PROBABILISTIC SOLUTION OF THE DIRICHLET PROBLEM FOR BIHARMONIC FUNCTIONS IN DISCRETE SPACE[1]

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Considering difference equations in discrete space instead of differential equations in Euclidean space, we investigate a probabilistic formula for the solution of the Dirichlet problem for biharmonic functions. This formula involves the expectation of a weighted sum of the pay-offs at the successive times at which the Markov chain is in the complement of the domain. To make the infinite sum converge, we use Borel's summability method. This is interpreted probabilistically by imbedding the Markov chain into a continuous time, discrete space Markov process.

1. Introduction. The probabilistic formula for the solution of the Dirichlet problem for harmonic functions is well known and has been extensively investigated (see e.g. Dynkin, 1965). A probabilistic formula for the function f which is biharmonic in a given domain and which is specified by the values of f and Δf on the boundary was discovered by Has'minski (1960) and independently by Helms (1967) (see also Athreya and Kurtz, 1973). A more difficult problem is to specify a biharmonic function f in terms of the values of f and its normal derivative on the boundary; that is, Dirichlet boundary conditions. Considering difference operators in discrete spaces instead of differential operators in Euclidean spaces, we investigate a probabilistic formula for the solution of the Dirichlet problem for biharmonic functions.

For the sake of comparison we begin by discussing the Dirichlet problem for harmonic functions. Let X_n, $n = 0, 1, \cdots$, be a Markov chain on a discrete state space E and let τ be the first time that X_n leaves a set Γ in E. Denote by A the generator of X_n; i.e., $Af(x) = \sum_{y \in E} p(x, y) f(y) m(y) - f(x)$ where $p(x, y)$ is the transition function for X_n and m is a positive measure on E (precise definitions are given in Section 3).

THEOREM 1. (Dynkin's formula). *Suppose that $P_x \tau < \infty$ for all $x \in \Gamma$. (P_x denotes expectation given that $X_0 = x$.) Then for every bounded real valued function f defined on E,*

$$(1.1) \qquad f(x) = P_x\{f(X_\tau) - \sum_{n=0}^{\tau-1} Af(X_n)\}, \quad x \in E.$$

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PROOF.   Formally the proof goes as follows:

$$f(x) = P_x\{f(X_\tau) - \sum_{n=0}^{\infty} 1_{n<\tau}(f(X_{n+1}) - f(X_n))\} \quad \text{(telescoping series)}$$

$$= P_x f(X_\tau) - \sum_{n=0}^{\infty} P_x 1_{n<\tau}(f(X_{n+1}) - f(X_n)) \quad \text{(Fubini's theorem)}$$

$$= P_x f(X_\tau) - \sum_{n=0}^{\infty} P_x 1_{n<\tau} Af(X_n) \quad \text{(Markov property)}$$

$$= P_x\{f(X_\tau) - \sum_{n=0}^{\infty} 1_{n<\tau} Af(X_n)\} \quad \text{(Fubini's theorem)}$$

$$= P_x\{f(X_\tau) - \sum_{n=0}^{\tau-1} Af(X_n)\}.$$

The proof is made rigorous by justifying the use of Fubini's theorem. For this it is sufficient to assume that f is bounded and $P_x\tau < \infty$. $\square$

We see from the first step of the proof that formula (1.1) is really a *summation by parts* formula.

Suppose now that $P_x\tau < \infty$ for all $x \in \Gamma$ and that f is a solution of the *Dirichlet problem for harmonic functions*:

$$(1.2) \qquad\qquad\qquad Af(x) = 0, \quad x \in \Gamma,$$

$$(1.3) \qquad\qquad\qquad f(x) = \varphi(x), \quad x \in \partial\Gamma,$$

where $\partial\Gamma = \{y \notin \Gamma : p(x, y) > 0$ for some $x \in \Gamma\}$ (that is, all points in the complement of Γ which are within one step of Γ) and φ is a given bounded function defined on $\partial\Gamma$. By formula (1.1), we see that

$$(1.4) \qquad\qquad\qquad f(x) = P_x\varphi(X_\tau), \quad x \in \Gamma \cup \partial\Gamma.$$

This proves uniqueness for the Dirichlet problem. Existence is proved by verifying that the function defined by the right hand side of (1.4) satisfies (1.2) and (1.3). We then have

THEOREM 2.   *Suppose that $P_x\tau < \infty$ for every $x \in \Gamma$ and that φ is a bounded function on $\partial\Gamma$. Then there exists one and only one function f which satisfies (1.2), (1.3) and this function is given by (1.4).*

Now consider the *Dirichlet problem for biharmonic functions*:

$$(1.5) \qquad\qquad\qquad A^2f(x) = 0, \quad x \in \Gamma,$$

$$(1.6) \qquad\qquad\qquad f(x) = \varphi(x), \quad x \in \partial^2\Gamma,$$

where $\partial^2\Gamma = \partial\Gamma \cup \partial(\Gamma \cup \partial\Gamma)$; i.e., all points in the complement of Γ which are within two steps of Γ. In the discrete case, specifying values on the thickened boundary $\partial^2\Gamma$ plays the role of specifying values and the normal derivative on the boundary. From the above discussion, we see that we need a formula like

$$(1.7) \qquad f(x) = P_x \sum_{n=0}^{\infty} \{U_n f(X_n) - V_n A^2f(X_n)\}, \quad x \in E,$$

where

$$(1.8) \qquad\qquad\qquad U_n = 0, \quad X_n \notin \partial^2\Gamma,$$

$$(1.9) \qquad\qquad V_n = 0, \quad X_n \notin \Gamma$$

(if we replace A^2 by A and put $U_n = 1_{\tau=n}$, $V_n = 1_{n<\tau}$, formula (1.7) becomes formula (1.1)). Then, as before, if f is a solution of the Dirichlet problem (1.5), (1.6), we see from (1.7) that

$$(1.10) \qquad\qquad f(x) = P_x \sum_{n=0}^{\infty} U_n \varphi(X_n).$$

This would prove uniqueness. Existence would be proved by verifying that the right hand side defines a solution.

Proceeding formally, we find in Section 2 explicit expressions for U_n and V_n in terms of the successive times τ_j in which X_n is in the complement of Γ:

$$(1.11) \qquad\qquad \tau_j = \inf\{n > \tau_{j-1} : X_n \notin \Gamma\}, \quad \tau_{-1} = -1.$$

After some simplification it turns out that formulas (1.7) and (1.10) can be written as

$$(1.12) \quad f(x) = P_x \sum_{j=0}^{\infty} (-1)^j M_{j-1}\{(\tau_j - \tau_{j-1})f(X_{\tau_j}) + \sum_{n=\tau_{j-1}}^{\tau_j-1} (n - \tau_{j-1})A^2 f(X_n)\},$$

$$(1.13) \qquad\qquad f(x) = P_x\{\varphi(X_{\tau_0}) - \sum_{j=0}^{\infty} (-1)^j M_j \tilde{A}\varphi(X_{\tau_j})\},$$

where

$$(1.14) \qquad\qquad M_j = \prod_{i=0}^{j} (\tau_i - \tau_{i-1} - 1),$$

and $\tilde{A}$ is the generator of the time changed process $\tilde{X}_j = X_{\tau_j}$, $j \geq 0$. (The process $\tilde{X}_j$ is the trace of X_n on Γ^c.) This means that $\tilde{A}h(x) = P_x h(\tilde{X}_1) - h(x)$.

It turns out that the right hand sides of formulas (1.12) and (1.13) are generally not absolutely convergent. In fact, in Section 2, we give an example involving 1-dimensional simple random walk which shows that formula (1.13) is absolutely convergent for every choice of the function φ if and only if the domain Γ consists of one point! Roughly the same holds true for simple random walk in higher dimensions.

In the case of a *symmetric* Markov chain (i.e., the operator A is "in divergence form"), these difficulties can be circumvented by using Borel's summability method (see e.g. Hardy (1949) page 80). This has a simple probabilistic interpretation. Indeed, let η_t, $t \geq 0$, be a Poisson process which is independent of the Markov chain X_n. We show in Theorems 3 and 4 that, subject to a few mild assumptions (see conditions 3.A, B), formulas (1.12) and (1.13) can be salvaged by replacing the sum on all $j \geq 0$ by a sum from $j = 0$ to $j = \eta_t$ and then passing to the limit as t tends to infinity (on the outside of the expectation).

Now that the Dirichlet problem for the square of the discrete Laplacian can be completely studied using simple random walk we can hope that, by refining the space and making a passage to the limit, the Dirichlet problem for the Laplacian squared (in Euclidean space) may be studied using Brownian motion.

By using exponential holding times, we can stretch X_j into a continuous time process Y_t such that X_j is the imbedded Markov chain. That is, $X_j = Y_{\sigma_j}$ where

the σ_j are the successive jump times of Y_t. Formula (1.13) can now be written as

$$f(x) = \lim_{t\to\infty} P_x\{\varphi(X_{\tau_0}) - \sum_{j:\sigma_j\le t} (-1)^j M_j \tilde{A}\varphi(X_{\tau_j})\}.$$

This formula may turn out to be useful for finding a formula for the Euclidean case.

The problem of finding probabilistic solutions of the Dirichlet problem for biharmonic functions actually arose out of the author's attempts to investigate the potential theory of certain two-parameter Markov random fields. In Section 6, we explain in what sense these fields are Markov and we explain the analogous boundary value problems. In this general setting, however, almost nothing has been proved.

2. Formal solution. We start by investigating formula (1.7). Assume for now that all sums and integrals converge absolutely so that we may apply Fubini's theorem. Also, assume that V_n is measurable with respect to the σ-algebra $\mathscr{F}_n$ generated by X_m, $m \le n$, (we will verify these assumptions later). As in the proof of Theorem 1, we interchange sum and integral, apply the Markov property, and interchange back to get

$$\begin{aligned}
(2.1) \qquad P_x \sum_{n=0}^{\infty} V_n A^2 f(X_n) &= P_x \sum_{n=0}^{\infty} V_n\{f(X_{n+2}) - 2f(X_{n+1}) + f(X_n)\} \\
&= P_x \sum_{n=0}^{\infty} (V_n - 2V_{n-1} + V_{n-2})f(X_n) - f(x),
\end{aligned}$$

where

$$(2.2) \qquad\qquad V_{-1} = 0 \quad\text{and}\quad V_{-2} = 1.$$

From (2.1) we see that for (1.7) to formally hold it is sufficient that

$$(2.3) \qquad\qquad V_n - 2V_{n-1} + V_{n-2} = U_n, \quad n \ge 0.$$

If we impose the requirement that

$$(2.4) \qquad\qquad \begin{aligned} U_n &= 0 \quad\text{if}\quad X_n \in \Gamma, \\ V_n &= 0, \quad\text{if}\quad X_n \notin \Gamma \end{aligned}$$

(which is a priori weaker than (1.8), (1.9)) then it is easy to see that the system (2.2), (2.3), (2.4) has one and only one solution and that this solution satisfies our measurability requirement: $V_n \in \mathscr{F}_n$.

LEMMA 1. *The solution of (2.2), (2.3), (2.4) is*

$$(2.5) \qquad U_n = \sum_{j=0}^{\infty} (-1)^j 1_{\tau_j=n} M_{j-1}(n - \tau_{j-1}), \quad n \ge 0,$$

$$(2.6) \qquad V_n = -\sum_{j=0}^{\infty} (-1)^j 1_{\tau_{j-1}<n<\tau_j} M_{j-1}(n - \tau_{j-1}), \quad n \ge 0,$$

where τ_j and M_j are defined by (1.11) and (1.14), respectively.

From (1.11) we see that $\tau_j \ge j$ and so the infinite sums in (2.5) and (2.6) actually terminate after $j = n$.

PROOF. It is obvious that the functions U_n and V_n defined by (2.5) and (2.6)

satisfy (2.4). To check that (2.3) is satisfied let us put $W_n = V_n - V_{n-1}$. From the definitions of U_n and V_n we see that

(2.7) $$W_n = U_n - \sum_{j=0}^{\infty} (-1)^j 1_{\tau_{j-1} < n \leq \tau_j} M_{j-1},$$

and so

(2.8) $$W_n - W_{n-1} = U_n - U_{n-1} + \sum_{j=0}^{\infty} (-1)^j (1_{\tau_j = n-1} - 1_{\tau_{j-1} = n-1}) M_{j-1}.$$

Now note that

$$M_{j-1}(\tau_j - \tau_{j-1}) = M_j + M_{j-1}$$

(this is the basic identity satisfied by M_j which makes everything work) and so we can rewrite (2.5) as

(2.9) $$U_n = \sum_{j=1}^{\infty} (-1)^j (1_{\tau_j = n} - 1_{\tau_{j-1} = n}) M_{j-1}.$$

Substituting (2.9) into (2.8) with $n - 1$ in place of n we get (2.3). □

LEMMA 2. *The functions U_n defined by (2.5) satisfy (1.8).*

PROOF. Suppose that $X_0 \in \Gamma$, $X_n \notin \Gamma \cup \partial^2 \Gamma$ and $n = \tau_j$. Then by the definition of $\partial^2 \Gamma$, j is greater than 1 and $\tau_{j-1} - \tau_{j-2} = 1$. Hence $M_{j-1} = 0$ and so $U_n = 0$. □

It is easy to see that formulas (1.12) and (1.13) are just rearrangements of formulas (1.7) and (1.10), respectively, with U_n and V_n defined by (2.5) and (2.6).

We now give an example which shows that the expectation on the right hand side of formula (1.12) does not exist (see also the remark at the end of Section 3). Let X_n be simple random walk on the integers $\mathbb{Z}$ (i.e., $Af(x) = \frac{1}{2}f(x + 1) - f(x) + \frac{1}{2}f(x - 1)$), Γ be the interval $\{1, 2, \cdots, a - 1\}$ and

$$f(x) = \begin{cases} \dfrac{x(x - a)}{a + 1}, & x \in \{0, 1, \cdots, a\}, \\ 1, & x \notin \{0, 1, \cdots, a\}. \end{cases}$$

Note that $\partial \Gamma = \{0, a\}$ and $\partial^2 \Gamma = \{-1, 0, a, a + 1\}$. It is easy to check that $Af(x) = 1/(a + 1)$ for $x \in \Gamma \cup \partial \Gamma$ and so

$$A^2 f(x) = 0, \quad x \in \Gamma.$$

Put

$$\sigma = \inf\{j : \tau_j - \tau_{j-1} = 1\}.$$

For $j < \sigma$, $f(X_{\tau_j}) = 0$ and, for $j = \sigma$, $f(X_{\tau_j}) = 1$. On the other hand, for $j > \sigma$, $M_{j-1} = 0$. Hence the right hand side of formula (1.12) becomes

$$\text{RHS}(1.12) = P_x(-1)^\sigma M_{\sigma-1}.$$

However, $(-1)^\sigma M_{\sigma-1}$ is not integrable with respect to the measure P_x. Indeed, let $((-1)^\sigma M_{\sigma-1})^+$ and $((-1)^\sigma M_{\sigma-1})^-$ denote the positive and negative parts of

$(-1)^\sigma M_{\sigma-1}$. Then

$$((-1)^\sigma M_{\sigma-1})^+ = \sum_{j\geq 0, \text{even}} \prod_{i=0}^{j-1} (\tau_i - \tau_{i-1} - 1) 1_{\sigma=j}$$

$$= \sum_{j\geq 0, \text{even}} \{\prod_{i=0}^{j-1} (\tau_i - \tau_{i-1} - 1) 1_{\tau_i-\tau_{i-1}>1}\} 1_{\tau_j-\tau_{j-1}=1}.$$

Applying the strong Markov property and using the fact that 0 and a play symmetric roles, we can write

$$(2.10) \qquad P_x((-1)^\sigma M_{\sigma-1})^+ = P_x\tau_0 \sum_{j\geq 2, \text{even}} r^{j-1} P_0\{\tau_1 = 1\}$$

where

$$r = P_0(\tau_1 - 1).$$

Of course $P_0\{\tau_1 = 1\} = \frac{1}{2}$. Since the function $g(x) = P_x\tau_0$ is the unique solution of

$$Ag(x) = -1, \quad x \in \Gamma,$$

$$g(x) = 0, \qquad x \in \partial\Gamma,$$

it is easy to see that

$$P_x\tau_0 = x(a - x).$$

Conditioning on the first step, we see that

$$r = \frac{1}{2} P_1\tau_0 = \frac{1}{2}(a - 1).$$

Hence the geometric series in (2.10) converges if and only if $a = 2$ which means that Γ contains only one point. In the same way,

$$P_x((-1)^\sigma M_{\sigma-1})^- = \frac{1}{2} P_x\tau_0 \sum_{j\geq 1, \text{odd}} r^{j-1}$$

which also diverges except when Γ consists of one point.

Note, however, that if we write $\sum_{j=0}^{\infty} r^j = 1/(1 - r)$ we see that the right hand side of (1.12) is formally equal to

$$\text{RHS}(1.12) = -\frac{1}{2} P_x\tau_0 \sum_{j=1}^{\infty} (-r)^{j-1} = -\frac{1}{2} P_x\tau_0 \frac{1}{1+r} = \frac{x(x-a)}{a+1} = f(x).$$

3. Symmetric Markov chains and the Q operator.

Let m be a strictly positive measure on the discrete state space E; that is, $m(x) > 0$ for all $x \in E$. A function $p(x, y)$, $x, y \in E$, is a *symmetric transition function* if

$$p(x, y) \geq 0, \quad \sum_y p(x, y)m(y) = 1, \quad p(x, y) = p(y, x).$$

Corresponding to every symmetric transition function $p(x, y)$, there is a *symmetric Markov chain* $X = (X_n, \mathscr{F}_n, \theta_n, P_x)$ defined on a probability space $(\Omega, \mathscr{F})$. The connection between X and $p(x, y)$ is expressed by the formula

$$P_x\{X_1 = y\} = p(x, y)m(y).$$

Let us remind the reader here that θ_n is the *shift operator*; that is, θ_n maps Ω into

Ω in such a way that

$$(3.1) \qquad X_m(\theta_n\omega) = X_{m+n}(\omega).$$

The *one step shift operator* θ_1 will be denoted simply by θ. The action of the shift operator on a random variable $Z(\omega)$ is defined by the formula

$$\theta_n Z(\omega) = Z(\theta_n\omega).$$

The fact that X is *Markov* means that for every stopping time τ,

$$(3.2) \qquad P_x Y\theta_\tau Z = P_x Y P_{X_\tau} Z$$

for all $\mathscr{F}_\tau$-measurable P_x-integrable random variables Y and all P_x-integrable random variables Z for which $P_x Z$ is a bounded function of x.

The *one step transition operator* P acts on bounded functions according to the formula

$$Pf(x) = P_x f(X_1).$$

The *generator* A is defined by the formula

$$Af(x) = (P - I)f(x),$$

where I is the identity operator. If $f(x) = P_x Z$, the Markov property implies that

$$(3.3) \qquad Af(x) = P_x\{\theta Z - Z\},$$

$$(3.4) \qquad A^2 f(x) = P_x\{\theta_2 Z - 2\theta_1 Z + Z\}.$$

For any set Γ in E, the boundary $\partial\Gamma$ was defined in Section 1 as the set of all points y not in Γ for which $p(x, y) > 0$ for some x in Γ. For the rest of this paper we consider a fixed set Γ which satisfies the following two properties:

3A. $\Gamma \cup \partial\Gamma$ is finite.
3B. $P_x \tau_0^2 < \infty$ for every $x \in \Gamma$.

It follows from the Markov property and condition 3B that each τ_k is finite a.s. P_x for every x.

The shift operator acts on the stopping times τ_j according to the following simple formulas

$$(3.5) \qquad \theta\tau_j = \begin{cases} \tau_j - 1, & X_0 \in \Gamma, \\ \tau_{j+1} - 1, & X_0 \notin \Gamma, \end{cases}$$

$$(3.6) \qquad \theta_{\tau_k}\tau_j = \tau_{j+k} - \tau_k.$$

We define an operator Q which acts on functions defined on $\partial\Gamma$ by the formula

$$(3.7) \qquad Qf(x) = P_x(\tau_1 - 1)f(X_{\tau_1}), \quad x \in \partial\Gamma,$$

(the right hand side makes sense since $\partial\Gamma$ is finite and, by 3A, $P_x(\tau_1 - 1)$ $= P_x P_{X_1}\tau_0 \le \sup_\Gamma P_y\tau_0$).

The operator Q is closely related to the random variables M_j. However, the $\tau_0 - \tau_{-1} - 1 = \tau_0$ factor in the definition of M_j plays a somewhat different role

than the other factors so it seems reasonable to introduce the random variables

$$(3.8) \qquad N_j = \prod_{i=1}^{j} (\tau_i - \tau_{i-1} - 1), \quad j \geq 0.$$

We have then that

$$(3.9) \qquad M_j = \tau_0 N_j, \quad j \geq 0,$$

$$P_x N_j f(X_{\tau_j}) = Q^j f(x), \quad j \geq 0, \quad x \in \Gamma^c.$$

PROPOSITION. *Q is a positive semi-definite self-adjoint operator on real* $L^2(\partial\Gamma, m)$.

PROOF. Put

$$f_e(x) = \begin{cases} f(x), & x \in \partial\Gamma, \\ 0 & x \notin \partial\Gamma. \end{cases}$$

Conditioning on the possible values of τ_1, we can write

$$Qf(x) = \sum_{n=1}^{\infty} nP_x\{\prod_{k=1}^{n} 1_\Gamma(X_k)\}f_e(X_{n+1}) = \sum_{n=1}^{\infty} nPP_\Gamma^{n-1}Pf_e(x)$$

where P_Γ is the one step transition operator for the process obtained from X_n by killing it at the first exit time from Γ:

$$(3.10) \qquad P_\Gamma f(x) = \sum_y 1_\Gamma(x)p(x, y)1_\Gamma(y)f(y)m(y).$$

Condition 3B is sufficient to guarantee that the operator

$$(3.11) \qquad G_\Gamma = \sum_{n=0}^{\infty} P_\Gamma^n$$

can be defined on all of $L^2(E, m)$. It is easy to check that

$$G_\Gamma^2 = \sum_{n=1}^{\infty} nP_\Gamma^{n-1},$$

and so

$$Qf = PG_\Gamma^2 Pf_e.$$

Since P and G_Γ are self-adjoint on $L^2(E, m)$, it follows that Q is self-adjoint on $L^2(\partial\Gamma, m)$ and

$$(f, Qf)_{\partial\Gamma} = (f_e, PG_\Gamma^2 Pf_e)_E = (G_\Gamma Pf_e, G_\Gamma Pf_e)_E \geq 0. \quad \square$$

As a consequence of the proposition, we can diagonalize the operator Q:

$$(3.12) \qquad Q = \sum_{\lambda \in \Lambda} \lambda \Pi_\lambda$$

where Λ is the (necessarily finite) spectrum of Q and Π_λ is the projection operator onto the eigenspace corresponding to the eigenvalue λ. Since Q is nonnegative, all the eigenvalues $\lambda \in \Lambda$ are nonnegative. Having the representation (3.12) is important because we can evaluate powers of the operator Q by

$$(3.13) \qquad Q^j = \sum_\Lambda \lambda^j \Pi_\lambda.$$

REMARK. In the case where X_n is simple random walk on $\mathbb{Z}^d$, the largest eigenvalue $\lambda_{\max}$ is roughly proportional to the diameter of the domain Γ.

4. Dynkin's formula for A^2.

We are now prepared to formulate and prove the "salvaged" version of formula (1.12) in the case where X_n is a symmetric Markov chain and conditions 3A and 3B are satisifed.

THEOREM 3. *Let η_t, $t \geq 0$, be a Poisson process which is independent of the Markov chain X_n. Then for every bounded function f and every x in E,*

$$(4.1) \quad f(x) = \lim_{t \to \infty} P_x \sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \{ (\tau_j - \tau_{j-1}) f(X_{\tau_j})$$
$$+ \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) A^2 f(X_m) \}.$$

PROOF. Put

$$(4.2) \quad f_n(x) = P_x \sum_{j=0}^{n} (-1)^j M_{j-1} \{ (\tau_j - \tau_{j-1}) f(X_{\tau_j}) + \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) A^2 f(X_m) \}.$$

Formula (3.9) and conditions 3A and 3B show that the right hand side makes sense.

Applying the Markov property and rearranging terms, we get

$$P_x M_{j-1} \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) A^2 f(X_m)$$
$$= P_x M_{j-1} \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) \{ f(X_{m+2}) - 2f(X_{m+1}) + f(X_m) \}$$
$$= P_x M_{j-1} \{ (\tau_j - \tau_{j-1} - 1) f(X_{\tau_j+1}) - (\tau_j - \tau_{j-1}) f(X_{\tau_j}) + f(X_{\tau_{j-1}+1}) \}.$$

Substituting this into (4.2) and collapsing the resultant telescoping sum, we see that

$$(4.3) \quad f_n(x) = P_x (-1)^n \tau_0 N_n f(X_{\tau_n+1}) + f(x).$$

By (3.9),

$$P_x \tau_0 N_n f(X_{\tau_n+1}) = P_x \tau_0 Q^n P f(X_{\tau_0})$$

and so by (3.13), we have

$$(4.4) \quad P_x \tau_0 N_n f(X_{\tau_n+1}) = \sum_{\lambda \in \Lambda} \lambda^n P_x \tau_0 \Pi_\lambda P f(X_{\tau_0}).$$

Substituting (4.4) into (4.3) we see that

$$\sum_{n=0}^{\infty} e^{-t} (t^n / n!) f_n(x) = f(x) + \sum_\lambda e^{-t(\lambda+1)} P_x \tau_0 \Pi_\lambda P f(X_{\tau_0}).$$

Since the eigenvalues are nonnegative, each term in the sum over λ vanishes as t goes to infinity. $\square$

5. The Dirichlet problem.

In the introduction we discussed the Dirichlet problem for biharmonic functions, i.e. functions which satisfy $A^2 f = 0$ in Γ. However, it is not much more difficult to study solutions of the equation $A^2 f = \psi$ where ψ is some function defined on Γ. We say that a function f is a *solution of*

the (non-homogeneous) Dirichlet problem for A^2 if

(5.1) $A^2 f = \psi, \quad \text{in} \quad \Gamma,$

(5.2) $f = \varphi, \quad \text{on} \quad \Gamma^c.$

THEOREM 4.   *There is one and only one function f which solves the Dirichlet problem (5.1), (5.2). It is given by the formula*

(5.3)
$$f(x) = \lim_{t \to \infty} P_x \sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \{ (\tau_j - \tau_{j-1}) \varphi(X_{\tau_j})$$
$$+ \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) \psi(x_m) \}.$$

PROOF.   Uniqueness follows immediately from Theorem 3. Put

(5.4) $g_t(x) = P_x \sum_{j=0}^{\eta_t} (-1)^j M_{j-1} (\tau_j - \tau_{j-1}) \varphi(X_{\tau_j}),$

(5.5) $h_t(x) = P_x \sum_{j=0}^{\eta_t} (-1)^j M_{j-1} \sum_{m=\tau_{j-1}}^{\tau_j - 1} (m - \tau_{j-1}) \psi(X_m).$

We will show that, as t tends to infinity, the functions g_t and h_t converge pointwise and that the limit functions, call then g and h, satisfy

(5.6) $A^2 g = 0 \quad \text{in} \quad \Gamma, \quad A^2 h = \psi \quad \text{in} \quad \Gamma,$

(5.7) $g = \varphi \quad \text{on} \quad \Gamma^c, \quad h = 0 \quad \text{on} \quad \Gamma^c.$

First we consider g_t. Substituting (3.9) and (3.13) into (5.4), we sum the resulting geometric sum and then the exponential sum to get

$$g_t(x) = P_x \varphi(X_{\tau_0}) + \sum_\lambda \frac{1 - e^{-t(1+\lambda)}}{1 + \lambda} P_x \tau_0 \Pi_\lambda (I - \tilde{P}) \varphi(X_{\tau_0})$$
$$+ \sum_\lambda e^{-t(1+\lambda)} P_x \tau_0 \Pi_\lambda \varphi(X_{\tau_0}),$$

where $\tilde{P}$ is the operator which acts on functions defined on $\partial \Gamma$ by the formula

(5.8) $\tilde{P} f(x) = P_x f(X_{\tau_1}), \quad x \in \partial \Gamma.$

Since each eigenvalue λ is nonnegative, the limit as t tends to infinity exists for each x and

(5.9) $g(x) = P_x \varphi(X_{\tau_0}) + \sum_\lambda \frac{1}{1 + \lambda} P_x \tau_0 \Pi_\lambda (I - \tilde{P}) \varphi(X_{\tau_0}).$

Since, for $X_0 \in \Gamma^c$, $\tau_0 = 0$, it is clear from (5.9) that g satisfies (5.7). To show that g satisfies (5.6), put

$$G_n(x) = P_x Z$$

where

$$Z = \sum_{j=0}^{n} (-1)^j M_{j-1} (\tau_j - \tau_{j-1}) \varphi(X_{\tau_j}).$$

By (3.5), we see that, for $X_0 \in \Gamma$ and $j \geq 0$, $\theta M_j = (\tau_0 - 1) N_j$. Hence

(5.10) $\theta Z - Z = -\varphi(X_{\tau_0}) - \sum_{j=1}^{n} (-1)^j N_{j-1} (\tau_j - \tau_{j-1}) \varphi(X_{\tau_j}), \quad X_0 \in \Gamma.$

On the other hand, for $X_0 \notin \Gamma$ and $j \geq 0$, $\theta M_j = N_{j+1}$. Consequently,

(5.11) $\quad \theta Z - Z = -\varphi(X_{\tau_0}) - \sum_{j=1}^{n+1} (-1)^j N_{j-1}(\tau_j - \tau_{j-1})\varphi(X_{\tau_j}), \quad X_0 \notin \Gamma$.

Combining (5.10) and (5.11) we have

$$\theta Z - Z = -\varphi(X_{\tau_0}) - \sum_{j=1}^{n} (-1)^j N_{j-1}(\tau_j - \tau_{j-1})\varphi(X_{\tau_j})$$

$$+ 1_{X_0 \notin \Gamma}(-1)^n N_n(\tau_{n+1} - \tau_n)\varphi(X_{\tau_{n+1}}).$$

Put $Z' = \theta Z - Z$. Then $\theta_2 Z - 2\theta_1 Z + Z = \theta Z' - Z'$. For $X_0 \in \Gamma$ and $j \geq 0$, it follows from (3.5) that $\theta N_j = N_j$ and so

$$\theta Z' - Z' = 1_{X_1 \notin \Gamma}(-1)^n N_n(\tau_{n+1} - \tau_n)\varphi(X_{\tau_{n+1}}).$$

By (3.3), we see that

$$A^2 G_n(x) = (-1)^n P_x 1_{\Gamma^c}(X_1) N_n(\tau_{n+1} - \tau_n)\varphi(X_{\tau_{n+1}}).$$

It follows from (3.9) and the Markov property that

$$A^2 G_n(x) = (-1)^n P_x 1_{\Gamma^c}(X_1) Q^n(Q + \tilde{P})\varphi(X_1).$$

Using the spectral representation (3.13), we get

(5.12) $\quad A^2 g_t(x) = \sum_n e^{-t}(t^n/n!)A^2 G_n(x) = \sum_\lambda e^{-t(1+\lambda)} P_x 1_{\Gamma^c}(X_1)\Pi_\lambda(Q + \tilde{P})\varphi(X_1)$.

It follows from condition 3A that A^2 is a finite difference operator. Hence $A^2 g(x) = \lim_{t\to\infty} A^2 g_t(x)$ which, according to (5.12), vanishes.

We now turn our attention to the function h. Using the fact that

$$P_x \sum_{m=0}^{\tau_1-1} m\psi(X_m) = PG_\Gamma^2\psi(x), \quad x \in \Gamma^c,$$

it follows from the Markov property that

$$h_t(x) = P_x \sum_{m=0}^{\tau_0-1} (m+1)\psi(X_m) - \sum_\lambda \frac{1 - e^{-t(1+\lambda)}}{1 + \lambda} P_x \tau_0 \Pi_\lambda PG_\Gamma^2\psi(X_{\tau_0}).$$

Hence the limit as t tends to infinity exists and we have

$$h(x) = P_x \sum_{m=0}^{\tau_0-1} (m+1)\psi(X_m) - \sum_\lambda \frac{1}{1 + \lambda} P_x \tau_0 \Pi_\lambda PG_\Gamma^2\psi(X_{\tau_0}).$$

To show that h satisfies (5.6), put

$$H_n(x) = P_x Z$$

where

$$Z = \sum_{j=0}^{n} (-1)^j M_{j-1} \sum_{m=\tau_{j-1}}^{\tau_j-1} (m - \tau_{j-1})\psi(X_m).$$

Again, using (3.5) and (3.8), we have

$$Z' = \theta Z - Z = -\sum_{m=0}^{\tau_0-1} \psi(X_m) - \sum_{j=1}^{n} (-1)^j N_{j-1} \sum_{m=\tau_{j-1}}^{\tau_j-1} (m - \tau_{j-1})\psi(X_m)$$

$$+ (-1)^n 1_{\Gamma^c}(X_0) N_n \sum_{m=\tau_n}^{\tau_{n+1}-1} (m - \tau_n)\psi(X_m).$$

Hence, for $X_0 \in \Gamma$,

$$\theta_2 Z - 2\theta_1 Z + Z = \theta Z' - Z'$$

$$= \psi(X_0) + 1_{\Gamma^c}(X_1)(-1)^n N_n \sum_{m=\tau_n}^{\tau_{n+1}-1} (m - \tau_n)\psi(X_m).$$

By (3.3), we see that

$$A^2 H_n(x) = \psi(x) + (-1)^n P_x 1_{\Gamma^c}(X_1) N_n \sum_{m=\tau_n}^{\tau_{n+1}-1} (m - \tau_n)\psi(X_m).$$

Arguing as we did for the function g, we see that the contribution from the second term above goes to zero and so we have

$$A^2 h(x) = \lim_{t\to\infty} \sum_{n=0}^{\infty} e^{-t}(t^n/n!)A^2 H_n(x) = \psi(x).$$

That $h = 0$ on Γ^c follows from the fact that, for $X_0 \in \Gamma^c$, $\tau_0 = 0$ and $M_j = 0$ for $j > 0$. $\square$

6. Random fields and boundary value problems.

In this section we discuss two-parameter random fields and the boundary value problems associated with them. As mentioned in the introduction, very few results have been obtained in this setting and it seems like an interesting area for future investigation.

We say that a random field $\mathbf{X_n}$, $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, with state space E is *Markov* if there exist linear operators A_1 and A_2 such that

(6.1) $$P_x\{f(\mathbf{X_{n+e_i}}) \mid \mathscr{F_n}\} = A_i f(\mathbf{X_n}) \quad \text{a.s.} \quad P_x$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $\mathscr{F_n}$ is the σ-algebra generated by $\mathbf{X_m}$, $\mathbf{m} \leq \mathbf{n}$ (which means $m_1 \leq n_1$ and $m_2 \leq n_2$). The analog of formula (1.1) is

(6.2) $$f(x) = P_x \sum_{\mathbf{n}} \{\mathbf{U_n} f(\mathbf{X_n}) - \mathbf{V_n} A_1 A_2 f(\mathbf{X_n})\}$$

where

(6.3) $$\mathbf{U_n} = 0 \quad \text{when} \quad \mathbf{X_n} \in \Gamma \quad \text{and} \quad \mathbf{V_n} = 0 \quad \text{when} \quad \mathbf{X_n} \notin \Gamma.$$

Proceeding as usual, we interchange sum and expectation, apply the Markov property (6.1), and interchange back. We then see that formula (6.2) formally holds if

(6.4) $$\mathbf{V_n} - \mathbf{V_{n-e_1}} - \mathbf{V_{n-e_2}} + \mathbf{V_{n-e_1-e_2}} = \mathbf{U_n}, \quad \mathbf{n} \in \mathbb{N}^2,$$

where we have put

(6.5) $$\mathbf{V}_{n_1,-1} = \mathbf{V}_{-1,n_2} = 0, \quad n_1, n_2 \geq 0,$$

$$\mathbf{V}_{-1,-1} = 1.$$

The system (6.3), (6.4), (6.5) has one and only one solution and this solution satisfies $\mathbf{V_n} \in \mathscr{F_n}$.

Suppose that f satisfies

(6.6) $$A_1 A_2 f(x) = 0, \quad x \in \Gamma,$$

(6.7) $$f(x) = \varphi(x), \quad x \notin \Gamma.$$

Then (6.2) shows that

(6.8) $$f(x) = P_x \sum_{\mathbf{n}} \mathbf{U_n} \varphi(\mathbf{X_n}).$$

This would prove uniqueness for the problem (6.6), (6.7) and existence would be established by verifying that (6.8) defines a solution. Of course, as before, we need to make sense of formulas (6.2) and (6.8) (absolute convergence is probably too much to ask). This most likely cannot be done in such a general setting so we now consider some special cases.

1. Let E be the product of two state spaces $E = E_1 \times E_2$ and let $\mathbf{X_n} = (X_{n_1}^1, X_{n_2}^2)$ be a pair of independent Markov chains observed at different times. Then for domains Γ of the form

$$\Gamma = \{x : \psi_1(x_1) + \psi_2(x_2) < 0\}$$

it is possible to give verifiable sufficient conditions for formula (6.2) to hold. This is the discrete time version of the problem studied in Vanderbei (1983).

2. Let $\mathbf{X_n} = X_{n_1 + n_2}$ where X_n is a Markov chain. Then A_1 and A_2 coincide with the generator A of X_n and formula (6.2) reduces to formula (1.7) with

$$U_m = \sum_{\mathbf{n} \in l_m} \mathbf{U_n}, \quad V_m = \sum_{\mathbf{n} \in l_m} \mathbf{V_n},$$

where

$$l_m = \{\mathbf{n} \in \mathbb{N}^2 : n_1 + n_2 = m\}.$$

Replacing the condition that $\mathbf{X_n} \in \Gamma$ (and $\mathbf{X_n} \notin \Gamma$) in (6.3) by the condition that $\mathbf{n}$ belongs to a fixed set $B \subset \mathbb{N}^2$ (and the condition that $\mathbf{n} \notin B$) we get a non-random difference equation which can be solved by computer. Having tried many sets B it seems that sets which are unions of the l_m are bad in the sense that $|U_n|$ and $|V_n|$ grow fast. However, no precise statement to this effect has been proved. Since $\{\mathbf{n} : \mathbf{X_n} \in \Gamma\}$ is a union of the l_m (over a random set of indices m), we conjecture that for the next random field formula (6.2) may hold under weaker assumptions.

3. Let $\mathbf{X_n} = \mathbf{X_0} + \sum_{i=1}^{n_1} \xi_i + \sum_{j=1}^{n_2} \eta_j$, where the ξ_i and the η_j are independent identically distributed random vectors with values in $\mathbb{Z}^d$. Then A_1 and A_2 coincide with the generator A of the Markov chain $X_n = X_0 + \sum_{i=1}^n \xi_i$.

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