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SOME LIMIT THEOREMS FOR PERCOLATION PROCESSES WITH NECESSARY AND SUFFICIENT CONDITIONS

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Let $t(x, y)$ be the passage time from x to y in Z^2 in a percolation process with passage time distribution F. If $x \in R^2$ it is known that $\int (1 - F(t))^4 \, dt < \infty$ is a necessary and sufficient condition for $t(0, nx)/n$ to converge to a limit in L^1 or almost surely. In this paper we will show that the convergence always occurs in probability (to a limit $\varphi(x) < \infty$) without any assumptions on F. The last two results describe the growth of the process in any fixed direction. We can also describe the asymptotic shape of $A_t = \{y : t(0, y) \le t\}$. Our results give necessary and sufficient conditions for $t^{-1}A_t \to \{x : \varphi(x) \le 1\}$ in the sense of Richardson and show, without any assumptions on F, that the Lebesgue measure of $t^{-1}A_t \Delta \{x : \varphi(x) \le 1\} \to 0$ almost surely. The last result can be applied to show that without any assumptions on F, the x-reach and point-to-line processes converge almost surely.

1. Introduction. In (1965) Hammersley and Welsh introduced the following model of the spread of a fluid through a porous medium. Consider a graph L with vertex set Z^2 and a set of edges E connecting all x and y with $\| x - y \| = 1$ (here and in what follows $\| x - y \| = | x_1 = y_1 | + | x_2 - y_2 |$). With each edge e (which we will also call a bond) there is associated an independent nonnegative random variable $X(e)$, with distribution F, which represents the amount of time it takes fluid to flow through the opening which the edge represents.

If $x, y \in Z^2$, let $t(x, y)$ be the minimum travel time over all paths from x to y. $t(x, y)$ is the first time fluid will appear at y if there is a source at x which begins operating at time 0. The object of the study of percolation processes is to determine the limiting behavior of $t(0, x)$ when $x \to \infty$. The first steps in doing this were taken by Hammersley and Welsh (1965) who observed that the passage time process was subadditive:

$$t(x, y) + t(y, z) \ge t(x, z) \qquad \text{for all} \quad x, y, z$$

and used this to show that if F has finite mean and $e_1 = (1, 0)$ then $t(0, ne_1)/n \to \gamma$ in probability where $\gamma = \inf Et(0, ne_1)/n$. This result was strengthened to almost sure and L^1 convergence by Kingman (1968) (see also (1973)) who obtained the result as a consequence of his subadditive ergodic theorem.

The result remained in this form for several years until it was observed (by several people independently) that if the distribution defined by $1 - G(t) = (1 - F(t))^4$ had finite mean then $Et(0, x) < \infty$ for all x so Kingman's result could be applied to show that $t(0, ne_1)/n$ converges almost surely and in L^1 to the same constant. If we insist on L^1 or almost sure convergence then this is the ultimate form of the result. It is easy to show that if G has infinite mean then $Et(0, x) = \infty$ for all $x \ne 0$ and $\limsup t(0, ne_1)/n = \infty$. From Theorem 1.1 below it follows that $\liminf t(0, ne_1)/n < \infty$ for any distribution F so we have almost sure or L^1 convergence if and only if the mean of G is finite.

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The last result suggests the problem of determining conditions for convergence in probability. Reh (1979) and Wierman (1980) have studied this problem. Wierman (1980) has shown that if $EX^{\delta} < \infty$ for some $\delta > 0$ then there is a finite constant γ so that $t(0, ne_1)/n \to \gamma$ in probability. To prove the last result Wierman picks k so that $k\delta > 1$ and then considers the passage time between k-point clusters to produce a finite mean so that Kingman's result can be applied. Because of the method of this proof it seems natural to conjecture that the same result is true in greater generality or, on the basis of the fact about the lim inf quoted above, that the "weak law" holds without any assumptions at all. The last statement is true and is our first theorem.

THEOREM 1. *Let F be the distribution of a nonnegative random variable. There is a finite constant γ (called the time constant) such that $t(0, ne_1)/n \to \gamma$ in probability and*

$$\liminf_{n\to\infty} t(0, ne_1)/n = \gamma \quad \text{almost surely.}$$

SKETCH OF PROOF. Pick M so that $1 - F(M) \le 1/4$ and call all bonds with travel time $\le M$ open, $> M$ closed. It follows from known results about percolation processes that with probability 1 there is an infinite connected set of open bonds and no infinite connected set of closed bonds, so for each x there is a minimal open circuit $\Delta(x)$ which is part of the infinite connected set of open bonds and contains x in its interior. Let $\bar{\Delta}(x)$ be the set of points on $\Delta(x)$ or in its interior. Define $\hat{t}(x, y)$ to be the minimum passage time from a point in $\bar{\Delta}(x)$ to a point in $\bar{\Delta}(y)$. Using ideas from the proof of Theorem 8.10 in Smythe and Wierman (1978) it is easy to show that $E\hat{t}(x, y) < \infty$ for all x, y (or if you are energetic, that all moments are finite), and in addition that the random variables $\hat{t}(x, y)$ are almost subadditive so Kingman's Theorem can be applied to show that $\hat{t}(0, ne_1)/n \to \gamma$ a.s. and in L^1 where $\gamma = \inf E\hat{t}(0, ne_1)/n$. To translate this into a result about t observe that if we let $u(x)$ be the sum of all passage times for bonds on $\Delta(x)$ or in its interior then

$$\hat{t}(x, y) \le t(x, y) \le \hat{t}(x, y) + u(x) + u(y) \tag{1.1}$$

and the distribution of $u(x)$ is independent of x so

$$t(0, ne_1)/n \to \gamma \quad \text{in probability}$$

and $\liminf_{n\to\infty} t(0, ne_1)/n = \gamma$ almost surely.

From the proof of Theorem 1 sketched above the reader can see that the modified passage time process $\hat{t}$ is almost the same as t, and $\hat{t}$ has all moments finite. Using this observation and techniques developed to prove Theorem 1 it is possible to extend many of the results proved in Smythe and Wierman (1978) under the assumption of finite mean. As an example of a theorem of this type we give the following result (see Smythe and Wierman (1978), Theorem 7.7 for the original version of this result and Reh (1979) and Wierman (1980) for generalizations).

THEOREM 2. *Let $p = F(0)$ and U_p be the Bernoulli distribution with $U_p(\{0\}) = p$ and $U_p(\{1\}) = 1 - p$. If the time constant for U_p is 0 then the time constant for F is 0.*

Theorem 1 shows that $t(0, ne_1) \sim \gamma n$. If we extend $t(0, x)$ to $x \in R^2$ by assigning x the passage time to the closest point in Z^2 (taking the minimum in case of ties), then it is easy to use the same proof to show that if x has rational coordinates $t(0, nx) \sim \varphi(x)n$. If we let $A_t = \{x : t(0, x) \le t\}$ then the results above suggest that $t^{-1}A_t$ should approach $\{x : \varphi(x) \le 1\}$ as $t \to \infty$. In (1973) Richardson proved a result of this type for a growth model which corresponds to a percolation process in which F has a geometric distribution. Richardson's result (when strengthened by Kesten's contribution to the discussion of Kingman (1973)) states that for all $\epsilon > 0$

$$P(\{x : \varphi(x) \le 1 - \epsilon\} \subset t^{-1}A_t \subset \{x : \varphi(x) \le 1 + \epsilon\} \quad \text{for all } t \text{ sufficiently large}) = 1 \tag{$*$}$$

The central part of Richardson's proof (Section III, pages 517–521) is based on subadditivity and the existence of second moments so it seems natural to conjecture that under appropriate conditions a similar result should hold for percolation processes.

To determine what these conditions should be, we observe that if $G(t) = 1 - (1 - F(t))^4$ has infinite second moment and we let $Y(z)$ be the minimum of the passage times on the four bonds which contain z, then for any $c < \infty$

$$\sum_{z\in(2Z)^2} P(Y(z) > c \| z \|) = \infty.$$

Since $t(0, z) \geq Y(z)$ it follows from the Borel Cantelli lemma that

$$P(\sup_{z\neq 0} t(0, z)/\| z \| = \infty) = 1,$$

so Richardson's theorem is false if G has an infinite second moment.

There is a second case in which Richardson's result cannot hold for trivial reasons. If $EX < \infty$ and $F(0) > p_T$ (see Smythe and Wierman (1978), pages 30 and 113) then $\gamma = 0$. An easy argument shows that if $\gamma = 0$ then $t(0, nx)/n \to 0$ almost surely for all $x \in R^2$ so $\varphi \equiv 0$ and $t^{-1}A_t$ will never contain $\{x : \varphi(x) \leq 1 - \epsilon\}$ for all $\epsilon \leq 1$. In this case $(*)$ fails trivially but this difficulty can be remedied by reformulating the result:

THEOREM 3. *Richardson's result is valid for percolation processes if and only if* $EY^2 < \infty$. *If* $\gamma > 0$, $(*)$ *holds. If* $\gamma = 0$ *then for every compact set* K

$$P(K \subset t^{-1}A_t \quad \text{for all } t \text{ sufficiently large}) = 1.$$

REMARK. This theorem generalizes a result of Schürger (1980) who has shown that in any dimension $d \geq 2$, $EY^{2d+\delta} < \infty$ is sufficient. While we have proved our result only for $d = 2$, it follows from the proof given in Section 3 that in any dimension $d \geq 2$, $EY^d < \infty$ is necessary and sufficient.

To describe the limit behavior of $t^{-1}A_t$ when $EY^2 = \infty$ we have to introduce another notion of convergence. One definition which is common to measure theory is to say $B_t \to B$ if $| B_t \triangle B | \to 0$ where $B_t \triangle B$ is the symmetric difference $(B_t - B) \cup (B - B_t)$ and $|\ \ |$ denotes Lebesgue measure. The next result shows that if we introduce an obvious modification for the case $\gamma = 0$ then $t^{-1}A_t$ always converges to $\{x : \varphi(x) \leq 1\}$ in this sense. (Here $\varphi(x) = \inf E\hat{t}(0, nx)/n$.)

THEOREM 4. *Let* F *be an arbitrary distribution. For any* $K < \infty$

$$| (t^{-1}A_t \triangle \{x : \varphi(x) \leq 1\}) \cap \{x : \| x \| \leq K\} | \to 0 \quad \text{almost surely.}$$

For any $\epsilon > 0$

$$P(t^{-1}A_t \subset \{x : \varphi(x) < 1 + \epsilon\} \text{ for all } t \text{ sufficiently large}) = 1.$$

To prove this result we let $\hat{A}_t = \{x : \hat{t}(0, x) \leq t\}$ and use ideas from the proof of Theorem 3 to show that $t^{-1}\hat{A}_t$ converges to $\{x : \varphi(x) \leq 1\}$ in the sense of Richardson. Since $\hat{t}(x, y) \leq t(x, y)$ we have $A_t \subset \hat{A}_t$ and this proves the second conclusion. To prove the first we observe that from (1.1) $t(0, x) \leq u(0) + \hat{t}(0, x) + u(x)$ and show that if $\eta > 0$ and $K < \infty$, $t^{-2} | \{x : \| x \| \leq Kt, u(x) > \eta t\} | \to 0$ almost surely.

Theorem 4 describes the asymptotic shape of a general percolation process. Just as Donsker's theorem can be used to find the limit distribution of various functionals of random walk paths, Theorem 4 may be used to obtain results for functionals of the set of occupied sites. By projecting $t^{-1}A_t$ and $\{x : \varphi(x) \leq 1\}$ onto the line $x_2 = 0$ we can obtain the following result about the x-reach process first proved by Smythe and Wierman in the finite mean case (see Theorem 6.3 in [11]).

THEOREM 5. *Let* $x_t = \sup\{m : t(0, (m, n)) \leq t \quad \text{for some} \quad n \in Z\}$. *As* $t \to \infty$

$$x_t/t \to 1/\varphi(e_1) \quad \text{almost surely.}$$

If we regard the x-reach process sample paths as functions then the inverse functions $b(m) = \inf\{t(0, (m, n)) : n \in Z\}$ give us the point-to-line process which was introduced by Hammersley and Welsh (1965) and has been studied by many authors (see Smythe and Wierman (1978) for a listing of results). From Theorem 5 it is easy to obtain one of the basic results about this process.

THEOREM 6. *As* $m \to \infty$

$$b(m)/m \to \varphi(e_1) \quad \textit{almost surely.}$$

In the case F has finite mean, this is a result of Wierman and Reh (1978). At first glance it seems this result should be a corollary of Kingman's subadditive ergodic theorem but it is not because the sequence $b(m)$ cannot be extended to be a stationary subadditive process. To overcome this difficulty Wierman and Reh introduce a sequence of approximating processes and the argument is rather involved so we think that obtaining results for the point-to-line process by "projecting Theorem 4" provides an attractive alternative to their proof.

By considering other functionals of the set of occupied sites and applying Theorem 4 it is possible to prove other results. An example of a result we can obtain in this way is

THEOREM 7. *Let* N_t *be the number of points in* A_t *with integer coordinates. As* $t \to \infty$

$$t^{-2}N_t \to \quad \textit{area of} \quad \{x : \varphi(x) \le 1\} \quad \textit{almost surely.}$$

The reader should observe that Theorems 5, 6, and 7 are valid for any distribution F. Theorems 1 and 2 are proved in Section 2, Theorems 3 and 4 are proved in Sections 3 and 4, and Theorems 5 and 6 are proved in Section 5.

Section 2. In this section we will prove Theorems 1 and 2. We begin by proving

THEOREM 1. *Let* F *be the distribution of a nonnegative random variable. There is a finite constant* γ *such that* $t(0, ne_1)/n \to \gamma$ *in probability and*

$$\liminf_{n\to\infty} t(0, ne_1)/n = \gamma \quad \textit{almost surely.}$$

The proof of this result is accomplished by defining a modification of the basic point-to-point process, showing that Kingman's subadditive ergodic theorem applies to this new process, and then comparing with the basic process.

To define the modified process we have to introduce some definitions. Pick M so that $1 - F(M) < 1/4$. Let $p = F(M)$ and $q = 1 - F(M)$. Let $\{X(e), e \in E\}$ be independent random variables with distribution F. Call a bond e *open* if $X(e) \le M$ and closed if $X(e) > M$. By Theorem 3.14 in Smythe and Wierman (1978) there exists, with probability one, an infinite connected set of open bonds which we call Λ, and there is no infinite connected set of closed bonds.

A self-avoiding circuit C is an alternating sequence of vertices and bonds of the form v_0, e_1, v_1, e_2, $\cdots$ e_n, v_n, where e_i connects v_{i-1} and v_i, $v_0, v_1, \cdots, v_{n-1}$ are all distinct, and $v_0 = v_n$. By viewing a circuit C as a curve in $\mathbb{R}^2$ we see that a self-avoiding circuit divides the plane into two connected components. Let C^0 denote the bounded component (colloquially called the inside of C) and let $\bar{C} = C \cup C^0$. A circuit will be called open (closed) if all its bonds are open (closed).

It follows from Lemma 3.6 in Smythe and Wierman (1978) that for each $z \in Z^2$ there exist infinitely many open circuits C with $z \in C^0$ and $C \subset \Lambda$. Let $\Delta(z)$ be the minimal open circuit with these properties. Let $\tilde{\Delta}(z) = \Delta(z) \cup (\Lambda \cap \Delta^0(C))$, i.e., $\tilde{\Delta}(z)$ consists of the circuit $\Delta(z)$ and all bonds inside $\Delta(z)$ which belong to the infinite open cluster Λ.

(Note: This somewhat peculiar definition is necessary for Lemma 2.6 below and is needed to correct an error on page 140 in Smythe and Wierman (1978).)

We can now introduce the modification of the basic passage time process needed to prove Theorem 1. Let

$$\hat{t}(x, y) = \inf\{t(x', y') : x' \in \bar{\Delta}(x),\ y' \in \bar{\Delta}(y)\}$$

$$u(x) = \textstyle\sum_{e\in\bar{\Delta}(x)} X(e),$$

and

$$v(x) = \textstyle\sum_{e\in\bar{\Delta}(x)} X(e).$$

The following result compares $t(x, y)$ to $\hat{t}(x, y)$.

LEMMA 2.1. *For all $x, y \in Z^2$ we have*

$$\hat{t}(x, y) \le t(x, y) \le \hat{t}(x, y) + u(x) + u(y).$$

PROOF. The first inequality is trivial, since $x \in \bar{\Delta}(x)$ and $y \in \bar{\Delta}(y)$. To prove the second, let r be a path from some vertex in $\bar{\Delta}(x)$ to some vertex in $\bar{\Delta}(y)$. Then $t(x, y) \le u(x) + t(r) + u(y)$ where $t(r)$ is the travel time along the path r. Since $\hat{t}(x, y)$ is the infimum of $t(r)$ over all such paths r we have $t(x, y) \le \hat{t}(x, y) + u(x) + u(y)$.

The distribution of $u(x)$ is independent of x so it is easy to see from Lemma 2.1 that t and $\hat{t}$ are almost the same when $\| x - y \|$ is large. The next step in our proof will be to show that Kingman's subadditive ergodic theorem can be applied to determine the limiting behavior of $\hat{t}(0, x)$ when $x \to \infty$. To apply Kingman's theorem we need to show that $E\hat{t}(0, x) < \infty$ for all x. To do this we begin by getting an estimate on $| \bar{\Delta}(z) |$, the number of bonds in $\bar{\Delta}(z)$.

For $z \in Z^2$ let $R_z(n)$ be the annular region in the plane bounded on the inside by the rectangle $z + [-n, n + 1] \times [-n - 1, n + 1]$ and on the outside by the rectangle $z + [-3n + 1, 3n] \times [-3n + 1, 3n]$.

LEMMA 2.2. *Let $\kappa_n(p) = \inf_{m\ge n} P(R_z(m)$ contains an open circuit C with z in its interior). Then $\kappa_n(p) > 0$ and $\kappa_n(p) \to 1$ as $n \to \infty$.*

PROOF. From Lemma 3.5 in Smythe and Wierman [11] $\kappa_n(p) \ge S_p(2n, 2n)^{12}(1 - (1 - S_p(2n, 2n))^{1/2})^{64}$ where $S_p(m, n)$ is the crossing probability for an $m \times n$ sponge (for a definition see [11], page 10). By Lemma 3.10, Theorem 3.13, Theorem 2.2, and equation 2.2 in [11], $S_p(n, n) \to 1$ as $n \to \infty$.

LEMMA 2.3. $E\,| \bar{\Delta}(z) |^m < \infty$ *for all* $m < \infty$.

PROOF. Suppose $z = 0$. Let $B(n) = \{y : \| y \| \le n\}$. If $B(n) \cap \Lambda = \varnothing$, then by Whitney's Theorem (Theorem 2.1 in Smythe and Wierman (1978)) there is a closed circuit C^* in the dual graph L^* (see Smythe and Wierman (1978) page 8 for definition) with $B(n)$ in its interior. C^* must contain at least one of the bonds e_k^* connecting $(k + \frac{1}{2}, \frac{1}{2})$ to $(k + \frac{1}{2}, -\frac{1}{2})$. Let D_k^* denote the connected set of closed bonds in L^* containing e_k^*. Since $q < \frac{1}{4}$, D_k^* has all moments finite (Corollary 3.7 and Theorem 3.8 in Smythe and Wierman (1978)). Thus

$$\begin{aligned} P(B(n) \cap \Lambda = \varnothing) &\le \textstyle\sum_{k=n}^{\infty} P(\exists \text{ a closed circuit } C^* \text{ in } L^* \text{ with } B(n) \text{ in its interior}, e_k^* \in C^*) \\ &\le \textstyle\sum_{k=n}^{\infty} P(| D_k^* | \ge 2k + 2n) \\ &\le \textstyle\sum_{k=n}^{\infty} E(| D_1^* |^{\alpha})/(2k + 2n)^{\alpha} \end{aligned}$$

for any $\alpha < \infty$. Comparison with an integral shows that if $\alpha > 1$

$$P(B(n) \cap \Lambda = \varnothing) \le c_\alpha/(4n-2)^{\alpha-1}$$

where $c_\alpha = E\,|D_1^*|^\alpha/2(\alpha-1)$.

The next step in proving Lemma 2.3 is to obtain an estimate on how far out we must go to find an open circuit. Let E_k be the event that $R_0(3^k)$ contains an open circuit with 0 in its interior. The events E_k are independent, so if $3^k \ge n$, $P[E_{k+1}^c \cap \cdots \cap E_{k+j}^c] \le (1-\kappa_n)^j$. Since $B(6 \cdot 3^i) \supset R_0(3^i)$ it follows that

$$\begin{aligned} P(\bar\Delta(0) \subset B(6 \cdot 3^{k+j})) &\ge P(B(3^k) \cap \Lambda \ne \varnothing, E_{k+1} \cup \cdots \cup E_{k+j}) \\ &\ge 1 - P(B(3^k) \cap \Lambda = \varnothing) - P(E_{k+1}^c \cap \cdots \cap E_{k+j}^c) \\ &\ge 1 - c_\alpha(4 \cdot 3^k - 2)^{1-\alpha} - (1-\kappa_n)^j. \end{aligned}$$

Letting $j = k$ above gives

$$P(\bar\Delta(0) \not\subset B(6 \cdot 9^k)) \le c_\alpha(4 \cdot 3^k - 2)^{1-\alpha} + (1-\kappa_n)^k.$$

Since $|\bar\Delta(0)| \ge (12 \cdot 9^k)^2 = (144)81^k$ implies $\bar\Delta(0) \not\subset B(6 \cdot 9^k)$ we have

$$P(|\bar\Delta(0)| \ge (144)81^k) \le c_\alpha(3^{-\alpha+1})^k + (1-\kappa_n)^k$$

(recall $\alpha > 1$).

Fix m. By Lemma 2.2 we can choose n large enough so that $81^m(1-\kappa_n) < 1$ and choose α large enough so that $81^m 3^{-\alpha+1} < 1$. The fact that $E\,|\bar\Delta(0)|^m < \infty$ now follows from

LEMMA 2.4. *Let Z be a nonnegative random variable such that $P(Z \ge b^k) \le Ac^k$ for all $k \ge 0$. If $b^m c < 1$ then $EZ^m < \infty$.*

PROOF. Since $b^m c < 1$ we have

$$\textstyle\sum_{k=0}^\infty \sum_{n=b^k}^{b^{k+1}-1} n^{m-1} P(Z \ge n) \le \sum_{k=0}^\infty b^{(k+1)m} A c^k = A b^m \sum_{k=0}^\infty (b^m c)^k < \infty.$$

At this point we have shown that $E\,|\bar\Delta(z)|^m < \infty$ for all m. From this we get

COROLLARY 2.5. *$E(v(z)^m) < \infty$ for all $m < \infty$.*

PROOF. $v(z) = \sum_{e \in \tilde\Delta(z)} X(e) \le M\,|\tilde\Delta(z)| \le M\,|\bar\Delta(z)|$.

The next step in the proof is to show that $\hat t(x, y)$ has finite moments of all orders. To do this we must prove a few facts about the sets $\tilde\Delta(z)$.

LEMMA 2.6. (i) *If $\tilde\Delta(x) \cap \tilde\Delta(y) = \varnothing$ then $\bar\Delta(x) \cap \bar\Delta(y) = \varnothing$*
(ii) *If $\|x - y\| = 1$ then $\tilde\Delta(x) \cap \tilde\Delta(y) \ne \varnothing$.*

PROOF. To prove (i), suppose $\bar\Delta(x) \cap \bar\Delta(y) \ne \varnothing$. If $\bar\Delta(x) \subset \bar\Delta(y)$ then by definition $\tilde\Delta(x) \subset \tilde\Delta(y)$. If $\bar\Delta(x) \not\subset \bar\Delta(y)$ then since we have supposed $\bar\Delta(x) \cap \bar\Delta(y) \ne \varnothing$ there are points a and b in $\bar\Delta(x)$ such that $a \in \bar\Delta(y)$ and $b \notin \bar\Delta(y)$. Since $a, b \in \bar\Delta(x)$ there is a polygonal curve entirely contained in $\bar\Delta(x)$ which connects a and b. Since $a \in \bar\Delta(y)$ and $b \notin \bar\Delta(y)$, the curve must intersect $\Delta(y)$ at some point c. Since $c \in \Delta(y)$, $c \in e$ for some $e \subset \tilde\Delta(y)$. By the choice of c, $e \subset \bar\Delta(x)$; so $\tilde\Delta(x) \cap \tilde\Delta(y) \supset \{e\}$ and (i) is proved. (ii) follows immediately from (i) since $\|x - y\| = 1$ implies $\bar\Delta(x) \cap \bar\Delta(y) \ne \varnothing$.

COROLLARY 2.7. *$E(\hat t(x, y))^m < \infty$ for all $m < \infty$.*

PROOF. Construct a sequence $z_0, \cdots, z_k$ so that $z_0 = x$, $z_k = y$, and $\|z_i - z_{i-1}\| = 1$ for $1 \le i \le k$. From (ii) of Lemma 2.6 it follows that there is a path r from $\tilde\Delta(x)$ to $\tilde\Delta(y)$ such that $r \subset \cup_{i=0}^k \tilde\Delta(z_i)$. From the definition of $\tilde\Delta$ it follows that

$$\hat{t}(x, y) \le M \sum_{i=0}^{k} |\tilde{\Delta}(z_i)| \le M \sum_{i=0}^{k} |\bar{\Delta}(z_i)|$$

and the desired conclusion follows from Lemma 2.3.

PROOF OF THEOREM 1. Let $\xi_{m,n} = \hat{t}(me_1, ne_1) + v(me_1)$. The process $\{\xi_{m,n}, 0 \le m < n\}$ is stationary under the shift $(m, n) \to (m + 1, n + 1)$ and it follows from Corollaries 2.5 and 2.7 that $E\xi_{m,n} < \infty$ so to apply Kingman's theorem it only remains to show that $\{\xi_{m,n}, 0 \le m < n\}$ is a subadditive process. Fix $l < m < n$. Let r_1 be a path from $\tilde{\Delta}(le_1)$ to $\tilde{\Delta}(me_1)$ and let r_3 be a path from $\tilde{\Delta}(me_1)$ to $\tilde{\Delta}(ne_1)$. There is a path r_2 which connects the endpoints of r_1 and r_3 and which is contained in $\tilde{\Delta}(me_1)$. The union of these three paths forms a path r from $\tilde{\Delta}(le_1)$ to $\tilde{\Delta}(ne_1)$ so

$$\hat{t}(le_1, ne_1) \le t(r) \le t(r_1) + v(me_1) + t(r_3).$$

Taking the infimum over all such paths r_1 and r_3 gives

$$\hat{t}(le_1, ne_1) \le \hat{t}(le_1, me_1) + v(me_1) + \hat{t}(me_1, ne_1).$$

Adding $v(le_1)$ to both sides of this inequality gives $\xi_{ln} \le \xi_{lm} + \xi_{mn}$ so $\xi_{m,n}$ is subadditive in the sense of Kingman (1973).

Applying Kingman's subadditive ergodic theorem (Theorem 1 in [6], see page 885) gives that there is a random variable ξ such that $\xi_{0,n}/n \to \xi$ almost surely and in L^1. The random variable ξ has $E\xi = \gamma$ where

$$\gamma = \inf_{n \ge 1} E\xi_{0,n}/n = \lim_{n\to\infty} E\xi_{0,n}/n.$$

Since $\xi_{0,n} = \hat{t}(0, ne_1) + v(0)$ and $Ev(0) < \infty$ it follows from the result for $\xi_{0,n}$ that $\hat{t}(0, ne_1)/n \to \xi$ almost surely and in L^1. To translate this into results about t we observe that

$$\hat{t}(0, ne_1) \le t(0, ne_1) \le \hat{t}(0, ne_1) + u(0) + u(ne_1)$$

so we have

$$\liminf_{n\to\infty} t(0, ne_1)/n = \xi \qquad \text{a.s.}$$

and since the distribution of $u(x)$ is independent of x, we also have that $t(0, ne_1)/n \to \xi$ in probability.

To complete the proof of Theorem 1 it only remains to show that $\xi = \gamma$ almost surely. To do this we observe that if $\mathscr{G}_n = \sigma(X(e): e \not\subset B(n))$ then $\xi = \liminf t(0, ne_1)/n$ is measurable with respect to $\cap \mathscr{G}_n$ and this implies that ξ is constant (to reduce this to a well known result order the bonds $e_1, e_2, \cdots$ in such a way that $\mathscr{G}_n = \sigma(X(e_j): j > k_n)$ where $k_n = |\{e: e \subset B(n)\}|$).

At this point we have completed the proof of Theorem 1 so we now begin the proof of

THEOREM 2. *Let $F(0) = p$ and U_p be the Bernoulli distribution with $U_p(\{0\}) = p$ and $U_p(\{1\}) = 1 - p$. If the time constant for U_p is 0 then the time constant for F is 0.*

PROOF. Define

$$F'(x) = \begin{cases} 0 & x < 0 \\ \dfrac{F(x) - p}{1 - p} & x \ge 0. \end{cases}$$

Let $\{X'(e)\}$ and $\{B(e)\}$ be independent i.i.d. families with distributions F' and U_p. If we let $X(e) \equiv X'(e)B(e)$ then we obtain an i.i.d. family with distribution F.

Let M be chosen so that $(1 - F'(M)) < 1/4$. Using the definitions given in the proof of Theorem 1 define circuits $\Delta(x)$ for the random variables $\{X(e)\}$ and circuits $\Delta'(x)$ for the random variables $\{X'(e)\}$. Since $X(e) = X'(e)B(e) \le X'(e)$, $X'(e) \le M$ implies $X(e) \le M$ and we have $\bar{\Delta}'(x) \supset \bar{\Delta}(x)$.

Let $\hat{t}(x, y)$ be the modified passage time process for the $\{X(e)\}$ and let $t^B(x, y)$ be the ordinary passage time process for the $\{B(e)\}$. Let n be an integer. From Theorem 4.10 in Smythe and Wierman (1978) it follows that there is a path r from 0 to ne_1 with

$$\textstyle\sum_{e\in r} B(e) = t^B(0, ne_1).$$

To relate $\hat{t}$ to t^B we make the following observation:

LEMMA 2.9.

$$\hat{t}(0, ne_1) \le \textstyle\sum_{e\in r} M|\bar{\Delta}(e)| 1_{\{B(e)=1\}}.$$

PROOF. The inequality is trivially true if $\tilde{\Delta}(0) \cap \tilde{\Delta}(ne_1) = 0$, since in this case $\hat{t}(0, ne_1)$ so we may assume $\tilde{\Delta}(0) \cap \tilde{\Delta}(ne_1) = \varnothing$ and hence that r contains a path, say r', from $\tilde{\Delta}(0)$ to $\tilde{\Delta}(ne_1)$. To demonstrate Lemma 2.9 we successively remove bonds $e \in r'$ *with* $B(e) = 1$ and replace them with the sets $\tilde{\Delta}(e)$. It is easy to see that if we do this then at each stage we have a connected set of bonds which intersects $\tilde{\Delta}(0)$ and $\tilde{\Delta}(ne_1)$ so that after we have removed all the bonds with $B(e) = 1$ we can produce a path r^* from $\tilde{\Delta}(0)$ to $\tilde{\Delta}(ne_1)$ such that each bond $e^* \in r^*$ has $B(e^*) = 0$ or $e^* \in \tilde{\Delta}(e')$ where $e' \in r'$ and $B(e') = 1$. Since

$$\tilde{t}(0, ne_1) \le \textstyle\sum_{e\in r^*} X(e) \le M \sum_{e\in r',\, B(e)=1} |\bar{\Delta}(e)|,$$

we have proved Lemma 2.9.

At this point it is easy to complete the proof of Theorem 2. Since $|\bar{\Delta}(e)| \le |\bar{\Delta}'(e)|$ and $|\bar{\Delta}'(e)|$ is independent of r and $\{B(e)\}$ we have

$$\begin{aligned} E(\tilde{t}(0, ne_1)) &\le ME(\textstyle\sum_{e\in r} |\bar{\Delta}'(e)| 1_{\{B(e)=1\}}) \\ &= ME(|\bar{\Delta}'(0)|)E(\textstyle\sum_{e\in r} 1_{\{B(e)=1\}}) \\ &= ME(|\bar{\Delta}'(0)|)E(t^B(0, ne_1)). \end{aligned}$$

Dividing by n and letting $n \to \infty$ now shows that if $\gamma(U_p) = \lim Et^B(0, ne_1)/n = 0$ then $\gamma(F) = \lim E\hat{t}(0, ne_1)/n = 0$.

Section 3. In this section we will prove

THEOREM 3.3. *Richardson's result is valid for percolation processes if and only if* $EY^2 < \infty$.

PROOF. The necessity of the condition was shown in the introduction so suppose $EY^2 < \infty$. The first step is to define the function $\varphi(x)$ which was mentioned in the introduction. To do this we need the following

LEMMA 3.1. *Suppose* $r > 0$. *If* $EY^r < \infty$ *then* $Et(0, z)^r < \infty$ *for all* $z \in Z^2$.

PROOF. $t(0, 0) = 0$ by definition so suppose $z \ne 0$. In this case we can construct four disjoint paths from 0 to z. Let T_1, T_2, T_3 and T_4 be the travel times for these four paths. If we suppose that the paths are numbered so that the fourth is the longest we have

$$P(t(0, z) > s) \le P(T_4 > s)^4.$$

Let B be the number of bonds in the 4th path. If $T_4 > s$ at least one of the bonds must have a travel time $> s/B$, i.e.,

$$P(T_4 > s) \le BP(X > s/B).$$

Combining this with the previous inequality gives

$$P(t(0, z) > s) \le B^4 P(X > s/B)^4 = B^4 P(Y > s/B)$$

which proves the desired result.

By hypothesis $EY^2 < \infty$ so it follows from Lemma 3.1 that $Et(0, x)^2 < \infty$ and $g(x) = Et(0, x) < \infty$ for all x. To define the function φ we will let

$$\varphi(x) = \lim_{n\to\infty} g(nx)/n.$$

To define the limit for all $x \in R^2$ we have to extend the domain of definition of g from Z^2 to R^2. To make life simple later in the proof we will construct a special extension which is Lipschitz continuous. Let $\bar{g}$ be the function from R^2 to R which has $\bar{g}(z) = g(z)$ for all $z \in Z^2$ and is linear all the triangular areas with vertices $((m, n), (m + 1, n), (m, n + 1))$ or $((m, + 1, n), (m, n + 1), (m + 1, n + 1))$.

Using the subadditivity and symmetry of $t(x, y)$ it is immediate that for all $x, y \in Z^2$

$$g(x) + g(y) \ge g(x + y)$$

$$g(x + y) + g(-y) \ge g(x)$$

$$g(-y) = g(y) \le \|y\| g(e_1).$$

Combining the last three formulas gives

$$|g(x + y) - g(x)| \le \|y\| g(e_1) \quad \text{for all} \quad x, y \in Z^2. \tag{3.1}$$

From the definition of the extension it follows that

$$|\bar{g}(x + y) - \bar{g}(x)| \le 2\|y\| g(e_1) \quad \text{for all} \quad x, y \in R^2. \tag{3.2}$$

(Observe that the inequality holds for any x when y is a small step parallel to either axis. The constant 2 comes from the fact that $\|(1, 0) - (0, 1)\| = 2$.)

With formula (2) it is easy to prove

LEMMA 3.2. *There is a function* φ *so that* $\lim_{n\to\infty} \frac{1}{n} \bar{g}(nx) = \varphi(x)$ *uniformly on compact subsets of* R^2.

PROOF. Suppose x has rational coordinates. Let $N = \min\{m \ge 1 : mx \in Z^2\}$. Let $a_k = \bar{g}(kNx)$. Since $Nx \in Z^2$, $a_k + a_l \ge a_{k+l}$ for all $k, l \ge 0$, i.e., the sequence a_k is subadditive, so it follows from an easy argument that

$$\lim_{k\to\infty} k^{-1} a_k = \inf_{k\ge1} k^{-1} a_k.$$

To extend the convergence to all values of n we observe that if $kN \le n < (k + 1)N$

$$|\bar{g}(nx) - \bar{g}(kNx)| \le 2\|Nx\| g(e_1)$$

so we have

$$\lim_{n\to\infty} n^{-1}\bar{g}(nx) = \inf_{n\ge1} n^{-1}\bar{g}(nx)$$

for all x with rational coordinates. To extend the convergence to all $x \in R^2$ and prove the uniformity we observe that if $h_n(x) = n^{-1}\bar{g}(nx)$

$$|h_n(x) - h_n(y)| = n^{-1}|\bar{g}(nx) - \bar{g}(ny)| \le 2g(e_1)\|x - y\|$$

so the functions h_n, $n \ge 1$ are equicontinuous and the Arzela-Ascoli theorem can be applied to conclude that every subsequence of h_n has a further subsequence which converges uniformly on compact sets. All the subsequential limits are continuous and agree on the rationals so we have proved Lemma 3.2.

At this point we have shown that $\bar{g}(nx)/n \to \varphi(x)$ uniformly on compact sets. To prove

Richardson's theorem we need to do this for the passage times $t(0, x)$ (which are assumed to be extended to $x \in R^2$ by assigning x the passage time to the nearest $z \in Z^2$, taking the minimum in the case of ties). The first step in doing this is to observe that the existence of radial limits for $x \in Q^2$ follows from Kingman's subadditive ergodic theorem.

Let $x \in Q^2$. Let $N = \inf\{m \geq 1 : mx \in Z^2\}$. Then $a_{m,n} = t(mNx, nNx)$ defines a subadditive process so it follows from Kingman's subadditive ergodic theorem (see Smythe and Wierman (1978), Section 5.1) that we have

$$(nN)^{-1}a_{0,n} \to \inf_{n\geq 1} Et(0, nNx)/nN \qquad \text{almost surely.}$$

To extend this result to conclude the almost sure convergence of $k^{-1}t(0, kx)$ observe that if $nN \leq k < (n+1)N$ it follows from the subadditivity and (2) that

$$E\,|\,t(0, kx) - t(0, nNx)\,| \leq Et(nNx, kx) \leq 2\,\|kx - nNx\|\,g(e_1).$$

An obvious estimate implies that for all $\epsilon > 0$

$$\textstyle\sum_{n=1}^{\infty} P(\,|\,t(0, (nN+j)x) - t(0, nNx)\,| > n\epsilon) < \infty$$

for $j = 0, \cdots N - 1$ so it follows from the Borel-Cantelli lemma that

$$k^{-1}t(0, kx) \to \inf_{n\geq 1} Et(0, nNx)/nN \qquad \text{almost surely.}$$

The last result asserts the almost sure existence of radial limits so at this point we are ready to consider $A_t = \{x \in R^2 : t(0, x) \leq t\}$ and prove Theorem 3. The statement of Theorem 3 is different in the cases $\gamma > 0$ and $\gamma = 0$ but the proofs are almost the same so we will carry out the argument for the case $\gamma > 0$ and leave it to the reader to check that the same proof works when $\gamma = 0$. In the case $\gamma > 0$ the statement we want to prove is

$$(3.3) \qquad P(\{x : \varphi(x) \leq 1 - \epsilon\} \subset t^{-1}A_t \subset \{x : \varphi(x) \leq 1 + \epsilon\} \quad \text{for all } t \text{ sufficiently large}) = 1.$$

To prove the first inclusion we will show that for each x with $\varphi(x) < 1$ there is a $\delta > 0$ (which will depend upon x) so that

$$(3.4) \qquad P(t^{-1}A_t \supset B_\delta(x) \quad \text{for all } t \text{ sufficiently large}) = 1$$

where $B_\delta(x) = \{\,\|x - y\| \leq \delta\}$. This is enough to prove the first inclusion in (3.3) since for all $\epsilon > 0$, $\{x : \varphi(x) \leq 1 - \epsilon\}$ is a compact set and hence can be covered by a finite number of sets $B_{\delta(x)}(x)$ where the x are chosen to have rational coordinates (the last fact is not immediate from topological considerations but follows from the choice of $\delta(x)$ given below).

To prove (3.4) for $x = 0$ show that there is a $K < \infty$ such that

$$(3.5) \qquad \textstyle\sum_z P(t(0, z) > K\|z\|) < \infty$$

(for then it follows by Borel-Cantelli that (3.4) holds when $\delta < 1/K$). To prove (3.5) we need to define the constant K and then find upper bounds on $P(t(0, z) > K\|z\|)$. To describe the constant we cover Z^2 with disjoint squares $z + (-5/2, 5/2]^2$ where $z \in (5Z)^2$. It is easy to define 4 disjoint paths from (0, 0) to (5, 0) which lie in $\{(0, 0), (5, 0)\} + (-5/2, 5/2]^2$.

1. (0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0)
2. (0, 0), (0, 1), (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (5, 0)
3. (0, 0), (0, −1), (1, −1), (2, −1), (3, −1), (4, −1), (5, −1), (5, 0)
4. (0, 0), (−1, 0), (−1, −1), (−1, −2), (0, −2), (1, −2), (2, −2), (3, −2), (4, −2), (5, −2), (6, −2), (6, −1), (6, 0), (5, 0).

By translation and rotation through an angle of $\pi/2$ we can extend the path construction above to any points $5x$ and $5y$ where $x, y \in Z^2$ and $\|x - y\| = 1$. Let $\tau(5x, 5y)$ be the minimum passage time between $5x$ and $5y$ over the four paths. It follows from the proof of

Lemma 3.1 that $E\tau(5x, 5y)^2 < \infty$ for all $x, y \in Z^2$. Using the variables $\tau(5x, 5y)$ we can get bounds on the passage times $t(0, 5z)$, $z \in Z^2$. Let $z_0, z_1, \cdots z_m$ be a sequence of distinct points with $z_0 = 0$, $z_m = z$ and $\| z_i - z_{i-1} \| = 1$ for $i = 1, \cdots m$. From the subadditivity and the definition of τ we have

$$t(0, 5z) \leq \sum_{i=1}^{m} t(5z_{i-1}, 5z_i) \leq \sum_{i=1}^{m} \tau(5z_{i-1}, 5z_i).$$

Let S_m denote the rightmost term of the expression above. The mean of S_m is $mE\tau(0, 5e_1)$. To compute the variance of S_m, we let $T_i = \tau(5z_{i-1}, 5z_i)$ and observe that the variables T_i and T_j are independent if $| i - j | > 1$. (This is true because the points z_i are distinct and all four paths from $5z_i$ to $5z_{i+1}$ lie in $\{5z_1, 5z_{i+1}\} + (-5/2, 5/2]^2$.) Using this observation we see that

$$E(\textstyle\sum_{i=1}^{k} (T_i - ET_i))^2 = \sum_{i=1}^{k} E(T_i - ET_i)^2 + 2 \sum_{i=1}^{k-1} E((T_i - ET_i)(T_{i+1} - ET_{i+1})).$$

The Cauchy-Schwartz inequality implies that

$$| E(T_i - ET_i)(T_{i+1} - ET_{i+1}) | \leq (\mathrm{Var}(T_i)\ \mathrm{Var}(T_{i+1}))^{1/2} = \mathrm{Var}(T_1)$$

so $E(\sum_{i=1}^{k} (T_i - ET_i))^2 \leq 3k\ \mathrm{Var}(T_1)$ and it follows from Chebyshev's inequality that

$$P(\textstyle\sum_{i=1}^{k} T_i > k(ET_1 + 1)) \leq 3\ \mathrm{Var}(T_1)/k. \tag{3.6}$$

To prove (3.5) we need to improve this estimate. To do this we observe that if $z \in Z^2 - \{0\}$ we can construct four disjoint paths from 0 to z in such a way that the longest path has at most $\| z \| + 8$ bonds. The paths to $z = (z_1, z_2)$ with $z_1, z_2 \geq 0$ are

1. $(0, 0) \to (z_1, 0) \to (z_1, z_2)$
2. $(0, 0) \to (0, 1) \to (z_1 - 1, 1) \to (z_1 - 1, z_2) \to (z_1, z_2)$
3. $(0, 0) \to (0, -1) \to (z_1 + 1, -1) \to (z_1 + 1, z_2) \to (z_1, z_2)$
4. $(0, 0) \to (-1, 0) \to (-1, 2) \to (z_1 - 2, 2) \to (z_1 - 2, z_2 + 1) \to (z_1, z_2 + 1) \to (z_1, z_2)$.

Here $x \to y$ denotes moving from x to y by the straight line path. The paths to points in other quadrants are defined by a clockwise rotation of these paths.

Let z be a point in Z^2 and let z_i^j, $i = 0, 1, \cdots n_j$ be the sequence of points used in path $j = 1, 2, 3, 4$. Let

$$U_j = \sum_{i=2}^{n_j-1} \tau(5z_{i-1}^j, 5z_i^j)$$

$$R_0 = \max_{1 \leq j < 4} \tau(0, 5z_1^j)$$

$$R_1 = \max_{1 \leq j \leq 4} \tau(5z_{n_j-1}^j, 5z).$$

For $i = 0$ and 1 we have

$$P(R_i > \| z \|) \leq 4P(T_1 > \| z \|).$$

To estimate $t(0, 5z)$ we observe that $t(0, 5z) \leq R_0 + R_1 + \min_{1 \leq j \leq 4} U_j$ so we have

$$\begin{aligned} &P(t(0, 5z) > (ET_1 + 4) \| z \|) \\ &\leq 8P(T_1 > \| z \|) + P(\min_{1 \leq j \leq 4} U_j > (ET_1 + 2) \| z \|). \end{aligned} \tag{3.7}$$

To estimate the second term we observe that from (3.6)

$$P(U_j > (ET_1 + 1)n_j) \leq 3\ \mathrm{Var}(T_1)/n_j$$

and from the path construction

$$P(U_j > (ET_1 + 2) \| z \|) \leq 3\ \mathrm{Var}(T_1)/(\| z \| - 2)$$

whenever $\| z \| \geq z_0 = \min\{n{:}ET_1 + 1)(n + 6) \leq (ET_1 + 2)n,\ n \geq 3\}$. Since the random variables U_1, U_2, U_3, and U_4 are independent it follows from this that

$$P(\min_{1\le j\le 4} U_j > (ET_1 + 2)\,\|z\|) \le (9\,\mathrm{Var}(T_1))^4\,\|z\|^{-4}$$

Combining this with (3.7) gives

$$(3.8)\qquad \begin{aligned} &P(t(0, 5z) > (ET_1 + 4)\,\|z\|) \\ &\le (9\,\mathrm{Var}(T_1))^4\,\|z\|^{-4} + 8P(T_1 > \|z\|) \end{aligned}$$

for all z with $\|z\| \ge z_0$.

To estimate the travel times to other points $x \in Z^2$ we have to add one more term. If $x \in 5z + (-5/2, 5/2]^2$ we can find 4 disjoint paths from $5z$ to x which lie in $5z + [-3, 3]^2$ (use the paths generated in the last construction). Let $R_2(x)$ be the minimum passage time from $5z$ to x along one of these four paths. From the subadditivity we have that if $x \in 5z + (-5/2, 5/2]^2$ then

$$t(0, x) \le t(0, 5z) + R_2(x).$$

To obtain an upper bound for $R_2(x)$ we let

$$\bar{R}_2 = \sup_{x\in[-2,2]^2} R_2(x).$$

It follows from the definition of $\bar{R}_2$ that

$$P(R_2(x) > \|z\|) \le P(\bar{R}_2 > \|z\|)$$

and from the proof of Lemma 3.1 that $\bar{R}_2$ has a finite second moment. Combining this with (3.8) shows that

$$(3.9)\qquad P(t(0, x) > (ET_1 + 5)\,\|z\|) \le RHS \text{ of } (3.8) + P(\bar{R}_2 > \|z\|)$$

for all $x \in 5z + (-5/2, 5/2]^2$ when $\|z\| \ge z_0$.

If we replace $\|z\|$ by $\|x\|/5$ we can reformulate the last result as

LEMMA 3.3. *There is an $x_0 < \infty$ so that if $\|x\| \ge x_0$ then*

$$\begin{aligned} P(t(0, x) > (ET_1 + 6)\,\|x\|/5) &\le (46\,\mathrm{Var}(T_1))^4\,\|x\|^{-4} \\ &\quad + 8P(T_1 > .1\|x\|) + P(\bar{R}_2 > .1\,\|x\|). \end{aligned}$$

Since T_1 and $\bar{R}_2$ have finite second moments it follows from Lemma 3.3 and the Borel Cantelli lemma that (3.5) holds when $K = (ET_1 + 6)/5$.

To prove (3.4) for $x \ne 0$ we observe that in order for $A_t \supset B_{\delta t}(xt)$ for all t sufficiently large it is necessary and sufficient that

$$(3.10)\qquad \begin{aligned} &\text{for all but a finite number of } z \in D_x = \bigcup_{t\ge 0} B_{\delta t}(xt) \\ &t(0, z) \le \sigma_z = \inf\{t : z \in B_{\delta t}(xt)\}. \end{aligned}$$

Pick x such that $\varphi(x) < 1$ and pick $\epsilon < (1 - \varphi(x))$. From the almost sure existence of radial limits we have

$$P(t(0, \sigma_z x) > (\varphi(x) + \epsilon)\sigma_z \text{ i.o.}) = 0.$$

The next step is to estimate $t(\sigma_z x, z)$. To do this we observe that $\|z - \sigma_z x\| = \delta\sigma_z$ so if we pick δ so that $(1 - \varphi(x) - \epsilon)/\delta > (ET_1 + 6)/5$ then it follows from Lemma 3.3 that if $\|z\|$ is sufficiently large

$$(*)\qquad \begin{aligned} &P(t(\sigma_z x, z) > (1 - \varphi(x) - \epsilon)\sigma_z) \\ &\le (46\,\mathrm{Var}(T_1))^4(\delta\sigma_z)^{-4} + 8P(T_1 > .1\delta\sigma_z) + P(\bar{R}_2 > .1\delta\sigma_z). \end{aligned}$$

At this point to prove (3.4) all we have to show it that the sum of the right-hand side over all $z \in D_x$ is finite. To do this we observe that since $z \in B_{\delta\sigma_z}(x\sigma_z)$ we have

$$\sigma_z(\|x\| - \delta) \leq \|z\| \leq \sigma_z(\|x\| + \delta)$$

so

$$\frac{\sigma_z\|x\|}{\|z\|}\left(1 - \frac{\delta}{\|x\|}\right) \leq 1 \leq \frac{\sigma_z\|x\|}{\|z\|}\left(1 + \frac{\delta}{\|x\|}\right)$$

and if δ was chosen $< \|x\|$ we have

$$\frac{\|x\|}{\|x\| + \delta} \leq \frac{\sigma_z\|x\|}{\|z\|} \leq \frac{\|x\|}{\|x\| - \delta}$$

Combining the last inequality with (*) shows that

$$\textstyle\sum_{z\in D_x} P(t(\sigma_z x, z) > (1 - \varphi(x) - \epsilon)\sigma_z) < \infty$$

so (3.10) follows from the Borel Cantelli lemma.

At this point we have shown (3.4) so the last detail which remains is to show

(3.11) $\qquad P(t^{-1}A_t \subset \{x:\varphi(x) \leq 1 + \epsilon\}$ for all t sufficiently large$) = 1$.

To do this we observe that if $\epsilon < 1$ and $t^{-1}A_t \cap \{x:\varphi(x) > 1 + \epsilon\} \neq \varnothing$ then $t^{-1}A_t \cap \{x:1 + \epsilon < \varphi(x) \leq 2\} \neq \varnothing$ so to prove (3.11) it suffices to show that for each x with $\varphi(x) > 1$ there is a $\delta > 0$ (which will depend upon x) so that

(3.12) $\qquad P((t^{-1}A_t) \cap B_\delta(x) = \varnothing$ for all t sufficiently large$) = 1$.

To do this we observe that from the previous part of the proof if we pick x so that $\varphi(x) > 1$, pick $\epsilon < (\varphi(x) - 1)/2$, and pick δ so that $(\varphi(x) - 1 - 2\epsilon)/\delta > (ET_1 + 6)/5$ then using the notation defined above we have

$$P(t(0, \sigma_z x) < (\varphi(x) - \epsilon)\sigma_z \quad \text{i.o.}) = 0$$

and

$$P(t(\sigma_z x, z) > (\varphi(x) - 1 - 2\epsilon)\sigma_z \quad \text{i.o.}) = 0.$$

From the subadditivity it follows that $t(0, z) > t(0, \sigma_z x) - t(z, \sigma_z x)$ so combining the last two results gives

$$P(t(0, z) \leq (1 + \epsilon)\sigma_z \quad \text{i.o.}) = 0.$$

If δ was chosen sufficiently small then

$$(1 + \epsilon)\sigma_z \geq \sigma_z' = \sup\{t:z \in B_{\delta t}(xt)\}$$

for all z sufficiently large so we have shown

(3.13) $\quad$ for all but a finite number of $z \in D_x$ we have $z \notin A_t$ whenever $z \in B_{\delta t}(xt)$

This completes the proof of (3.12) and Theorem 3.

Section 4. In this section we will prove

THEOREM 4. *Let F be an arbitrary distribution. For any $K < \infty$*

$$|(t^{-1}A_t \Delta \{x:\varphi(x) \leq 1\} \cap \{x:\|x\| \leq K\}| \to 0 \quad \textit{almost surely.}$$

Furthermore for any $\epsilon > 0$

$$P(t^{-1}A_t \subset \{x:\varphi(x) < 1 + \epsilon\} \quad \textit{for all } t \textit{ sufficiently large}) = 1.$$

The proof will be done in two parts. In the first part we show that Richardson's theorem holds for $\hat{A}_t = \{x:\hat{t}(0, x) \le t\}$ (here and in what follows we will assume that passage time processes are extended from Z^2 to R^2 in the same way t was.) In the second (much shorter) part we use the result for $\hat{A}_t$ and the inequalities

$$\hat{t}(0, x) \le t(0, x) \le u(0) + \hat{t}(0, x) + u(x) \qquad \text{for all} \quad x \in Z^2$$

to obtain the two results in Theorem 4.

To prove Richardson's theorem for $\hat{A}_t$ we will follow the outline of the proof of Theorem 3. The first step in that outline is to show that $\lim E\hat{t}(0, nx)/n$ exists. To do this we consider $\tilde{t}(0, x) = \hat{t}(0, x) + v(x)$ (for notation see Section 2). This process is subadditive and has $E\tilde{t}(0, x) < \infty$ so the argument given in the proof of Theorem 3 shows

$$\lim_{n\to\infty} E\tilde{t}(0, nx)/n \text{ exists for all } \quad x \in R^2$$

and the convergence is uniform on compact subsets of R^2. Since $Ev(x) = Ev(0) < \infty$ the result above implies that

$$\text{(4.1)} \qquad \lim_{n\to\infty} E\hat{t}(0, nx)/n \quad \text{exists for all} \quad x \in R^2$$

and the convergence occurs in the same sense.

At this point we have completed the first step. The second step is to prove

$$\text{(4.2)} \qquad \hat{t}(0, nx)/n \to \varphi(x) \quad \text{almost surely for all} \quad x \in Q^2$$

but this result is contained in Section 2, so there is nothing to show, and we are ready to consider $\hat{A}_t = \{x \in R^2:\hat{t}(0, x) \le t\}$. As in Section 3 we will carry out the proof only in the case $\gamma = \varphi(1, 0) > 0$ and leave the case $\gamma = 0$ to the reader. In the case $\gamma > 0$ the statement we want to prove is

$$\text{(4.3)} \quad P(\{x:\varphi(x) \le 1 - \epsilon\} \subset t^{-1}\hat{A}_t \subset \{x:\varphi(x) < 1 + \epsilon\} \quad \text{for all } t \text{ sufficiently large}) = 1.$$

The third step in the outline is to prove the first inclusion. To do this we will show that for each x with $\varphi(x) < 1$ there is a $\delta > 0$ (which will depend upon x) so that

$$\text{(4.4)} \qquad P(t^{-1}\hat{A}_t \supset B_\delta(x) \quad \text{for all } t \text{ sufficiently large}) = 1.$$

This is sufficient to prove the first inclusion for reasons indicated in Section 3.

The key to the proof of (4.4) for A_t was the inequality given in Lemma 3.3. Our next goal is to obtain a similar result for $\tilde{t}$. To do this we will again construct several disjoint paths from 0 to x. The paths we construct in this proof will be slightly different since we have very little control over how large the circuits $\tilde{\Delta}(x)$ in the definition of $\tilde{t}$ can be. Let M be the truncation level used in Section 2 (i.e., M is chosen so that $1 - F(M) \le 1/4$) and call all bonds with travel time $\le M$ open, $>M$ closed. As we observed in Section 2 this implies there exists, with probability one, an infinite connected set of open bonds and there is no infinite connected set of closed bonds. This implies that for each $z \in Z^2$ there is a minimal self-avoiding circuit $c(z)$ of open bonds which contains z in its interior. Let $\tilde{c}(z)$ be the set of bonds which are on $c(z)$, or in the interior and part of the infinite connected set of open bonds. For technical reasons which will become evident in the proof of Lemma 4.2 we have to use $\tilde{c}$ instead of $\tilde{\Delta}$ in this section.

To get estimates on the travel time from 0 to z we will construct special paths from 0 to z using the $\tilde{c}(z)$. To construct these paths we need the following

LEMMA 4.1. *Let $z_0, z_1, \cdots, z_m$ be a self-avoiding path from 0 to z. Then $\cup_{i=0}^m \tilde{c}(z_i)$ contains a path from $\bar{\Delta}(0)$ to $\bar{\Delta}(z)$.*

PROOF. Let $I = \{i: c^0(z_i) \not\subset c^0(z_j) \text{ for all } j \ne i\}$. It is easy to see that all the z_i are contained in the interior of some $c(z_j)$ with $j \in I$. To prove Lemma 4.1 we will show that there is an $I' \subset I$ so that $\cup_{c\in I'} \tilde{c}(c_i)$ is connected and intersects $\bar{\Delta}(0)$ and $\bar{\Delta}(z)$. To construct

I' start with the smallest $i_0 \in I$ so that $0 \in c^0(z_{i_0})$. To pick the next index find the smallest j so that $z_j \notin c^0(z_{i_0})$ and then pick the smallest i_1 so that $i_1 \in I$ and $z_j \in c^0(z_{i_1})$. (Observe that this guarantees $c^0(z_{i_0}) \cap c^0(z_{i_1}) \neq \varnothing$ and that neither of the sets $c^0(z_{i_0})$ and $c^0(z_{i_1})$ contains the other). We can continue this procedure, each time picking the point with least index which is not contained in the interior of any previously chosen circuit, until the point z is contained in the interior of some circuit. At this point we stop and let $I' = \{i_0, \cdots, i_k\}$ be the set of indices generated.

It is easy to see that $\tilde{R} = \cup_{j=0}^{k} \tilde{c}(z_{i_j})$ is connected. To prove this observe that if $j \le k$ it follows from the construction that $c^0(z_{i_j}) \not\subset c^0(z_{i_{j-1}})$, $c^0(z_{i_j}) \not\supset c^0(z_{i_{j-1}})$, and $c^0(z_{i_j}) \cap c^0(z_{i_{j-1}}) \neq \varnothing$ so repeating the path argument in the proof of Lemma 2.6 now shows that $\tilde{c}(z_{i_{j-1}}) \cap \tilde{c}(z_{i_j}) \neq \varnothing$ for $j = 1, \cdots k$. Since each $\tilde{c}(z_i)$ is connected this shows that $\tilde{R}$ is connected.

To complete the proof of Lemma 4.1 it suffices to show that $\tilde{R}$ intersects $\bar{\Delta}(0)$ and $\bar{\Delta}(z)$, but this is trivial. *If* $\tilde{c}(z_{i_0}) \cap \bar{\Delta}(0) \neq \varnothing$ this is true and if $\tilde{c}(z_{i_0}) \cap \bar{\Delta}(0) \neq \varnothing$, $c(z_{i_0})$ must contain $\bar{\Delta}(0)$ in its interior so we will have $\tilde{c}(z_{i_0}) \supset \bar{\Delta}(0)$ (thanks to the inclusion of all the interior bonds which are part of the infinite connected set of open bonds).

The construction in Lemma 4.1 shows how to associate with each self-avoiding path $z_0, \cdots, z_m$ from 0 to z a route from $\bar{\Delta}(0)$ to $\bar{\Delta}(z)$ through a connected set of open bonds. The next step is to obtain estimates on the length of this route. To do this we observe that the route we constructed is contained in $\cup_{i=0}^{m} \tilde{c}(z_i)$and the individual terms in the sum are small $-$ $\tilde{c}(z) \subset \bar{\Delta}(z)$ so Lemma 2.3 implies that

$$(4.5) \qquad E\,|\,\tilde{c}(z)\,|^r < \infty \quad \text{for all} \quad r < \infty,\ z \in Z^2$$

where $|\,\tilde{c}(z)\,|$ is the number of bonds in $\tilde{c}(z)$. To obtain estimates on$\sum m_{c=0}^{m}\ |\,\tilde{c}(z_i)\,|$ we need an estimate on the covariances.

LEMMA 4.2. *For any positive $q < \infty$, and $x, y \in Z^2$*

$$|\,\mathrm{Cov}(|\tilde{c}(x)|, |\,\tilde{c}(y)\,|)\,| \le 2^{1-q} E\,|\,\tilde{c}(0)\,|^{q+3}\,\|x - y\|^{-q}.$$

PROOF. Let $\mathscr{B}_k(x)$ be the σ-algebra generated by the $X(e)$ with $e \subset B_k(x) = \{y: \|x - y\| \le k\}$. Now any circuit which contains x in its interior and has length i must lie in $B_i(x)$ so if $\|x - y\| \ge i + j$, $\{|\,\tilde{c}(x)\,| = i\}$ and $\{|\,c(y)\,| = j\}$ are independent. Combining this observation with the definition of the covariance gives that

$$\mathrm{Cov}(|\,\tilde{c}(x)\,|, |\,\tilde{c}(y)\,|) = \textstyle\sum_{i,j;\,i+j>\|x-y\|} ij(P(|\,\tilde{c}(x)\,| = i, |\,\tilde{c}(y)\,| = j)$$
$$-\textstyle\sum_{i,j;\,i+j>\|x-y\|} iP(|\,\tilde{c}(x)\,| = i)jP(|\,\tilde{c}(y)\,| = j).$$

If $\|\,x - y\,\| = k$ each of the sums above is smaller in absolute value than

$$2 \textstyle\sum_{i\ge k/2} \sum_{j=1}^{i} ijP(|\,\tilde{c}(x)\,| = i)$$
$$\le 2 \textstyle\sum_{i\ge k/2} i^3 P(|\,\tilde{c}(x)\,| = i)$$
$$= 2E\,|\,\tilde{c}(0)\,|^{q+3}/(k/2)^q.$$

To obtain the conclusion stated in Lemma 4.2 observe that $|\,\mathrm{Cov}(|\tilde{c}(x)|, |\,\tilde{c}(y)\,|)\,|$ is smaller than the last expression above.

Using Lemma 4.2 it is easy to obtain a bound for $\sum_{i=0}^{n}\ |\,\tilde{c}(z_i)\,|$. For simplicity we will confine our attention to a special type of paths (geodesics).

LEMMA 4.3. *Let $z_0, \cdots z_m$ be a path with $\|\,z_i - z_j\,\| = |\,i - j\,|$ for all i, j and let $L_m = \sum_{i=1}^{m} |\,\tilde{c}(z_i)\,|$. If $\mu = E\,|\,\tilde{c}(0)\,|$ then*

$$P(L_m > (\mu + 1)(m + 1)) \le A_1/(m + 1)$$

where $A_1 = (\mathrm{Var}(|\,\tilde{c}(0)\,|) + E\,|\,\tilde{c}(0)\,|^5(\pi^2/6))$.

PROOF. Let $\mu = E|\tilde{c}(0)|$, $\tau_i = |\tilde{c}(z_i)| - \mu$, and $S_m = \sum_{i=0}^{m} \tau_i$.

$$ES_m^2 = (m+1)E\tau_0^2 + 2\sum_{0\le i<j\le m} E\tau_i\tau_j.$$

To estimate the second term we observe that

$$\left|\sum_{0\le i<j\le m} E\tau_i\tau_j\right| \le m \sum_{k=1}^{\infty} |E\tau_0\tau_k|$$

and from Lemma 4.2 (with $q = 2$) the right-hand side is

$$\le (m/2)\, E|\tilde{c}(0)|^5(\pi^2/6)$$

so we have shown $ES_m^2 \le A_1(m+1)$. The desired result now follows from the fact that

$$P(L_m > (\mu+1)(m+1)) = P(S_m > (m+1)) \le ES_m^2/(m+1)^2.$$

To prove (4.4) we need to improve Lemma 4.3 so that the error is $O(m^{-3})$. To do this we need to construct three paths from 0 to z which are well separated. Defining these paths in general requires the discussion of several cases so we will consider first the case $z = (n, 0)$ with $n > 0$. In this case we let k be the smallest integer $\ge n^{1/4}$ and define the paths to be

1. $(0, 0) \to (0, 2k) \to (n, 2k) \to (n, 0)$
2. $(0, 0) \to (n, 0)$
3. $(0, 0) \to (0, -2k) \to (n, -2k) \to (n, 0)$

where again $x \to y$ denotes moving from x to y by the straight line path. Let

$$R_0 = \sum_{j=-2k}^{2k} |\tilde{c}(0, j)|,$$

$$R_1 = \sum_{j=-2k}^{2k} |\tilde{c}(n, j)|,$$

and

$$U_h = \sum_{i=1}^{n-1} |\tilde{c}(i, (4-2h)k)| \quad \text{for} \quad h = 1, 2, 3.$$

From Lemma 4.1 it follows that there is a path from 0 to $(n, 0)$ with length at most

$$V = R_0 + R_1 + \min_{1\le h\le 3} U_h.$$

To estimate R_0 and R_1 we use

LEMMA 4.4 *If S_m is the sum of m identically distributed nonnegative random variables $X_1, \cdots, X_m$ then*

$$ES_m^4 \le 4m^4 EX_1^4.$$

PROOF. If we expand $(\sum_{i=1}^{m} X_i)^4$ we obtain m^4 terms of the form $EX_{i_1}X_{i_2}X_{i_3}X_{i_4}$. The terms are each

$$\le E(\max_{1\le j\le 4} X_{i_j})^4 \le \int_0^\infty (4x^3)4(1 - F(x))\, dx = 4EX_1^4,$$

so we have proved the desired result.

Using Lemma 4.4 and Chebyshev's inequality gives

$$P(S_m > m^\beta) \le ES_m^4/m^{4\beta} \le 4E\,|\tilde{c}(0)|^4/m^{4(\beta-1)}.$$

Setting $\beta = 4$ and $m = 5n^{1/4}$ (there are $2k + 1 \le 2n^{1/4} + 3$ terms in the sum defining R_0) gives

$$P(R_0 > 625n) \le A_2/n^3 \tag{4.6}$$

where $A_2 = 4E\,|\tilde{c}(0)|^4/5^{12}$.

To estimate min U_h we need to show that the circuits associated with the three paths are almost independent. To do this we break the plane into three regions $D_1 = \{x : x_2 \ge n^{1/4}\}$, $D_2 = \{x : n^{1/4} > x_2 > -n^{1/4}\}$, and $D_3 = \{x : -n^{1/4} \ge x_2\}$. The probability that some circuit on one of the paths $(1,(4-2h)n^{1/4}) \to (n-1,(4-2h)n^{1/4})$ lies outside its region is no more than $3nP(|\tilde{c}(0)| > n^{1/4})$ which by Chebyshev's inequality is no more than $3E\,|\tilde{c}(0)|^{16}n^{-3}$.

To compare the U_h with independent variables we will use a special construction. Generate by independent mechanisms three sets of passage times for the whole plane and let $\bar{U}_h$ be the total lengths of the circuits associated with the three paths in their respective planes. By taking region D_h from plane h we can define a fourth plane. Let U_h be the total length of the circuits for the three paths in this plane. If we let Ω' be the event that all the circuits in the definition of the $\bar{U}_h$ lie in their respective regions then on Ω', $U_h = \bar{U}_h$ for $h = 1, 2, 3$. From this observation it follows that

$$P(\min_{1\le h\le 3} U_h > (\mu+1)n, \Omega')$$
$$= P(\min_{1\le h\le 3} \bar{U}_h > (\mu+1)n, \Omega')$$
$$\le P(U_1 > (\mu+1)n)^3$$

so applying Lemma 4.3 and results in the last two paragraphs gives that

$$P(\min_{1\le h\le 3} U_h > (\mu+1)n) \le A_3 n^{-3} \tag{4.7}$$

where $A_3 = (3E\,|c(0)|^{16} + A_1^3)$. Combining this with (4.6) and the definition of V shows

$$P(V > (\mu + 1251)n) \le A_4 n^{-3}$$

where $A_4 = 2A_2 + A_3$. (This estimate can obviously be improved but is sufficient for our purposes.)

The last inequality says that with a probability which is at least $1 - A_4 n^{-3}$ there is a path of length $\le (\mu + 1251)n$ from $\bar{\Delta}(0)$ to $\bar{\Delta}(ne_1)$ which lies in the set of bonds with travel time $\le M$. Since $\hat{t}(0, ne_1)$ is the minimum passage time from $\bar{\Delta}(0)$ to $\bar{\Delta}(ne_1)$ we have

$$P(\hat{t}(0, ne_1) > (\mu + 1251)Mn) \le A_4 n^{-3}.$$

To prove (4.4) we need to generalize this result to all $z \in Z^2$. It is easy to see that this can be done by generalizing the proof given above. If k is the smallest integer $> \|z\|^{1/4}$, $z_1 > 2k$, and $z_2 \ge 0$ we define the three paths from 0 to z by

1. $(0, 0) \to (0, 2k) \to (z_1 - 2k, 2k) \to (z_1 - 2k, z_2) \to (z_1, z_2)$
2. $(0, 0) \to (z_1, 0) \to (z_1, z_2)$
3. $(0, 0) \to (0, -2k) \to (z_1 - 2k, -2k) \to (z_1 + 2k, z \to (z_1, z_2)$.

If $z_1\, z_2 \ge 0$ but $z_1 < 2k$ we replace the first path by $(0, 0) \to (z_1 - 2k, 0) \to (z_1 - 2k, z_2) \to (z_1, z_2)$ and define the other two paths as above. As before the paths to points in other quadrants are defined by clockwise rotation.

The last paragraph describes the paths to be used in the proof. If we define R_0, R_1, U_1, U_2, and U_3 as we did above (letting R_0 and R_1 be the contributions of $(0, -2k) \to (0, 2k)$ and $(z_1 - 2k, z_2) \to (z_1 + 2k, z_2)$ respectively it follows from the arguments above that R_0 and R_1 are both $\le 625\,\|z\|$ with probability $1 - 2A_2\|z\|^{-3}$ and the U_h are almost independent. Since the U_h are derived from geodesics, Lemma 4.3 and the observations above imply that

$$P(\hat{t}(0, z) > (\mu + 1251)M\,\|z\|) \le A_4\|z\|^{-3}. \tag{4.8}$$

We can now prove (4.4) by proceeding as in Section 3 using equation (4.8) instead of Lemma 3.3. To prove (4.4) with $x = 0$ we observe that if $K > (\mu + 1251)M$ then (4.8) implies that

$$\textstyle\sum_z P(\hat{t}(0, z) > K\,\|z\|) < \infty$$

so it follows from the Borel-Cantelli lemma that (4.4) holds for $x = 0$ when $\delta < 1/K$.

To prove (4.4) for $x \ne 0$ we observe that in order for $\hat{A}_t \supset B_{\delta t}(xt)$ for all t sufficiently large it is necessary and sufficient that

(4.9) $$\begin{gathered}\text{for all but a finite number of } z \in \cup_{t\ge 0} B_{\delta t}(xt) \\ \hat{t}(0, z) \le \sigma_z = \inf\{t : z \in B_{\delta t}(xt)\}.\end{gathered}$$

Pick x such that $\varphi(x) < 1$ and pick $\epsilon < (1 - \varphi(x))/2$. From the almost sure existence of radial limits we have

$$P(\hat{t}(0, \sigma_z x) > (\varphi(x) + \epsilon)\sigma_z \quad \text{i.o.}) = 0.$$

To estimate $\hat{t}(0, z)$ we recall that from Section 2

$$\hat{t}(0, z) \le \hat{t}(0, \sigma_z x) + v(\sigma_z x) + \hat{t}(\sigma_z x, z).$$

To bound the second term we observe that from Corollary 2.5 $Ev(\sigma_z x)^3 = Ev(0)^3 < \infty$ so the Borel-Cantelli lemma implies that

$$P(v(\sigma_z x) > \epsilon\sigma_z \quad \text{i.o.}) = 0.$$

To estimate the third term pick $\delta > 0$ so that $(1 - \varphi(x) - 2\epsilon)/\delta > (\mu + 1251)M$ and observe that it follows from (4.8) and the observation that $\| \sigma_z x - z \| = \delta\sigma_z$ that we have

$$P(\hat{t}(\sigma_z x, z) > (1 - \varphi(x) - 2\epsilon)\sigma_z) \le A_4(\delta\sigma_z)^{-3}.$$

Using the fact that $\| z \|/\| x \| + \delta \le \sigma_z \le \| z \|/\| x \| - \delta$ (see Section 3) it follows from the Borel-Cantelli lemma that

$$P(\hat{t}(\sigma_z x, z) > (1 - \varphi(x) - 2\epsilon)\sigma_z \quad \text{i.o.}) = 0.$$

Combining the last five results proves (4.9) and completes the proof of (4.4).

The final step in proving Richardson's theorem for $\hat{A}_t$ is to prove (3.12) of Section 3 with $\hat{A}_t$ instead of A_t. As in Section 3 pick x so that $\varphi(x) > 1$, pick ϵ so that $\varphi(x) - 1 - 3\epsilon > 0$, and pick δ so that $(\varphi(x) - 1 - 3\epsilon)/\delta > (\mu + 1251)M$. Since

$$\hat{t}(0, z) \ge \hat{t}(0, \sigma z_x) - v(z) - \hat{t}(z, \sigma_z x)$$

the desired result follows (as it did in Section 3) from the fact that

$$P(\hat{t}(0, \sigma_z x) < (\varphi(x) - \epsilon)\sigma_z \quad \text{i.o.}) = 0$$
$$P(v(z) > \epsilon\sigma_z \quad \text{i.o.}) = 0$$

and $$P(\hat{t}(z, \sigma_z x) > (\varphi(x) - 1 - 3\epsilon)\sigma_z \quad \text{i.o.}) = 0.$$

We have now established

(4.10) $$P(\{x : \varphi(x) \le 1 - \epsilon\} \subset t^{-1}\hat{A}_t \subset \{x : \varphi(x) \le 1 + \epsilon\} \text{ for all } t \text{ sufficiently large}) = 1$$

and completed Part I of the proof of Theorem 4.

The second part of the proof is to show that (4.10) implies the conclusions of Theorem 4. As always we will deal only with the case $\varphi \not\equiv 0$. Since $A_t \subset \hat{A}_t$ the second conclusion of Theorem 4 is an immediate consequence of (4.10). To prove the first statement in Theorem 4 we observe that for any $\epsilon > 0$

$$\begin{aligned}| t^{-1}A_t \Delta \{x : \varphi(x) \le 1\} &\le |\{x : \varphi(x) \le 1 - \epsilon\} - t^{-1}A_t| \\ &+ |\{x : 1 - \epsilon < \varphi(x) < 1 + \epsilon\}| + | t^{-1}A_t \cap \{x : \varphi(x) > 1 + \epsilon\}).\end{aligned}$$

It follows from the second conclusion of Theorem 4 that the third term $\to 0$ a.s. The second term $=((1 + \epsilon)^2) - (1 - \epsilon)^2| \{x : \varphi(x) \le 1\}|$ which is small if ϵ is small so it suffices to show that

$$\limsup_{t\to\infty} |\{x : \varphi(x) \le 1 - \epsilon\} - t^{-1}A_t| = 0 \quad \text{a.s.}$$

for each $\epsilon > 0$. Since $t(0, y) \le u(0) + \hat{t}(0, y) + u(y)$

$$\{x : \varphi(x) \le 1 - \epsilon\} - t^{-1}A_t \subset \{x : \varphi(x) \le 1 - \epsilon,\ u(0) + \hat{t}(0, xt) + u(xt) > t\}.$$

For any positive $\delta < \epsilon$, it follows from (4.10) that for almost every ω

$$\{x: \varphi(x) \leq 1 - \epsilon, u(0) + \hat{t}(0, xt) > (1 - \delta)t\} = \varnothing$$

for all t sufficiently large, so

$$\limsup_{t\to\infty} |\{x: \varphi(x) \leq 1 - \epsilon\} - t^{-1}A_t| \leq \limsup_{t\to\infty} |\{x: \varphi(x) \leq 1 - \epsilon, u(xt) \geq \delta t\}|.$$

If we pick K large enough so that $\{x: \varphi(x) \leq 1\} \subset [-K, K]^2$ then

$$|\{x: \varphi(x) \leq 1 - \epsilon, u(tx) > \delta t\}| \leq \frac{1}{(2Kt)^2} \sum_{z\in[-Kt,Kt]^2} 1_{\{u(z)>\delta t\}}$$

so we are done once we prove

LEMMA 4.5. $\lim_{t\to\infty} \frac{1}{(2Kt)^2} \sum_{z\in[-Kt,Kt]^2} 1_{\{u(z)>t\}} = 0$ a.s.

PROOF. For $N > 0$ let $S_N(z) = \sum_{e\subset B_N(z)} X(e)$, and note that

$$1_{\{u(z)>\delta t\}} \leq 1_{\{S_N(z)>\delta t\}} + 1_{\{|\bar{\Delta}(z)|>N\}}.$$

If we let

$$\xi = \limsup_{t\to\infty} \frac{1}{(2Kt)^2} \sum_{z\in[-Kt,Kt]^2} 1_{\{u(z)>\delta t\}}$$

$$\xi_1^{(N)} = \limsup_{t\to\infty} \frac{1}{(2Kt)^2} \sum_{z\in[-Kt,Kt]^2} 1_{\{S_N(z)>\delta t\}}$$

$$\xi_2^{(N)} = \limsup_{t\to\infty} \frac{1}{(2Kt)^2} \sum_{z\in(-Kt,Kt)^2} 1_{\{|\bar{\Delta}(z)|>N\}},$$

then $\xi \leq \xi_1^{(N)} + \xi_2^{(N)}$.

The first step is to show $\xi_1^{(N)} = 0$ a.s. To see this, note that for any J, $\delta t \geq J$ for all t sufficiently large. Hence it follows from the multiparameter ergodic theorem that

$$\xi_1^{(N)} \leq \limsup_{t\to\infty} \frac{1}{(2Kt)^2} \sum_{z\in[-Kt,Kt]^2} 1_{\{S_N\geq J\}} = \xi(J) \quad \text{a.s.}$$

where $\xi(J)$ is a random variable with $E\xi(J) = P(S_N \geq J)$ (see Dunford (1951) or Smythe (1976)). This implies that $\xi_1^{(N)}$ is a nonnegative random variable with mean smaller than $P(S_N \geq J)$ for any J, so $\xi_1^{(N)} = 0$.

Repeating the argument above shows that $\xi_2^{(N)}$ is a finite random variable with $E\xi_2^{(N)} = P(|\bar{\Delta}(0)| \geq N)$. Since $\xi \leq \xi_1^{(N)} + \xi_2^{(N)}$ for all N this implies that $\xi = 0$ almost surely.

Section 5. In this section we will prove Theorems 5 and 6. The first result concerns the x-reach process $x_t = \sup\{m: t(0, (m, n) \leq t \text{ for some } n \in Z\}$.

THEOREM 5. *As* $t \to \infty$

$$x_t/t \to 1/\varphi(1, 0) \quad \textit{almost surely}.$$

PROOF. $1/\varphi(1, 0) = \sup\{y: \varphi(y, 0) \leq 1\}$ so it follows from the first conclusion of Theorem 4 that

$$\liminf_{t\to\infty} x_t/t \geq 1/\varphi(1, 0).$$

To prove the opposite inequality we observe that from Theorem 4 we have that for all $\epsilon > 0$

$$P(t^{-1} A_t \subset \{x : \varphi(x) \leq 1 + \epsilon\} \quad \text{for all } t \text{ sufficiently large}) = 1$$

so to complete the proof it suffices to show that

$$1/\varphi(1, 0) = \sup\{y : \varphi(y, x) \leq 1 \quad \text{for some} \quad x \in R\}.$$

If $\varphi(1, 0) = 0$ this is obvious so suppose $\varphi(1, 0) = \gamma > 0$. Let $x, y \in R$ be chosen so that $\varphi(y, x) \leq 1$. From symmetry it follows that $\varphi(y, -x) \leq 1$. Since φ is a norm the triangle inequality implies

$$\begin{aligned} \varphi(y, 0) &= \varphi(½(y, x) + ½(y, -x)) \\ &\leq ½\varphi(y, x) + ½\varphi(y, -x) \leq 1 \end{aligned}$$

and so $y \leq 1/\varphi(1, 0)$. Since (y, x) is an arbitrary point with $\varphi(y, x) \leq 1$ this proves the desired result.

The next result concerns the point to line process $b(m) = \inf\{t(0, (m, n)) : n \in Z\}$.

THEOREM 6. *As* $m \to \infty$

$$b(m)/m \to \varphi(1, 0) \quad \textit{almost surely}$$

PROOF. $b(m) \leq u(0) + \hat{t}(0, (m, 0)) + v(m, 0)$ so it follows from results in Section 2 that

$$\limsup_{m\to\infty} b(m)/m \leq \varphi(1, 0).$$

If $\varphi(1, 0) = 0$, this is the desired result so suppose $\varphi(1, 0) = \gamma > 0$. From Theorem 4 we have that for all $\epsilon > 0$

$$P(t^{-1}A_t \subset \{x : \varphi(x) < 1 + \epsilon\} \quad \text{for all } t \text{ sufficiently large}) = 1$$

and from the proof of Theorem 5 we have

$$1/\gamma = \sup\{y : \varphi(y, x) \leq 1 \quad \text{for some} \quad x \in R\}.$$

Combining the last two results shows

$$\liminf_{m\to\infty} b(m)/m \geq \varphi(1, 0)/(1 + \epsilon).$$

Since ϵ was arbitrary the proof is complete.

NOTE ADDED IN PROOF. We have recently learned that Gunnar Brånvall (Uppsala University Department of Mathematics Report No. 11, December 1979) has independently proved Theorem 3 stated above. His work also includes "weak moment" versions (assuming $EX^\delta < \infty$ for some $\delta > 0$) for our other theorems.

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