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RENEWAL THEORY FOR MARKOV CHAINS ON THE REAL LINE[1]

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Standard renewal theory is concerned with expectations related to sums of positive i.i.d. variables,

$$S_n = \sum_{i=1}^n Z_i.$$

We generalize this theory to the case where $\{S_i\}$ is a Markov chain on the real line with stationary transition probabilities satisfying a drift condition. The expectations we are concerned with satisfy generalized renewal equations, and in our main theorems, we show that these expectations are the unique solutions of the equations they satisfy.

1. Introduction. One method of describing renewal theory is as follows. Let $\{S_i\}_{i\geq 0}$ be a random walk with initial position $S_0 = s$. For a function h, define the function

$$R(s) = E_s[\textstyle\sum_{i=0}^\infty h(S_i)].$$

Then $R(s)$ satisfies the *renewal equation*

$$R(s) = h(s) + E_s[R(S_1)].$$

In our generalization of renewal theory we let the sequence $\{S_i\}_{i\geq 0}$ be a Markov chain on the real line with stationary transition probabilities. This type of process will be called a *generalized random walk*, or GRW, to distinguish it from an ordinary random walk in which the *increments* or *steps*, $Z_i = S_i - S_{i-1}$, are i i.d. random variables. The starting position of the GRW will be called s.

For a *continuation set* $\mathscr{C}$, we define an *extended Markov stopping time*

$$N = \inf\{i : S_i \notin \mathscr{C}\}.$$

The *stopping set*, $\mathscr{S}$, will be the complement of $\mathscr{C}$. For a function h, we will define the function R as

$$R(s) = E_s[\textstyle\sum_{i=0}^N h(S_i)].$$

If $R(s)$ exists, we can condition on the value of S_1 and obtain the following generalized renewal equation

$$R(s) = h(s) + 1_{\mathscr{C}}(s)E_s[R(S_1)]. \tag{1.1}$$

To insure that $R(s)$ exists, we restrict our study to GRW's satisfying one of the following two conditions.

C1: There exist positive constants a and b such that for all starting positions, $s \in \mathbb{R}$, we have

$$E_s[e^{-aZ}] \leq e^{-b}.$$

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C2: For some constant $k \geq 2$ there exist positive constants μ_0 and M such that for all starting positions, $s \in \mathbb{R}$, we have

$$E_s[Z] \geq \mu_0$$

and

$$E_s[|Z - E_s[Z]|^k] \geq M^k.$$

Both of these conditions imply that $S_n \to \infty$ with probability one and that $E_s\#\{n : S_n < c\} < \infty$. Processes satisfying C1 or C2 share many properties with random walks with positive drift.

Some of the results presented have applications to sequential hypothesis testing. In simple versus simple testing, the sequence of log likelihood ratios $\{\ln(dP_n/dQ_n)\}$ is a random walk (P or Q). In a design problem where the experimenter chooses an experiment at each stage based on the current likelihood ratio, the same sequence is a GRW. If the experiments are chosen from a finite set, then C1 holds. Error probabilities and expected sample sizes for this problem can be expressed in the form of $R(s)$ (for example, $E_sN = E_s \sum_{i=1}^{N} 1_{\mathscr{C}}(S_i)$). For more details see Keener (1980a).

Sections 2 and 3 study GRW's satisfying C1 and C2 respectively. Theorems 2.1 and 3.4 assert that R is the unique solution to the renewal equation (1.1) in appropriate function classes. If we define the linear operator L by $LR(s) = R(s) - 1_{\mathscr{C}}(s)E_sR(S_1)$, then the renewal equation is $LR = h$ and these theorems show that L is invertable. This can be useful in assessing the accuracy of an approximation $\hat{R}$ of R. Since $R - \hat{R} = L^{-1}(h - L\hat{R})$ and L^{-1} is continuous, $\hat{R}$ will be close to R provided $L\hat{R}$ is close to h. This idea is applied in Keener (1980a) to obtain approximations for expected sample sizes and error probabilities in the testing problem mentioned. Bounds for the accuracy of the approximations are found using Theorem 2.2 which gives an upper bound for $L^{-1}h$ for certain functions h.

Other theorems in Section 3 give bounds for probabilities and moments for processes satisfying conditions related to C2.

2. First drift condition. Conditions C1 and C2 both imply that $\{S_i\}$ drifts to $+\infty$. To verify this for C1 we have the following lemma.

LEMMA 2.1. *If the* GRW *satisfies* C1 *then*

$$E_s[\#\{i : S_i \leq \lambda\}] \leq \frac{a}{b}(\lambda - s) + (1 + e^{-b})^{-1} \quad \text{if} \quad \lambda \geq s$$

$$\leq e^{a(\lambda - s) - b}/(1 - e^{-b}) \qquad \text{if} \quad \lambda < s.$$

PROOF. We can assume that $s = 0$. By induction, C1 implies that

$$E_s[e^{-aS_j}] \geq e^{-bj} \quad \text{for all} \quad j \in \mathbb{N}, \tag{2.1}$$

and from this it follows that

$$P_s(S_j \leq \lambda) \leq e^{-bj + a\lambda}.$$

Monotone convergence now implies

$$E_s[\#\{i : S_i \leq \lambda\}] = \textstyle\sum_{j=0}^{\infty} P_s(S_j \leq \lambda) \leq \sum_{j=0}^{\infty} 1 \wedge e^{-bj + a\lambda}.$$

If $\lambda < 0$, the sum may be taken from 1 to ∞, giving the desired result. If $\lambda \geq 0$, the sum is

$$\lceil\left(\frac{a\lambda}{b}\right) + \exp\left(-\lceil\left(\frac{a\lambda}{b}\right) b + a\lambda\right) \Big/ (1 - e^{-b}),$$

where $\lceil(x)$ is the ceiling of x, i.e. the least integer $\geq x$. This expression is less than the desired result.

An immediate consequence of Lemma 1 is the following corollary.

COROLLARY 2.1. *If the* GRW *satisfies* C1 *and if* $\mathscr{I}$ *is an interval of length* λ, *then*

$$E_s[\#\{i: S_i \in \mathscr{I}\}] \le \left[\frac{a}{b}\lambda + (1-e^{-b})^{-1}\right] P_s[\exists i \text{ s.t. } S_i \in \mathscr{I}].$$

PROOF. By Lemma 2.1, the result is obvious if $s \in \mathscr{I}$. If $s \notin \mathscr{I}$, we use the Markov property and condition on the first time the GRW enters $\mathscr{I}$ to obtain the desired result.

The following theorem gives conditions under which $R(s)$ is finite and shows that $R(s)$ is the only solution of the renewal equation (1.1) which is bounded on finite intervals and has reasonable behavior as $s \to \pm\infty$.

THEOREM 2.1. *Let the* GRW *satisfy* C1 *and let* $\{e^{-AS_j}\}$ *be a supermartingale for some* $A \ge a$. *Let* h *be a non-negative function such that* $1_{\mathscr{I}}(x)h(x)/(1+e^{-Ax})$ *is bounded and* $1_{\mathscr{C}}(x)h(x)/(1+e^{-Ax})$ *is directly Riemann integrable*[2]. *Then* $R(s)$ *is finite for all* $s \in \mathbb{R}$. *If* $1_{\mathscr{C}}(x)$ *has a limit as* x *approaches* $+\infty$ *and* $-\infty$, *then* $R(s)$ *is the only solution of the renewal equation (1.1), which is bounded on finite intervals and satisfies*

$$\lim_{s\to\pm\infty} 1_{\mathscr{C}}(s)R(s)/(1+e^{-As}) = 0. \tag{2.2}$$

To facilitate the proof of this theorem, we have the following technical lemma.

LEMMA 2.2. *Let* $\{m_i\}_{i\in\mathbb{Z}}$ *be a sequence of positive constants satisfying* $\sum_{i\in\mathbb{Z}} m_i < \infty$, *and let* $\{n_i\}_{i\in\mathbb{Z}}$ *be positive random variables depending on a parameter* s. *If there exist positive constants* A *and* K *such that*

$$\begin{aligned} E_s(n_i) &\le K && \text{for } i \ge s \\ &\le Ke^{A(i-s)} && \text{for } i < s, \end{aligned}$$

then

$$E_s\left[\sum_{i\in\mathbb{Z}} n_i m_i (1+e^{-A(i-1)})\right] < \infty, \tag{2.3}$$

and

$$\lim_{s\to\pm\infty} E_s\left[\sum_{i\in\mathbb{Z}} \frac{n_i m_i (1+e^{-A(i-1)})}{1+e^{-As}}\right] = 0. \tag{2.4}$$

PROOF OF LEMMA 2.2. From the bounds on $E[n_i]$ we have

$$E_s[n_i(1+e^{-A(i-1)})] \le K(1+e^{-A(s-1)}).$$

Hence

$$\sum_{i\in\mathbb{Z}} m_i E_s[n_i(1+e^{-A(i-1)})] \le K(1+e^{-A(s-1)}) \sum_{i\in\mathbb{Z}} m_i < \infty,$$

and (2.3) follows. To prove (2.4) we note that

$$\lim_{s\to\pm\infty} \frac{E_s[n_i]}{1+e^{-As}} = 0 \quad \text{for all} \quad i \in \mathbb{Z}.$$

Applying dominated convergence for sums, the limit of the sum is equal to the sum of the limits and (2.4) is established.

PROOF OF THEOREM 2.1. For $k \in \mathbb{Z}$ define

$$\mathscr{I}_k = [k-1, k)$$

[2] See page 362 of Feller (1966) for a discussion of direct Riemann integrability as it relates to renewal theory.

and

$$n_k = \#\{j : S_j \in \mathscr{I}_k\}.$$

Define the extended stopping times

$$T_k = \inf\{i : S_i \in \mathscr{I}_k\}.$$

Since $\{e^{-AS_j}\}$ is a positive supermartingale, an optional stopping theorem for positive supermartingales (page 267 of Karlin and Taylor, 1975) implies that

$$e^{-As} \geq E_s[1_N(T_k)e^{-AS_{T_k}}] \geq e^{-kA}P_s(\exists i : S_i \in \mathscr{I}_k).$$

Corollary 2.1 now implies that

$$(2.5)\qquad \begin{aligned} E[n_k] &\leq \frac{a}{b} + (1 - e^{-b})^{-1} && \text{if} \quad k \geq s \\ &\leq \left(\frac{a}{b} + (1 - e^{-b})^{-1}\right)e^{A(k-s)} && \text{if} \quad k < s. \end{aligned}$$

We now define

$$m_k = \sup_{x \in \mathscr{I}_k} 1_{\mathscr{C}}(x)h(x)/(1 + e^{-Ax}),$$

which implies that

$$\sup_{x \in \mathscr{I}_k} 1_{\mathscr{C}}(x)h(x) \leq m_k(1 + e^{-A(k-1)}).$$

By the integrability condition, $\sum_{i \in \mathbb{Z}} m_i < \infty$, and using (2.5) we see that the conditions of Lemma 2.2 are satisfied. We now observe that

$$(2.6)\qquad \textstyle\sum_{i=0}^{N} h(S_i) \leq h(S_{\mathcal{N}})1_N(N) + \sum_{K \in \mathbb{Z}} n_k m_k (1 + e^{-A(k-1)}).$$

$R(s)$ will be finite if the expectation of the right-hand side of this equation is finite. Using Equation (2.3) in Lemma 2.2, we only need to show that $E[h(S_{\mathcal{N}})1_N(N)] < \infty$. Let

$$M = \sup_{x \in \mathscr{S}} h(x)/(1 + e^{-Ax}).$$

Using the same optional stopping theorem for supermartingales, we have

$$E_s[h(S_N)1_N(N)] \leq ME[(1 + e^{-AS_N})1_N(N)] \leq (1 + e^{-As})M < \infty,$$

and hence $R(s)$ is finite.

We now show that $R(s)$ satisfies (2.2). Using (2.4) in Lemma 2.2, and Equation (2.6), it is sufficient to show that

$$(2.7)\qquad \lim_{s \to \pm\infty} E_s[1_{\mathscr{C}}(s)1_N(N)h(S_N)/(1 + e^{-As})] = 0.$$

We deal first with the limit as $s \to +\infty$. If $\lim_{s \to +\infty} 1_{\mathscr{C}}(s) = 0$, then the result is obvious. If $\lim_{s \to +\infty} 1_{\mathscr{C}}(s) = 1$, then the conditions imposed on h imply that there exists a constant M' such that when $N < \infty$,

$$h(S_N) \leq M'e^{-AS_N}.$$

By optional stopping we have

$$\lim_{s \to +\infty} E_s[1_{\mathscr{C}}(s)1_N(N)h(S_N)/(1 + e^{-As})] \leq \lim_{s \to +\infty} M'E_s[1_N(N)e^{-AS_N}] \leq \lim_{s \to +\infty} M'e^{-s} = 0.$$

To verify (2.7) as $s \to -\infty$, we note that the result is obvious if $\lim_{s \to -\infty} 1_{\mathscr{C}}(s) = 0$. If $\lim_{s \to -\infty} 1_{\mathscr{S}}(s) = 1$ then h is bounded on $\mathscr{S}$ which implies (2.7) and completes our proof that R satisfies (2.2). To complete our proof, we need to show uniqueness. Since (1.1) and (2.2) are linear in R we can assume without loss of generality that $h = 0$. Let G be an arbitrary function which is bounded on finite intervals and such that

$$G(s) = 1_{\mathscr{C}}(s)E_s[G(S_1)] \tag{2.8}$$

and

$$\lim_{s\to\pm\infty} 1_{\mathscr{C}}(s)G(s)/(1 + e^{-As}) = 0. \tag{2.9}$$

We must show that $G = 0$. (2.8) and (2.9) imply that

$$|G(s)| \le K(1 + e^{-As}).$$

Iterating (2.8) gives

$$|G(s)| \le E_s[|G(S_k)|] \quad \text{for all} \quad k \in N.$$

We now partition the line into the intervals $(-\infty, -\lambda)$, $[-\lambda, \lambda]$ and (λ, ∞) and get

$$\begin{aligned} |G(s)| &\le E_s[|G(S_k)|(1_{(-\infty,-\lambda)}(S_k) + 1_{[-\lambda,\lambda]}(S_k) + 1_{(\lambda,\infty)}(S_k))] \\ &\le \sup_{x<-\lambda} e^{Ax} |G(x)|\, E_s[e^{-AS_k}1_{(-\infty,\lambda)}(S_k)] \\ &\quad + KE_s[(1 + e^{-AS_k})1_{[-\lambda,\lambda]}(S_k)] + \sup_{x>\lambda} |G(x)| \\ &\le e^{-As}\sup_{x<-\lambda} e^{Ax} |G(x)| + KE_s[e^{-a(S_k-\lambda)} + e^{A\lambda+a(\lambda-S_k)}] \\ &\quad + \sup_{x>\lambda} |G(x)|. \end{aligned}$$

Equation (2.1) now gives

$$|G(s)| \le e^{-As}\sup_{x<-\lambda} e^{Ax} |G(x)| + K(e^{-a(s-\lambda)-bk} + e^{A\lambda+a(\lambda-s)-bk}) + \sup_{x>\lambda} |G(x)|.$$

If we now take $\lambda = \sqrt{k}$ and let $k \to \infty$, we have $G(s) = 0$ which completes the proof.

We will close this section by deriving bounds for the magnitude of $R(s)$ for certain functions h. If we define the renewal measure for our GRW as

$$U_s(A) = E_s[\#\{j \le N : S_j \in A\}],$$

then $R(s)$ can be expressed as the integral

$$R(s) = \int h(x)\, dU_s.$$

From Corollary 2.1, we know that if $\mathscr{I}$ is an interval of length λ then

$$U_s(\mathscr{I}) \le \frac{a}{b}\lambda + (1 - e^{-b})^{-1}.$$

Using this we can construct the following bounds for $R(s)$.

THEOREM 2.2. *Under the conditions of Theorem* 2.1, *if* $h(x) = 0$ *for* $x < \lambda$ *and* $h(x)$ *is integrable and non-increasing for* $x \ge \lambda$, *then for all* s

$$R(s) \le h(\lambda)(1 - e^{-b})^{-1} + \frac{a}{b}\int_\lambda^\infty h(x)\, dx. \tag{2.10}$$

If $h(x) = 0$ *for* $x > \lambda$ *and* $h(x)$ *is integrable and non-decreasing for* $x \le \lambda$, *then for all* s

$$R(s) \le h(\lambda)(1 - e^{-b})^{-1} + \frac{a}{b}\int_{-\infty}^\lambda h(x)\, dx. \tag{2.11}$$

PROOF. To establish (2.10), we use the fact that $h(x) = 0$ for $x < \lambda$ and integration by parts to get

$$R(s) = \int h(x)\, d(U_s([\lambda, x])) = -\int U_s([\lambda,x])\, dh(x)$$

$$\leq -\int \left(\frac{a}{b}\lambda(x-\lambda) + (1-e^{-b})^{-1}\right) dh(x)$$

$$= h(\lambda)(1-e^{-b})^{-1} + \frac{a}{b}\int_\lambda^\infty h(x)\, dx.$$

Equation (2.11) can be established the same way.

Corollary 2.1 and Lemma 2.1 can be used to construct sharper bounds for $U_s(\mathcal{J})$. These bounds can be used to construct an improved bound on $R(s)$ for a given s, but will not improve our global bounds.

3. Second drift condition. Our main goal in this section will be to prove a theorem similar to Theorem 2.1 for GRW's satisfying C2 instead of C1. This result is useful because C2 is often a weaker condition than C1. Unfortunately, C2 is more difficult to work with than C1, and we need several preliminary results before we can prove our main theorem. The following lemma will be used in many of the proofs to follow.

LEMMA 3.1. *If $E|X|^k \leq M^k$ for some $k > 2$ and $EX = 0$ then*

$$E|1+X|^k \leq (1+M)^k - kM \leq 1 + \frac{k(k-1)}{2}M^2(1+M)^{k-2}.$$

PROOF. By Taylor's theorem with remainder we have

$$E|1+X|^k = E\left[1 + kX + \int_0^1 k(k-1)X^2|1+X-Xy|^{k-2}y\, dy\right]$$

$$= 1 + \int_0^1 k(k-1)E[X^2|1+X-Xy|^{k-2}]y\, dy$$

$$\leq 1 + \int_0^1 k(k-1)M^2(E|1+X-Xy|^{k(k-2)/k}y\, dy.$$

Applying the Minkowski inequality

$$E|1+X|^k \leq 1 + \int_0^1 k(k-1)M^2(1+(1-y)M)^{k-2}y\, dy = (1+M)^k - kM.$$

The last inequality in the statement of Lemma 3.1 follows from Taylor's theorem with Lagrange's form of the remainder.

Using Lemma 3.1, we now have the following result which bounds the magnitude of kth absolute moments of terms in a martingale. This result generalizes a theorem due to Brillinger (1962).

THEOREM 3.1. *Let $\{S_n, \mathcal{F}_n\}_{n\geq 0}$ be a martingale satisfying*

$$E|S_{n+1} - S_n|^k \leq M^k$$

for some $k > 2$. Then

$$E|S_n - S_0|^k \leq (M\gamma\sqrt{n})^k$$

where

$$\gamma = 18k(k/(k-1))^{1/2}.$$

PROOF. Burkholder's (1973) square function inequality gives

$$E\,|S_n - S_0|^k \leq \gamma^k E\,(\textstyle\sum_{j=1}^n (S_j - S_{j-1})^2)^{k/2}$$

and the result follows from Minkowsky's inequality.

COMMENT. A proof of this result under the stronger moment condition that

$$E(|S_{n+1} - S_n|^k \mid \mathscr{F}_n) \leq M^k$$

is given in Keener (1980b). The constant in that result, $(k-1)^{1/2}\exp((k-2)/2)$, is smaller than γ unless k is greater than 10. A proof of the result for Markov chains on a countable state space is given in Doob (1953).

To state our next two theorems in their proper generality, we will replace C2 by the following condition for processes $\{S_i\}_{i\geq 0}$ where S_i is measurable with respect to $\mathscr{F}_i$ and $\{\mathscr{F}_i\}_{i\geq 0}$ is an increasing family of σ-algebras.

C3: For constants $k \geq 2$, $\mu_0 > 0$ and $M > 0$, we have for all $i \geq 0$

$$E(Z_{i+1} \mid \mathscr{F}_i) \geq \mu_0$$

and

$$E[\,|Z_{i+1} - E(Z_{i+1} \mid \mathscr{F}_i)|^k \mid \mathscr{F}_i] \leq M^k$$

where

$$Z_i = S_i - S_{i-1}.$$

THEOREM 3.2. *If* $\{S_j\}$ *satisfies* C3 *and* $S_0 = s > 0$ *then*

$$P[\inf_{j\geq 0} S_j \leq 0] \leq (1 + as)^{-k/2},$$

where

$$a = M^{-1}\left[\left(1 + \frac{2\mu_0}{(k+1)M}\right)^{1/(k-1)} - 1\right] > 0.$$

PROOF. This theorem is a generalization of a result in Dubins and Savage (1976). We assume without loss of generality that $M = 1$. Define $f(x)$ as

$$\begin{aligned} f(x) &= (1 + ax)^{-1} \quad \text{for} \quad x \geq 0 \\ &= 1 \qquad\qquad\quad \text{for} \quad x \leq 0. \end{aligned}$$

A change of notation in Equation (9.4.5) of Dubins and Savage (1976) leads to

$$f(x) - f(y) \leq af^2(y)(y - x + a(x-y)^2) \quad \text{for all} \quad x \in \mathbb{R} \tag{3.1}$$

provided $y > 0$. This can be checked directly after noting that the right-hand side has negative derivative for $x < 0$. We now let Z be any random variable satisfying

$$EZ = \mu \geq \mu_0$$

and

$$E\,|Z - \mu|^k \leq 1.$$

Using (3.1) and Lemma 3.1 we see that for positive x

$$
\begin{aligned}
E[f(x+Z)^{k/2}] &\le E[|f(x+\mu)+af^2(x+\mu)(\mu-Z+a(Z-\mu)^2)|^{k/2}] \\
&\le f(x+\mu)^{k/2}\{(E[|1+af(x+\mu)(\mu-Z)|^{k/2}])^{2/k} \\
&\quad + (E[|a^2f(x+\mu)(Z-\mu)^2|^{k/2}])^{2/k}\}^{k/2} \\
&\le f(x+\mu)^{k/2}\{(E[|1+af(x+\mu)(\mu-Z)|^{k}])^{1/k}+a^2f(x+\mu)\}^{k/2} \\
&\le f(x+\mu)^{k/2}\left\{\left[1+\frac{k(k-1)}{2}a^2f^2(x+\mu)(1+af(x+\mu))^{k-2}\right]^{1/k}\right. \\
&\quad \left. + a^2f(x+\mu)\right\}^{k/2} \\
&\le f(x+\mu)^{k/2}\left\{1+\frac{k-1}{2}a^2f^2(x+\mu)(1+af(x+\mu))^{k-2}+a^2f(x+\mu)\right\}^{k/2} \\
&\le f(x+\mu)^{k/2}\left\{1+\frac{k-1}{2}a^2f(x+\mu)(1+a)^{k-2}+a^2f(x+\mu)\right\}^{k/2}. \\
&= \left\{f(x+\mu)-f'(x+\mu)\left[\left(\frac{k-1}{2}\right)a(1+a)^{k-2}+a\right]\right\}^{k/2} \\
&= \left\{f(x+\mu)-f'(x+\mu)\left[\left(\frac{k-1}{2}\right)[(1+a)^{k-1}-(1+a)^{k-2}]+a\right]\right\}^{k/2} \\
&\le \left\{f(x+\mu)-f'(x+\mu)\left[\left(\frac{k-1}{2}\right)((1+a)^{k-1}-1)+(1+a)^{k-1}-1\right]\right\}^{k/2} \\
&= \{f(x+\mu)-\mu_0 f'(x+\mu)\}^{k/2} \\
&\le f(x)^{k/2}.
\end{aligned}
$$

For negative x, this result follows trivially, and applying these results it easily follows that $\{f(S_j)^{k/2}\}_{j\ge 0}$ is a non-negative supermartingale. We now define the Markov time T as the first n for which $S_n \le 0$. By an optional stopping theorem for non-negative supermartingales (Karlin and Taylor, 1975, page 267) we have

$$f(s)^{k/2} \ge E[f(S_T)^{k/2}1_N(T)] = P[\inf_{j\ge 0} S_j \le 0].$$

The next lemma is needed to make use of Theorem 3.1.

LEMMA 3.2. *If* $EX=\mu\ge 0$ *and* $E|X-\mu|^k \le 1$ *then for* $\lambda > 0$

$$P(X\le -\lambda) \le \frac{(1+\lambda)^k - k\lambda}{(1+\lambda^2)^k}.$$

PROOF. By Lemma 3.1 we have

$$\frac{(1+\lambda)^k-k\lambda}{\lambda^k} \ge E[|X-\mu-1/\lambda|^k] \ge P(X-\mu\le -\lambda)(\lambda+1/\lambda)^k \ge P(X\le -\lambda)(\lambda+1/\lambda)^k.$$

Using this lemma, we have the following bound for the expected number of steps a process satisfying C3 takes from an interval of length μ_0.

LEMMA 3.3. *If $\{S_i\}$ satisfies* C3 *and $S_0 = s$ and if we define*

$$\mathscr{J} = [\lambda - \mu_0, \lambda]$$

and

$$K = 2 + \sum_{n=1}^{\infty} \frac{(M\gamma\sqrt{n+1} + n\mu_0)^k - kn\mu_0(M\gamma\sqrt{n+1})^{k-1}}{(M^2\gamma^2(n+1) + n^2\mu_0^2)^k} (M\gamma\sqrt{n+1})^k,$$

then

$$\begin{aligned} E[\#\{i : S_i \in \mathscr{J}\}] &\le K && \text{if } \lambda \ge s \\ &\le K(1 + a(s-\lambda))^{-k/2} && \text{if } \lambda \le s, \end{aligned}$$

where a is defined as in Theorem 3.2.

PROOF. We begin by showing that

$$E[\#\{i : S_i \le s + \mu_0\}] \le K. \tag{3.2}$$

If we define

$$\delta_n = s + \sum_{i=1}^{n} E[Z_i \mid \mathscr{F}_{i-1}],$$

then $\{S_i - \delta_i\}$ is a martingale satisfying the conditions of Theorem 3.1. Application of the theorem gives

$$E[|S_n - \delta_n|^k] \le (M\gamma\sqrt{n})^k.$$

By condition C3 we know that

$$\delta_n \ge s + n\mu_0,$$

and using Lemma 3.2, we have for $n \ge 1$

$$\begin{aligned} P(S_n \le s + \mu_0) &\le P(S_n - \delta_n \le -(n-1)\mu_0) \\ &\le \frac{(M\gamma\sqrt{n} + (n-1)\mu_0)^k - k(n-1)\mu_0(M\gamma\sqrt{n})^{k-1}}{(M^2\gamma^2 n + (n-1)^2\mu_0^2)^k} (M\gamma\sqrt{n})^k. \end{aligned}$$

Equation (3.2) now follows by monotone convergence and the theorem is true if $s \in \mathscr{J}$. If $s \notin \mathscr{J}$, we condition on the time the process first enters $\mathscr{J}$ and obtain

$$E[\#\{i : S_i \in \mathscr{J}\}] \le KP[\exists i \text{ s.t. } S_i \in \mathscr{J}).$$

Application of Theorem 3.2 now finishes the proof.

Let C_k be the set of non-negative measurable functions on $\mathbb{R}$ which are bounded on $[-1, \infty)$ and satisfy

$$\int_{-\infty}^{-1} \frac{dy}{|y|} \sup_{x \le y} \frac{h(x)}{|x|^{k-1}} < \infty.$$

THEOREM 3.3. *If $h \in C_k$ then there exists a measurable function $g > h$ such that if $\{S_i, \mathscr{F}_i\}$ satisfies* C3, *$\{g(S_i), \mathscr{F}_i\}$ is a supermartingale. If $h(x) = 0$ for sufficiently large x, then g can be chosen so that $\lim_{x\to\infty} g(x) = 0$.*

To facilitate the proof of this theorem we have

LEMMA 3.4. *Let $g(x, \alpha) = (x^-)^\alpha$ and let Z be an arbitrary random variable such that*

$$EZ = \mu \ge \mu_0 \tag{3.3}$$

and

$$E\,|\,Z-\mu\,|^k \le M^k. \tag{3.4}$$

Then for $1<\alpha<k$ *and* $k>2$,

$$Eg(s+Z)-g(s)\le c_1/(s+\mu_0)^{k-\alpha} \quad \textit{for} \quad s\ge 0 \tag{3.5}$$

$$\le c_2 \quad \textit{for} \quad c_3<s<0 \tag{3.6}$$

$$\le 0 \quad \textit{for} \quad s\le c_3, \tag{3.7}$$

where c_1, c_2 *and* c_3 *depend on* μ_0, M *and* k.

PROOF. By ordinary calculus

$$((x+y)^-)^\alpha y^{k-\alpha}\le |\,x\,|^k\frac{\alpha^\alpha}{k^k}(k-\alpha)^{k-\alpha}$$

for $x\in\mathbb{R}$ and $y\in\mathbb{R}^+$. Choosing $x=Z-\mu$ and $y=s+\mu$ and taking expectations gives (3.5). To prove (3.7), we note that

$$Eg(s+Z)\le Eg(s+\mu_0+Z-\mu)\le E\,|\,s+\mu_0+Z-\mu\,|^\alpha\le[E\,|\,s+\mu_0+Z-\mu\,|^k]^{\alpha/k}. \tag{3.8}$$

Using Lemma 3.1 and assuming $s+\mu_0\le M+(M^2/\mu_0)(k-1)2^{k-3}$ we have

$$Eg(s+Z)\le\left\{|\,s\,|^k-\mu_0k\,|\,s+\mu_0\,|^{k-1}+\frac{k(k-1)}{2}M^2 2^{k-2}(|\,s+\mu_0\,|^{k-2}+M^{k-2})\right\}^{\alpha/k}$$
$$\le|\,s\,|^\alpha$$

which proves (3.7). (3.6) holds because (3.8) implies

$$Eg(s+Z)\le|\,|\,s+\mu_0\,|+M\,|^\alpha\le|\,1+|\,c_3\,|+\mu_0+M\,|^k \quad \text{for} \quad c_3<s<0.$$

PROOF OF THEOREM 3.3. We assume without loss of generality that $h(x)/|\,x\,|^{k-1}$ is nondecreasing on $(-\infty,-1)$ and that $h(x)=0$ for $x\ge-1$. For $t<1$

$$\frac{h(t)}{|\,t\,|^{k-1}}\le 32\int_0^\infty dx\int_{e^{1/x}}^\infty dyh(-y)\,|\,ty\,|^{-x}/y^k \tag{3.9}$$

because

$$32\int_0^\infty dx\int_{e^{1/x}}^\infty dy\,1_{\mathbb{R}^-}(y+t)\,|\,ty\,|^{-x}/y\ge 1.$$

Equation (3.9) implies that for $t<-1$

$$h(t)<ct^-+A(t) \tag{3.10}$$

where

$$c=32\int_0^\infty dx\int_{e^{1/x}}^\infty dyh(-y)y^{-x-k}$$

and

$$A(t)=32\int_0^{k-2}dx(t^-)^{k-1-x}\int_{e^{1/x}}^\infty dyy^{-x-k}h(-y).$$

Since A and c are non-negative, (3.10) holds for all t. Let Z be any random variable satisfying (3.3) and (3.4). Using Lemma 3.4, we see that

$$E[c(t+Z^-) + A(t+Z) - ct^- - A(t)] \le B(t)$$

where

$$\begin{aligned} B(t) &= 0 && \text{for} \quad t < c_3 \\ &= 2cc_3 && \text{for} \quad c_3 \le t \le 0 \\ &= c_1\left\{32 \int_0^{k-2} dx(t+\mu_0)^{-x-1} \int_{e^{1/x}}^{\infty} dy y^{-x-k} h(-y) + \frac{c}{(t+\mu_0)^{k-1}}\right\} && \text{for} \quad t > 0. \end{aligned}$$

Observing that

$$\begin{aligned} \int_0^\infty dt \int_0^{k-2} dx(t+\mu_0)^{-x-1} \int_{e^{1/x}}^\infty dy y^{-x-k} h(-y) &\le (1+\mu_0^{2-k}) \int_0^\infty \frac{dx}{x} \int_{e^{1/x}}^\infty dy y^{-x-k} h(-y) \\ &= (1+\mu_0^{2-k}) \int_1^\infty x^{-k}\, h(-x)\, dx \int_1^\infty \frac{dy}{y} e^{-y} < \infty \end{aligned}$$

we see that B is directly Riemann integrable. We now let f be defined as in Theorem 3.2. Choosing $\delta = \mu_0/3$ and using Lemma 3.2 we see that there exists $\varepsilon > 0$ such that for any random variable Z satisfying $EZ = \mu \ge \mu_0$ and $E\,|\,Z - \mu\,|^k \le M^k$ we have

$$E[f(s+Z) - f(s)] < -\varepsilon \quad \text{for} \quad -\delta \le s \le 0. \tag{3.11}$$

Since $B(t)$ is directly Riemann integrable, there exist constants a_i such that

$$B(t) < a_i \quad \text{for} \quad i\delta \le t \le (i+1)\delta \tag{3.12}$$

and

$$\textstyle\sum_{i\in\mathbb{Z}} a_i < \infty.$$

Let

$$C(t) = \textstyle\sum_{i\in\mathbb{Z}} \dfrac{a_i}{\varepsilon} f(t - (i+1)\delta).$$

C is positive and (3.11), (3.12) imply that

$$E(C(t+Z) - C(t)) < -B(t)$$

for all Z such that $EZ = \mu \ge \mu_0$, $E\,|\,Z-\mu\,|^k < M^k$. Hence $\{c(S_i^-) + A(S_i) + C(S_i), \mathscr{F}_i\}$ is a positive supermartingale whenever $\{S_i, \mathscr{F}_i\}$ satisfies C3 and we are done.

After so many preliminaries we can now establish our main result.

THEOREM 3.4. *Let $\{S_i\}$ be a* GRW *satisfying* C2 *and let h be a non-negative measurable function such that $1_{\mathscr{S}}(x)h(x) \in C_k$ and $1_{\mathscr{C}}(x)h(x)/(1+(x^-)^{k/2})$ is directly Riemann integrable. Then $R(s)$ is finite for all $s \in \mathbb{R}$. If $1_{\mathscr{C}}(x)$ has a limit as x approaches $+\infty$ and $-\infty$, then $R(s)$ is the only solution of the renewal equation* (1.1) *which is bounded on finite intervals and satisfies*

$$\lim_{s\to\pm\infty} 1_{\mathscr{C}}(s)R(s)/(1+(s^-)^{k/2}) = 0. \tag{3.13}$$

PROOF. This proof is similar to the proof of Theorem 2.1. To simplify notation we will assume without loss of generality that $\mu_0 = 1$. We define

$$\mathscr{I}_k = [k-1, k), \qquad n_k = \#\{j : S_j \in \mathscr{I}_k\},$$

and

$$m_k = \sup_{x\in \mathscr{I}_k} 1_{\mathscr{C}}(x)h(x)/(1 + (x^-)^{k/2}).$$

It follows that

$$= \sup_{x\in \mathscr{I}_k} 1_{\mathscr{C}}(x)h(x) \le m_k(1 + 1_{\mathbb{R}^-}(k-1)\,|\,k-1\,|^{k/2}).$$

By the integrability condition, $\sum_{i\in\mathbb{Z}} m_i < \infty$. We now observe that

$$\sum_{i=0}^{N} h(S_i) \le h(S_N)1_N(N) + \sum_{k\in\mathbb{Z}} n_k m_k(1 + 1_{\mathbb{R}^-}(k-1)\,|\,k-1\,|^{k/2}). \tag{3.14}$$

Now by Lemma 3.3

$$E_s[n_k] \le K(1_{\mathbb{R}^+}(k-s) + 1_{\mathbb{R}^-}(k-s)(1 + a(s-k))^{-k/2}).$$

From this it follows that

$$E_s[n_k](1 + 1_{\mathbb{R}^-}(k-1)\,|\,k-1\,|^{k/2})/(1 + (s^-)^{k/2})$$

is a bounded function of k and s which approaches zero as $s \to \pm\infty$. Hence

$$E_s[\textstyle\sum_{k\in\mathbb{Z}} n_k m_k(1 + 1_{\mathbb{R}^-}(k-1)\,|\,k-1\,|^{k/2})] < \infty$$

and

$$\lim_{s\to\pm\infty} \frac{E_s[\sum_{k\in\mathbb{Z}} n_k m_k(1 + 1_{\mathbb{R}^-}(k-1)\,|\,k-1\,|^{k/2})]}{(1 + (s^-)^{k/2})} = 0.$$

Using (3.14), we see that $R(s)$ will be finite provided $E_s[h(S_N)1_N(N)] < \infty$, and $R(s)$ will satisfy (3.13) provided

$$\lim_{s\to\pm\infty} 1_{\mathscr{C}}(s)E_s[h(S_N)1_N(\mathrm{N})]/(1 + 1_{\mathbb{R}^-}(s)\,|\,s\,|^{k/2}) = 0. \tag{3.15}$$

Using Theorem 3.3 we choose a function $g(s) > h(s)1_{\mathscr{S}}(s)$ such that $\{g(S_i)\}$ is a non-negative supermartingale. By optional stopping

$$E_s[h(S_N)1_N(N)] \le g(s) < \infty.$$

If $\lim_{s\to+\infty} 1_{\mathscr{C}}(s) = 1$, then g can be chosen so that $\lim_{s\to+\infty} g(s) = 0$. Hence (3.15) holds as $s \to +\infty$. (3.15) holds as $s \to -\infty$ because $1_{\mathscr{S}}(s)h(s)$ is bounded if $\lim_{s\to-\infty} 1_{\mathscr{S}}(s) = 1$.

To complete our proof we need to show uniqueness. We can assume without loss of generality that $h = 0$. Let G be an arbitrary function which is bounded on finite intervals and satisfies

$$G(s) = 1_{\mathscr{C}}(s)E_s[G(S_1)], \tag{3.16}$$

and

$$\lim_{s\to\pm\infty} 1_{\mathscr{C}}(s)G(s)/(1 + (s^-)^{k/2}) = 0. \tag{3.17}$$

We must show that $G(s) = 0$. Iterating (3.16) gives

$$|\,G(s)\,| \le E_s[\,|\,G(S_n)\,|\,]. \tag{3.18}$$

Since h is zero and G is bounded on finite intervals, (3.17) implies that there exists a constant K such that

$$|\,G(s)\,| \le K(1 + (s^-)^{k/2}). \tag{3.19}$$

Equations (3.18) and (3.19) now give for $x > 0$

$$\begin{aligned} |\,G(s)\,| &\le E_s[K(1 + (S_n^-)^{k/2})1_{[-\lambda,\lambda]}(S_n)] + \sup_{x>\lambda}|\,G(x)\,| \\ &\quad + \sup_{x<-\lambda}(|\,G(x)\,|\,|\,x\,|^{k/2})E_s[\,|\,S_n\,|^{k/2}1_{(-\infty,\lambda)}(S_n)] \\ &\le K(1 + \lambda^{k/2})P_s(S_n \le \lambda) + \sup_{x>\lambda}|\,G(x)\,| + \sup_{x<-\lambda}(|\,G(x)\,|\,|\,x\,|^{-k/2}) \\ &\quad \cdot \{E_s[\,|\,S_n\,|^k 1_{(-\infty,-\lambda)}(S_n)]P_s(S_n < -\lambda)\}^{1/2}. \end{aligned} \tag{3.20}$$

We define δ_n as

$$\delta_n = s + \sum_{i=1}^{n} E_s[Z_i \mid S_{i-1}].$$

By Condition C2, $\delta_n \geq s + n$ and Theorem 3.1 gives

$$E_s[\,|S_n - \delta_n|^k] \leq (M\gamma\sqrt{n})^k. \tag{3.21}$$

We now take n large enough that $\delta_n > \sqrt{n}$. Then for $S_n < -\lambda$ we will have $|S_n| < |S_n - \delta_n|$ implying

$$E_s[\,|S_n|^k 1_{(-\infty,-\lambda)}(S_n)] \leq (M\gamma\sqrt{n})^k.$$

We now let $\lambda = \sqrt{n}$ in (3.20) and obtain

$$|G(s)| \leq K(1 + n^{k/4})P_s(S_n - \delta_n \leq \sqrt{n} - n - s) + \sup_{x>\sqrt{n}} |G(x)|$$
$$+ \sup_{x<-\sqrt{n}} |G(x)|\,|x|^{-k/2}\{(M\gamma\sqrt{n})^k P_s(S_n - \delta_n \leq -\sqrt{n} - n - s)\}^{1/2}.$$

If we let $n \to \infty$ in this expression, we can use Lemma 3.2 and (3.21) to conclude that $G(s) = 0$ and our proof is complete.

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