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EXPONENTIAL MOMENTS OF VECTOR VALUED RANDOM SERIES AND TRIANGULAR ARRAYS[1]

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We prove the finiteness of certain exponential moments of Banach space valued random series and triangular arrays. We also prove integrability results for Poisson measures on Banach spaces.

1. Introduction. This paper contributes to the study of integrability properties of independent series and triangular arrays of random vectors taking values in a Banach space.

For the case of random series, previous research on this question has been carried out in Hoffmann-Jorgensen [4], [5] Ch. II.4, Jain and Marcus [6], Kwapień [9] and Kuelbs [8]. Let $\{X_j\}$ be independent random vectors taking values in a Banach space, $S_n = \sum_{j=1}^n X_j$. In the case when $S_n \to S$ a.s., we give necessary and sufficient conditions for the finiteness of the expectation of certain exponential functions of $\|S\|$ (Theorem 2.8); we also obtain results on the exponential integrability of $\sup_n \|S_n\|$ even when $S_n \nrightarrow$ a.s. (Theorem 2.9). Theorems 2.3, 2.8 and 2.9 improve results in [6] (Theorems 3.8 and 3.11) and in [5] (Corollary II.4.8); in particular, Theorems 2.8 and 2.9 answer a question posed in [5], page 4. Our approach is somewhat different from that of [4], [5] and [6]; we use truncation and inequalities which involve the exponential moments directly, instead of inequalities which make possible the comparison of the tails of $\sup_j \|X_j\|$ and $\sup_n \|S_n\|$, as done in the aforementioned papers. (Although we deal only with exponential moments, we remark that the results on integrability of powers of $\|S\|$ and $\sup_n \|S_n\|$ in [4] and [6] can also be proved by our method.)

In Section 3 we prove the finiteness of first-order exponential moments of uniformly bounded triangular arrays (Theorems 3.1 and 3.2). As corollaries of Theorem 3.2, we obtain two results on the integrability of Poisson measures: Corollary 3.3 improves a result of Yurinskii [10] and Corollary 3.4 extends to Banach spaces a result of Kruglov [7] for the case of Poisson measures on Hilbert space.

In what follows, B will denote a separable Banach space.

2. Exponential moments of random series. We recall the following well-known inequality.

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LEMMA 2.1. *Let X, Y be independent B-valued random vectors, $X \in L^1(B)$, $EX = 0$. Let $f : R+ \to R$ be a convex function such that $Ef(\|X\|) < \infty$ and $Ef(\|Y\|) < \infty$. Then $Ef(\|Y\|) \leqslant Ef(\|X + Y\|)$.*

The next lemma is the exponential form of the converse Kolmogorov inequality (de Acosta and Samur [2]). It is essential for what follows.

LEMMA 2.2. *Let $\{X_j : j = 1, \cdots, n\}$ be independent B-valued random vectors, $S_n = \sum_{j=1}^n X_j$, and assume $\|X_j\| \leqslant c < \infty$ a.s., $j = 1, \cdots, n$. Then*

(a) *if $EX_j = 0, j = 1, \cdots, n$, then for all $\lambda > 0$, $t > 0$,*

$$E(\exp \lambda\|S_n\|) \leqslant e^{\lambda(t+c)}P\{\sup_{1\leqslant k\leqslant n}\|S_k\| > t\}E(\exp \lambda\|S_n\|) + e^{\lambda t};$$

(b) *if each X_j is symmetric, then for all $\lambda > 0$, $t > 0$,*

$$E(\exp \lambda\|S_n\|) \leqslant 2e^{\lambda(t+c)}P\{\|S_n\| > t\}E(\exp \lambda\|S_n\|) + e^{\lambda t}.$$

PROOF. (a) Let $A_k = \{\|S_j\| \leqslant t, j < k;\ \|S_k\| > t\}$, $A = \cup_{k=1}^n A_k = \{\sup_{1\leqslant k\leqslant n}\|S_k\| > t\}$. Then

$$\begin{aligned} E\{(\exp \lambda\|S_n\|)I_A\} &= \textstyle\sum_{k=1}^n E\{(\exp \lambda\|S_n\|)I_{A_k}\} \\ &\leqslant \textstyle\sum_{k=1}^n E\{(\exp \lambda(\|S_k\| + \|S_n - S_k\|))I_{A_k}\} \\ &= \textstyle\sum_{k=1}^n \big(E\{(\exp \lambda\|S_k\|)I_{A_k}\}\big)\big(E(\exp \lambda\|S_n - S_k\|)\big). \end{aligned}$$

By Lemma 2.1, $E(\exp \lambda\|S_n - S_k\|) \leqslant E(\exp \lambda\|S_n\|)$. Since for $\omega \in A_k$ we have $\|S_k(\omega)\| \leqslant \|S_{k-1}(\omega)\| + \|X_k(\omega)\| \leqslant t + c$, it follows that

$$\begin{aligned} E\{(\exp \lambda\|S_n\|)I_A\} &\leqslant e^{\lambda(t+c)}E(\exp \lambda\|S_n\|)\textstyle\sum_{k=1}^n P(A_k) \\ &= e^{\lambda(t+c)}E(\exp \lambda\|S_n\|)P(A). \end{aligned}$$

Since $E\{(\exp \lambda\|S_n\|)I_{A^c}\} \leqslant e^{\lambda t}P(A^c)$, part (a) follows by addition. To prove (b), we use P. Lévy's inequality: $P(A) \leqslant 2P\{\|S_n\| > t\}$. □

We shall use the notation $M = \sup_n\|S_n\|$.

THEOREM 2.3. *Let $\{X_j : j \in N\}$ be independent B-valued random vectors, $S_n = \sum_{j=1}^n X_j$. Assume* (a) *$\|X_j\| \leqslant c < \infty$ for all $j \in N$;*

(b) *$S_n \to S$ almost surely in B.*

Then for every $\lambda > 0$, $E(\exp \lambda\|S\|) < \infty$, $E(\exp \lambda M) < \infty$ and $\lim_n E(\exp \lambda\|S_n\|) = E(\exp \lambda\|S\|)$.

PROOF. Assume first that each X_j is symmetric. For $m \leqslant n$, let $A_{m,n} = \{\|S_n - S_m\| > t\}$. By Lemma 2.2, we have

$$E(\exp \lambda\|S_n - S_m\|) \leqslant 2e^{\lambda(t+c)}E(\exp \lambda\|S_n - S_m\|)P(A_{m,n}) + e^{\lambda t}.$$

Choose m so that $P(A_{m,n}) \leqslant (\frac{1}{4})e^{-\lambda(t+c)}$ for all $n \geqslant m$. Then

$$(\tfrac{1}{2})E(\exp \lambda\|S_n - S_m\|) \leqslant e^{\lambda t} \qquad \text{for all} \quad n \geqslant m$$

and, therefore, $E(\exp \lambda\|S - S_m\|) \leqslant 2e^{\lambda t}$. Since $E(\exp \lambda\|S_m\|) < \infty$ and

$E(\exp \lambda\|S\|) \leqslant E(\exp \lambda\|S_m\|) \cdot E(\exp \lambda\|S - S_m\|)$, it follows that $E(\exp \lambda\|S\|) < \infty$. By P. Lévy's inequality and Fubini's theorem also $E(\exp \lambda M) < \infty$.

A standard desymmetrization procedure (see, e.g., [4] or [6]) completes the proof of the integrability statements. The convergence statement follows by a well-known argument. ▯

The following example exhibits a real-valued independent random series such that $|X_j| \leqslant 1$ a.s. for all j, $S_n \to S$ a.s. but $E(\exp|S|^p) = \infty$ for all $p > 1$. Thus the above integrability result is best possible when one considers functions of the norm of the form $g(x) = \exp(\lambda\|x\|^p)$ $(x \in B)$.

EXAMPLE 2.4. Let $0 \leqslant p_j \leqslant 1$. Let $\{X_j : j \in N\}$ be independent random variables with $\mathcal{L}(X_j) = (1 - p_j)\delta_0 + p_j\delta_1$. Then for any $p > 0$, $n \in N$,

$$E(\exp S_n^p) = \Sigma_{k=0}^n(\exp k^p)P\{S_n = k\}$$
$$\geqslant (\exp n^p)\Pi_{j=1}^n p_j.$$

Now take $p_j = j^{-2}$ for $j \in N$ and $p > 1$; then S_n converges a.s. by the three series theorem but

$$E(\exp S_n^p) \geqslant \exp(n^p)(n!)^{-2} \to \infty \quad \text{as} \quad n \to \infty.$$

If in Theorem 2.3 we merely assume that $\{S_n\}$ is stochastically bounded, then we obtain a weaker result.

THEOREM 2.5. *Let $\{X_j : j \in N\}$ be independent B-valued random vectors, $S_n = \Sigma_{j=1}^n X_j$. Assume* (a) $\|X_j\| \leqslant c < \infty$ *for all* $j \in N$;
(b) $\{S_n\}_{n \in N}$ *is stochastically bounded.*
Then $M < \infty$ a.s. *and there exists $\varepsilon > 0$ such that $E(\exp \varepsilon M) < \infty$.*

PROOF. Assume first that $\{X_j\}$ is symmetric. Choose $t > 0$ so that $\sup_n P\{\|S_n\| > t\} < \frac{1}{8}$ and then $\varepsilon > 0$ small enough so that $\exp(\varepsilon(t + c)) < 2$. Then Lemma 2.2 implies: for all n,

$$\left(\tfrac{1}{2}\right)E(\exp \varepsilon\|S_n\|) \leqslant e^{\varepsilon t}.$$

By Lévy's inequality and Fubini's theorem we have: for all n

$$E(\exp \varepsilon M_n) \leqslant 2E(\exp \varepsilon\|S_n\|),$$

where $M_n = \sup_{k \leqslant n}\|S_k\|$. Now monotone convergence implies $(E \exp \varepsilon M) \leqslant 4e^{\varepsilon t}$. Again, a standard desymmetrization procedure ends the proof. ▯

The following lemma appears in [3]. We give a different proof. For a B-valued random vector X, $\tau > 0$, let $X_\tau = XI_{\{\|X\| \leqslant \tau\}}$, $X^\tau = X - X_\tau$.

LEMMA 2.6. *Let $\{X_j\}_{1 \leqslant j \leqslant n}$ be independent symmetric B-valued random vectors, $S_n = \Sigma_{j=1}^n X_j$. Let $S_{n,\tau} = \Sigma_{j=1}^n X_{j\tau}$. Then for every measurable convex subset K of B,*

$$P\{S_{n,\tau} \notin K\} \leqslant 2P\{S_n \notin K\}.$$

PROOF. It is easily verified that if X is symmetric, then $\mathcal{L}(X_\tau, -X^\tau) = \mathcal{L}(X_\tau, X^\tau)$.

It follows that

$$\begin{aligned}\mathcal{L}(X_{1\tau}, -X_1^\tau; X_{2\tau}, -X_2^\tau; \cdots; X_{n\tau}, -X_n^\tau) &= \mathcal{L}(X_{1\tau}, -X_1^\tau) \otimes \cdots \otimes \mathcal{L}(X_{n\tau}, -X_n^\tau) \\ &= \mathcal{L}(X_{1\tau}, X_1^\tau) \otimes \cdots \otimes \mathcal{L}(X_{n\tau}, X_n^\tau) \\ &= \mathcal{L}(X_{1\tau}, X_1^\tau; \cdots; X_{n\tau}, X_n^\tau),\end{aligned}$$

and, consequently, $\mathcal{L}(S_{n,\tau} - S_n^\tau) = \mathcal{L}(S_{n,\tau} + S_n^\tau) = \mathcal{L}(S_n)$, where $S_n^\tau = \sum_{j=1}^n X_j^\tau$. Now $P\{S_{n,\tau} \notin K\} \leqslant P\{S_{n,\tau} + S_n^\tau \notin K\} + P\{S_{n,\tau} - S_n^\tau \notin K\} = 2P\{S_n \notin K\}$. $\square$

LEMMA 2.7. *Let $\{X_j\}_{j\geqslant 1}$ be independent symmetric B-valued random vectors, $S_n = \sum_{j=1}^n X_j$.*

(a) *Suppose $\{S_n\}$ converges almost surely. Then for every $\tau > 0$, $\{S_{n,\tau}\}$ converges almost surely.*

(b) *Suppose $\{S_n\}$ is stochastically bounded. Then for every $\tau > 0$, $\{S_{n,\tau}\}$ is stochastically bounded.*

PROOF. (a) By well known results, (see, e.g. [4]), it is enough to prove that $\{S_{n,\tau}\}$ is Cauchy in probability. But Lemma 2.6 implies that for every $\varepsilon > 0$, n, $m \in N$ with $n > m$,

$$P\{\|S_{n,\tau} - S_{m,\tau}\| > \varepsilon\} \leqslant 2P\{\|S_n - S_m\| > \varepsilon\}.$$

(b) is similar. $\square$

We shall now consider exponential moments in the unbounded case. Let us remark that if $0 < p < 1$, then $f(x) = \exp(\lambda x^p)(x \geqslant 0)$ is not convex and therefore Lemma 2.1 does not apply to f. However, one may easily prove the following substitute inequality: if X, Y are independent B-valued rv's, $0 < p \leqslant 1$ and $\lambda > 0$, then

$$E(\exp \lambda\|X + Y\|^p) \geqslant E(\exp \lambda\|X\|^p) \cdot E(\exp(-\lambda\|Y\|^p)). \tag{2.1}$$

We shall write $V = \sup_j\|X_j\|$.

THEOREM 2.8. *Let $\{X_j : j \in N\}$ be independent B-valued random vectors, $S_n = \sum_{j=1}^n X_j$. Assume $S_n \to S$ a.s., and let $\lambda > 0$, $0 < p \leqslant 1$. Then the following conditions are equivalent:*

(1) $E(\exp \lambda V^p) < \infty$;

(2) $E(\exp \lambda\|S\|^p) < \infty$;

(3) $E(\exp \lambda M^p) < \infty$.

Moreover, if any of (1)–(3) *is satisfied, then* $\lim_n E(\exp \lambda\|S_n\|^p) = E(\exp \lambda\|S\|^p)$.

PROOF. Assume first that each X_j is symmetric.

(1) $\Rightarrow$ (2). Let $f(x) = \exp(\lambda x^p)$ for $x \geqslant 0$ and let $\tau > 0$. Define $A_n = \{\sup_{1\leqslant j\leqslant n}\|X_j\| > \tau\}$, $A_{n,k} = \{\sup_{1\leqslant j\leqslant k-1}\|X_j\| \leqslant \tau; \|X_k\| > \tau\}$. Then, if $S_n^{(k)} = S_n$

$-X_k$, using $f(x+y) \leqslant f(x)f(y)(x \geqslant 0, y \geqslant 0)$, we have

$$(2.2) \qquad \begin{aligned} E\{f(\|S_n\|)I_{A_n}\} &= \Sigma_{k=1}^n E\{f(\|S_n\|)I_{A_{n,k}}\} \\ &\leqslant \Sigma_{k=1}^n E\{f(\|S_n^{(k)}\| + \|X_k\|)I_{\{\|X_k\|>\tau\}}\} \\ &\leqslant \Sigma_{k=1}^n Ef(\|S_n^{(k)}\|) \cdot E\{f(\|X_k\|)I_{\{\|X_k\|>\tau\}}\} \\ &\leqslant (\sup_{1\leqslant k\leqslant n} Ef(\|S_n^{(k)}\|))F(\tau) \end{aligned}$$

where $F(\tau) = \Sigma_{k=1}^\infty E\{f(\|X_k\|)I_{\{\|X_k\|>\tau\}}\}$.

Since $S_n \to S$ a.s., it follows that for some $r > 0$, $\inf_{j\in N} P\{\|X_j\| \leqslant r\} > (\frac{1}{2})$. Let $a = 2\exp(\lambda r^p)$; then by (2.1)

$$(2.3) \qquad Ef(\|S_n^{(k)}\|) \leqslant aEf(\|S_n\|).$$

Next, let us observe that $Ef(V) < \infty$ implies $\lim_{\tau\to\infty} F(\tau) = 0$. In fact, according to [6], Lemma 3.1, there exists $\alpha > 0$ such that $F(\alpha) < \infty$. The claim follows then from the Fatou-Lebesgue lemma (for series).

By (2.2) and (2.3),

$$\begin{aligned} Ef(\|S_n\|) &= E\{f(\|S_n\|)I_{A_n}\} + E\{f(\|S_n\|)I_{A_n^c}\} \\ &\leqslant aF(\tau)Ef(\|S_n\|) + Ef(\|S_{n,\tau}\|), \end{aligned}$$

where $S_{n,\tau} = \Sigma_{k=1}^n X_k I_{\{\|X_k\|\leqslant\tau\}}$.

Now choose τ so that $aF(\tau) < \frac{1}{2}$; then Lemma 2.7 and Theorem 2.3 imply

$$(\tfrac{1}{2})\sup_n Ef(\|S_n\|) \leqslant \sup_n Ef(\|S_{n,\tau}\|) < \infty,$$

and, therefore, $Ef(\|S\|) < \infty$.

(2) $\Rightarrow$ (3). This follows from Lévy's inequality and Fubini's theorem.

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). By ([2], Lemma 2.3), $P\{V > t\} \leqslant 2P\{\|S\| > t\}$ for all $t > 0$. The result follows now from Fubini's theorem.

By well-known, elementary arguments, in order to prove the convergence statement it is enough to show that

$$(2.4) \qquad \lim_{t\to\infty} \sup_n E\{f(\|S_n\|)I_{\{\|S_n\|>t\}}\} = 0.$$

Now (using the notation in part (1) $\Rightarrow$ (2)), for $t > 0$, $\tau > 0$

$$\begin{aligned} E\{f(\|S_n\|)I_{\{\|S_n\|>t\}}\} &\leqslant E\{f(\|S_n\|)I_{\{\|S_n\|>t\}}I_{A_n^c}\} + E\{f(\|S_n\|)I_{A_n}\} \\ &\leqslant E\{f(\|S_n\|)I_{\{\|S_n\|>t\}}\} + aF(\tau)Ef(\|S_n\|) \\ &\leqslant (Ef^2(\|S_{n,\tau}\|))^{\frac{1}{2}}(P\{\|S_n\| > t\})^{\frac{1}{2}} + aF(\tau)Ef(\|S_n\|) \\ &\leqslant (E(\exp\beta\|S_{n,\tau}\|))^{\frac{1}{2}}(P\{\|S_n\| > t\})^{\frac{1}{2}} + \rho F(\tau) \end{aligned}$$

for an appropriate $\beta > 0$, where $\rho = a\sup_n Ef(\|S_n\|) < \infty$. By Theorem 2.3, for any $\tau > 0$

$$\sup_n E(\exp\beta\|S_{n,\tau}\|) < \infty;$$

therefore, given $\varepsilon > 0$, one may find $\tau > 0$ so that $\rho F(\tau) < \varepsilon/2$ and then $t > 0$ so that the first summand is less than $\varepsilon/2$ for all n. This proves (2.4).

The general (nonsymmetric) case follows by a standard desymmetrization procedure, similar to those presented in [4] and [6] (see also Theorem 3.1 below for desymmetrization in a slightly different situation). $\square$

REMARK. In connection with (2) $\Rightarrow$ (1) above, let us remark that it is a standard procedure in similar contexts to use the inequality $P\{V > t\} \leqslant 2P\{\|S\| > t/2\}$, which is deduced at once from Lévy's inequality and the obvious fact $V \leqslant 2M$. This somewhat weaker inequality would not suffice for our present purposes.

An analogous argument, using Theorem 2.5 and the ideas in the proof of Theorem 2.8, yields the following result. We omit the proof.

THEOREM 2.9. *Let $\{X_j : j \in N\}$ be independent B-valued random vectors, $S_n = \sum_{j=1}^n X_j$ and assume that $\{S_n\}$ is stochastically bounded. Let $V = \sup_j \|X_j\|$, $M = \sup_n \|S_n\|$, $\lambda > 0$.*

(1) *Let $0 < p < 1$. Then $E(\exp \lambda V^p) < \infty$ if and only if $E(\exp \lambda M^p) < \infty$.*

(2) *If $E(\exp \lambda V) < \infty$, then $E(\exp \varepsilon M) < \infty$ for some $\varepsilon > 0$.*

3. Triangular arrays and the integrability of Poisson measures. The methods we have used also yield integrability results for vector-valued triangular arrays. The following result improves Theorem 2.3 (1) of [1].

THEOREM 3.1. *Let $\{X_{nj} : j = 1, \cdots, k_n; n \in N\}$ be a row-wise independent triangular array of B-valued random vectors, $S_n = \sum_j X_{nj}$. Assume*

(a) $\|X_{nj}\| \leqslant c < \infty$ a.s. *for all n, j;*

(b) *$\{S_n\}$ is stochastically bounded.*

Then there exists $\varepsilon > 0$ such that $\sup_n E(\exp \varepsilon \|S_n\|) < \infty$.

PROOF. Assume first that $\{X_{nj}\}$ is symmetric. Then the proof is entirely similar to that of Theorem 2.5.

The desymmetrization technique is a variant of that used in the case of series. The details are as follows. Let $\{X'_{nj}\}$ be a row-wise independent triangular array, independent of $\{X_{nj}\}$ and such that $\mathcal{L}(X'_{nj}) = \mathcal{L}(X_{nj})$ for all n, j, and let $S'_n = \sum_j X'_{nj}$.

If $\{S_n\}$ is stochastically bounded, then so is $\{S_n - S'_n\}$, and the result for the symmetric case implies: for some $\varepsilon > 0$,

$$D = \sup_n Ef(\|S_n - S'_n\|) < \infty,$$

where $f(x) = \exp(\varepsilon x)(x \in R)$. Let $g_n(y) = Ef(\|S_n - y\|), y \in B$. By Fubini's theorem, for all n

$$\int g_n(y)\, d\mathcal{L}(S_n)(y) = Ef(\|S_n - S'_n\|) \leqslant D.$$

Let $r > 0$ be such that $\sup_n P\{\|S_n\| > r\} < \frac{1}{2}$, and let $B_r = \{x \in B : \|x\| \leqslant r\}$. Then $B_r \cap \{y : g_n(y) \leqslant 3D\} \neq \phi$. In fact, if $B_r \subset \{y : g_n(y) > 3D\}$, then

$$D \geqslant \int_{B_r} g_n(y)\, d\mathcal{L}(S_n)(y) \geqslant (3D)\left(\tfrac{1}{2}\right) = \left(\tfrac{3}{2}\right)D,$$

impossible. For each n, choose $y_n \in B_r$ so that $g_n(y_n) \leqslant 3D$. Then for all n,

$$Ef(\|S_n\|) \leqslant Ef(\|S_n - y_n\| + \|y_n\|) = f(\|y_n\|)g_n(y_n) \leqslant (\exp(\varepsilon r))(3D). \qquad \square$$

We may prove a better result under more restrictive conditions on the triangular array $\{X_{nj}\}$. We refer to [1] for definitions and basic facts on Poisson measures.

THEOREM 3.2. *Let $\{X_{nj}\}$ be a row-wise independent infinitesimal triangular array of B-valued random vectors. Assume*

(a) $\|X_{nj}\| \leqslant c < \infty$ a.s. *for all n, j;*

(b) $\mathcal{L}(S_n) \to_w \delta_z * c_\tau \operatorname{Pois} \mu$ *for some $z \in B$ and $\tau > 0$ and some Lévy measure μ.*

Then for every $\lambda > 0$, $\sup_n(E(\exp \lambda\|S_n\|) < \infty$.

PROOF. As in Theorem 3.1, it is enough to prove the result in the symmetric case. Thus we may assume $\mathcal{L}(S_n) \to_w \operatorname{Pois} \mu$, where μ is a symmetric Lévy measure.

By Lemma 2.2 we have for any $\alpha > 0$, $\tau > 0$, $n \in N$ (taking $t = c$)

$$E(\exp \alpha\|S_{n,\tau}\|) \leqslant 2e^{2\alpha t}P\{\|S_{n,\tau}\| > t\}E(\exp \alpha\|S_{n,\tau}\|) + e^{\alpha t}. \tag{3.1}$$

Since $\operatorname{Pois}(\mu|B_r) \to_w \delta_0$ as $r \to 0$, given $\varepsilon > 0$ we may choose $\tau > 0$ so that $\operatorname{Pois}(\mu|B_\tau)(\overline{B_t^c}) < \varepsilon/2$ and τ is a continuity radius of μ. By [1], Theorem 2.10, $\mathcal{L}(S_{n,\tau}) \to_w \operatorname{Pois}(\mu|B_\tau)$ and therefore there exists n_0 such that $n \geqslant n_0$ implies

$$P\{\|S_{n,\tau}\| \geqslant t\} < \varepsilon.$$

Taking $\varepsilon = (\frac{1}{4})\exp(-2\alpha t)$ we obtain from (3.1): for all $n \geqslant n_0$,

$$\left(\tfrac{1}{2}\right)E(\exp \alpha\|S_{n,\tau}\|) \leqslant e^{\alpha t},$$

and, therefore, $\sup_n E(\exp \alpha\|S_{n,\tau}\|) < \infty$. We proceed now to bound $E(\exp \alpha\|S_n^{(\tau)}\|)$, where $S_n^{(\tau)} = S_n - S_{n,\tau}$. Let $\phi_{nj} = I_{\{\|X_{nj}\| > \tau\}}$, $\phi_n = \Sigma_j \phi_{nj}$. Then $\|S_n^{(\tau)}\| \leqslant \phi_n \max_j \|X_{nj}^\tau\| \leqslant c\phi_n$.

By the independence of $\{\phi_{nj} : j = 1, \cdots, k_n\}$,

$$\begin{aligned} E(\exp(\alpha\|S_n^{(\tau)}\|)) &\leqslant E\left(\exp(\alpha c \Sigma_j \phi_{nj})\right) \\ &= \Pi_j E(\exp(\alpha c \phi_{nj})) \\ &= \Pi_j\left(e^{\alpha c}P\{\phi_{nj} = 1\} + P\{\phi_{nj} = 0\}\right) \\ &= \Pi_j\left(1 + (e^{\alpha c} - 1)P\{\|X_{nj}\| > \tau\}\right) \\ &\leqslant \exp\left((e^{\alpha c} - 1)\Sigma_j P\{\|X_{nj}\| > \tau\}\right). \end{aligned}$$

By [1], Theorem 2.2, $\sup_n \Sigma_j P\{\|X_{nj}\| > \tau\} < \infty$. Therefore,

$$\sup_n E(\exp(\alpha\|S_n^{(\tau)}\|)) < \infty.$$

Given $\lambda > 0$, take $\alpha = 2\lambda$; then, since $S_n = S_{n,\tau} + S_n^{(\tau)}$, we have for all n

$$\begin{aligned} E(\exp(\lambda\|S_n\|)) &\leqslant E(\exp(\lambda\|S_{n,\tau}\| + \lambda\|S_n^{(\tau)}\|)) \\ &\leqslant (E(\exp(\alpha\|S_{n,\tau}\|)))^{\frac{1}{2}}(E(\exp(\alpha\|S_n^{(\tau)}\|)))^{\frac{1}{2}}. \qquad \square \end{aligned}$$

As a consequence of Theorem 3.2, we obtain an improvement of the following result of Yurinskii [10]: if μ is a Lévy measure on B with bounded support then $\int \exp(\varepsilon\|x\|)(c_\tau \text{ Pois } \mu)(dx) < \infty$ for some $\varepsilon > 0$.

COROLLARY 3.3. *Let μ be a Lévy measure on B such that $\mu(B_r^c) = 0$ for some $r > 0$. Then for all $\lambda > 0$, all $\tau > 0$,*

$$\int \exp(\lambda\|x\|)(c_\tau \text{ Pois } \mu)(dx) < \infty.$$

PROOF. Let $\{X_{nj}\}$ be an infinitesimal triangular array such that $\mathcal{L}(S_n) \to_w c_\tau \text{ Pois } \mu$; say, let $\{X_{nj} : j = 1, \cdots, n\}$ be independent with $\mathcal{L}(X_{nj}) = c_\tau \text{ Pois}(\mu/n)$.

By [1], Theorem 2.10, for every δ which is a continuity radius of μ we have $\mathcal{L}(S_{n,\delta} - ES_{n,\delta}) \to_w c_\delta \text{ Pois}(\mu|B_\delta)$; if $\delta \geq r$, then $\mu|B_\delta = \mu$ and if $Y_{nj} = X_{nj\delta} - EX_{nj\delta}$, $T_n = \sum_j Y_{nj}$, we have

$$\mathcal{L}(T_n) \to_w c_\delta \text{ Pois } \mu.$$

By Theorem 3.2, $\sup_n E(\exp \lambda\|T_n\|) < \infty$; therefore, by a standard argument $\int \exp(\lambda\|x\|)(c_\delta \text{ Pois } \mu)(dx) < \infty$. □

REMARK. If B is Hilbert space, Kruglov [7] has a sharper result: if μ is a Lévy measure with bounded support, then $\int f(x)(c_\tau \text{ Pois } \mu)(dx) < \infty$, where $f(x) = \exp(\alpha\|x\|\log(1 + \|x\|))$ for some $\alpha > 0$.

Our next proposition generalizes to Banach spaces a result of Kruglov [7] for Poisson measures on Hilbert space.

COROLLARY 3.4. *Let $\phi : B \to R^+$ be a continuous function such that $\phi(x + y) \leq c\phi(x)\phi(y)$ for all $x, y \in B$. Let μ be a Lévy measure on B. Then $\int \phi d(\mu|B_r^c) < \infty$ for some (for all) $r > 0$ if and only if for some (for all) $\tau > 0$ $\int \phi d(c_\tau \text{ Pois } \mu) < \infty$.*

PROOF. If ϕ is a function with the stated property, then it easily follows that there exist $\alpha > 0$, $\beta > 0$ such that for all $x \in B$

$$\phi(x) \leq \alpha \exp(\beta\|x\|). \tag{3.2}$$

Assume $\int \phi d(\mu|B_r^c) < \infty$. Since

$$\begin{aligned}\int \phi d(c_\tau \text{ Pois } \mu) &= \int\int \phi(x + y) d(c_\tau \text{ Pois}(\mu|B_r))(x) d(c_\tau \text{ Pois}(\mu|B_r^c))(y)\\ &\leq c \int \phi d(c_\tau \text{ Pois}(\mu|B_r)) \cdot \int \phi d(c_\tau \text{ Pois}(\mu|B_r^c)),\end{aligned}$$

in view of (3.2) and Corollary 3.3 it is enough to prove that $\int \phi d \text{ Pois } \nu < \infty$, with $\nu = \mu|B_r^c$ (recall that $\mu(B_r^c) < \infty$). But the usual expansion gives

$$\begin{aligned}e^{\|\nu\|} \int \phi d \text{ Pois } \nu &= \sum_{k=0}^\infty (k!)^{-1} \int \phi d\nu^k\\ &= \phi(0) + \sum_{k=1}^\infty (k!)^{-1} \int \phi(x_1 + \cdots + x_k)\nu(dx_1) \cdots \nu(dx_k)\\ &\leq \phi(0) + \sum_{k=1}^\infty (k!)^{-1} c^{k-1} \int \phi(x_1) \cdots \phi(x_k)\nu(dx_1) \cdots \nu(dx_k)\\ &= \phi(0) + \sum_{k=1}^\infty (k!)^{-1} c^{k-1} \left(\int \phi d\nu\right)^k\\ &= \phi(0) + c^{-1}\{\exp(c \textstyle\int \phi d\nu) - 1\} < \infty.\end{aligned}$$

To prove the converse statement, just use the above expansion to obtain, for any $t > 0$

$$\int \phi d(\mu|B_t^c) \leqslant \exp\{\mu(B_t^c)\} \int \phi d \operatorname{Pois}(\mu|B_t^c) < \infty. \qquad \square$$

Let us point out that as particular examples of functions ϕ we have

$$\phi(x) = \exp(\alpha\|x\|^p) \qquad \text{for} \quad \alpha > 0, \qquad 0 < p \leqslant 1$$

and

$$\phi(x) = 2^p(2 + \|x\|^p) \qquad \text{for} \quad p > 0 \quad (\text{with} \quad c = 1.)$$

NOTE ADDED IN PROOF. It is possible to prove versions of Theorems 2.8 and 2.9 for functions ϕ as in Corollary 3.4. Thus one may unify in a single statement the results in [4] and [6] on the integrability of vector valued random series in the case of arbitrary powers of the norm and the results of the present paper in the case of exponential functions of the norm. An investigation along these lines for triangular arrays will be carried out in a forthcoming joint paper with E. Giné.

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