and let F_1^j denote the product of j copies of F_1.

LEMMA 2.1. *If Condition C is satisfied and if I is any compact interval, then for any $j \geqq 1$, $Q_{k,z} \circ \pi_{kj}^{-1}$ converges strongly to F_1^j uniformly with respect to $z \in I$. Moreover, letting $T_{kj} = Y_{k1} + \cdots + Y_{kj}$ for $1 \leqq j \leqq k-1$,*

$$\lim \int |T_{kj}|\, dQ_{k,z} = E|S_j - j\mu|$$

as $k \to \infty$ uniformly with respect to $z \in I$ for each $j \geqq 1$.

PROOF. If I is given, then there is a $k_0 = k_0(I)$ for which $g_k{}^*$ is positive and continuous on I for $k \geqq k_0$. It is then easily seen that for $k \geqq k_0$ and $z \in I$, $Q_{k,z} \circ \pi_{kj}^{-1}$ has density

$$(2.4)\qquad \begin{aligned} h_{kj}(y_1, \cdots, y_j \mid z) &= \Big[\Big(\frac{k}{k-j}\Big)^{\frac{1}{2}} \prod_{i=1}^{j} f(y_i + \mu + \sigma k^{-\frac{1}{2}} z) \\ &\quad \times g_{k-j}^*\Big\{\Big(\frac{k-j}{k}\Big)^{\frac{1}{2}} z - \sigma^{-1}(k-j)^{-\frac{1}{2}} \textstyle\sum_{i=1}^{j} y_i\Big\}\Big]\Big/ g_k{}^*(z) \end{aligned}$$

with respect to Lebesgue measure. Thus if z_k, $k \geqq 1$, is any sequence from I, then

$$\lim h_{kj}(y_1, \cdots, y_j \mid z_k) = \textstyle\prod_{i=1}^{j} f(y_i + \mu)$$

for a.e. $(y_1, \cdots, y_j)$. The first assertion of the lemma now follows from Scheffe's theorem (Lehmann (1959), page 351).

Let M_k and m_k denote the maximum of $g_k{}^*$ and the minimum of $g_k{}^*(z)$ for $z \in I$, respectively. Then $h_{kj}(y_1, \cdots, y_j \mid z) \leqq 2M_{k-j} m_k{}^{-1} \prod_{i=1}^{j} f(y_i + \mu + k^{-\frac{1}{2}}\sigma z)$ for $y_1, \cdots, y_j \in R^j$, $z \in I$, and k sufficiently large, so

$$\textstyle\int T_{kj}^2 \, dQ_{k,z} \leqq 2M_{k-j} m_k{}^{-1} j\sigma^2 \{1 + jk^{-1}z^2\}$$

is uniformly bounded for $z \in I$ and k sufficiently large. The lemma now follows from the convergence of moments theorem (Loève (1963), page 184).

It is easily seen that the equation

$$(2.5)\qquad \mu\lambda = cV(\lambda)$$

has a unique solution $\lambda = \lambda(c)$ for c sufficiently large, and it is known that $t_c/\lambda \to 1$ w.p. 1 as $c \to \infty$ (Siegmund (1967)).

In our next lemma we let $T_{kj} = Y_{k1} + \cdots + Y_{kj}$ for $1 \leqq j < k$.

LEMMA 2.2. *Suppose that Condition* C *is satisfied and let I be any compact interval. If $k = k_c \to \infty$ as $c \to \infty$ in such a manner that $k \sim \lambda$, then*

$$\lim_{m\to\infty} \limsup_{c\to\infty} Q_{k,z}\{T_{kj} \geqq cV(j) - jk^{-1}[cV(k+1) - y] \text{ for some } j \leqq k - m\} = 0$$

uniformly with respect to $z \in I$ for any $y \in R$.

The proof of Lemma 2.2 is similar to that of Lemma 2.1, but longer and much more technical. We defer it to Section 6.

LEMMA 2.3. *Suppose that Condition* C *is satisfied and let I be any compact interval. Let*

$$\psi_{k,c}(y, z) = Q_{k,z}\{T_{kj} \leqq cV(j) - jk^{-1}[cV(k+1) - y],\ 1 \leqq j \leqq k\}$$

for $y \in R$ and $z \in R$. If $k \to \infty$ as $c \to \infty$ in such a manner that $k \sim \lambda$, then $\lim \psi_{k,c}(y, z) = \psi(y)$ uniformly with respect to $z \in I$ for every $y \in R$, where $\psi(y) = P\{S_j \geqq (j+1)\alpha\mu - y$, for all $j \geqq 0\}$, as in (1.4).

PROOF. By Lemma 2.2, $\psi_{k,c}(y, z) = \psi^*_{k,m}(y, z) + \varepsilon(c, m)$, where $\psi^*_{k,m}(y, z) = Q_{k,z}\{T_{kj} \leqq cV(j) - jk^{-1}[cV(k+1) - y]$, for $k - m \leqq j \leqq k\}$ and $\lim_{m\to\infty} \limsup_{c\to\infty} |\varepsilon(c, m)| = 0$, uniformly on compacts (in z) for each fixed y. By symmetry, we also have

$$\psi^*_{k,m}(y, z) = Q_{k,z}\{-T_{kj} \leqq cV(k - j) - (k - j)k^{-1}[cV(k + 1) - y], \text{ for } 0 \leqq j \leqq m\}.$$

Moreover,

$$cV(k - j) - (k - j)k^{-1}[cV(k + 1) - y] \to (1 - \alpha)\mu j - \alpha\mu$$

as $c \to \infty$ for each fixed j. Thus, by Lemma 2.1, $\psi^*_{k,m}(y, z) \to P\{S_j \geqq (j + 1)\alpha\mu - y$, for $0 \leqq j \leqq m\}$ as $c \to \infty$ for each fixed m. The lemma now follows by letting $m \to \infty$, since $P\{S_j \leqq (j + 1)\alpha\mu - y$, for some $j \geqq m\} \to 0$ as $m \to \infty$.

3. A renewal theorem. In this section we will suppose that X_1 has a finite positive mean μ and a finite positive variance σ^2. Let I be any compact interval and let

$$J_0 = \{k \geqq 0 : y_k \in I\} \qquad \text{and} \qquad J_1 = \{k \geqq 0 : y_k \notin I\},$$

where y_k is as in (2.2c). Further, let

$$W_c^i(a) = \sum_{k \in J_i} P\{t > k, cV(k + 1) - S_k < a\}, \qquad i = 0, 1, \tag{3.1}$$

for $a \geqq 0$, so that $W_c = W_c^0 + W_c^1$. The dependence of W_c^0 and W_c^1 on I will be suppressed in the notation.

LEMMA 3.1. *If Condition C is satisfied and if I is any finite interval, then*

$$\lim W_c^0(b) - W_c^0(a) = (\beta\mu^{-1} \textstyle\int_a^b \psi(z)\, dz)(\int_I \phi(z)\, dz)$$

as $c \to \infty$ for $0 \leqq a < b < \infty$, where ϕ denotes the standard normal density and ψ is as in (1.4).

PROOF. Let I_k be as in (2.2b). Then

$$W_c^0(b) - W_c^0(a) \leqq \sum_{k \in J_0} \textstyle\int_{I_k} \psi_{k,c}(b, z) g_k^*(z)\, dz, \tag{3.2}$$

where g_k^* denotes a density for S_k^* and $\psi_{k,c}$ is as in Lemma 2.3. As $c \to \infty$, $k \sim \lambda$ uniformly with respect to $k \in J_0$ and $g_k^*(z) \to \phi(z)$ uniformly with respect to $k \in J_0$ and $z \in R$, so that

$$\textstyle\int_{I_k} \psi_{k,c}(b, z) g_k^*(z)\, dz \sim \psi(b)\sigma^{-1}k^{-\frac{1}{2}}(b - a)\phi(y_k)$$

uniformly with respect to $k \in J_0$. Since $y_k - y_{k+1} \sim (1 - \alpha)\mu\sigma^{-1}k^{-\frac{1}{2}}$ uniformly with respect to $k \in J_0$, it now follows easily that

$$\limsup W_c^0(b) - W_c^0(a) \leqq \beta\mu^{-1}(b - a)\psi(b) \textstyle\int_I \phi(y)\, dy$$

as $c \to \infty$. A similar lower bound with $\psi(a)$ replacing $\psi(b)$ may be obtained for the lim inf, and the lemma then follows from the Riemann integrability of ψ.

LEMMA 3.2. *Suppose that Condition C is satisfied and that* $E|X_1^3| < \infty$. *Then*

$$\lim_{I \uparrow R} \limsup_{c\to\infty} W_c^1(b) - W_c^1(a) = 0$$

for $0 \leqq a < b < \infty$. *Moreover, if I is any finite interval, then there is a constant* $B = B_I$ *for which* $W_c^1(y) \leqq B(1 + y)$ *for all* $y \geqq 0$ *and all* $c > 0$.

The proof of Lemma 3.2 is somewhat similar to that of Lemma 3.1, but longer and more technical. We defer it to Section 6.

THEOREM 3.1. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with finite positive mean* μ, *finite positive variance* σ^2, *and a finite third moment. Suppose also that Condition* C *is satisfied. Then*

$$\lim W_c(a) = \beta\mu^{-1} \int_0^a \psi(y)\, dy$$

for $0 \leqq a < \infty$ *as* $c \to \infty$. *Here* $\beta = 1/(1 - \alpha)$ *and* ψ *is as in* (1.4).

PROOF. The theorem follows directly from Lemmas 3.1 and 3.2 by letting $c \to \infty$ and $I \uparrow R$ in that order.

It is clear from Theorem 3.1 that if z is a continuous function with compact support in $[0, \infty)$, then

$$\lim \int_0^\infty z(y)\, dW_c(y) = \beta\mu^{-1} \int_0^\infty z(y)\psi(y)\, dy \tag{3.3}$$

as $c \to \infty$. A more interesting class of functions is covered by the following theorem.

THEOREM 3.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied. If* z *is a nonincreasing, nonnegative, integrable function on* $[0, \infty)$, *then* (3.3) *holds.*

PROOF. To establish Theorem 3.2, one integrates the left side of (3.3) by parts, applies the dominated convergence theorem, and then integrates the limit by parts. The dominating function is supplied by taking $I = \varnothing$ in the second conclusion of Lemma 3.2.

4. On the excess over the boundary. In this section we will find and study the asymptotic distribution of the excess over the boundary

$$R_c = S_{t_c} - cV(t_c)$$

as $c \to \infty$. We denote the distribution function of R_c by H_c.

THEOREM 4.1. *If the hypotheses of Theorem* 3.1 *are satisfied, then* H_c *converges weakly to a limit* H *as* $c \to \infty$, *where* H *has density*

$$h(y) = \beta\mu^{-1}P\{S_j \geqq j\alpha\mu + y, \ \text{for all} \ j \geqq 1\}\,. \tag{4.1}$$

Moreover, if $E|X_1^{k+1}| < \infty$, *then the first* k *moments of* H_c *converge to those of* H.

PROOF. Let F denote the distribution function of X_1. Then,

$$H_c(a) = \int_0^\infty [F(a + y) - F(y)]\, dW_c(y) \tag{4.2}$$

for $0 \leqq a < \infty$. Moreover, for each $a > 0$, the integrand in (4.2) is the difference of two functions to which Theorem 3.2 applies, so that $H_c(a)$ converges to

$$H(a) = \beta\mu^{-1} \int_0^\infty [F(a + y) - F(y)]\psi(y)\, dy$$

as $c \to \infty$. That $H' = h$ follows from standard manipulations, but it is not clear that h is a density. To see this observe that

$$\int_0^\infty h(y)\, dy = \beta\mu^{-1}E\{M^+\},$$

where $M = \min\{S_j' : j \geqq 1\}$ with $X_i' = X_i - \alpha\mu$ and $S_j' = X_1' + \cdots + X_j'$. Now $M = X_1' + \min\{M_1, 0\}$, where M_1 has the same distribution as M, so that $E\{M\} = (1 - \alpha)\mu - E\{M^-\}$—that is, $E\{M^+\} = (1 - \alpha)\mu$.

The proof of the second assertion in Theorem 4.1 is similar to that of the first.

We need the following result which is due to Siegmund (1968) in the case that $V(x) = x^\alpha$ and has been extended by Gut (1974), pages 299–300.

THEOREM 4.2. *Let $X_1, X_2, \cdots$ be* i.i.d. *with finite positive mean μ and finite positive variance σ^2. Then the distribution of*

$$t_c^* = \lambda^{-\frac{1}{2}}(t_c - \lambda)$$

converges weakly to the normal distribution with mean 0 *and variance $\tau^2 = \beta^2\sigma^2\mu^{-2}$ as $c \to \infty$.*

THEOREM 4.3. *If the hypotheses of Theorem* 3.1 *are satisfied, then t_c^* and R_c are asymptotically independent. That is,*

$$\lim P\{t_c^* \in I, R_c \in J\} = \left(\tau^{-1} \int_I \phi(y\tau^{-1})\, dy\right)\left(\int_J h(y)\, dy\right) \tag{4.3}$$

as $c \to \infty$ for all intervals I and J.

PROOF. Let

$$s_c^* = \sigma^{-1}(t_c - 1)^{-\frac{1}{2}}[cV(t_c) - \mu(t_c - 1)];$$

then it follows easily from Taylor's theorem and Theorem 4.2 that $s_c^* - \tau^{-1}t_c^* \to 0$ in probability as $c \to \infty$, so it will suffice to show that s_c^* and R_c are asymptotically independent. If I and $J = [a, b]$ are finite intervals, then

$$P\{s_c^* \in I, R_c \in J\} = \int_0^\infty [F(b + y) - F(a + y)]\, dW_c^0(y),$$

where W_c^0 is as in (3.1). It now follows easily from Lemma 3.1 and an argument similar to that given in the proof of Theorem 4.1 that

$$\lim P\{s_c^* \in I, R_c \in J\} = \left(\int_I \phi\, dy\right)\left(\int_J h\, dy\right).$$

The extension to unbounded intervals is routine.

We will now relate the asymptotic distribution H to the distribution F of X_1. Let $X_i' = X_i - \alpha\mu$ and $M = \min\{S_j' : j \geqq 1\}$, as above. Further, let F_α and ϕ_α denote the distribution function and characteristic function of X_1' and let G and w denote the distribution function of M and the characteristic function of $M^- = \min\{M, 0\}$, respectively. Then $h = \beta\mu^{-1}(1 - G)$ on $[0, \infty)$, and a result of Spitzer (1960) asserts that

$$w(t) = \exp\{\textstyle\sum_{k=1}^{\infty} k^{-1} \int_{-\infty}^{0} (e^{itx} - 1)\, dF_{\alpha}{}^{*k}(x)\}\,, \tag{4.4}$$

where * denotes convolution. See also Feller (1966), page 576.

THEOREM 4.4. *Let* $X_1, X_2, \cdots$ *be* i.i.d. *with finite positive mean* μ *and let* H *be as in Theorem* 4.1. *Then the characteristic function of* H *is given by*

$$\hat{H}(t) = \beta\mu^{-1}\left(\frac{\phi_{\alpha}(t) - 1}{it}\right) w(t)$$

for $t \neq 0$, *where* w *is as in* (4.4). *If, in addition,* X_1 *has finite, positive variance* σ^2, *then the mean of* H *is*

$$\nu = \frac{\sigma^2 + (1-\alpha)^2\mu^2}{2(1-\alpha)\mu} - \textstyle\sum_{k=1}^{\infty} k^{-1} E\{(S_k - k\alpha\mu)^-\}\,. \tag{4.5}$$

The first assertion follows from the identity $M = X_1' + \min\{M_1, 0\}$, where $M_1 = \min\{S'_{j+1} - S_1' : j \geqq 1\}$ has the same distribution as M and is independent of X_1'. The second then follows by differentiation. We omit the details.

5. On the expectation of t_c. In this section we will derive an asymptotic expression for the expectation of the first passage time t_c. We suppose throughout that $X_1, X_2, \cdots$ are i.i.d. with $\mu > 0$ and $0 < \sigma^2 < \infty$. We will also suppose that $V(x) = x^{\alpha}$, $x > 0$, in which case

$$\lambda = c^{\beta}\mu^{-\beta}$$

for $c > 0$. We will need to know when powers of $t_c{}^* = \lambda^{-\frac{1}{2}}(t_c - \lambda)$ are uniformly integrable.

THEOREM 5.1. *If* $E|X_1|^{\gamma} < \infty$ *for some* $\gamma > \max\{4, \beta\}$, *then* $t_c{}^{*2}$ *is uniformly integrable with respect to* $c > 0$.

The proof of Theorem 5.1 will be given in Section 7.

THEOREM 5.2. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied, that* $V(x) = x^{\alpha}$, $x > 0$, *and that* $E|X_1|^{\gamma} < \infty$ *for some* $\gamma > \max\{4, \beta\}$. *Then*

$$E\{t_c\} = c^{\beta}\mu^{-\beta} + \beta\mu^{-1}\nu - \tfrac{1}{2}\alpha\beta^2\sigma^2\mu^{-2} + o(1)$$

as $c \to \infty$, *where* ν *is as in* (4.5).

PROOF. By Wald's lemma we have $\mu E\{t_c\} = cE\{t_c{}^{\alpha}\} + E\{R_c\}$. This may also be written as

$$\begin{aligned} E\{t_c - \lambda\} &= \lambda^{1-\alpha} E\{t_c{}^{\alpha} - \lambda^{\alpha}\} + \mu^{-1}E\{R_c\} \\ &= \lambda^{1-\alpha} E\{\alpha\lambda^{\alpha-1}(t_c - \lambda) - \tfrac{1}{2}\alpha(1-\alpha)\lambda_1{}^{\alpha-2}(t_c - \lambda)^2\} + \mu^{-1}E\{R_c\}\,, \end{aligned}$$

where $|\lambda_1 - \lambda| \leqq |t_c - \lambda|$. Equivalently,

$$E\{t_c - \lambda\} = \beta\mu^{-1}E\{R_c\} - \tfrac{1}{2}\alpha E\{(\lambda \cdot \lambda_1{}^{-1})^{2-\alpha} t_c{}^{*2}\}\,.$$

Now $E\{R_c\} \to \nu$, $|\lambda/\lambda_1| \to 1$, and $t_c{}^{*2}$ is uniformly integrable, so it will suffice to show that $|\lambda/\lambda_1|$ is bounded. It is clear that $|\lambda/\lambda_1|$ is bounded on $\{t_c \geqq \frac{1}{2}\lambda\}$. Moreover, on $\{t_c < \frac{1}{2}\lambda\}$, we have

$$\tfrac{1}{2}\alpha(1-\alpha)\lambda_1^{\alpha-2} = \{\lambda^\alpha - t_c^\alpha + \alpha\lambda^{\alpha-1}(t_c - \lambda)\}(t_c - \lambda)^{-2} \leqq 4\lambda^{\alpha-2}.$$

6. Proofs. In this section we will present the proofs of Lemmas 2.2 and 3.2. We suppose throughout this section that $X_1, X_2, \cdots$ are i.i.d. with a positive mean μ and a finite positive variance σ^2.

In order to prove Lemma 2.2, we need some auxiliary results. The first of these is an invariance principle for conditional distributions which may be of minor interest in its own right. In the notation of Section 2, let $T_{kj} = Y_{k1} + \cdots + Y_{kj}$, $j < k$, where Y_{ki} are the coordinate functions on R^{k-1}. Further, let Z_k be a continuous function on [0, 1] for which $Z_k(0) = 0 = Z_k(1)$ and

$$Z_k(jk^{-1}) = \sigma^{-1}k^{-\frac{1}{2}}T_{kj}, \qquad j = 1, \cdots, k-1,$$

and Z_k is linear on each of the intervals $[(j-1)k^{-1}, jk^{-1}]$, $j = 1, \cdots, k-1$. It is easily seen that Z_k is a measurable mapping from R^{k-1} into $C[0, 1]$, when both spaces are endowed with their Borel sigma algebras. Let $Q_{k,z}$ be as in (2.3), let Q_0 denote the distribution of a Brownian bridge in $C[0, 1]$, and let d denote the Prokhorov distance between probability measures on the Borel sets of $C[0, 1]$ (Billingsley (1968), pages 237–238).

LEMMA 6.1. *If Condition C is satisfied and if I is any compact interval, then* $\lim d(Q_{k,z} \circ Z_k^{-1}, Q_0) = 0$ *uniformly with respect to $z \in I$ as $k \to \infty$.*

PROOF. It will suffice to show that if $z_k \to z \in R$, then $Q_k^* = Q_{k,z_k} \circ Z_k^{-1}$ converges weakly to Q_0. That the finite dimensional distributions of Q_k^* converge (strongly) to those of Q_0 follows from the local limit theorem for densities (Feller (1966), pages 489–490) by an argument which is similar to that given in Steck (1957).

To show that Q_k^*, $k \geqq 1$, is tight, it will suffice to show that for every $\varepsilon > 0$, there is an $n = n_\varepsilon$ for which

$$\text{(6.1)} \qquad \sup_{k \geqq n_\varepsilon} Q_{k,z_k}\{|T_{kj}| > \varepsilon k^{\frac{1}{2}}, \text{ for some } j \leqq k\delta\} = o(\delta)$$

as $\delta \to 0$ (Billingsley (1968), page 56). Given $\varepsilon > 0$, let $\delta > 0$ and let $m = [k\delta]$ be the greatest integer which is less than or equal to $k\delta$. Further, let A_k be the set of $y \in R^m$ for which $|T_{m+1,j}| > \varepsilon k^{\frac{1}{2}}$ for some $j \leqq m$, so that the left side of (6.1) is simply $Q_{k,z_k} \circ \pi_{km}^{-1}(A_k)$. If h_{km} is as in (2.4), then, as in the proof of Lemma 2.1, there is a constant B for which

$$h_{km}(y_1, \cdots, y_m \mid z_k) \leqq B \prod_{i=1}^m f(y_i + \mu + \sigma k^{-\frac{1}{2}} z_k)$$

for all $y = (y_1, \cdots, y_m) \in R^m$ and all sufficiently large k. It then follows easily that

$$\begin{aligned} Q_{k,z_k} \circ \pi_{km}^{-1}(A_k) &\leqq BP\{|S_j - j\mu - j\sigma k^{-\frac{1}{2}} z_k| > \varepsilon k^{\frac{1}{2}}, \text{ for some } j \leqq m\} \\ &\leqq BP\{|S_j - j\mu| > \tfrac{1}{2}\varepsilon k^{\frac{1}{2}}, \text{ for some } j \leqq m\} \end{aligned}$$

for k sufficiently large and δ sufficiently small. Relation (6.1) now follows from standard arguments (Billingsley (1968), page 69).

LEMMA 6.2. *Suppose that Condition C is satisfied. Let $F_j \subset \mathbf{B}(R^{k-1})$ be the smallest sigma algebra with respect to which Y_{ki}, $i = j, \cdots, k-1$, are measurable. If I is any compact interval, then $\{j^{-1}T_{kj}, \mathbf{F}_j, j = 1, \cdots, k-1\}$ is a reverse martingale on the probability space $(R^{k-1}, \mathbf{B}(R^{k-1}), Q_{k,z})$ for all $z \in I$ for k sufficiently large.*

Lemma 6.2 follows easily from considerations of symmetry.

We will now prove Lemma 2.2, which asserts that if Condition C is satisfied, and if $k = k_c \sim \lambda$ as $c \to \infty$, then $\lim_{m\to\infty} \limsup_{c\to\infty} Q_{k,z}\{T_{kj} \geqq cV(j) - jk^{-1}[cV(k+1) - y]$, for some $j \leqq k - m\} = 0$ uniformly in $z \in I$ for any compact interval I and any $y \in R$. We divide the range $j \leqq k - m$ into $j \leqq k\delta$, $k\delta < j < k(1-\varepsilon)$, and $k(1-\varepsilon) \leqq j \leqq k - m$, where $0 < \delta < \varepsilon < \frac{1}{2}$. Since $v(j) = V(j) - jk^{-1}V(k+1)$ is concave on $k\delta \leqq j \leqq k(1-\varepsilon)$ for c sufficiently large, we have $\min\{v(j) : k\delta \leqq j \leqq k(1-\varepsilon)\} \geqq \min\{v(k\delta), v(k(1-\varepsilon))\} = m(c)$, say, for c sufficiently large. It is easily seen that $m(c) \sim v(k\delta) \sim V(k)[\delta^\alpha - \delta]$ as $c \to \infty$ for δ sufficiently small. In particular, $ck^{-\frac{1}{2}}m(c) \to \infty$ as $c \to \infty$, so that

$$\lim Q_{k,z}\{T_{kj} \geqq cv(j), \text{ for some } j, k\delta \leqq j \leqq k(1-\varepsilon)\} = 0 \tag{6.2}$$

uniformly with respect to $z \in I$ as $c \to \infty$ by Lemma 6.1.

From (1.2) it is easily seen that $j^{-1}V(j)$ is nonincreasing in j for j sufficiently large. Let $u(j) = j^{-1}V(j) - k^{-1}V(k)$; then $\min\{u(j) : j \leqq k\delta\} \sim u(k\delta) \sim k^{-1}V(k)[\delta^{\alpha-1} - 1]$ as $c \to \infty$. Thus, for c sufficiently large and δ sufficiently small

$$\begin{aligned} Q_{k,z}\{T_{kj} &\geqq cV(j) - jk^{-1}[cV(k+1) - y], \text{ for some } j \leqq k\delta\} \\ &\leqq Q_{k,z}\{j^{-1}T_{kj} \geqq \tfrac{1}{2}cV(k)\delta^{\alpha-1}k^{-1}, \text{ for some } j \leqq k\delta\} \\ &\leqq \frac{2k\delta^{1-\alpha}}{cV(k)} \int |y| h_{k1}(y)\, dy \leqq 4\mu^{-1}\delta^{1-\alpha} E|X_1 - \mu| \end{aligned} \tag{6.3}$$

by Lemmas 2.1 and 6.2 and the martingale inequality.

Finally, if ε is sufficiently small and m is sufficiently large, then

$$cV(k-j) - (k-j)k^{-1}[cV(k+1) - y] \geqq \tfrac{1}{2}j\mu(1-\alpha)$$

for $m \leqq j \leqq k\varepsilon$ and c sufficiently large. It now follows from symmetry, Lemma 6.2, and the martingale inequality that

$$\begin{aligned} Q_{k,z}\{T_{kj} &\geqq cV(j) - jk^{-1}[cV(k+1) - y], \\ &\qquad \text{for some } j, k(1-\varepsilon) \leqq j \leqq k - m\} \\ &\leqq Q_{k,z}\{-T_{kj} \geqq \tfrac{1}{2}j\mu(1-\alpha), \text{ for some } j, m \leqq j \leqq k\varepsilon\} \\ &\leqq 2\beta\mu^{-1} \int |m^{-1}T_{km}|\, dQ_{k,z} \end{aligned} \tag{6.4}$$

for all $z \in I$, if c is sufficiently large (and m is sufficiently large and ε is sufficiently small). By Lemma 2.1, the right side of (6.4) tends to zero as $c \to \infty$ and

$m \to \infty$ (in that order). Thus, Lemma 2.2 follows from (6.2), (6.3), and (6.4) by letting $c \to \infty$, $\delta \to 0$, and $m \to \infty$ (in that order).

To prove Lemma 3.2 we will also need some preparation. We will use the fact that if $E|X_1^3| < \infty$, then there is a constant B for which

$$|g_k^*(z) - \phi(z)| \leqq Bk^{-\frac{1}{2}}(1 + |z^3|)^{-1} \tag{6.5}$$

for all $z \in R$ for k sufficiently large, where g_k^* denotes the density of S_k^*. See Petrov (1964). In particular, it follows from (6.5) that $g^* = \sup\{g_k^* : k \geqq 1\}$ is integrable.

LEMMA 6.3. *Let* $y_k = \sigma^{-1}k^{-\frac{1}{2}}[cV(k+1) - k\mu]$, $k \geqq 1$, *as in* (2.2c). *Then there is a sequence* ε_k, $k \geqq 1$, *for which* $\varepsilon_k \to 0$ *as* $k \to \infty$ *and*

$$y_k - y_{k+1} \geqq \frac{1}{(k+1)^{\frac{1}{2}}} \left\{ \frac{\mu}{2\sigma} - (\alpha - \tfrac{1}{2} - 2\varepsilon_k) \frac{cV(k+1)}{\sigma(k+1)} \right\} \qquad k \geqq 1 .$$

PROOF.

$$\begin{aligned} y_k - y_{k+1} &= c\sigma^{-1}\{k^{-\frac{1}{2}}(k+1)^\alpha - (k+1)^{-\frac{1}{2}}(k+2)^\alpha\}L(k+1) \\ &\quad + c\sigma^{-1}(k+1)^{-\frac{1}{2}}(k+2)^\alpha\{L(k+1) - L(k+2)\} \\ &\quad + \mu\sigma^{-1}\{(k+1)^{\frac{1}{2}} - k^{\frac{1}{2}}\} \\ &= \Delta_1 + \Delta_2 + \Delta_3 , \qquad \text{say.} \end{aligned}$$

Simple convexity arguments show that $-\Delta_1 \leqq c\sigma^{-1}V(k+1)(k+1)^{\alpha-\frac{3}{2}}[\alpha - \frac{1}{2}]$ and that $\Delta_3 \geqq \frac{1}{2}\mu\sigma^{-1}(k+1)^{-\frac{1}{2}}$ for $k \geqq 1$. By (1.2) there is a sequence ε_k, $k \geqq 1$ for which $\varepsilon_k \to 0$ as $k \to \infty$ and

$$|\Delta_2| \leqq c\sigma^{-1}(k+1)^{-\frac{3}{2}}(k+2)^\alpha L(k+1)\varepsilon_k \leqq 2c\sigma^{-1}(k+1)^{-\frac{3}{2}}V(k+1)\varepsilon_k$$

for $k \geqq 1$. Lemma 6.3 follows easily.

We will now give the proof of Lemma 3.2, which asserts that if Condition C is satisfied and if $E|X_1^3|$ is finite, then $\lim_{I \uparrow R} \limsup_{c\to\infty} W_c^1(b) - W_c^1(a) = 0$ for $0 \leqq a < b < \infty$ and that $W_c^1(y) \leqq B(1+y)$ for all $y \geqq 0$ and all $c > 0$ for some constant B (see (3.1)). Let

$$(\alpha - \tfrac{1}{2})^+ < \gamma < \tfrac{1}{2} \qquad \text{and} \qquad 2\delta = 1 - \gamma^{-1}(\alpha - \tfrac{1}{2})^+ .$$

Then, it will suffice to prove the first assertion of the lemma in the special case that $b - a \leqq \delta\mu/4\sigma$, for the general case may be reduced to a finite sum of terms to which the special case applies.

Let $J_{11} = \{k \in J_1 : k \leqq 2\gamma\lambda\}$ and $J_{12} = J_1 - J_{11}$. It follows easily from the fact that $k^{-1}V(k)$ is eventually nonincreasing that $cV(k) \leqq k\mu/2\gamma$ for $k \geqq 2\gamma\lambda$ and c sufficiently large. Thus

$$y_k - y_{k+1} \geqq \frac{\mu}{2\sigma}[1 - \gamma^{-1}(\alpha - \tfrac{1}{2} - 2\varepsilon_k)](k+1)^{-\frac{1}{2}} \geqq \frac{\delta\mu}{2\sigma k^{\frac{1}{2}}}$$

for $k \geqq 2\gamma\lambda$ and c sufficiently large by Lemma 6.3. Thus for c sufficiently large and $b - a \leqq \delta\mu/4\sigma$, the intervals $I_k = (y_k - \sigma^{-1}k^{-\frac{1}{2}}b, y_k - \sigma^{-1}k^{-\frac{1}{2}}a]$, $k \in J_{12}$, are disjoint. Consequently (with the obvious notational conventions)

$$W_c^{12}(b) - W_c^{12}(a) \leqq \sum_{k \in J_{12}} \int_{I_k} g_k^*(z)\, dz \leqq \int_{(I' \pm 1)} g^*(z)\, dz\,, \tag{6.6}$$

which is independent of a, b, and c and tends to zero as $I \uparrow R$.

To estimate W^{11} let k_0 be so large that $k^{-1}V(k)$ is nonincreasing for $k \geqq k_0$. If $\alpha < \alpha' < 1$, it is then the case that $ck^{-1}V(k) \geqq \mu(2\gamma)^{\alpha'-1}$ for $k_0 \leqq k \leqq 2\gamma\lambda + 1$ for c sufficiently large. Consequently,

$$cV(k+1) - k\mu \geqq k\mu\{(2\gamma)^{\alpha'-1} - 1\} = k\mu\gamma^*\,, \qquad \text{say,}$$

for $k_0 \leqq k \leqq 2\gamma\lambda + 1$ for c sufficiently large. Given $b > 0$, let

$$k_1 = k_1(b) = \max\left\{\frac{2b}{\mu\gamma^*}\,,\ k_0\right\}.$$

Then for $k \geqq k_1(b)$, we have

$$\begin{aligned} W^{11}(b) &\leqq \sum_{j \leqq 2\gamma\lambda} P\{S_j > cV(j+1) - b\} \\ &\leqq \sum_{j \leqq k} P\{S_j > cV(j+1) - b\} + \sum_{j > k} \left(\frac{2}{j\mu\gamma^*}\right)^3 E|S_j - j\mu|^3 \end{aligned} \tag{6.7}$$

which tends to zero as $c \to \infty$ and $k \to \infty$ (in that order). Moreover, letting $k = k_1(b)$ in (6.7), we find that $W^{11}(b) \leqq A(1 + b)$ for $b \geqq 0$, where A is independent of b and c. Thus (6.6) and (6.7) combine to establish Lemma 3.2.

7. Uniform integrability. In this section we will give the proof of Theorem 5.1. We suppose throughout that $X_1, X_2, \cdots$ are i.i.d. with positive mean and finite positive variance and that $V(x) = x^\alpha$, $x > 0$.

LEMMA 7.1. *If either $\alpha < \frac{1}{2}$ or $E|X_1|^\gamma < \infty$ for some $\gamma > \beta = 1/(1 - \alpha)$, then $\lim \lambda P\{t_c \leqq \alpha\lambda\} = 0$ as $c \to \infty$.*

PROOF. We use the easily verified fact that

$$P\{\max_{k \leqq n}(S_k - k\mu) > y\} \leqq KP\{S_n - n\mu > y\}$$

for $n \geqq 1$ and $y > 0$, where $K^{-1} = \inf\{P[S_k - k\mu > 0] : k \geqq 1\} > 0$ (cf. Loève (1963), pages 247–248). Let $n \geqq 2$ be an integer and let $L = L(n, c)$ be an integer for which $n^{L-1} < \alpha\lambda \leqq n^L$. Further, let $\delta = 1 - \alpha^{1-\alpha}$. Then

$$\begin{aligned} P\{t_c \leqq \alpha\lambda\} &\leqq \sum_{i=1}^{L} P\{S_k - k\mu > \delta c n^{\alpha(i-1)}, \text{ for some } k \leqq n^i\} \\ &\leqq K \sum_{i=1}^{L} P\{S_{n^i} - n^i\mu > \delta c n^{\alpha(i-1)}\}\,. \end{aligned} \tag{7.1}$$

Let A_k be the event that $|S_k^*| > \delta ck^{-\frac{1}{2}}$ and let

$$r(c) = \max_{k \leqq \lambda} \int_{A_k} |S_k^*|^2\, dP\,.$$

If $\alpha < \frac{1}{2}$, then $r(c) \to 0$ as $c \to \infty$ by the uniform integrability of $|S_k^*|^2$, $k \geqq 1$, and the right side of (7.1) does not exceed

$$K(\delta c)^{-2} \sum_{i=1}^{L} n^{i-2\alpha(i-1)} \cdot r(c) = c^{-2}r(c)O[n^{(1-2\alpha)L}] = o(\lambda^{-1})$$

as $c \to \infty$. Similarly, if $\alpha \geqq \frac{1}{2}$ and if $E|X_1|^\gamma$ is finite for some $\gamma > \beta$, then the right side of (7.1) does not exceed

$$K(\delta c)^{-\gamma} \sum_{i=1}^{L} n^{-\alpha\gamma(i-1)} E|S_{n^i} - n^i\mu|^\gamma = O\{Lc^{-\gamma}\} = o(\lambda^{-1})$$

as $c \to \infty$.

LEMMA 7.2. *If* $E|X_1|^\gamma < \infty$ *for some* $\gamma > 4$, *then* $\lim \int_{\{t_c > 4\lambda\}} t_c^2 \, dP = 0$ *as* $c \to \infty$.

PROOF. The integral in question does not exceed

$$32\lambda^2 P\{t_c > 4\lambda\} + \sum_{j \geqq 4\lambda} 8jP\{t_c > j\}.$$

Letting $\delta = 1 - 4^{\alpha-1}$, we find that $P\{t_c > j\} \leqq P\{S_j - j\mu \leqq -j\delta\mu\} \leqq Bj^{-\frac{1}{2}\gamma}$ for all $j \geqq 4\lambda$ for some constant B. The result follows easily.

We will now prove Theorem 5.1, which asserts that if $V(x) = x^\alpha$ for $x > 0$ and if $E|X_1|^\gamma$ is finite for some $\gamma > \max\{4, \beta\}$, then t_c^{*2}, $c > 0$, are uniformly integrable. By Lemmas 7.1 and 7.2, it will suffice to show that there is a function J for which $yJ(y)$ is integrable over $(0, \infty)$ and

$$P\{|t_c^*| > y, \alpha\lambda < t_c < 4\lambda\} \leqq J(y)$$

for all $y > 0$ and all $c > 0$. Let

$$J_0(y) = \sup_{j \geqq 1} P\{|S_j^*| > y\}$$

for $y > 0$. It is easily seen that $yJ_0(y)$ is integrable. Given $y > 0$, let n be the greatest integer which is $< \lambda - y\lambda^{\frac{1}{2}}$. Then since $cx^\alpha - \mu x$ is decreasing $x \geqq \alpha^\beta\lambda$, we have

$$(7.2) \qquad P\{t_c > \alpha\lambda, t_c^* < -y\} \leqq P\{S_j - j\mu > cj^\alpha - j\mu, \text{ for some } j \leqq n\}$$
$$\leqq KP\{S_n - n\mu > cn^\alpha - n\mu\}$$

for $n \geqq \alpha\lambda$; and, of course, the left side of (7.2) is zero if $n < \alpha\lambda$. Let $h(x) = cx^\alpha - \mu x$ for $x > 0$. Then h is concave and $h(\lambda) = 0$, so that

$$h(n) \geqq h'(n)(n - \lambda) \geqq \mu[1 - \alpha\lambda^{1-\alpha}n^{\alpha-1}]y\lambda^{\frac{1}{2}} \geqq \delta\mu y n^{\frac{1}{2}},$$

where $\delta = 1 - \alpha^\alpha > 0$. It follows that

$$P\{t_c > \alpha\lambda, t_c^* < -y\} \leqq KJ_0(\delta\mu y)$$

for $y > 0$. A similar, somewhat simpler, argument will show that $P\{t_c \leqq 4\lambda, t_c^* > y\} \leqq J_0[\frac{1}{2}(1 - \alpha)y]$ for $y > 0$ to complete the proof of Theorem 5.1.

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