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ON SOME TWO-SEX POPULATION MODELS

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Let M_t be the number of males and F_t the number of females present at time t in a population where births take place at rates which at time t are $mR(M_t, F_t)$ and $fR(M_t, F_t)$ for males and females, respectively. Assume that R has the form $R(M, F) = (M + F)h(M/(M + F))$ with h sufficiently smooth at $m/(m + f)$. A Malthusian parameter λ and a random variable W such that $e^{-\lambda t}M_t \to mW$, $e^{-\lambda}F_t \to fW$ a.s. are exhibited, the rate of convergence is found in form of a central limit theorem and a law of the iterated logarithm and an asymptotic expansion of the reproductive value function $\tilde{V}(M, F) = E(W|M_0 = M, F_0 = F)$ is given. Also some discussion of an associated set of deterministic differential equations is offered and the stochastic model compared to the solutions.

1. Introduction. A number of deterministic and stochastic models describing the development of a population with two interacting sexes have been considered in the literature. See, for example, the surveys by Keyfitz (1971) and Pollard (1971, 1973 Chapter 7, 1977) and the extensive list of references therein. The treatment of these models has, however, intrigued demographers for quite a while, and there appears to have been considerable difficulty in handling sex, as opposed to other relevant features of the population, such as age.

It is, of course, of interest to discuss which features such models should incorporate in order to be of use in applications. In the present paper we follow a different path and attempt to answer some crucial mathematical questions about models which, although too simple to be of any great practical applicability, do incorporate the feature of genuine sex interaction in its purest form. That is, we disregard phenomena such as death, formation and dissolution of couples (marriages) and the structure of the population according to age, parity, location, etc. The state of the population at time t therefore is completely described by the number M_t of males and the number F_t of females present, or, equivalently, by the total population size $N_t = M_t + F_t$ and the sex ratio, which we represent by $X_t = M_t/N_t$.

In deterministic theory, going back to Kendall (1949), the development of (M_t, F_t) is usually described by a system of differential equations,

$$\dot{M}_t = mR(M_t, F_t), \qquad \dot{F}_t = fR(M_t, F_t). \tag{1.1}$$

Here R is the *marriage function* and m and f the *male and female birth rates*. Our discussion starts in Section 2 with a brief review of some of the suggestions for

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explicit forms of R and of the general discussion of properties of R. The main point is to introduce the basic assumption,

$$R(M, F) = (M + F)h\left(\frac{M}{M + F}\right) = Nh(X), \tag{1.2}$$

used in the rest of the paper. It states that for given sex ratio, R is linear in the total population size.

In Section 3, we then study the equations (1.1), in particular the behaviour of the solutions $t \to \infty$. Let $z = m/(m + f)$ be the relative proportion of male births, and let $\lambda = (m + f)h(z)$. If $X_0 = z$, it follows at once from (1.1) and (1.2) that $X_t = z$, $N_t = N_0 e^{\lambda t}$ solves (1.1). In general, one might hope that $X_t \to z$ sufficiently fast to ensure exponential growth at rate λ in the sense that

$$e^{-\lambda t} M_t \to m V_0(M_0, F_0), \qquad e^{-\lambda t} F_t \to f V_0(M_0, F_0) \tag{1.3}$$

for some function V_0 of the initial population. Indeed, this is so. More precisely, we find that

$$V_0(M, F) = (M + F)h_0^*\left(\frac{M}{M + F}\right) = Nh_0^*(X) \tag{1.4}$$

and we give an explicit expression for h_0^* in terms of h. In demographic terms, λ is the *Malthusian parameter* of the model and $V_0(M, F)$ the *reproductive value* of a population of M males and F females. See, for example, Fisher (1930).

The rest of the paper then deals with a stochastic version of the model. This is a pure birth process, where individuals are born at rates which at time t are $mR(M_t, F_t)$ and $fR(M_t, F_t)$ for males and females, respectively. In Section 4, we first show the stochastic analogue of (1.3),

$$e^{-\lambda t} M_t \to m W_0, \qquad e^{-\lambda t} F_t \to f W_0 \text{ a.s.} \tag{1.5}$$

(with $0 < W_0 < \infty$ a.s.), and next find the rate of convergence in (1.5) in the form of a central limit theorem and a law of the iterated logarithm. Our final result, proved in Section 5, then gives a stochastic version of (1.4), viz.,

(1.6)

$$\tilde{V}_0(M, F)/\,(M + F) \to h_0^*(x) \qquad \text{as } M \to \infty, \qquad F \to \infty, \qquad \frac{M}{M + F} \to x,$$

where $\tilde{V}_0(M, F) = E(W_0 | M_0 = M, F_0 = F)$ is a natural extension of the reproductive value function to the stochastic model.

The precise assumptions (essentially smoothness conditions on h) for the above results are given in the body of the paper and Section 6 contains a concluding discussion, incorporating bibliographical remarks.

2. The marriage function. Some of the explicit forms of $R(M, F)$, suggested by Kendall (1949) and others are: MF (*random mating*); M (*male marriage dominance*); F (*female marriage dominance*); $(M + F)/2$ (*arithmetic mean*); $(MF)^{\frac{1}{2}}$ (*geometric mean*); $MF/(M + F)$ (*harmonic mean*); and $M \wedge F$ (*minimum*). Most of these models are based on certain intuitive ideas concerning the mating mechanism, while the motivation for others, such as the geometric mean model, seems more to be mathematical convenience, for example, that equations related to (1.1) can be solved explicitly. The analysis by Kendall makes it reasonable to exclude the random mating model since it leads to infinite population size in finite time (and is hard to interpret in large populations). As a first motivation for (1.2), one can then note that (1.2) holds in the remaining examples, with h as specified in the following table:

$R(M, F)$	M	F	$\frac{M+F}{2}$	$(MF)^{\frac{1}{2}}$	$\frac{MF}{M+F}$	$M \wedge F$
$h(x)$	x	$1-x$	$\frac{1}{2}$	$(x(1-x))^{\frac{1}{2}}$	$x(1-x)$	$x \wedge (1-x)$
$h^*(x)$ $(z = \frac{1}{2})$	$2x$	$2(1-x)$	1	$\frac{1}{2} + (x(1-x))^{\frac{1}{2}}$	$2(x(1-x))^{\frac{1}{2}}$	$2[x \wedge (1-x)]$

(the function h^* differs by a constant from h_0^* of Section 1 and is specified in Section 3). The second motivation for (1.2) is provided by axiomatic discussions such as those of Pollard (1971) and Fredrickson (1971), based upon certain logical rules for marriage and leading to requirements of a more general type, among which (1.2). One could think of the sexes being uniformly distributed in the population and of each individual having a limited milieu, within which the partner is chosen (this limited milieu should be compared to the reasoning behind random mating, implying that within a short period of time the number of contacts of any individual with the opposite sex could in principle be arbitrary large). Seen from the standpoint of one sex only (say the male sex) the idea of a limited milieu would lead to

$$R(M, F) = M\phi\left(\frac{M}{M+F}\right). \tag{2.1}$$

Formulations (1.2) and (2.1) are, of course, equivalent, the correspondence being $h(x) = x\phi(x)$.

The precise conditions on h are stated in the respective theorems. They are little restrictive, essentially smoothness conditions like Hölder continuity on suitable intervals $I \subseteq (0, 1)$,

$$|h(x_1) - h(x_2)| \leqslant c|x_1 - x_2|^p \qquad x_1, x_2 \in I \tag{2.2}$$

(with $0 \leqslant p \leqslant 1$). In order to avoid trivialities, we also need

$$h(x) > 0, \qquad 0 < x < 1. \tag{2.3}$$

Further axiomatic discussions such as those in the above references would limit the class of functions h somewhat, however. For example, it would not seem unreasonable to require that

$$R(M, F_1) \geqslant R(M, F_2), \qquad F_1 \geqslant F_2, \tag{2.4}$$

$$\lim_{F \to \infty} R(M, F) = cM \text{ with } 0 < c < \infty. \tag{2.5}$$

Note that in the formulation (2.1), these axioms correspond to $\phi(x)\uparrow c$ as x decreases from 1 to 0. From either formulation, it is easy to conclude that

$$h(x_2) \leqslant h(x_1)\frac{x_2}{x_1} \text{ when } 0 < x_1 < x_2, \tag{2.6}$$

$$h(x) \leqslant cx = h'(0)x,$$

$$h(x_2) \leqslant h(x_1)\frac{1-x_2}{1-x_1} \text{ when } 0 < x_1 < x_2, \tag{2.7}$$

$$h(1-y) \leqslant dy = -h'(1)y$$

(with $0 < d < \infty$). Here formula (2.7) is derived by interchanging the role of males and females. Beyond the highly unrealistic models corresponding to arithmetic mean or one of the sexes being marriage dominant, this would exclude also the geometric mean model. However, these models are formally included in what follows since none of the conditions (2.4)–(2.7) come up.

3. The deterministic differential equations. The existence and uniqueness of a set (M_t, F_t) of solutions to (1.1), given the initial values (M_0, F_0), is well known assuming Lipschitz type conditions ((2.2) with $p = 1$), see Kamke (1962) Chapter III or argue directly from (3.1), (3.2) below. In the present section, we study the asymptotic behaviour of this set of solutions as $t \to \infty$ with (M_0, F_0) fixed. Passing from the variables (M_t, F_t) to (N_t, X_t), equations (1.1) can be written as

$$\dot{N}_t = N_t h(X_t)(m+f) \tag{3.1}$$

$$\dot{X}_t = (z - X_t)h(X_t)(m+f). \tag{3.2}$$

It also follows from (1.1) that the derivative of $fM_t - mF_t$ vanishes so that $fM_t - mF_t = fM_0 - mF_0$, which is equivalent to

$$N_t(z - X_t) = N_0(z - X_0). \tag{3.3}$$

Therefore, if X_t is known, N_t can be computed from (3.1) or (3.3). In this manner, the investigation of (1.1) reduces to the study of (3.2).

Assume without loss of generality that $0 < X_0 < z$. Then, by (3.2), $X_0 \leqslant X_t \leqslant z$ for all t.

Choosing $\beta_1 \geqslant \beta_2 > 0$ such that $\beta_1 \geqslant h(x)(m+f) \geqslant \beta_2$ when $X_0 \leqslant x \leqslant z$, we get

$$\beta_1(z - X_t) \geqslant \dot{X}_t \geqslant \beta_2(z - X_t), \; (z - X_0)e^{-\beta_1 t} \leqslant z - X_t \leqslant (z - X_0)e^{-\beta_2 t}.$$

Thus, $X_t \uparrow z$ and $X_t < z$ for all $t < \infty$. Define

$$k(y) = h(z)\frac{1}{z-y}\left(\frac{1}{h(z)} - \frac{1}{h(y)}\right), \; h^*(x) = e^{\int_x^z k(y)\,dy}.$$

Note that (2.2) with $p = 1$ ensures the integrability of k at z. We can rewrite (3.2) as

$$\lambda = \dot{X}_s \frac{h(z)}{h(X_s)} \frac{1}{z - X_s} = \frac{\dot{X}_s}{z - X_s} - \dot{X}_s k(X_s),$$

and integration from 0 to t yields

$$\lambda t = -\log(z - X_t) + \log(z - X_0) + \log h^*(X_t) - \log h^*(X_0),$$

$$z - X_t = (z - X_0)\frac{h^*(X_t)}{h^*(X_0)} e^{-\lambda t}, \tag{3.4}$$

$$N_t = N_0 \frac{h^*(X_0)}{h^*(X_t)} e^{\lambda t}. \tag{3.5}$$

Here (3.5) is obtained by combining (3.4) with (3.3). Note that since $X_t \to z$, we also have $h^*(X_t) \to h^*(z) = 1$ so that (3.5) contains (1.3) as a corollary with $h_0^*(x) = h^*(x)/(m + f)$ in (1.4).

Noting that (3.4) and (3.5) follow by symmetry if $z < X_0 < 1$ and are trivial if $X_0 = z$, we have proved the first part of the following result.

THEOREM 1. *Assume that* $0 < X_0 < 1$ *and that conditions* (1.2), (2.3) *and* (2.2) *with* $p = 1$ *and* I *the closed interval with endpoints* X_0 *and* z *hold. Let* (N_t, X_t) *be solutions of* (3.1), (3.2) *corresponding to a set* (M_t, F_t) *of solutions to* (1.1). *Then* $X_t \to z$ *monotonically and* $e^{-\lambda t} N_t \to N_0 h^*(X_0)$. *More precisely,*

$$X_t = z + \frac{X_0 - z}{h^*(X_0)} e^{-\lambda t} + O(e^{-2\lambda t}), N_t = N_0 h^*(X_0) e^{\lambda t} + O(1). \tag{3.6}$$

Furthermore, if h *has a derivative* $h'(z)$ *at* z, *then*

$$X_t = z + \frac{X_0 - z}{h^*(X_0)} e^{-\lambda t} + \left(\frac{X_0 - z}{h^*(X_0)}\right)^2 \frac{h'(z)}{h(z)} e^{-2\lambda t} + o(e^{-2\lambda t}), \tag{3.7}$$

$$N_t = N_0 h^*(X_0) e^{\lambda t} - N_0(X_0 - z)\frac{h'(z)}{h(z)} + o(1). \tag{3.8}$$

To complete the proof, note first that (3.6) follows immediately from (3.4), (3.5) once we observe that as $x, y \to z$, then $k(y) = O(1)$, $h^*(x) = 1 + O(x - z)$. If $h'(z)$ exists, these estimates can be strengthened to $k(y) = -h'(z)/h(z) + o(1)$, $h^*(x) = 1 + (x - z)h'(z)/h(z) + o(x - z)$ and (3.7), (3.8) follow.

REMARKS. Of course, further assumptions on well-behaviour of h at z will yield further refinements of (3.6), (3.7), (3.8). In connection with the minimum model with $z = \frac{1}{2}$, we note also that existence of one-sided derivatives at z suffices for (3.7), (3.8), if one replaces $h'(z)$ by the left derivative for $X_0 < z$ and the right derivative for $X_0 > z$.

In Section 2, the function h^* has been computed (with $z = \frac{1}{2}$) for the various examples considered there. From the above considerations, a straightforward method to obtain explicit solutions is to compute h^*, solve (3.4) for X_t and insert in (3.3). For example, in the harmonic mean model, (3.4) yields a quadratic equation for X_t. More elegant methods may, of course, exist in this and other specific examples.

Even if no explicit form of h is assumed, some information may still be obtained concerning the properties of h^*. Of particular interest is the behaviour of h^* at one of the boundaries, say at 0. As was argued in Section 2, the typical case is (2.6). If,

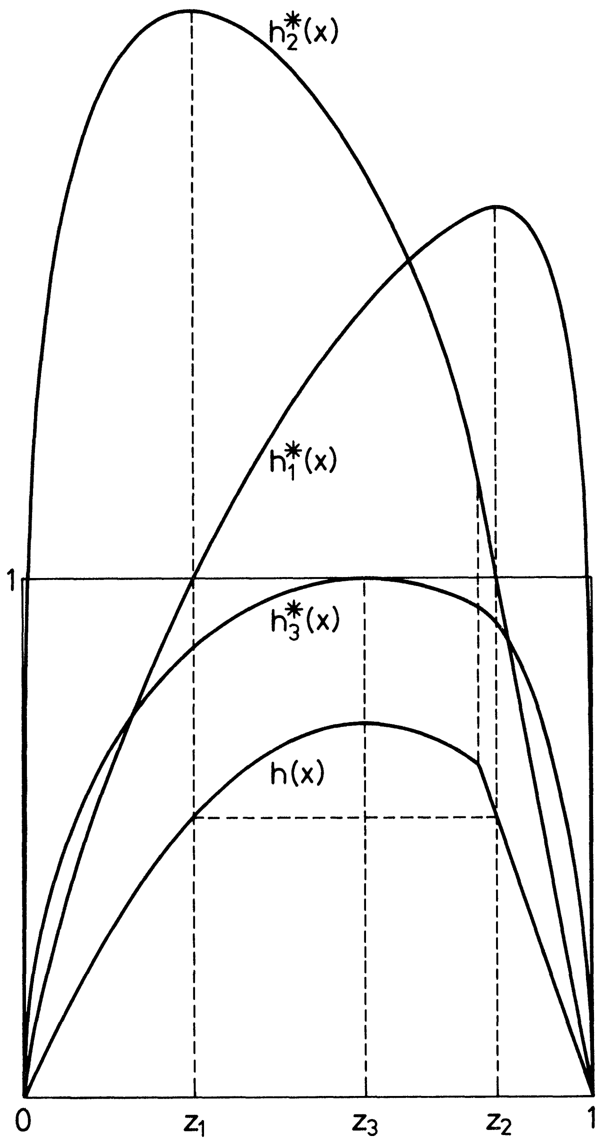

Fig. 1.

furthermore, $h(x) = cx + O(x^2)$ as $x\downarrow 0$, then $h^*(x) \cong dx^\alpha$, where $\alpha = h(z)/cz$. Note that by (2.6), $\alpha \leqslant 1$, with $\alpha = 1$ if and only if h is linear on $[0, z]$. As is seen in the geometric mean example, $h^*(x)$ may have a nonzero limit as $x\downarrow 0$ if (2.6) is violated. This type of behaviour does not correspond nicely to intuition and it will occur if and only if $\int_0^\varepsilon 1/h(y)\,dy < \infty$. Also the (typically unique) point y at which h^* attains its maximum, has a simple description as a solution of $h(y) = h(z)$. The situation is illustrated in the following figure.

For the same h, we have taken three values z_1, z_2, z_3 of z and plotted the corresponding h^*-functions h_1^*, h_2^*, h_3^*. Note that the z_i have been chosen such that $h(z_1) = h(z_2)$ and that h attains its maximum at z_3.

4. Limiting behaviour of the stochastic model as $t \to \infty$. The process $(M_t, F_t)_{t\geqslant 0}$ in question is a time-homogenous continuous time Markov process with state space $\{1, 2, \cdots\} \times \{1, 2, \cdots\}$, where the only possible transitions from state (M, F) are to $(M + 1, F)$ or $(M, F + 1)$, with intensities $mR(M, F)$, respectively $fR(M, F)$. We let $\tau(n)$ be the time of the nth birth (male or female) and $\tau(0) = 0$. The process is then completely described by two independent sequences $Y_1', Y_2', \cdots, V_0, V_1, \cdots$ of random variables, where the Y_k' are i.i.d. $0 - 1$ variables with $P(Y_k' = 1) = z$ and the V_k are i.i.d. with $P(V_k > v) = e^{-v}$, in the following way: $M_{\tau(0)}, M_{\tau(1)}, \cdots$ is a random walk, i.e., $M_{\tau(n)} = M_0 + Y_1' + \cdots + Y_n'$. Also, $N_{\tau(n)} = N_0 + n$, $F_{\tau(n)} = F_0 + n - Y_1' - \cdots - Y_n'$, and the sojourn times $U_k = \tau(k + 1) - \tau(k)$ are given by

$$U_k = \mu_k V_k, \text{ where } \mu_k = \frac{1}{(m + f)R(M_{\tau(k)}, F_{\tau(k)})}.$$

Note that conditional upon $\mathcal{H} = \sigma(Y_1', Y_2', \cdots)$, the U_k are independent and exponentially distributed with $E(U_k|\mathcal{H}) = \mu_k$. It will be convenient to consider the centered variables $Y_k = Y_k' - z$ instead of the Y_k' themselves. Then

(4.1)
$$X_{\tau(n)} = \frac{M_0 + Y_1 + \cdots + Y_n + nz}{N_0 + n} = z + \frac{Y_1 + \cdots + Y_n}{N_0 + n} + \frac{M_0 - N_0 z}{N_0 + n},$$

$$Y_1 + \cdots + Y_n = O(n^{\frac{1}{2}+\varepsilon}) \text{ for all } \varepsilon > 0, \tag{4.2}$$

using the law of the iterated logarithm for (4.2).

Our first result is (1.5) with $W_0 = W/(m + f)$:

THEOREM 2. *Assume that conditions* (1.2), (2.3), *and* (2.2) *with* $p > 0$ *and* I *containing a neighbourhood of* z *hold and let* $\lambda = (m + f)h(z)$. *Then there exists a random variable* W *such that* $0 < W < \infty$ *and*

$$X_t = z + o(1), \qquad N_t = e^{\lambda t}W + o(e^{\lambda t}) \text{ a.s. as } t \to \infty. \tag{4.3}$$

PROOF. Combining (4.1), (4.2) and (2.2), one obtains $h(X_{\tau(k)}) = h(z) + O(k^{-\delta})$, where $\delta = p(\frac{1}{2} - \varepsilon)$,

$$\mu_k = \frac{1}{(m+f)(N_0+k)h(X_{\tau(k)})} = \frac{1}{\lambda(N_0+k)} + O(k^{-1-\delta}). \tag{4.4}$$

Thus $\Sigma_0^\infty \operatorname{Var}(U_k|\mathcal{H}) = \Sigma_0^\infty \mu_k^2$ converges a.s., and conditioning upon $\mathcal{H}$, it follows by standard criteria for convergence of sums of independent mean zero variables that $\Sigma_0^\infty\{U_k - \mu_k\}$ converges a.s. Also from (4.4) and the well-known relation

$$\Sigma_{k=1}^n k^{-1} = \log n + \text{Euler's constant} + O\left(\frac{1}{n}\right), \tag{4.5}$$

it follows that

$$\lambda\Sigma_{k=0}^n \mu_k - \log(N_0+n) = \Sigma_{k=n_0}^{N_0+n} k^{-1} - \log(N_0+n) + \Sigma_{k=0}^n O(k^{-1-\delta})$$

has a limit as $n \to \infty$. Therefore W is well defined by

(4.6)

$$\lambda\tau(n+1) = \lambda\Sigma_{k=0}^n\{U_k - \mu_k\} + \lambda\Sigma_{k=0}^n \mu_k = -\log W + \log(N_0+n) + o(1)$$

and $e^{-\lambda\tau(n+1)}N_{\tau(n+1)} \to W$. Choosing the paths to be right-continuous, we have $N_t = N_{\tau(n)}$ when $\tau(n) \leqslant t < \tau(n+1)$, so that $e^{-\lambda t}N_t = e^{-\lambda\tau(n)}N_{\tau(n)} \cdot e^{-\lambda(t-\tau(n))} \to W$ as $t \to \infty$, since $0 \leqslant t - \tau(n) \leqslant \tau(n+1) - \tau(n) \to 0$ by (4.6). Similarly, $X_{\tau(n)} \to z$, as is obvious from (4.1) and (4.2); and $X_t \to z$ from $X_t = X_{\tau(n)}$ when $\tau(n) \leqslant t < \tau(n+1)$.

We next show that (from the point of view of distribution) the remainder terms in (4.3) are of magnitude $e^{-\lambda t/2}$ and $e^{\lambda t/2}$ respectively. This should be compared to relations (3.6), (3.7), (3.8) for the deterministic model.

THEOREM 3. *In addition to the conditions of Theorem* 2, *suppose that*

$$h(x) = h(z) + (x-z)h'(z) + O((x-z)^2) \text{ as } x \to z \tag{4.7}$$

and write

$$X_t = z + \frac{A_t}{(We^{\lambda t})^{\frac{1}{2}}}, N_t = We^{\lambda t} + (We^{\lambda t})^{\frac{1}{2}}B_t.$$

Then (i) *the limiting distribution of* (A_t, B_t) *exists and is the two-dimensional normal distribution with mean zero and covariance matrix*

$$\begin{pmatrix} \gamma_1^2 & \rho \\ \rho & \gamma_2^2 \end{pmatrix} = \begin{bmatrix} z(1-z) & -z(1-z)\dfrac{h'(z)}{h(z)} \\ -z(1-z)\dfrac{h'(z)}{h(z)} & 1 + 2z(1-z)\left(\dfrac{h'(z)}{h(z)}\right)^2 \end{bmatrix}$$

and (ii) *for all* $(\alpha, \beta) \neq (0, 0)$ *and*

$$C_t = \alpha A_t + \beta B_t, \sigma^2 = \alpha^2\gamma_1^2 + \beta^2\gamma_2^2 + 2\alpha\beta\rho,$$

$$\limsup_{t\to\infty} C_t/(2\sigma^2 \log t)^{\frac{1}{2}} = 1, \liminf_{t\to\infty} C_t/(2\sigma^2\log t)^{\frac{1}{2}} = -1 \text{ a.s.}$$

Proof. We first remark that the central limit theorem for A_t alone in (i) as well as the case $\beta = 0$ in (ii) are almost immediate from similar results on sums of independent random variables by reference to (4.1) and (4.3). The main new difficulty entering here is to obtain precise estimates of the remainder term Δ_n (say) in (4.6). The notation used will indicate that $\Gamma_1, \Gamma_2, \cdots$ are (finite) constants or random variables adding up to $\log W$, that $\Delta_n^1, \Delta_n^2, \cdots$ are remainder terms of the same magnitude as Δ_n, and that $E_n^1, E_n^2, \cdots$ are remainder terms of lower magnitude. First, let $\Delta_n^1 = (Y_1 + \cdots + Y_n)/n$ and use (4.1), (4.2), (4.7) to write

$$h(X_{\tau(k)}) = h(z) + \Delta_k^1 h'(z) + O(k^{-(1-\varepsilon)}),$$

$$\mu_k = \tilde{\mu}_k + O(k^{-(2-\varepsilon)}) \text{ where } \tilde{\mu}_k = \frac{1}{\lambda(N_0 + k)} - \frac{1}{\lambda}\frac{h'(z)}{h(z)}\frac{\Delta_k^1}{N_0 + k},$$

$$\lambda\Sigma_{k=n+1}^{\infty}\{U_k - \mu_k\} = \Sigma_{k=n+1}^{\infty}\frac{V_k - 1}{N_0 + k} + \lambda\Sigma_{k=n+1}^{\infty}\varepsilon_k(V_k - 1) = \Delta_n^2 + E_n^1$$

(say),

where $\varepsilon_k = \mu_k - 1/\lambda(N_0 + k) = O(k^{-(\frac{3}{2}-\varepsilon)})$. Since $\Sigma_0^{\infty} k^{1+2\varepsilon}\varepsilon_k^2 < \infty$, it follows that $\Sigma_0^{\infty} k^{\frac{1}{2}+\varepsilon}\varepsilon_k(V_k - 1)$ converges and using Abel's lemma, e.g. in the form of part (i) of Lemma 2 of Asmussen (1976), we can conclude that $E_n^1 = O(n^{-\frac{1}{2}-\varepsilon})$. Next, in the formula

$$\lambda\Sigma_{k=0}^{n}\mu_k = \lambda\Sigma_{k=0}^{n}\{\mu_k - \tilde{\mu}_k\} + \Sigma_{k=0}^{n}\frac{1}{N_0 + k} - \frac{h'(z)}{h(z)}\Sigma_{k=0}^{n}\frac{\Delta_k^1}{N_0 + k},$$

the first term can be written as $\Gamma_1 + E_n^2$, where

$$\Gamma_1 = \lambda\Sigma_{k=0}^{\infty}\{\mu_k - \tilde{\mu}_k\},\ E_n^2 = -\lambda\Sigma_{k=n+1}^{\infty}\{\mu_k - \tilde{\mu}_k\}$$
$$= \Sigma_{k=n+1}^{\infty}O(k^{-(2-\varepsilon)}) = O(n^{-(1-\varepsilon)}),$$

the middle term, using (4.5), can be written as $\log(N_0 + n + 1) + \Gamma_2 + E_n^3$, where Γ_2 is constant and $E_n^3 = O(n^{-1})$, and the last term as $\Gamma_3 + \tilde{\Delta}_n^3 + \tilde{\Delta}_n^4$, where

$$\kappa(n) = \Sigma_{k=n}^{\infty}\frac{1}{k(N_0 + k)},\ \Gamma_3 = -\frac{h'(z)}{h(z)}\Sigma_{k=1}^{\infty}\kappa(k)Y_k,$$

$$\tilde{\Delta}_n^3 = \frac{h'(z)}{h(z)}\Sigma_{k=n+1}^{\infty}\kappa(k)Y_k,\ \tilde{\Delta}_n^4 = \frac{h'(z)}{h(z)}\Sigma_{k=1}^{n}\kappa(n + 1)Y_k.$$

Note that $\kappa(n) = n^{-1} + O(n^{-2})$, which makes Γ_3 well defined and $\tilde{\Delta}_n^3 \to 0$, $\tilde{\Delta}_n^4 \to 0$ a.s. It will be slightly more convenient to work with Δ_n^3, Δ_n^4, defined as above by replacing $\kappa(n)$, respectively $\kappa(n + 1)$, by n^{-1}. Then it is easy to see that $\tilde{\Delta}_n^3 + \tilde{\Delta}_n^4 = \Delta_n^3 + \Delta_n^4 + E_n^4$, where $E_n^4 = o(n^{-1})$. Combining the above estimates with (4.6), it follows as the first step of the proof that

$$\lambda\tau(n + 1) = \log(N_0 + n + 1) - \log W + \Delta_n + E_n, \tag{4.8}$$

where $-\log W = \lambda\Sigma_0^{\infty}\{U_k - \mu_k\} + \Gamma_1 + \Gamma_2 + \Gamma_3$, $\Delta_n = -\Delta_n^2 + \Delta_n^3 + \Delta_n^4$ and $E_n = -E_n^1 + E_n^2 + E_n^3 + E_n^4 = o(n^{-\frac{1}{2}-\varepsilon})$ a.s.

If $n \to \infty$, $t \to \infty$ such that $\tau(n) \leqslant t < \tau(n+1)$, then $We^{\lambda t}/(n-1) \to 1$, so that we can replace the normalizing factors $(We^{\lambda t})^{\frac{1}{2}}$ by $(n-1)^{\frac{1}{2}}$. Furthermore,

$$We^{\lambda t} = We^{\lambda\tau(n)}\{1 + O(\tau(n+1) - \tau(n))\} = We^{\lambda\tau(n)} + O(n\mu_n V_n)$$

$$= We^{\lambda\tau(n)} + O(V_n) = We^{\lambda\tau(n)} + O(\log n) = We^{\lambda\tau(n)} + O(t),$$

using the Borel-Cantelli lemma to estimate V_n. Therefore, the assertions of Theorem 3 are equivalent to that (i) the limiting distribution as $t \to \infty$ of

$$(A_n', B_n') = \left(n^{\frac{1}{2}}(X_{\tau(n+1)} - z), n^{-\frac{1}{2}}\left(N_{\tau(n+1)} - We^{\lambda\tau(n+1)}\right)\right)$$

(with $n = n(t)$ as above) exists and is as asserted, and (ii) lim sup $C_n'/(2\sigma^2 \log \log n)^{\frac{1}{2}} = 1$ a.s., lim inf $= -1$ a.s., where $C_n' = \alpha A_n' + \beta B_n'$. We claim that

$$A_n'' = n^{\frac{1}{2}}\Delta_n^1 = A_n' + O\left(n^{-\frac{1}{2}}\right), B_n'' = -n^{\frac{1}{2}}\Delta_n = B_n' + O(n^{-\varepsilon})$$

and that, therefore, we can consider A_n'', B_n'', $C_n'' = \alpha A_n'' + \beta B_n''$ rather than A_n', B_n', C_n'. Indeed, for A_n'' this follows from (4.1), (4.2), while for B_n'', inserting (4.8) in the definition of B_n', the claim boils down to $n^{\frac{1}{2}}\Delta_n^2 = O(n^{-\varepsilon})$. From the proof below of the law of the iterated logarithm for C_n'', it follows by taking $\alpha = 0$, that even $n^{\frac{1}{2}}\Delta_n^2 = O$ (log log $n/n^{\frac{1}{2}}$), but the provisional bound $O(n^{-\varepsilon})$ could also easily be derived directly.

By the Cramér-Wold device (Feller (1971) page 522), the central limit theorem will follow if we can show that C_n'' is asymptotically normal with mean zero and variance σ^2. The second step in the proof is thereby completed by reducing it to the study of C_n'', which is simply a sum of independent mean zero variables. Indeed,

$$C_n'' = n^{\frac{1}{2}}\left[\tau_1 n^{-1}\Sigma_{k=1}^n Y_k + \tau_2\Sigma_{k=n+1}^{\infty}\frac{Y_k}{k} + \tau_3\Sigma_{k=n+1}^{\infty}\frac{V_k - 1}{N_0 + k}\right], \tag{4.9}$$

where $\tau_1 = \alpha - \beta\dfrac{h'(z)}{h(z)}$, $\tau_2 = -\beta\dfrac{h'(z)}{h(z)}$, $\tau_3 = \beta$. Note that

$$\sigma_n^2 = \operatorname{Var} C_n'' = z(1-z)\left(\tau_1^2 + \tau_2^2\right) + \tau_3^2 + o(1) = \sigma^2 + o(1)$$

and the central limit theorem for C_n'' as $n \to \infty$ (nonrandom) follows easily from standard criteria adapted to infinite sums (in fact, one can even estimate the rate of convergence, see (4.10) below). That this still holds subject to the random indexing $n = n(t)$ is (noting that $e^{-\lambda t}n(t) \to W$) an assertion of the type of Anscombe's theorem. The details of proof, which involve a use of maximal inequalities similar to the one in the use of the law of the iterated logarithm below, can be found in a forthcoming monograph by the author and H. Hering.

Summing the third moments in (4.9) and using the Berry-Esseen theorem yields

$$\sup_{-\infty<c<\infty}|P(C_n'' \leqslant c\sigma_n) - \Phi(c)| = O\left(n^{-\frac{1}{2}}\right). \tag{4.10}$$

This and related estimates will be a main tool in the proof of the law of the iterated logarithm for C_n''. To this end, we note that if $D_1, D_2, \cdots$ are random variables such that

$$\Sigma_{r=1}^{\infty}\sup_{-\infty<d<\infty}|P(D_r \leqslant d) - \Phi(d)| < \infty, \tag{4.11}$$

then $\limsup D_r/(2\log r)^{\frac{1}{2}} \leqslant 1$ a.s., with $= 1$ if the D_r are independent. Indeed, from well-known tail estimates of Φ and (4.11) it follows easily that $\Sigma P(D_r > \eta(2\log r)^{\frac{1}{2}})$ converges for $\eta > 1$ and diverges for $\eta < 1$. The details have been spelled out in Lemma 1 of Asmussen (1977). Letting first $D_r = C''_{\Theta^r}/\sigma_{\Theta^r}$ with $1 < \Theta < \infty$, (4.11) follows at once from (4.10) so that $\limsup C''_{\Theta^r}/(2\sigma^2 \log r) \leqslant 1$. Let $\Theta^r < n \leqslant \Theta^{r+1}$. Then it follows after some elementary calculations that

$$C_n'' \leqslant \Theta^{\frac{1}{2}}C''_{\Theta^r} + |\tau_1|\Theta^{\frac{1}{2}}\left\{\frac{|Y_1 + \cdots + Y_{\Theta^r}|}{\Theta^{r/2}}\left(1 - \frac{1}{\Theta}\right) + M_r^1\right\} + |\tau_2|M_r^2 + |\tau_3|M_r^3,$$

where

$$M_r^1 = \Theta^{-r/2}\max_{\Theta^r \leqslant n \leqslant \Theta^{r+1}}|\Sigma_{k=\Theta^r+1}^{n} Y_k|,$$

$$M_r^2 = \Theta^{(r+1)/2}\max_{\Theta^r \leqslant n \leqslant \Theta^{r+1}}|\Sigma_{k=\Theta^r+1}^{n}\frac{Y_k}{k}|,$$

$$M_r^3 = \Theta^{(r+1)/2}\max_{\Theta^r \leqslant n \leqslant \Theta^{r+1}}|\Sigma_{k=\Theta^r+1}^{n}\frac{V_k - 1}{N_0 + k}|.$$

Letting $\alpha^i(\Theta) = \limsup M_r^i/(2\log r)^{\frac{1}{2}}$ and using the law of the iterated logarithm for $Y_1 + \cdots + Y_{\Theta^r}$, we get

$$\limsup_{n\to\infty} C_n''/\ (2\log\log n)^{\frac{1}{2}} = \limsup_{r\to\infty}\ \max_{\Theta^r \leqslant n \leqslant \Theta^r+1} C_n''/\ (2\log r)^{\frac{1}{2}}$$

$$\leqslant \Theta^{\frac{1}{2}}\sigma + |\tau_1|(\Theta z(1 - z))^{\frac{1}{2}}\left(1 - \frac{1}{\Theta}\right) + |\tau_1|\Theta^{\frac{1}{2}}\alpha^1(\Theta) + |\tau_2|\alpha^2(\Theta) + |\tau_3|\alpha^3(\Theta).$$

To prove the lim sup $\leqslant 1$ part of (ii), it is thus sufficient to show that $\alpha^i(\Theta) \to 0$ as $\Theta\downarrow 1$. The $\alpha^i(\Theta)$ are estimated by the same method, which we exemplify for $i = 3$. Let

$$w_r^2 = \Theta^{r+1}\Sigma_{k=\Theta^r+1}^{\Theta^{r+1}}\frac{1}{(N_0 + k)^2}, D_r = w_r^{-1}\Theta^{(r+1)/2}\Sigma_{k=\Theta^r+1}^{\Theta^{r+1}}\frac{V_k - 1}{N_0 + k}$$

and note that $w_r^2 \to \Theta - 1$, Var $D_r = 1$. Using the Berry-Esseen theorem, one can easily prove (4.11) so that $\Sigma P(|D_r| > \eta(2\log r)^{\frac{1}{2}})$ converges for $\eta > 1$. If $\xi > \eta(\Theta - 1)^{\frac{1}{2}}$, then $\xi > \eta w_r$ eventually and thus, using a version of Levy's inequality,

$$\Sigma_{r=1}^{\infty}P\left(M_r^3 > \xi(2\log r)^{\frac{1}{2}} + w_r 2^{1/2}\right) \leqslant \Sigma_{r=1}^{\infty}2P\left(w_r|D_r| > \xi(2\log r)^{\frac{1}{2}}\right) < \infty.$$

Thus $\alpha^3(\Theta) \leqslant \xi$, $\alpha^3(\Theta) \leqslant (\Theta - 1)^{\frac{1}{2}}$ and the claim follows.

In the proof of lim sup $\geqslant 1$, we approximate C''^{2r}_{Θ} by

$$D_r' = \Theta^r\left[\tau_1\Theta^{-2r}\Sigma_{k=\Theta^{2r-1}+1}^{\Theta^{2r}}Y_k + \tau_2\Sigma_{k=\Theta^{2r}+1}^{\Theta^{2r+1}}\frac{Y_k}{k} + \tau_3\Sigma_{k=\Theta^{2r}+1}^{\Theta^{2r+1}}\frac{V_k - 1}{N_0 + k}\right].$$

Then it is easy to check that $w_r^2 = \operatorname{Var} D_r' \to \sigma^2(1 - 1/\Theta)$ as $r \to \infty$ and, using the Berry-Esseen theorem, that (4.11) holds for $D_r = w_r^{-1} D_r'$. Thus, since the D_r are independent, lim sup $D_r'/(2\sigma^2 \log r)^{\frac{1}{2}} \geqslant (1 - 1/\Theta)^{\frac{1}{2}}$. Furthermore

$$C_\Theta''^{2r} = D_r' + \Theta^r\left[\tau_1 \Theta^{-2r} \Sigma_{k=1}^{\Theta^{2r-1}} Y_k + \tau_2 \Sigma_{k=\Theta^{2r+1}+1}^{\infty} \frac{Y_k}{k} + \tau_3 \Sigma_{k=\Theta^{2r+1}+1}^{\infty} \frac{V_k - 1}{N_0 + k}\right].$$

Estimating $T_r = \Theta^r \Sigma_{\Theta^{2r+1}+1}^{\infty}(V_k - 1)/(N_0 + k)$ as above or appealing to Chow and Teicher (1973) one can prove that lim sup$|T_r|/(2 \log r)^{\frac{1}{2}} \leqslant 1/\Theta^{\frac{1}{2}}$. Similar estimates of the two other terms under the bracket can be obtained and yields

$$\limsup_{n\to\infty} C_n''/\,(2\sigma^2 \log\log n)^{\frac{1}{2}} \geqslant \lim_{r\to\infty} C_\Theta''2r/\,(2\sigma^2 \log r)^{\frac{1}{2}}$$

$$\geqslant \left(1 - \frac{1}{\Theta}\right)^{\frac{1}{2}} - \frac{(|\tau_1| + |\tau_2|)(z(1 - z))^{\frac{1}{2}} + |\tau_3|}{\sigma\Theta^{\frac{1}{2}}}.$$

Letting $\Theta\uparrow\infty$ completes the proof of lim sup = 1 and the proof of lim inf = − 1 follows similarly or by symmetry.

REMARK. One-sided analogues of (4.7) do not suffice to determine the behaviour of B_t (but clearly of A_t) in Theorem 3. This should be compared to a remark in Section 3 on the deterministic case. The behaviour of B_t, say in the minimum model with $z = \frac{1}{2}$, could however be studied with similar methods.

5. An asymptotic formula for the reproductive value $\tilde{\mathbf{V}}(\mathbf{M}, \mathbf{F}) = \mathbf{E}(\mathbf{W}|\mathbf{M}_0 = \mathbf{M}, \mathbf{F}_0 = \mathbf{F})$. Besides the relation to the concept of reproductive value of a population, the function $\tilde{V}$ is of considerable theoretical interest. Thus we have:

PROPOSITION 1. *The process* $e^{-\lambda t}\tilde{V}(M_t, F_t)$ *is a nonnegative martingale* w.r.t. $\mathcal{F}_t = \sigma(M_s, F_s;\ 0 \leqslant s \leqslant t)$ *and* $e^{-\lambda t}\tilde{V}(M_t, F_t) \to W$ a.s. *Furthermore,* $\tilde{V}$ *solves the difference equation*

$$(5.1)\qquad \lambda\tilde{V}(M, F) = (M + F)h\left(\frac{M}{M + F}\right)$$
$$\left[m\tilde{V}(M + 1, F) + f\tilde{V}(M, F + 1) - (m + f)\tilde{V}(M, F)\right].$$

PROOF. The first assertion follows from general martingale theory since

$$E(W|\mathcal{F}_t) = E^{M_t, F_t}e^{-\lambda t}W = e^{-\lambda t}\tilde{V}(M_t, F_t)$$

(here and in the following $E^{M, F}$ denotes expectation in a process with $M_0 = M$, $F_0 = F$). The martingale property is equivalent to $\lambda\tilde{V} = A\tilde{V}$, where A is the infinitesimal generator of the transition semigroup, and this equation is simply (5.1).

In the deterministic case, $e^{-\lambda t}V_0(M_t, F_t)$ was constant, cf. (3.5), and the form of V_0 was derived from equations (1.1). The counterparts of these equations in the stochastic case are

$$(5.2)\qquad \dot{E}M_t = mER(M_t, F_t), \qquad \dot{E}F_t = fER(M_t, F_t),$$

which cannot be reduced by the same methods. We leave it as an open question whether equations (5.1) or (5.2) are of any use for the study of V and use instead the methods of Section 4 to prove the following result:

THEOREM 4. *Suppose that conditions* (1.2), (2.3) *and* (2.2) *with* $p > 0$ *and* I *containing a neighbourhood of* z *and of* $x \in (0, 1)$ *hold. Then* $\tilde{V}(M_0, F_0)/N_0 \to h^*(x)$ (*with* h^* *defined as in Section* 3) *when*

$$M_0 \to \infty,\ F_0 \to \infty \text{ in such a way that } X_0 = \frac{M_0}{M_0 + F_0} \to x. \tag{5.3}$$

PROOF. We use the notation of Section 3, with the same sequence $Y_1, Y_2, \cdots$ for all M_0, F_0. The constants in the inequalities are always independent of M_0, F_0 (but many depend on x). Let

$$W_{\tau(n+1)} = N_{\tau(n+1)}e^{-\lambda\tau(n+1)} = (N_0 + n + 1)\Pi_{k=0}^{n}e^{-\lambda\mu_k V_k}.$$

Conditioning upon $\mathcal{H}$ yields

(5.4)

$$E^{M_0, F_0}(W_{\tau(n+1)}|\mathcal{H}) = (N_0 + n + 1)\Pi_{k=0}^{n}\left(1 - \frac{1}{1/\lambda\mu_k + 1}\right)$$

$$= (N_0 + n + 1)\Pi_{k=0}^{n}\left(1 - \frac{\lambda}{(N_0 + k)h(X_{\tau(k)})(m + f) + \lambda}\right).$$

The idea of the proof is to observe that $W_{\tau(n)} \to W$, prove that indeed,

$$E^{M_0, F_0}W = \lim_{n\to\infty}E^{M_0, F_0}E^{M_0, F_0}(W_{\tau(n+1)}|\mathcal{H}) \tag{5.5}$$

and show that for large M_0, F_0, we can replace $X_{\tau(k)}$ in (5.4) by its expected value $(M_0 + kz)/(N_0 + k) = x_k$ (say). The asymptotic expression for $\tilde{V}$ will then come out by elementary calculus. To this end, define for some fixed $\varepsilon > 0$

$$T = \sup\{n : |Y_1 + \cdots + Y_n| > n^{\frac{1}{2}+\varepsilon}\}$$

$$C_n(M_0, F_0) = \Pi_{k=0}^{T\wedge n}\left(1 - \frac{\lambda}{1/\mu_k + \lambda}\right), \quad C_\infty(M_0, F_0) = \Pi_{k=0}^{T}\left(1 - \frac{\lambda}{1/\mu_k + \lambda}\right),$$

$$D_n(M_0, F_0) = \Pi_{k=T\wedge n+1}^{n}\left(1 - \frac{\lambda}{1/\mu_k + \lambda}\right).$$

Note that the right-hand side of (5.4) is $(N_0 + n + 1)C_n(M_0, F_0)D_n(M_0, F_0)$ and that $T < \infty$ a.s. by the law of the iterated logarithm. We shall need below the fact that even $ET^\beta < \infty$ for all $\beta > 0$. See, for example, the more general results by Strassen (1965). For C_n, the elementary estimates

$$C_\infty(M_0, F_0) \leqslant C_n(M_0, F_0) \leqslant 1,\ C_\infty(M_0, F_0) \to 1 \text{ subject to (5.3)} \tag{5.6}$$

will suffice, while more care is needed when treating D_n. Preparing for an expansion of $\log D_n$, we first note that for $k > T$ it follows from (2.2) that

$$h(X_{\tau(k)}) = h\left(\frac{M_0 + Y_1' + \cdots + Y_k'}{N_0 + k}\right)$$

$$= h\left(x_k + \frac{Y_1 + \cdots + Y_k}{N_0 + k}\right) = h(x_k) + E_k^1$$

where $|E_k^1| \leqslant \gamma_1 k^\delta/(N_0 + k)^p$, $\delta = (\frac{1}{2} + \varepsilon)p$. Also from (5.3) and (2.3), we must have $h(x_k)(m + f) \geqslant \xi$ for some $\xi > 0$ and all M_0, F_0, k. Without loss of generality, we can assume that $|E_k^1| + \lambda/(N_0 + k) < \xi/2$ (say) for all N_0, k and it then follows for $k > T$ that

$$\text{(5.7)} \qquad \frac{\lambda}{1/\mu_k + \lambda} = \frac{\lambda}{N_0 + k} \cdot \frac{1}{h(x_k)(m + f) + E_k^1 + \lambda/(N_0 + k)}$$

$$= \frac{h(z)}{(N_0 + k)h(x_k)} + E_k^2,$$

where $|E_k^2| \leqslant \gamma_2 k^\delta/(N_0 + k)^{1+p}$. Write further

$$\text{(5.8)} \qquad \Sigma_{k=0}^{T\wedge n} \frac{h(z)}{(N_0 + k)h(x_k)} = E^3, \Sigma_{k=0}^{n} \frac{1}{N_0 + k} = \log \frac{N_0 + n}{N_0} + E^4.$$

Then $0 \leqslant E^3 \leqslant \gamma_3 \log(T + N_0)/N_0, |E^4| \leqslant \gamma_4/N_0$. Assume without loss of generality $0 < x \leqslant z, 0 < X_0 \leqslant z$ so that $x_k\uparrow$ and let $I_k = [x_{k-1}, x_k)$, $l(y) = 1/h(z) - 1/h(y)$. The Lebesgue measure $m(I_k)$ of I_k is

$$m(I_k) = \frac{N_0(z - X_0)}{(N_0 + k)(N_0 + k - 1)} = \frac{N_0(z - X_0)}{(N_0 + k)^2} + E_k^5$$

where $|E_k^5| \leqslant \gamma_5 m(I_k)/N_0$. By (2.2),

$$\sup_{y_1, y_2 \in I_k} \left| \frac{l(y_1)}{z - y_1} - \frac{l(y_2)}{z - y_2} \right| \leqslant E^6$$

where $E^6 \leqslant \gamma_6/N_0^p$. Therefore,

$$\text{(5.9)} \quad \Sigma_{k=0}^{n} \frac{l(x_k)}{N_0 + k} = \Sigma_{k=0}^{n} \frac{N_0(z - X_0)}{(N_0 + k)^2} \frac{l(x_k)}{z - x_k} = \Sigma_{k=1}^{n} m(I_k) \frac{l(x_k)}{z - x_k} + E_8$$

$$= \int_{x_0}^{x_n} \frac{l(y)}{z - y}\, dy + E^8 + E^9 = \int_{x_0}^{z} \frac{l(y)}{z - y}\, dy + E^{10} + E_n^{11}$$

where $|E^8| \leqslant \gamma_8/N_0, |E^9| \leqslant E^6, |E^{10}| = |E^8 + E^9| \leqslant \gamma_{10}/N_0^p, |E_n^{11}| \leqslant \gamma_{11}N_0^p/(N_0 + n)^p$. Combining (5.8), (5.9) yields

$$\begin{aligned}\Sigma_{k=T\wedge n+1}^n \frac{\lambda}{(N_0 + k)h(x_k)(m + f)} &= \Sigma_{k=0}^n \frac{h(z)}{(N_0 + k)h(x_k)} - E^3\\ &= \log\frac{N_0 + n}{N_0} + E^4 - h(z)\Sigma_{k=0}^n \frac{l(x_k)}{N_0 + k} - E^3\\ &= \log\frac{N_0 + n}{N_0} - \log h^*(x_0) + E^{12} - h(z)E_n^{11} - E^3\end{aligned}$$

where $|E^{12}| \leqslant \gamma_{12}/N_0^p$. Using (5.7),

$$\begin{aligned}\log D_n(M_0, F_0) &\leqslant -\Sigma_{k=T\wedge n+1}^n \frac{\lambda}{1/\mu_k + \lambda}\\ &= -\Sigma_{k=T\wedge n+1}^n \frac{\lambda}{(N_0 + k)h(x_k)(m + f)} + E^{13},\end{aligned}$$

where $|E^{13}| \leqslant \Sigma_0^\infty |E_k^2| \to 0$ subject to (5.3), say by dominated convergence. Combining with $C_n \leqslant 1$, we have thus proved that

$$\begin{aligned}&E^{M_0, F_0}(W_{\tau(n+1)}|\mathcal{H})\\ &\qquad \leqslant (N_0 + n + 1) \cdot 1 \cdot \frac{N_0}{N_0 + n} h^*(X_0)e^{-E^{12} + E^{13} + h(z)E_n^{11}} \cdot \left(\frac{T + N_0}{N_0}\right)^{\gamma_3},\\ &E^{M_0, F_0}W \leqslant \liminf_{n\to\infty} E^{M_0, F_0}E^{M_0, F_0}(W_{\tau(n+1)}|\mathcal{H})\\ &\qquad \leqslant N_0 h^*(X_0)e^{-E^{12} + E^{13}}E\left(\frac{T + N_0}{N_0}\right)^{\gamma_3}.\end{aligned}$$

When (5.3) holds it follows by dominated convergence that lim sup $\tilde{V}(M_0, F_0)/N_0 \leqslant h^*(x)$.

To obtain the lim inf $\geqslant$ – part of Theorem 4, we first prove that for fixed M_0, F_0,

$$\text{(5.10)} \qquad \sup_n E^{M_0, F_0}W_{\tau(n+1)}^2 < \infty.$$

By uniform integrability, this is enough to ensure (5.5). Let $\check{\mu}_k = 2\mu_k$ and define $\check{C}_n$, $\check{E}^i$, $\check{\gamma}_i$, etc., as above, repeating the estimates with μ_k replaced by $\check{\mu}_k$. Then essentially one has to multiply the main terms by 2, while the order of magnitude of the $\check{E}^i$ and the E^i are the same. We obtain

$$\begin{aligned} E^{M_0, F_0} W^2_{\tau(n+1)} &= (N_0 + n + 1)^2 E^{M_0, F_0} \textstyle\prod_{k=0}^n E^{M_0, F_0}(e^{-2\lambda U_k} \mid \mathcal{H}) \\ &\leqslant (N_0 + n + 1)^2 E^{M_0, F_0} \check{D}_n(M_0, F_0) \\ &\leqslant (N_0 + n + 1)^2 \left(\frac{N_0}{N_0 + n} \right)^2 \\ &\qquad \times h^*(X_0)^2 e^{-\check{E}^{12} + \check{E}^{13} + h(z)\check{E}^{11}_n} E \left(\frac{T + N_0}{N_0} \right)^{\check{\gamma}_3}. \end{aligned}$$

From this and $ET^{\check{\gamma}_3} < \infty$ (5.10) is immediate and we get from (5.5), (5.6)

$$\begin{aligned} E^{M_0, F_0} W &\geqslant \lim_{n\to\infty} (N_0 + n + 1) E^{M_0, F_0} C_\infty(M_0, F_0) D_n(M_0, F_0) \\ &\geqslant \lim_{n\to\infty} (N_0 + n + 1) \frac{N_0}{N_0 + n} \\ &\qquad \times h^*(X_0) e^{-E^{12} + E^{13} + h(z) E^{11}_n} E^{M_0, F_0} C_\infty(M_0, F_0) \\ &= N_0 h^*(X_0) e^{-E^{12} + E^{13}} E^{M_0, F_0} C_\infty(M_0, F_0). \end{aligned}$$

When (5.3) holds $E^{M_0, F_0} C_\infty \to 1$ by (5.6) and the lim inf $\geqslant$ -part of the theorem is proved.

6. Concluding remarks. We first mention possible extensions. Though it would be of interest to generalize the model to allow for deaths, formation of couples, etc., not all questions have been settled even for the present class of models; e.g., we should have liked to have obtained asymptotic expansions for the variance of W similar to those of Theorem 4 and more terms in the expansion of the mean. Besides their intrinsic interest, these questions come up in connection with population projection (prediction) and a comparison of Theorem 3 with finer limit theorems for branching processes (see Asmussen (1977) and the references therein).

One generalization at least seems easy for most parts of the paper. That is, to weaken (1.2) so that it need only hold in some asymptotic sense and/or to replace the linear factor $N = M + F$ by a more general function of N, say sublinear which would lead to subexponential growth. This would probably be an important step towards making the model more realistic.

Surprisingly few results similar to those of the present paper seem to appear in the demographic literature. Indeed, treatments such as those of Yntema (1954) and Goodman (1953, 1968) deal with models corresponding to arithmetic mean and marriage dominance of one sex, i.e., with no genuine sex interaction. After the present paper was first submitted for publication, the author's attention was drawn to a series of notes by Yellin and Samuelson (1974) and Samuelson (1977a, b, 1978a, b). They give some discussion of the deterministic case, in part related to Section 3 but in part also using different assumptions on the marriage function.

The main treatment of stochastic models is that (in discrete time) of Kesten (1970, 1972; see also his 1971 survey), whose main results essentially are similar in form to Theorem 2. That our proof here is simpler and that Theorems 3 and 4 go somewhat further, should be considered in light of the fact that our model is much more specific than the general formulation of Kesten. It is, however, of considerable interest to ask whether the present models are imbeddable as discrete skeletons $(M_{n\delta}, F_{n\delta})_{n=0,1,2,\dots}$ in the set-up of Kesten. As far as we can see this is not the case. More specifically, the assumption (1.6) of Kesten (1972) will not hold if $h(x) \to 0$ at the boundary, while the assumption (6.3) of his 1970 paper would imply that X_δ is close to z no matter the value of X_0. This might be reasonable in some discrete time models, but is clearly not the case here.

The methods used here are rather different from the standard ones for one-sex branching processes, which rely essentially on martingales similar to $e^{-\lambda t}\tilde{V}(M_t, F_t)$ and the independence of different individuals. Some ideas related to those of the proof of Theorem 2 can be found in Athreya and Karlin (1968). See also Barbour (1975). In connection with the tail sums in the proof of Theorem 3, see Chow and Teicher (1973) and Barbour (1974).

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