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# ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE

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Let $\{X_k : k \geqq 1\}$ be a sequence of i.i.d.rv with $E(X_i) = 0$ and $E(X_i^2) = \sigma^2$, $0 < \sigma^2 < \infty$. Set $S_n = X_1 + \cdots + X_n$. Let $Y_n(t)$ be $S_k/\sigma n^{\frac{1}{2}}$ for $t = k/n$ and suitably interpolated elsewhere. This paper gives a generalization of a theorem of Iglehart which states weak convergence of $Y_n(t)$, conditioned to stay positive, to a suitable limiting process.

**1. Introduction.** Let $\{X_i\}_{i \in N}$ be a sequence of i.i.d.rv with $E(X_i) = 0$ and $E(X_i^2) = \sigma^2$ where $0 < \sigma^2 < \infty$. Let $S_k = X_1 + \cdots + X_k$ and $Y_n(t)$ be the continuous process on $[0, 1]$ for which $Y_n(k/n) = S_k/\sigma n^{\frac{1}{2}}$ and which is linearly interpolated elsewhere.

It is well known (see e.g., [2]) that $Y_n(t)$ converges weakly in $(C[0, 1], \rho)$ to the Brownian motion process, where $C[0, 1]$ is the set of continuous functions on $[0, 1]$ and $\rho$ the supremum metric.

Let now $C^+ = \{f \in C : f(t) \geqq 0 \text{ for } t \in [0, 1]\}$. We have $P(Y_n \in C^+) > 0$ for each $n$. So the definition of conditional probabilities is elementary. Let $Y_n^+$ be the $Y_n$-process conditioned to stay positive. That is for all Borel-sets $A \subset C[0, 1]$ we set $P(Y_n^+ \in A) = P(Y_n \in A \mid Y_n \in C^+)$. We remark that $C^+$ is a null set for the measure of the Brownian motion. Iglehart proved [3] weak convergence of the $Y_n^+$ process to the Brownian meander process $W^+$ which is defined by

$$W^+(t) = \left|\frac{1}{(1-\tau)^{\frac{1}{2}}} W(\tau + (1-\tau)t)\right|, \qquad 0 \leqq t \leqq 1 \tag{1.1}$$

with $W$ the Brownian process and $\tau = \sup\{t \in [0, 1] : W(t) = 0\}$. (Notice that $\tau < 1$ a.s.)

Iglehart assumed $E|X_i|^3 < \infty$ and $X_i$ nonlattice or integer valued with span 1. It is shown in this paper that these extra assumptions are superfluous. Iglehart calculates the finite-dimensional distributions and proves tightness. Then he identifies the process with (1.1) for which Belkin [1] calculated the finite dimensional distributions. The proof given here requires no computation. It is based on identifying $\lim_{n\to\infty} Y_n^+(t) = W(T+t) - W(T) = W^+(t)$ for an appropriate random time $T$ and uses only the continuous mapping theorem (Theorem 5.1 in [2]).

**2. Notations and preliminary lemmas.** For $s \in (0, \infty]$ let $C^s$ be the set of

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continuous functions on $[0, s]$ (or $[0, \infty)$ for $s = \infty$) and $\mathscr{B}^s$ the smallest $\sigma$-algebra such that the mappings $C^s \ni f \to f(t) \in \mathbb{R}$ are measurable.

Let $P^s$ be the measure of the Brownian motion on $(C^s, \mathscr{B}^s)$.

$$T^s: C^s \to \bar{\mathbb{R}}^+ = [0, \infty] \quad \text{is the mapping with}$$

$$T^s(f) = \inf\{t: f(u) \geqq f(t) \text{ for } t \leqq u \leqq t + 1 \leqq s\}, \quad (\inf \varnothing = \infty). \tag{2.1}$$

We set $T = T^\infty$ and $P = P^\infty$ for simplicity.

LEMMA 2.1. *For all $s \in (0, \infty]$ $T^s$ is $\mathscr{B}^s$-measurable.*

PROOF. If $v = s - (u + 1) > 0$ then $\{T^s \leqq u\} = \bigcap_{n \geqq 1/v} \{f \in C^s$: there exists a rational $r \leqq u + 1/n$ with $f(r) < \min_{1 \leqq i \leqq n-1} f(r + i/n) + 1/n\}$, which is easily seen to belong to $\mathscr{B}^s$.

LEMMA 2.2. $P(T < \infty) = 1$.

PROOF. Let $A_\varepsilon = \{f \in C^1$: ex. $s \leqq 1 - \varepsilon$ with $f(s) \leqq f(u)$ for $s \leqq u \leqq s + \varepsilon\}$. Now we have $A_\varepsilon^c \downarrow \{f \in C^1: f$ nonincreasing$\}$ as $\varepsilon \downarrow 0$. We infer $P(A_\varepsilon) \uparrow 1$ for $\varepsilon \downarrow 0$. If $\varphi : C^\infty \to C^\infty$ is defined by $\varphi(f)(t) = \varepsilon^{-\frac{1}{2}} f(\varepsilon t)$ then $\varphi$ is measure preserving (see [5] page 246) and $\varphi(A_\varepsilon) \subset \{T < \infty\}$ so $P(T < \infty) \geqq P(A_\varepsilon)$ for all $\varepsilon > 0$.

LEMMA 2.3. The following three statements are true for all $s \in (0, \infty]$.

$$P^s(f(T^s) = f(T^s + 1)) = 0; \tag{2.2}$$

$$P^s(T^s = s - 1) = 0; \tag{2.3}$$

$$P^s(\text{ex. } u \in (0, 1) \text{ with } f(T^s) = f(T^s + u)) = 0. \tag{2.4}$$

PROOF. We set $m(t) = \min_{0 \leqq s \leqq t} W(t)$. $D(t) = W(t) - m(t)$ has the same finite-dimensional distributions as $|W(t)|$ (see [5] page 193). Observe now that $T^s = \inf\{t \leqq s - 1: m(t) = m(t + 1)\}$. Now $T^s = s - 1$ implies $D(s - 1) = 0$ which has $P$ measure 0. This proves (2.3).

Let $U = \{$ex. $u < v < w$ with $m(u) = m(v) = m(w)$ and $D(u) = D(v) = D(w) = 0\}$. Then $U \subset \bigcup_{r, s \in Q} \{\min_{0 \leqq t \leqq r} W(t) = \min_{r \leqq t \leqq r+s} W(t)\}$ and the last has $P$ measure 0. This proves (2.4).

It suffices to prove (2.2) for $s = \infty$. With probability one, the hitting time process $\{T_{-x}: x \geqq 0\}$ ($T_{-x} = \inf\{t: W(t) = -x\}$) has no jumps of length one. This follows from its Lévy decomposition (see Section 1.7 of [4]). Together with $P(U) = 0$ this yields (2.2).

LEMMA 2.4. *For each $s \in (0, \infty]$ $T^s$ is a continuous $P^s$* a.e. *on $(C^s, \rho)$.*

PROOF. By (2.3) it suffices to consider the case $s = \infty$. Let $f$ be such that $T(f) < \infty$ and $f$ does not belong to the null sets defined in (2.2)—(2.4).

(I) We first prove that for all $\delta > 0$ there exists an $\varepsilon > 0$ with

$$T(f') \leqq T(f) + \delta \qquad \text{when} \quad \rho(f, f') < \varepsilon.$$

By (2.2) there is as $\tau < \delta$ so that

$$\inf_{T+1\leqq u\leqq T+1+\tau} f(u) > f(T) .$$

Now (2.4) gives $\varepsilon = \frac{1}{3}(\inf_{T+\tau\leqq u\leqq T+\tau+1} f(u) - f(T)) > 0$.

If $\rho(f, f') < \varepsilon$ and $\gamma'$ is such that $T(f) \leqq \gamma' \leqq T(f) + \tau$ and $f'(\gamma') = \inf_{T\leqq u\leqq T+\tau} f'(u)$ then $T(f') \leqq \gamma' \leqq T(f) + \delta$.

(II) To show the other inequality note that

$$\lim_{n\to\infty} (\inf\{T(f') : \rho(f, f') < 1/n\}) = \lambda \leqq T(f) .$$

Let $\{f_n\}_{n\in N}$ be a sequence with $\rho(f, f_n) \leqq 1/n$ and $\lim_{n\to\infty} T(f_n) = \lambda$. Let $\varepsilon > 0$. By the continuity of $f$ and the uniform convergence of $f_n$, there exists $n_0$ such that for $n \geqq n_0$ we have:

$$\begin{aligned}\inf_{\lambda\leqq u\leqq\lambda+1} f(u) &\geqq \inf_{T(f_n)\leqq u\leqq T(f_n)+1} f(u) - \varepsilon \\ &\geqq \inf_{T(f_n)\leqq u\leqq T(f_n)+1} f_n(u) - 2\varepsilon \\ &\geqq f_n(T(f_n)) - 2\varepsilon \geqq f(T(f_n)) - 3\varepsilon \geqq f(\lambda) - 4\varepsilon .\end{aligned}$$

So $\inf_{\lambda\leqq u\leqq\lambda+1} f(u) \geqq f(\lambda)$ which implies $T(f) \leqq \lambda$ completing the proof of Lemma 2.4.

Let $u$ be the function in $C^1$ which is everywhere equal $-1$. We define a map $\Phi_s : C^s \to C^1$

$$\begin{aligned}\Phi_s(f)(t) &= f(T^s(f) + t) &&\text{for } T^s(f) < \infty \\ &= u &&\text{for } T^s(f) = \infty .\end{aligned}$$

We write $\Phi = \Phi_\infty$ for simplicity.

A straightforward conclusion of Lemma 2.4 is

LEMMA 2.5. *For each* $s \in (0, \infty]$ $\Phi_s$ *is continuous* $P^s$ a.s. *on* $(C^s, \rho)$.

**3. Sums of independent random variables conditioned to stay positive.** Let $X_1, X_2, \cdots,$ be i.i.d.rv with $E(X_i) = 0$; $E(X_i^2) = \sigma^2 < \infty$ $(\sigma^2 > 0)$ and $S_k = \sum_{j=1}^k X_j$. $T_n = \inf\{k : S_{k+i} \geqq S_k \text{ for } i = 1, \cdots, n\}$. Clearly $T_n < \infty$ holds a.s. We set $Z_k = S_{T_n+k} - S_{T_n}$.

LEMMA 3.1. *For each sequence of real numbers* $a_1, \cdots, a_n$

$$P(S_k \leqq a_k, k = 1, \cdots, n \mid S_k \geqq 0, k = 1, \cdots, n) = P(Z_k \leqq a_k, k = 1, \cdots, n) . \tag{3.1}$$

PROOF. This is an easy consequence of the independence and identical distribution of the $X_i$:

If $B_j = \bigcup_{s=0}^{j-1} \{S_s \leqq S_r \text{ for } s+1 \leqq r \leqq \min(j, s+n)\}$ we have

$$\begin{aligned}&P(S_{T_n+k} - S_{T_n} \leqq a_k \text{ for } k = 1, \cdots, n) \\ &\quad = \sum_{j=0}^\infty P(S_{j+k} - S_j \leqq a_k \text{ for } k = 1, \cdots, n \mid T_n = j) P(T_n = j)\end{aligned}$$

$$= \sum_{j=0}^{\infty} P(S_{j+k} - S_j \leqq a_k \text{ for } k = 1, \cdots, n \mid S_{j+k} \geqq S_j \text{ for } k = 1, \cdots, n \text{ and } B_j^c) P(T_n = j)$$
$$= P(S_k \leqq a_k, k = 1, \cdots, n \mid S_k \geqq 0, k = 1, \cdots, n) \quad \text{since } T_n < \infty \text{ a.s.}$$

We set $Y_n(k/n) = (1/n^{\frac{1}{2}}\sigma)S_k$ for $k \geqq 0$ and $Y_n(t)$ linearly interpolated.

Let $Q_n$ be the probability measure defined on $(C^\infty, \mathscr{B}^\infty)$ by this process. Let $\Pi_s: C^\infty \to C^s$ be the projection map and $\Phi$, $C^+$ defined as above. We remark that $P^s = P\Pi_s^{-1}$.

Let $Q_n\Pi_1^{-1}(dx \mid C^+)$ be the probability measure on $C^1$ which is defined by

$$Q_n\Pi_1^{-1}(A \mid C^+) = Q_n(\Pi_1^{-1}(A \cap C^+))/Q_n(\Pi_1^{-1}(C^+))$$

for $A \in \mathscr{B}^1$.

THEOREM 3.2. *The probability measures* $Q_n\Pi_1^{-1}(dx \mid C^+)$ *converge weakly to* $P\Phi^{-1}$ *(on* $(C^1, \rho)$*).*

PROOF. We have proved in Lemma 3.1 that

$$Q_n\Pi_1^{-1}(dx \mid C^+) = Q_n\Phi^{-1}(dx) \quad \text{holds.} \tag{3.2}$$

Now by Donsker's theorem (see [2]), $Q_n\Pi_s^{-1}$ converges weakly to $P^s$ for $s < \infty$. With regard to Lemma 2.5 we have for $s < \infty$

$$Q_n(\Phi_s\Pi_s)^{-1} \to P^s\Phi_s^{-1} \quad \text{weakly.} \tag{3.3}$$

(Theorem 5.1 in [2].)

Let $A$ be a continuity set in $\mathscr{B}^1$, that is $P\Phi^{-1}(\partial A) = 0$. We are going to show that

$$\lim_{n\to\infty} Q_n\Phi^{-1}(A) = P\Phi^{-1}(A)\,. \tag{3.4}$$

The theorem then follows. (3.4) doesn't follow directly from (3.3) because we have there the assumption $s < \infty$. Set

$$D = \{f \in C^1 : \min_{0 \leqq t \leqq 1} f(t) \geqq -\tfrac{1}{2}\}\,.$$

Without loss of generality we can assume $A \subset D$. (If not: replace $A$ by $A \cap D$ noticing $Q_n\Phi^{-1}(D^c) = P\Phi^{-1}(D^c) = P\Phi^{-1}(\partial D) = 0$).

Let $\varepsilon > 0$ be given. According to Lemma 2.2 we have $P(T < \infty) = 1$. So there exists a real number $c > 0$ such that $P(T \leqq c - 1) \geqq 1 - \varepsilon$.

We choose $n_0$ such that for $n \geqq n_0$

$$|Q_n\Pi_c^{-1}(T^c < \infty) - P^c(T^c < \infty)| \leqq \varepsilon\,. \tag{3.5}$$

(According to Lemma 2.4 $\{T^c < \infty\}$ is a continuity set with respect to $P^c$. (3.5) then follows by Donsker's theorem.)

We infer from (3.5) and the setting of $c$:

$$P(\Phi_c\Pi_c \neq \Phi) \leqq \varepsilon\,, \tag{3.6}$$

$$(3.7)\qquad Q_n(\Phi_c\Pi_c \neq \Phi) \leqq 2\varepsilon\,.$$

(We have $\{\Phi_c\Pi_c = \Phi\} \cap \{T < \infty\} = \{T^c\Pi_c < \infty\} = \{T \leqq c-1\}$.)

We choose $n_1 \geqq n_0$ such that for $n \geqq n_1$

$$(3.8)\qquad |Q_n(\Phi_c\Pi_c)^{-1}(A) - P^c\Phi_c^{-1}(A)| \leqq \varepsilon\,.$$

(The element $u$ doesn't belong to $\partial A$ because we assumed $A \subset D$. It is easily seen that $(\Phi_c\Pi_c)^{-1}(\partial A) \subset \Phi^{-1}(\partial A)$ holds, so we infer that $P(\Phi_c\Pi_c)^{-1}(\partial A) = P^c\Phi_c^{-1}(\partial A) = 0$ and the existence of an $n_1$, such that (3.8) holds then follows from (3.3).)

For $n \geqq n_1$ we have:

$$\begin{aligned}|Q_n\Phi^{-1}(A) - P\Phi^{-1}(A)| &\leqq |Q_n\Phi^{-1}(A) - Q_n(\Phi_c\Pi_c)^{-1}(A)| \\ &\quad + |Q_n(\Phi_c\Pi_c)^{-1}(A) - P^c\Phi_c^{-1}(A)| \\ &\quad + |P(\Phi_c\Pi_c)^{-1}(A) - P\Phi^{-1}(A)| \\ &\leqq Q_n(\Phi \neq \Phi_c\Pi_c) + \varepsilon + P(\Phi \neq \Phi_c\Pi_c) \leqq 4\varepsilon\,.\end{aligned}$$

So $\lim_{n\to\infty} Q_n\Phi^{-1}(A) = P\phi^{-1}(A)$ which is (3.4) and the proof is complete.

So far we have proved that $Y_n^+$ converges weakly to $P\Phi^{-1}$ which is $W(T+t) - W(T)$ $0 \leqq t \leqq 1$. It remains to identify $W(T+.) - W(T)$ with the Brownian meander $W^+$. But this clearly follows from Iglehart's result. We give a sketch of a proof using the methods of the present paper: Let $X_i = \pm 1$ each with probability $\frac{1}{2}$. Set $\mu_n = \inf\{k \leqq n:$ the sequence $S_k, \cdots, S_n$ does not change sign$\}$ and let $\nu_n = n - \mu_n$ (remark that $\nu_n \geqq 1$). We define $\tilde{Y}_n(t)$ as follows: $\tilde{Y}_n(k/\nu_n) = (1/\nu_n)^{\frac{1}{2}}|S_{\mu_n+k}|$ for $0 \leqq k \leqq \nu_n$ and linearly interpolated elsewhere. $\tilde{Y}_n(\cdot)$ has the same distribution as $Y_{\nu_n}^+(\cdot)$ where $\{Y_k^+\}_{k\in\mathbf{N}}$ and $\nu_n$ are independent. Define $\tau'\colon C^1 \to [0, 1]$ by $\tau'(f) = \inf\{t \in [0, 1]: f(s)$ does not change sign for $s \in [t, 1]\}$. Further, define $\Psi\colon C^1 \to C^1$ by $\Psi(f)(t) = |(1-\tau')^{-\frac{1}{2}}f(\tau' + (1-\tau')t)|$ for $\tau' \in [0, 1)$, and $\Psi(f)$ identically zero for $\tau' = 1$. We then have $\tilde{Y}_n = \Psi(Y_n)$, which is identical in law to $Y_{\nu_n}^+$. Now $\tau' = \tau = \sup\{t \in [0, 1]: f(t) = 0\}$ $P^1$-a.s. (This can be proved in the same way as the statements of Lemma 2.3). So $W^+$ has the same distribution as $\Psi(W)$. It can be shown by the same methods as in Lemma 2.4 and 2.5 that $\Psi$ is $P^1$-a.s. continuous on $(C^1, \rho)$. The continuous mapping theorem implies $\tilde{Y}_n \to W^+$ and so $Y_{\nu_n}^+ \to W^+$ in distribution. By Theorem 3.2 $Y_n^+ \to W(T+.) - W(T)$. Clearly $\nu_n \to \infty$ in distribution. This is sufficient for $Y_{\nu_n}^+ \to W(T+\cdot) - W(T)$ because $\{Y_n^+\}$ and $\nu_n$ are independent. It follows that $W^+$ and $W(T+\cdot) - W(T)$ have the same distribution.

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