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SOME STABILITY RESULTS FOR VECTOR VALUED RANDOM VARIABLES

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This paper explores the strong law of large numbers in the infinite dimensional setting. It is shown that under several classical conditions—such as the Kolmogorov condition—the strong law holds if and only if the weak law holds.

1. Introduction. Let B denote a real vector space, $\mathbb{B}$ a sigma algebra of subsets of B, and $\|\cdot\|$ a semi-norm on B. We say the triple $(B, \mathbb{B}, \|\cdot\|)$ is a *linear measurable space* if (i) addition and scalar multiplication are $\mathbb{B}$ measurable operations on B, (ii) for all $t \geq 0$ we have $\{x \in B : \|x\| \leq t\} \in \mathbb{B}$, and (iii) there exists a subset F of the $\mathbb{B}$ measurable linear functionals on B such that

$$\|x\| = \sup_{f \in F}|f(x)| \qquad x \in B \tag{1.1}$$

(actually property (iii) is not used in this paper).

Examples of linear measurable spaces include the situation where B is a real separable Banach space, $\mathbb{B}$ denotes the Borel subsets of B, and $\|\cdot\|$ is the norm on B. Another important example consists of $B = D[0, T]$ where $D[0, T]$ denotes the real-valued functions on $[0, T]$ which are right continuous on $[0, T]$ and have left-hand limits on $(0, T]$. In this case $\mathbb{B}$ consists of the minimal sigma-algebra making the maps $x \to x(t)$, $0 \leq t \leq T$, measurable and the norm is the sup-norm, $\|x\| = \sup_{0 \leq t \leq T}|x(t)|$.

Now assume $(B, \mathbb{B}, \|\cdot\|)$ is a linear measurable space, and $\{X_j : j \geq 1\}$ is a sequence of independent $(B, \mathbb{B})$-valued random variables. We say $\{X_j\}$ satisfies the *strong law of large numbers* (SLLN) if

$$P(\lim_n\|(X_1 + \cdots + X_n)/n\| = 0) = 1. \tag{1.2}$$

If the $\{X_j : j \geq 1\}$ are real valued, then necessary and sufficient conditions for the strong law of large numbers which might be considered "satisfactory" are rather recent [6], [7]. These results are easily extended to the case where the $\{X_j\}$ takes values in a finite dimensional vector space, but for random variables with values in an infinite dimensional vector space the situation is much less desirable.

To be sure, the infinite dimensional case is substantially different. In fact, except for the case where the $\{X_j\}$ are independent indentically distributed (i.i.d.), only rather unsatisfactory sets of sufficient conditions are known when the $\{X_j\}$ takes

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values in an arbitrary separable Banach space. For example, we have

THEOREM A ([1]). *Let* $\{X_j : j \geqslant 1\}$ *be independent random variables with values in a real separable Banach space. If* $EX_j = 0$ $(j \geqslant 1)$,

(*i*) $\Sigma_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty$ *and* $\Sigma_{j=1}^{n} \frac{(E\|X_j\|^2)^{\frac{1}{2}}}{n} \to 0$ *as* $n \to \infty$, *or*
(*ii*) $\Sigma_{j=1}^{n} \text{ess sup}\|X_j\|/n \to 0$ *as* $n \to \infty$,

then $\{X_j\}$ *satisfies the* SLLN.

In (i) it is the second part which is unusually restrictive, and, of course, (ii) is extremely restrictive. Nevertheless, the fact remains that the conditions (i) and (ii) are best possible in the sense that if either is weakened the resulting statement is no longer true for all Banach spaces. For example, if $\Sigma_{j=1}^{n}(E\|X_j\|^2)^{\frac{1}{2}}/n \not\to 0$ *as* $n \to \infty$, then the Kolmogorov condition

$$\Sigma_{j=1}^{\infty} E\|X_j\|^2/j^2 < \infty \tag{1.3}$$

need not imply the SLLN for $\{X_j\}$ unless B is a type-2 Banach space.

What we have, then, is that the SLLN in an arbitrary Banach space is considerably different from the situation for real valued random variables. The purpose of this paper is to provide further sufficient conditions for the SLLN which hold for random variables with values in any measurable linear space. For example, if $\{X_j\}$ is a centered sequence of independent random variables with values in any linear space and having variances such that

$$\Sigma_{j=1}^{\infty} \frac{E\|X_j\|^2}{j^2} < \infty, \tag{1.4}$$

then we show that the SLLN is equivalent to $(S_n/n) \to_{\text{p}} 0$.

From a practical point of view our results are still somewhat unsatisfactory in that our assumptions are not entirely in terms of the individual summands. Nevertheless, finding good sufficient conditions for the SLLN without restrictions on the range space is a very delicate problem, and what is accomplished here in many cases reduces the problem to the study of the weak law of large numbers.

An application to the strong law of large numbers in Banach spaces of type p is made in Section 4.

2. Some lemmas.

LEMMA 2.1. *Let* $X = \{X_j : j \geqslant 1\}$ *and* $X' = \{X_j' : j \geqslant 1\}$ *be two sequences of independent random variables in the linear measurable space* $(B, \mathbb{B}, \|\cdot\|)$ *such that* X *and* X' *are independent copies of each other. Then*

$$\Sigma_{j=1}^{n} X_j/n \to_{\text{a.s.}} 0 \tag{2.1a}$$

iff

$$\Sigma_{j=1}^{n} \frac{(X_j - X_j')}{n} \to_{\text{a.s.}} 0 \quad \text{and} \quad \Sigma_{j=1}^{n} \frac{X_j}{n} \to_{\text{p}} 0. \tag{2.1b}$$

PROOF. Obvious.

LEMMA 2.2. *Let* $\{X_j : j \geq 1\}$ *be independent, symmetric random variables with values in the linear measurable space* $(B, \mathbb{B}, \|\cdot\|)$. *Then*

$$\frac{S_n}{n} \to_{\text{a.s.}} 0 \quad \text{iff} \quad \frac{S_{2^{n+1}} - S_{2^n}}{2^n} \to_{\text{a.s.}} 0.$$

PROOF. See [8], page 159, for a method which handles this easily.

LEMMA 2.3. *Let* $\{X_j : j \geq 1\}$ *be independent symmetric random variables with values in the linear measurable space* $(B, \mathbb{B}, \|\cdot\|)$ *such that*

$$\|X_j\| \leq j, \quad j \geq 1, \tag{2.2a}$$

and

$$\frac{S_n}{n} \to_{\text{p}} 0. \tag{2.2b}$$

Then

$$E\left\|\frac{S_n}{n}\right\| \to 0. \tag{2.3}$$

PROOF. To prove (2.3) fix $\varepsilon > 0$. By (2.2b) there exists n_0 such that $\sup_{n \geq n_0} P(\|S_n\| \geq n\varepsilon) \leq \frac{1}{24}$. From [2] or [4], Lemma 5.4, we have that

$$\begin{aligned}\int_0^A P(\|S_n\| \geq nt)\,dt &= 3\int_0^{A/3} P(\|S_n\| \geq 3nt)\,dt \\ &\leq 3\left[4\int_0^{A/3} P^2(\|S_n\| \geq nt)\,dt + \int_0^{A/3} P(N_n \geq nt)\,dt\right] \\ &\leq 12\varepsilon + \frac{12}{24}\int_0^A P(\|S_n\| \geq nt)\,dt + 12\int_0^{A/3} P(N_n \geq nt)\,dt \\ &\leq 24\varepsilon + 24\int_0^{A/3} P(N_n \geq nt)\,dt, \qquad \text{for} \quad n \geq n_0,\end{aligned}$$

where $N_n = \max_{1 \leq j \leq n}\|X_j\|$. By (2.2a) $N_n \leq n$, and hence

$$E\left\|\frac{S_n}{n}\right\| \leq 24\varepsilon + 24\int_0^1 P(N_n \geq nt)\,dt.$$

Now for $t > 0$,

$$\begin{aligned}P(N_n \geq nt) &\leq P(\max_{1 \leq k \leq n}\|S_k\| \geq nt/2) \\ &\leq (\text{by Lévy's inequality})\ 2P(\|S_n\| \geq nt/2) \to 0,\end{aligned}$$

as $n \to \infty$. Hence by the bounded convergence theorem

$$\int_0^1 P(N_n \geq nt)\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, by the arbitrariness of ε, $\lim_{n\to\infty} E\|S_n/n\| = 0$.

3. Stability results for vector valued random variables. The main results of the paper are presented here, and are expressed in terms of the quantities

$$\Lambda(n, p) = \sum_{j \in I(n)} E\|X_j\|^p / 2^{(n+1)p}$$

where

$$I(n) = \{j : 2^n < j \leqslant 2^{n+1}\}.$$

To prove Theorem 1 we employ the Erdös double truncation technique (see, e.g., [8]) and the exponential inequalities of [5].

THEOREM 1. *Let $\{X_j : j \geqslant 1\}$ be independent random variables with values in the linear measurable space $(B, \mathbb{B}, \|\cdot\|)$ such that*

(3.1a) $$\frac{X_j}{j} \to_{\text{a.s.}} 0$$

(3.1b) *for some* $p \in [1, 2]$ *and some* $r \in (0, \infty)$ $\Sigma_{n=1}^{\infty} \Lambda(n, p)^r < \infty$.

Then $(S_n/n) \to_{\text{p}} 0$ iff $(S_n/n) \to_{\text{a.s.}} 0$.

PROOF. Assume (3.1a, b) and $(S_n/n) \to_{\text{p}} 0$. By Lemmas 2.1 and 2.2 it suffices to assume that $\{X_j ; j \geqslant 1\}$ is symmetric and prove that

(3.2) $$\frac{S_{2^{n+1}} - S_{2^n}}{2^n} \to_{\text{a.s.}} 0.$$

Next we note that by symmetry and (3.1a) it suffices to assume that

(3.3) $$\|X_j\| \leqslant j.$$

Since under (3.3) $\Lambda(n, 2) \leqslant 2^p \Lambda(n, p)$, it suffices to prove the theorem for $p = 2$. Let $\Lambda(n) = \Lambda(n, 2)$. For this purpose fix $\varepsilon > 0$, $N \in \mathbb{N}$ = natural numbers, $N \geqslant 2$, and set

$$X_j' = X_j I\left[\|X_j\| \leqslant \Lambda(n)^{\frac{1}{4}} 2^{n+1}\right] \qquad j \in I(n)$$

$$U_n^1 = \|\Sigma_{j \in I(n)} X_j'\|$$

$$U_n^2 = \|\Sigma_{j \in I(n)} X_j I\left[\|X_j\| > \frac{2^{n+1}\varepsilon}{N}\right]\|$$

and

$$U_n^3 = \|\Sigma_{j \in I(n)} X_j I\left[\Lambda(n)^{\frac{1}{4}} 2^{n+1} < \|X_j\| \leqslant \frac{\varepsilon}{N} 2^{n+1}\right]\|.$$

To prove (3.2) we need only prove that

(3.4.k) $$\Sigma_{n=1}^{\infty} P\left(U_n^k > \varepsilon 2^{n+1}\right) < \infty \qquad \text{for} \quad k = 1, 2, 3.$$

To prove (3.4.1) we let $b_n = \varepsilon 2^{n+1} \Lambda(n)^{\frac{1}{8}}$, $\varepsilon_n = 2^{n+1}\varepsilon / b_n = \Lambda(n)^{-\frac{1}{8}}$ and $c_n = (1/\varepsilon)\Lambda(n)^{\frac{1}{8}}$. Then

(3.5) $$\|X_j'\| \leqslant \Lambda(n)^{\frac{1}{4}} 2^{n+1} = c_n b_n \qquad j \in I(n),$$

and hence by using the estimate $1 + (\varepsilon c/3) + ((\varepsilon c)^2/4 \cdot 3) + \cdots \leqslant e^{\varepsilon c}$ in the proof of Lemma 2.1 of [5], we have for $n \geqslant n_0(\varepsilon)$ (see 3.7 below)

$$(3.6) \qquad P(U_n^1 > 2^{n+1}\varepsilon) = P\left(\frac{U_n^1}{b_n} > \frac{2^{n+1}\varepsilon}{b_n}\right) = P\left(\frac{U_n^1}{b_n} > \varepsilon_n\right)$$

$$\leqslant \exp\left\{-\varepsilon_n^2 + \frac{\varepsilon_n}{2b_n} EU_n^1 + \left(\frac{\varepsilon_n^2}{2}\right)(e^{\varepsilon_n c_n})\Sigma_{j \in I(n)} \frac{E\|X_j'\|^2}{b_n^2}\right\}$$

$$\leqslant \exp\left\{-\varepsilon_n^2\left[1 - \frac{EU_n^1}{\varepsilon 2^{n+2}} - \frac{1}{2} e^{1/\varepsilon} \frac{\Lambda(n)^{\frac{3}{4}}}{\varepsilon^2}\right]\right\}$$

$$\leqslant \exp\left\{-\frac{\varepsilon_n^2}{2}\right\},$$

since

$$(3.7a) \qquad \lim_n \frac{EU_n^1}{2^n} = 0 \quad \text{by (3.3) and Lemma 2.3}$$

$$(3.7b) \qquad \varepsilon_n c_n = \frac{1}{\varepsilon}$$

$$(3.7c) \qquad \Sigma_{j \in I(n)} \frac{E\|X_j'\|^2}{b_n^2} \leqslant \Sigma_{j \in I(n)} \frac{E\|X_j\|^2}{b_n^2} = \frac{\Lambda(n)^{\frac{3}{4}}}{\varepsilon} \to 0.$$

Therefore,

$$\Sigma_{n \geqslant n_0(\varepsilon)} P(U_n^1 > 2^{n+1}\varepsilon) \leqslant \Sigma_n \exp(-\varepsilon_n^2/2)$$

$$\leqslant \Sigma_n \exp\left(-\frac{\Lambda(n)^{-\frac{1}{4}}}{2}\right) \leqslant \Sigma_n 2^{4r}(4r+1)!\Lambda(n)^r < \infty.$$

Now (3.4.2) follows since (3.1a) implies that only finitely many of the terms in U_n^2 are nonzero. Finally (3.4.3) follows since

$$P(U_n^3 > 2^{n+1}\varepsilon) \leqslant P(\exists \text{ at least } N\ j\text{'s in } I(n) \text{ such that } \|X_j\| > 2^{n+1}\Lambda(n)^{\frac{1}{4}})$$

$$\leqslant [\Sigma_{j \in I(n)} P(\|X_j\| > 2^{n+1}\Lambda(n)^{\frac{1}{4}})]^N \leqslant \left[\frac{\Lambda(n)}{\Lambda(n)^{\frac{1}{2}}}\right]^N = \Lambda(n)^{N/2}$$

which is summable if N is chosen so that $N/2 \geqslant r$.

COROLLARY 1. *Let $\{X_j : j \geqslant 1\}$ be independent random variables with values in the linear measurable space $(B, \mathbb{B}, \|\cdot\|)$ such that*

$$(3.8) \qquad \Sigma_{j=1}^{\infty} \frac{E\|X_j\|^p}{j^p} < \infty \quad \text{for some} \quad 1 \leqslant p \leqslant 2.$$

Then $(S_n/n) \to_p 0$ iff $(S_n/n) \to_{\text{a.s.}} 0$.

PROOF. (3.1a) follows from (3.8) and (3.1b) holds with $r = 1$ by (3.8).

REMARK. Theorem A (i) follows from Corollary 1 since $\sum_{j=1}^{n}((E\|X_j\|^2)^{\frac{1}{2}}/n) \to 0$ implies $(E\|S_n\|/n) \to 0$, which implies $(S_n/n) \to_{\mathrm{p}} 0$. On the real line (3.8) and $EX_j = 0$ $(j \geqslant 1)$ trivially implies $(S_n/n) \to_{\mathrm{p}} 0$, although (3.8) need not imply $\sum_{j=1}^{n}((E\|X_j\|^2)^{\frac{1}{2}}/n) \to 0$.

COROLLARY 2 ([9]). *If $\{X_j : j \geqslant 1\}$ is a sequence of independent random variables taking values in the separable Banach space B such that*

(3.9a) $$EX_j = 0, \qquad j \geqslant 1,$$

(3.9b) $$\{X_j\} \quad \text{is tight, and}$$

(3.9c) $$\sup_j E\|X_j\|^p < \infty \qquad \text{for some} \quad p > 1,$$

then $(S_n/n) \to_{\mathrm{a.s.}} 0$.

PROOF. Combining (3.9b, c) we have for every $\varepsilon > 0$ that there exists a compact set K such that $\int_{X_j \notin K} \|X_j\| \, dP < \varepsilon$. Hence by [3], Theorem 2.4, we have $E\|S_n/n\| \to 0$ and thus $(S_n/n) \to_{\mathrm{p}} 0$. Thus Corollary 1 applies and the corollary is proved.

If one imposes stronger conditions on the X_j's, then one can obtain the conclusion of Theorem 1 under hypotheses which are sometimes weaker than (3.1b). In what follows we let $LLj = \log(\log(j \vee e^e))$.

THEOREM 2. *Let $X_1, X_2, \cdots$ be independent random variables with values in the linear measurable space $(B, \mathbb{B}, \|\cdot\|)$ such that*

(3.10a) $$\|X_j\| \leqslant Mj/LLj \qquad \text{for some constant} \quad M < \infty, \quad j \geqslant 1,$$

and

(3.10b) $$\textstyle\sum_{n \geqslant 1} \exp\{-\varepsilon/\Lambda(n)\} < \infty \qquad \text{for all} \quad \varepsilon > 0 \quad \text{where} \quad \Lambda(n) = \Lambda(n, 2).$$

Then

$$\frac{S_n}{n} \to_{\mathrm{p}} 0 \quad \text{iff} \quad \frac{S_n}{n} \to_{\mathrm{a.s.}} 0.$$

REMARK. Condition (a) of Theorem 2 is necessary even on the real line. That is, if (3.10a) is replaced by

(3.10a)′ $$\|X_j\| \leqslant \Gamma(j)j/LLj \qquad j \geqslant 1$$

where $\Gamma(j) \to \infty$ as $j \to \infty$, then there exists a sequence of independent, symmetric, real valued random variables such that (3.10a)′ and (3.10b) hold, yet $(S_n/n) \not\to_{\mathrm{a.s.}} 0$. Such an example is given by Prokhorov [7], and is easily modified so that we also have $(S_n/n) \to_{\mathrm{p}} 0$ and still $(S_n/n) \not\to_{\mathrm{a.s.}} 0$.

Furthermore, under (3.10a) condition (3.10b) is necessary and sufficient for independent mean zero real-valued random variables to satisfy the strong law of large numbers [7], but it is easy to produce an example in the infinite dimensional setting which satisfies (a) and $(S_n/n) \to_{\mathrm{a.s.}} 0$, yet (b) fails.

The example is in the Banach space c_0. Let $X_n = (n/LLn)\varepsilon_n e_n$ where $\{\varepsilon_n\}$ is a sequence of independent random variables such that $P(\varepsilon_j = \pm 1) = \frac{1}{2}$ and $\{e_n\}$ is the usual basis in c_0. Then (a) holds and $(S_n/n) \to_{\text{a.s.}} 0$ in c_0, yet (b) fails.

PROOF. The proof proceeds in a manner similar to that of Theorem 1. As before we can assume the X_j's are symmetric, so it suffices to prove (3.2).

First we note by the method used to prove the exponential inequality of [5] we get: if $\|X_j\| \leqslant c_n b_n$ $(1 \leqslant j \leqslant n)$, then

$$P\left(\frac{\|S_n\| - E\|S_n\|}{2b_n} > \varepsilon_n\right) \leqslant \exp\left\{-\varepsilon_n^2 + \frac{\varepsilon_n^2}{2}\sum_{j=1}^n \frac{E\|X_j\|^2}{b_n^2} e^{\varepsilon_n c_n}\right\}. \tag{3.11}$$

Since $(S_n/n) \to_{\text{p}} 0$ and $\|X_j\| \leqslant Mj/LLj$ we have by Lemma 2.3 that $E\|S_n/n\| \to 0$. Hence $E\|(S_{2^{n+1}} - S_{2^n})/2^{n+1}\| \to 0$, and we have (3.2) by showing

$$\sum_n P\left(\|S_{2^{n+1}} - S_{2^n}\| - E\|S_{2^{n+1}} - S_{2^n}\| > \varepsilon 2^{n+1}\right) < \infty \tag{3.12}$$

for every $\varepsilon > 0$ $(\varepsilon \leqslant 1)$.

To verify (3.13) we divide the natural numbers into two sets. Fix $\delta > 0$, $\varepsilon > 0$. First we consider the case

$$\Lambda(n) LL2^{n+1} \leqslant \left[\frac{\varepsilon}{2(1+\delta)}\right]^2 \exp\left(\frac{-2(1+\delta)^2 M}{\varepsilon}\right). \tag{3.13}$$

Let

$$b_n = \frac{\varepsilon 2^{n+1}}{(1+\delta)(2LL2^{n+1})^{\frac{1}{2}}}, \qquad \varepsilon_n = \frac{2^{n+1}\varepsilon}{b_n} = (1+\delta)(2LL2^{n+1})^{\frac{1}{2}},$$

and

$$c_n = \frac{2^{\frac{1}{2}}(1+\delta)M}{\varepsilon(LL2^{n+1})^{\frac{1}{2}}}.$$

Then from (3.11) we have

$$\begin{aligned} &P\left(\|S_{2^{n+1}} - S_{2^n}\| - E\|S_{2^{n+1}} - S_{2^n}\| > \varepsilon 2^{n+1}\right) \\ &\qquad \leqslant \exp\left\{-\varepsilon_n^2\left[1 - \frac{\Lambda(n)2^{2n+2}}{2b_n^2}\exp\left(2(1+\delta)^2 M/\varepsilon\right)\right]\right\}. \end{aligned} \tag{3.14}$$

Using (3.13) we have

$$\frac{\Lambda(n)2^{2n+2}}{2b_n^2}\exp\left(2(1+\delta)^2 M/\varepsilon\right) \leqslant \tfrac{1}{4}$$

so for such n

$$\begin{aligned} &P\left(\|S_{2^{n+1}} - S_{2^n}\| - E\|S_{2^{n+1}} - S_{2^n}\| > \varepsilon 2^{n+1}\right) \\ &\qquad \leqslant \exp\left(-\tfrac{3}{2}(1+\delta)^2 LL2^{n+1}\right). \end{aligned} \tag{3.15}$$

The remaining case is

$$(3.16)\qquad \Lambda(n)LL2^{n+1} > \left[\frac{\varepsilon}{2(1+\delta)}\right]^2 \exp(-2(1+\delta)^2 M/\varepsilon) \equiv \lambda(\varepsilon, \delta).$$

Let $\tau(\varepsilon, \delta) = [\lambda(\varepsilon, \delta)/2\varepsilon]^{\frac{1}{2}}$, $b_n = \Lambda^{\frac{1}{2}}(n)(2^{n+1}/\tau(\varepsilon, \delta))$, $\varepsilon_n = \varepsilon 2^{n+1}/b_n = \varepsilon\tau(\varepsilon, \delta)/\Lambda^{\frac{1}{2}}(n)$, and $c_n = \tau(\varepsilon, \delta)/\Lambda^{\frac{1}{2}}(n)LL2^{n+1}$. Then $b_n c_n = 2^{n+1}/LL2^{n+1}$ and $\varepsilon_n c_n = \varepsilon\tau^2(\varepsilon, \delta)/\Lambda(n)LL2^{n+1} \leqslant \frac{1}{2}$ by (3.16). Hence from (3.11) we have for such n that

$$(3.17)\qquad P(\|S_{2^{n+1}} - S_{2^n}\| - E\|S_{2^{n+1}} - S_{2^n}\| > \varepsilon 2^{n+1})$$
$$\leqslant \exp\left\{-\frac{7}{8}\varepsilon_n^2\right\} \leqslant \exp\left\{-\frac{7}{8}\frac{\varepsilon^2\tau^2(\varepsilon, \delta)}{\Lambda(n)}\right\}$$

because

$$e^{\varepsilon_n c_n}\frac{\Lambda(n)2^{2n+2}}{2b_n^2} \leqslant e^{\frac{1}{2}}\frac{\Lambda(n)}{2}\frac{\tau^2(\varepsilon, \delta)}{\Lambda(n)} = \frac{\lambda(\varepsilon, \delta)e^{\frac{1}{2}}}{4\varepsilon} \leqslant \frac{1}{8}$$

when $\varepsilon \leqslant 1$. Combining (3.15) and (3.17) we have (3.12) holding since (3.10b) is at our disposal. Thus the theorem is proved.

Applying Lemma 2.1 of [5] and the methods used in Theorem 1 we can also prove the following result.

THEOREM 3. *Let $X_1, X_2, \cdots$ be independent random variables with values in the linear measurable space $(B, \mathbb{B}, \|\cdot\|)$ such that*

$$(3.18)\qquad P(\|X_j\| \leqslant \Lambda) = 1 \qquad j \geqslant 1.$$

Let $\{a_n : n \geqslant 1\}$ be a sequence of positive numbers such that

$$a_n = \Gamma_n(2nLLn)^{\frac{1}{2}}$$

where $\Gamma_n \nearrow \infty$, and there exists an increasing subsequence $\{n_k : k \geqslant 1\}$ satisfying

$$(3.19)\qquad \text{(i) } 0 < \liminf_k \frac{n_k}{n_{k+1}} \leqslant \limsup_k \frac{n_k}{n_{k+1}} < 1,$$
$$\text{(ii) } \liminf_k \frac{\Gamma n_k}{\Gamma^{\frac{1}{2}}_{n_{k+1}}} = \infty.$$

Then

$$\frac{S_n}{a_n} \to_{\mathrm{p}} 0 \quad \text{iff} \quad \frac{S_n}{a_n} \to_{\mathrm{a.s.}} 0.$$

Of course, if $a_n = n^r$ $(r > \frac{1}{2})$, then the above theorem holds.

4. An application to spaces of type p-Rademacher. As mentioned in the introduction the above results are not completely satisfactory, since our conditions are not on the individual terms. However, if one imposes restrictions on the geometry of the space, the results become less unsatisfactory.

A Banach space is said to be of *type p-Rademacher* if there exists a constant $c < \infty$ such that whenever $X_1, \cdots, X_n \in B$ are independent, with $EX_j = 0$ and $E\|X_j\|^p < \infty$, then

$$(4.1) \qquad E\|X_1 + \cdots + X_n\|^p \leqslant c\Sigma_{j=1}^n E\|X_j\|^p.$$

(For further information on these spaces see [3].)

THEOREM 4. *Let F be type p-Rademacher* $1 \leqslant p \leqslant 2$. *Then*

$$(4.2a) \qquad X_j/j \to 0 \quad \text{a.s.}$$

For some $q \in [1, 2]$ *and some* $r \in (0, \infty)$

$$(4.2b) \qquad \Sigma_{n=1}^{\infty}\Lambda(n, q)^r < \infty$$

and

$$(4.2c) \qquad \Sigma_{j=1}^n \frac{E\|X_j\|^p}{n^p} \to 0$$

imply $S_n/n \to 0$ a.s.

PROOF. By Theorem 1 we need only show that (4.2c) implies $(S_n/n) \to_p 0$. But (4.1) and (4.2c) imply $E\|(X_1 + \cdots + X_n)/n\|^p \leqslant c\Sigma_{j=1}^n(E\|X_j\|^p/n^p) \to 0$. Hence $(S_n/n) \to_p 0$.

Theorem 2 can be recast in a similar manner.

It is easy to construct an example to see that Theorem 4 is actually an improvement of the SLLN obtained from Theorem 2.1 of [3] when $1 \leqslant p \leqslant 2$. That is, let $X_j = \varepsilon_j x_j$ where $\{\varepsilon_j\}$ is a sequence of independent random variables with $P(\varepsilon_j = \pm 1) = \frac{1}{2}$, and assume $\{x_j\}$ is a fixed sequence in a type p-Rademacher space F such that $\|x_j\|^p = \frac{2^{n(p-1)}}{n}$ for $2^n \leqslant j < 2^{n+1}$. Then

$$(4.3) \qquad \Sigma_{j=1}^{\infty} \frac{E\|X_j\|^p}{j^p} = \infty,$$

$$(4.4) \qquad \frac{X_j}{j} \to 0 \text{ a.s.},$$

$$(4.5) \qquad \Sigma_{n=1}^{\infty}\Lambda(n,p)^2 < \infty,$$

and

$$(4.6) \qquad \lim_{n\to\infty}\Sigma_{j=1}^n \frac{E\|X_j\|^p}{n^p} = 0.$$

Combining (4.4), (4.5), and (4.6) we see from Theorem 4 that $\{X_j\}$ satisfies the SLLN, but (4.3) prevents Theorem 2.1 of [3] from applying.

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