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THE EXISTENCE OF REGULAR CONDITIONAL PROBABILITIES: NECESSARY AND SUFFICIENT CONDITIONS

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Different necessary and sufficient conditions for the existence of regular conditional probabilities are found for the cases of countably generated, countably separated, and complete probability spaces. Perfection is *n*. and *s*. for countably generated spaces, "almost pre-standardness" for the countably generated and countably separated cases, and discreteness for complete spaces. Several different forms of the regular conditional probability property must be distinguished.

1. Introduction. A number of distinct regular conditional probability concepts appear in the literature, and it is of considerable interest to probabilists to find conditions under which a space will have one or another of the corresponding properties.

Many sufficient conditions are known for a measure space to have a regular conditional probability property but except for Pachl (1978), page 161, the search for necessary conditions has so far yielded only isolated counterexamples (Dieudonné, 1948; Doob, 1953, page 624; Halmos, 1950, page 210f). We distinguish product, quotient and subfield forms of the rcp property, and show that for the first two it is necessary that the measure be perfect. Perfection is both necessary and sufficient for countably generated spaces. A complete measure space has the product or quotient rcp property iff it is discrete.

2. Regular conditional probability concepts. All measures in this paper are (countably-additive) probabilities. Given a measure space $(Y, \mathbf{T}, \mu)$ and a measurable space $(X, \mathbf{S})$, a function f: $Y \to X$ is *measurable* if $f^{-1}(\mathbf{S}) \subset \mathbf{T}$. It is *$\mu$-measurable* if $f^{-1}(\mathbf{S}) \subset \mathbf{T}^*$, the completion of $\mathbf{T}$ with respect to μ.

Given $(Y, \mathbf{T}, \mu)$ and $(X, \mathbf{S})$, a *kernel* is a function of the form ν: $Y \times \mathbf{S} \to [0, 1]$ satisfying two conditions: (i) $\nu(y, \cdot)$ is a measure on $(X, \mathbf{S})$ for each $y \in Y$, and (ii) $\nu(\cdot, F)$ is measurable for each $F \in \mathbf{S}$.

A *$\mathscr{D}$-kernel* is a function of the same form satisfying (i) but with $\nu(\cdot, F$ having to be only μ-measurable for each $F \in S$.

Let $(X \times Y, \mathbf{S} \times \mathbf{T}, \lambda)$ be a product measure space. A *product regular conditional probability* (rcp) is a kernel ν: $Y \times \mathbf{S} \to [0, 1]$ satisfying

$$\lambda(F \times E) = \int_E \nu(y, F)\lambda_y(dy) \tag{1}$$

for all $E \in \mathbf{T}$, $F \in \mathbf{S}$, where λ_y is the Y-marginal of λ. $(X, \mathbf{S}, \mu)$ has the *product*

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regular conditional probability property (rcpp) if any space $(X \times Y, \mathbf{S} \times \mathbf{T}, \lambda)$ for which μ is the X-marginal has a product rcp.

Let $(R, \mathbf{B})$ be the real line with Borel field. Given $(X, \mathbf{S}, \mu)$ and measurable f: $X \to R$, a *quotient rcp* is a kernel ν: $R \times \mathbf{S} \to [0, 1]$ satisfying

$$\mu(F \cap f^{-1}(E)) = \int_E \nu(t, F)\mu_R(dt) \quad (2)$$

for all $E \in \mathbf{B}$, $F \in \mathbf{S}$, where μ_R is the induced measure on $(R, \mathbf{B})$. A *$\mathscr{D}$-quotient rcp* is the same, except that ν need be only a D-kernel (with respect to measure μ_R). $(X, \mathbf{S}, \mu)$ has the *quotient rcpp* if there is a quotient rcp for every measurable f: $X \to R$. It has the *$\mathscr{D}$-quotient rcpp* if there is a D-quotient rcp for every such function. (The D stands for Doob, who introduced this weakened rcp concept in the subfield context (Doob, 1953, pages 26, 29).)

Let $\mathbf{T}$ be a sub-σ-field of $(X, \mathbf{S}, \mu)$. A *subfield rcp* is a kernel ν: $X \times \mathbf{S} \to [0, 1]$ (with $\nu(\cdot, F)$ $\mathbf{T}$-measurable, all $F \in \mathbf{S}$) satisfying

$$\mu(F \cap E) = \int_E \nu(x, F)\mu(dx) \quad (3)$$

for all $F \in \mathbf{S}$, $E \in \mathbf{T}$. $(X, \mathbf{S}, \mu)$ has the *subfield rcpp* if there is a subfield rcp for every $\mathbf{T} \subset \mathbf{S}$.

This completes our roster of rcp concepts. We now give some simple properties of them.

THEOREM 1. *$(X, \mathbf{S}, \mu)$ has the product rcpp iff it has the following property: For every measurable space $(Y, \mathbf{T})$, measure space $(Z, \mathbf{U}, \rho)$, and measurable functions f: $Z \to Y$, g: $Z \to X$, such that μ is the measure induced from ρ by g, there exists a kernel ν: $Y \times \mathbf{S} \to [0, 1]$ satisfying*

$$\rho(g^{-1}(F) \cap f^{-1}(E)) = \int_E \nu(y, F)\rho_y(dy) \quad (4)$$

for all $E \in \mathbf{T}$, $F \in \mathbf{S}$, where ρ_y is the induced measure on $(Y, \mathbf{T})$.

PROOF. *If.* Let $(Z, \mathbf{U}, \rho) = (X \times Y, \mathbf{S} \times \mathbf{T}, \lambda)$, f and g the projections on Y, X. The marginal μ is induced from λ. Hence there exists ν satisfying (4), which reduces to (1) on noting that $\rho_y = \lambda_y$ and

$$\lambda(g^{-1}(F) \cap f^{-1}(E)) = \lambda((F \times Y) \cap (X \times E)) = \lambda(F \times E).$$

Only if. Given $(Z, \mathbf{U}, \rho)$, $(Y, \mathbf{T})$, f and g, let λ be the measure induced on $(X \times Y, \mathbf{S} \times \mathbf{T})$ from ρ by the mapping $z \to (g(z), f(z))$. Then μ is the X-marginal of λ, so there exists kernel ν satisfying (1). But (1) reduces to (4), since $\lambda_y = \rho_y$, and the inverse image of $F \times E$ is $f^{-1}(E) \cap g^{-1}(F)$. □

THEOREM 2. *The following implications hold among the various rcpp's:*

product ⇒ quotient ⇒ D-quotient

↘ subfield

PROOF. quotient $\Rightarrow$ D-quotient is obvious. The other statements follow from Theorem 1, on noting that quotient and subfield rcpp are the special cases $(Z, \mathbf{U}, \rho) = (X, \mathbf{S}, \mu)$, g the identity, and in addition $(Y, \mathbf{T}) = (R, \mathbf{B})$ for quotient; and $Y = X$, f the identity for subfield. □

Given a sequence $G_1, G_2, \cdots$ of subsets of space Y, the *Marczewski function* $m: Y \to R$ is given by $m(y) = \sum_n 2 \cdot 3^{-n} I_n(y)$, where $I_n(y) = 1$ if $y \in G_n$, $= 0$ otherwise. (Marczewski, 1938).

In the definition of quotient rcpp, the restriction of mappings to the real line can be relaxed considerably.

THEOREM 3. *Let $(X, \mathbf{S}, \mu)$ have the quotient (or D-quotient) rcpp, let $(Y, \mathbf{T})$ be a measurable space with $\mathbf{T}$ countably generated, and let $f: X \to Y$ be measurable. Then*

(i) *There exists a kernel (or D-kernel) ν: $Y \times \mathbf{S} \to [0, 1]$ satisfying (2) with $(Y, \mathbf{T})$ in place of $(R, \mathbf{B})$.*

(ii) *Let f: $X \to Y$ be merely μ-measurable. Then a kernel (or D-kernel) still exists.*

PROOF. (i) Let $G_1, G_2, \cdots$ be a sequence in Y generating $\mathbf{T}$, and let m: $Y \to R$ be the Marczewski function. The composite function $m \circ f$: $X \to R$ is measurable, so there exists a (D-)kernel ν': $R \times \mathbf{S} \to [0, 1]$. Now define ν: $Y \times \mathbf{S} \to [0, 1]$ by the rule: $\nu(y, F) = \nu'(m(y), F)$. Clearly ν is a (D-)kernel. By countable generation, for any $E \in \mathbf{T}$ there is a Borel set H with $E = m^{-1}(H)$, and it follows easily from this observation that (2) holds for ν with $(Y, \mathbf{T})$ replacing $(R, \mathbf{B})$.

(ii) Given f, there is a measurable g: $X \to Y$ such that $g = f$ μ—a.s., by $\mathbf{T}$ countably generated. Let ν be a (D-)quotient rcp for g. Then it remains a (D-)quotient rcp for f. □

3. Necessary condition for regular conditional probability.

LEMMA 1. *Given $(X, \mathbf{S}, \mu)$, let ν: $R \times \mathbf{S} \to [0, 1]$ be a D-quotient rcp for measurable f: $X \to R$. Then there is a Borel set B, with $\mu((f^{-1}(B)) = 1$, such that $\nu(t, f^{-1}(t)) = 1$, all $t \in B$.*

PROOF. Let $E_1, E_2, \cdots$ enumerate the intervals with rational endpoints. From (2) we obtain $\mu(f^{-1}(E_n)) = \int_{E_n} \nu(t, f^{-1}(E_n))\mu_R(dt)$, so that $\nu(t, f^{-1}(E_n)) = 1$, μ_R—a.s. on $t \in E_n$. Also (2) yields $\int_{E_n^c} \nu(t, f^{-1}(E_n))\mu_R(dt) = 0$ (E_n^c is the complement of E_n), so that $\nu(t, f^{-1}(E_n)) = 0$ a.s. on $t \in E_n^c$. Choose $B_n \in \mathbf{B}$, $\mu_R(B_n) = 1$, so that

$$\begin{aligned} \nu(t, f^{-1}(E_n)) &= 1 \quad \text{on} \quad B_n \cap E_n, \\ &= 0 \quad \text{on} \quad B_n \cap E_n^c. \end{aligned}$$

Let $B = \cap B_n$. Then $B \in \mathbf{B}$, $\mu_R(B) = 1$. Let $t \in B$. For each n, let $A_n = E_n$ if $t \in E_n$, $A_n = E_n^c$ if $t \in E_n^c$. Then $\nu(t, f^{-1}(A_n)) = 1$, all n. Hence $1 = \nu(t, \cap_n f^{-1}(A_n)) = \nu(t, f^{-1}(\cap A_n)) = \nu(t, f^{-1}(t))$. □

$(X, \mathbf{S}, \mu)$ is a *perfect* measure space if, for any measurable $f: X \to R$, there is a real Borel set $E \subset f(X)$ such that $\mu(f^{-1}(E)) = 1$.

Many conditions equivalent to perfection are known (Ryll-Nardzewski, 1953; Sazonov, 1965; Ramachandran, 1979, I & II). The concept was introduced by Gnedenko and Kolmogorov (1949).

THEOREM 4. *If $(X, \mathbf{S}, \mu)$ has the D-quotient rcpp, then it is perfect.*

PROOF. Given measurable $f: X \to R$, we must find a Borel set $B \subset f(X)$ with $\mu(f^{-1}(B)) = 1$. Take the B in Lemma 1. If $t \in B$, then $\nu(t, f^{-1}(t)) = 1$, so $f^{-1}(t)$ is not empty. Thus $t \in f(X)$. □

4. A note on disintegrations. Given measure spaces $(X, \mathbf{S}, \mu)$, $(Y, \mathbf{T}, \rho)$ a *quasi-kernel* is a family of measures $\nu(t, \cdot): \mathbf{S}_t \to [0, 1]$, $\mathbf{S}_t$ is a sub-σ-field of $\mathbf{S}$ for each $t \in Y$, such that, for all $F \in \mathbf{S}$, we have $F \in S_t$, ρ—a.s., and $\nu(\cdot, F)$ (defined ρ—a.s.) is $\mathbf{T}$-measurable.

Product, quotient, and subfield disintegration properties may be defined exactly as the corresponding rcp properties, except that quasi-kernels replace kernels. (See Ramachandran, 1979, Part II, pages 83, 88). The rcp properties imply the corresponding disintegration properties. In particular, the D-quotient rcpp implies the quotient disintegration property, since quotient and D-quotient are logically equivalent for disintegrations.

THEOREM 4′. *Quotient disintegration property implies perfection.*

PROOF. Identical to Theorem 4, since the proof of Lemma 1 works for quotient disintegrations. □

Theorem 2.2 of Pachl (1978), page 161, yields the result that the product disintegration property implies compactness of $(X, \mathbf{S}, \mu)$ (which in turn implies perfection). Neither this result nor Theorem 4′ includes the other.

We shall not discuss disintegrations further.

5. Almost pre-standard measures. The *atoms* (or $\mathbf{S}$-*atoms*) of the countably generated $(X, \mathbf{S})$ are the smallest non-empty measurable sets. $(X, \mathbf{S})$ is *standard* if it is Borel-isomorphic to a real Borel set. (Any Borel subspace of any complete separable metric space is standard (Parthasarathy, 1967, pages 7–14).) It is *pre-standard* if the space of its atoms, with the natural induced Borel structure, is standard.

Finally, $(X, \mathbf{S}, \mu)$ is *almost pre-standard* if there exists $X_0 \in \mathbf{S}$ with $\mu(X_0) = 1$ and $(X, \mathbf{S})$ pre-standard when restricted to X_0. ($\mathbf{S}$ itself need not be countably generated).

THEOREM 5. *If $(X, \mathbf{S}, \mu)$ is almost pre-standard then it has the product rcpp.*

PROOF. First do the special case $(X, \mathbf{S}) = (R, \mathbf{B})$. Given $(R \times Y, \mathbf{B} \times \mathbf{T}, \lambda)$

with R-marginal μ, let $F_r = \{t \mid t < r\}$ for each rational r, and define λ_{yr}: $\mathbf{T} \to [0, 1]$ by $\lambda_{yr}(E) = \lambda[F_r \times E]$. Then $\lambda_{yr} \leq \lambda_y$, the Y-marginal. Let $h_r = d\lambda_{yr}/d\lambda_y$. There exists $Y_0 \in \mathbf{T}$, with $\lambda_y(Y_0) = 1$, such that, for all $y \in Y_0$, $h_r(y)$ is monotone in r, $\to 1$ as $r \to \infty$, and $\to 0$ as $r \to -\infty$. This distribution function on the rationals determines a unique measure on $(R, \mathbf{B})$. Let $\nu(y, \cdot)$ be this measure, all $y \in Y_0$, and define $\nu(y, \cdot) = \mu$ for $y \in Y_0^c$. It is easily verified that ν is a product rcp for λ (cf. Doob, 1953, page 30).

Next, let $(X, \mathbf{S})$ be a Borel subset of R. Map into R, determine ν as above and make an adjustment on a Y-null set to guarantee $\nu(y, X) = 1$ for all $y \in Y$. This is a product rcp.

Now the general case. Let $\mathbf{S}_0 = \mathbf{S}$ restricted to X_0 be pre-standard, where $\mu(X_0) = 1$. Let g: $X_0 \to R_0 \in \mathbf{B}$ be a Borel isomorphism from the $\mathbf{S}$-atoms of X_0 to R_0. Given $(X_0 \times Y, \mathbf{S}_0 \times \mathbf{T}, \lambda)$, the map $(x, y) \to (g(x), y)$ induces a measure λ' on $(R_0 \times Y)$. Let ν': $Y \times \mathbf{B} \to [0, 1]$ be a product rcp for λ'. Then define ν: $Y \times \mathbf{S}_0 \to [0, 1]$ by $\nu(y, F) = \nu'(y, g(F))$, and extend it to $Y \times \mathbf{S}$ by letting $\nu(y, X_0^c) = 0$ for all y. Then ν is a product rcp for λ, as may be easily verified. □

6. The countably generated case.

THEOREM 6. *Given $(X, \mathbf{S}, \mu)$, with $\mathbf{S}$ countably generated. The following conditions are logically equivalent*:

(i) *almost pre-standard,*
(ii) *product rcpp,*
(iii) *quotient rcpp,*
(iv) *D-quotient rcpp,*
(v) *perfect.*

PROOF. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) by Theorems 5, 2, and 4. It remains only to show that (v) ⇒ (i). Let $G_1, G_2, \cdots$ be a sequence generating $\mathbf{S}$, and let m: $X \to R$ be the corresponding Marczewski function. Let $E \subset m(X)$ be a Borel set with $\mu(m^{-1}(E)) = 1$. Then m is a Borel isomorphism between E and the atoms of $m^{-1}(E)$. Thus $(X, \mathbf{S}, \mu)$ is almost pre-standard (cf. Ryll-Nardzewski, 1953, page 129). □

7. The subfield regular conditional probability property.

Subfield rcpp has been an orphan to this point. However, it works well with the assumption of perfection.

THEOREM 7. *If $(X, \mathbf{S}, \mu)$ has the subfield rcpp and is perfect, then it has the quotient rcpp.*

PROOF. Let f: $X \to R$ be measurable, and let $\mathbf{T} = f^{-1}(\mathbf{B})$. $\mathbf{T} \subset \mathbf{S}$, so there exists a kernel ν': $X \times \mathbf{S} \to [0, 1]$ satisfying (3). Let $E_0 \subset f(X)$ be a Borel set with $\mu(f^{-1}(E_0)) = 1$. Now define $\nu: R \times \mathbf{S} \to [0, 1]$ as follows:

$$\text{if} \quad t \in E_0^c, \quad \nu(t, F) = \mu(F)$$

$$\text{if} \quad t \in E_0, \quad \nu(t, F) = \nu'(x, F), \quad \text{where} \quad f(x) = t.$$

This definition is sound. For suppose $f(x_1) = f(x_2) = t$. Then x_1, x_2 are in the same $\mathbf{T}$-atom. Since $\nu'(\cdot, F)$ is $\mathbf{T}$-measurable we must have $\nu'(x_1, F) = \nu'(x_2, F)$, all $F \in \mathbf{S}$. $\nu(\cdot, F)$ is Borel-measurable by the argument of Doob, 1953, page 603; hence, ν is a kernel. Finally, from (3) we get

$$\mu(f^{-1}(E) \cap F) = \int_{f^{-1}(E)} \nu'(x, F)\mu(dx) = \int_{f^{-1}(E)} \nu(f(x), F)\mu(dx)$$

$$= \int_E \nu(t, F)\mu_R(dt) \quad \text{for all} \quad E \in \mathbf{B}, \quad F \in \mathbf{S},$$

verifying (2). Thus ν is a quotient rcp. □

Theorem 2 may now be strengthened.

COROLLARY. *product rcpp* $\Rightarrow$ *subfield rcpp and perfect* $\Rightarrow$ *quotient rcp* $\Rightarrow$ *D-quotient rcpp.*

PROOF. From Theorems 2, 4 and 5. □

8. The countably separated case. A measurable space $(X, \mathbf{S})$ is *countably separated* if there is a sequence $F_1, F_2, \cdots$ in $\mathbf{S}$ such that any pair of points x_1, $x_2 \in X$ separated by some $\mathbf{S}$-set are separated by some F_n.

Equivalently, $\mathbf{S}$ has a countably generated sub-σ-field with the same atoms. (Note we do not assume that the atoms are singletons). It follows that every countably generated space is countably separated.

THEOREM 8. *Let* $(X, \mathbf{S}, \mu)$ *be countably separated. Then the following conditions are logically equivalent*:

(i) *almost pre-standard,*
(ii) *product rcpp,*
(iii) *subfield rcpp and perfect,*
(iv) *quotient rcpp.*

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold in general, by Theorems 5, 4, 7 and 2.

(iv) $\Rightarrow$ (i). Let $\mathbf{T} \subset \mathbf{S}$ be countably generated with the same atoms as $\mathbf{S}$. By Theorem 3 there is a quotient rcp $\nu: X \times \mathbf{S} \to [0, 1]$ with $\nu(\cdot, F)$ $\mathbf{T}$-measurable, all $F \in \mathbf{S}$. By Musiał (1972), Corollary 2, page 12 it follows that there exists $X_0 \in \mathbf{T}$, with $\mu(X_0) = 1$, such that, inside X_0, $\mathbf{S} = \mathbf{T}$ and so is countably generated.

Since μ must also be perfect (Theorem 4), it follows that it is almost pre-standard (Theorem 6). □

REMARK. Comparing this result with Theorem 6, we see that the former equivalents "perfect" and "D-quotient rcpp" have dropped out. Indeed, Lebesgue measure complete on [0, 1] is countably separated and perfect, but not almost pre-standard. The agument above for D-quotient rcpp yields the existence of an almost pre-standard $\mathbf{T}$ with $\mathbf{T} \subset \mathbf{S} \subset \mathbf{T}^*$. (This last condition may be shown to be logically equivalent to perfection, for countably separated spaces.)

9. Discrete measure spaces. Given $(X, \mathbf{S}, \mu)$, a *μ-atom* is a set $F \in \mathbf{S}$ such that $\mu(F) > 0$, and, for all $G \subset F$, $G \in S$, either $\mu(G) = \mu(F)$ or $\mu(G) = 0$. μ is *discrete* if there is a countable partition of X into μ-atoms.

THEOREM 9. *If $(X, \mathbf{S}, \mu)$ is discrete, then it has the product rcpp.*

PROOF. Let $X_1, X_2, \cdots$ be a partition of X into μ-atoms (the sequence may be finite). Given $(X \times Y, \mathbf{S} \times \mathbf{T}, \lambda)$ with X-marginal μ, define $\lambda_{yn}: \mathbf{T} \to [0, 1]$ by $\lambda_{yn}(E) = \lambda(X_n \times E)$. Then $\lambda_{yn} \leq \lambda_y$, the Y-marginal, so we may write $g_n = d\lambda_{yn}/d\lambda_y$. Then g_n: $Y \to R$ is measurable, and, except for a λ_y-null set $N \in \mathbf{T}$, $\sum_n g_n = 1$ and $g_n \geq 0$. Define ν: $Y \times \mathbf{S} \to [0, 1]$ as follows:

$$\nu(y, F) = \mu(F) \qquad \text{if} \quad y \in N;$$
$$\nu(y, F) = \textstyle\sum_n g_n(y)\mu(F \cap X_n)/\mu(X_n) \quad \text{if} \quad y \in N^c.$$

It is easily verified that ν is a product rcp. □

10. Complete measure spaces. A much deeper result than Theorem 9 is that, for complete measure spaces, discreteness is a necessary condition for the existence of the rcpp.

THEOREM 10. *Let $(X, \mathbf{S}, \mu)$ be a complete measure space with the quotient rcpp. Then μ is discrete.*

PROOF. The proof of Theorem 10 is long, and we begin with several lemmas.

LEMMA 2. *Lebesgue measure complete on the unit interval does not have the quotient rcpp.*

PROOF. Let f be the natural injection into $(R, \mathbf{B})$, and let ν be a quotient rcp. By Lemma 1, there exists $B \in \mathbf{B}$, with $\mu(B) = 1$, such that $\nu(t, \{t\}) = 1$, all $t \in B$. Hence, $\nu(t, F) = 1$ if $t \in F$, $= 0$ if $t \in F^c$, for all $t \in B$. Taking F to be a non-Borel set $\subset B$, one sees that $\nu(\cdot, F)$ cannot be (Borel) measurable. □

REMARK. $\nu(\cdot, F)$ *is* Lebesgue-measurable for all F. Hence the argument above leaves open the possibility that Lebesgue measure has the D-quotient rcpp. See Section 11 below. (The proof of Lemma 2 is from Ramachandran I, 1979, page 46; the result itself was mentioned by Sazonov, 1965, page 242. It also follows at once from the remark after Theorem 8).

The next two lemmas are stated both for the quotient rcpp and the D-quotient rcpp, the latter version being needed later.

LEMMA 3. *Let $(X, \mathbf{S}, \mu)$ have the (D-)quotient rcpp, let $g: X \to (Y, \mathbf{T})$ be measurable and induce measure ρ on $(Y, \mathbf{T})$. Then $(Y, \mathbf{T}, \rho)$ has the (D-)quotient rcpp.*

PROOF. Let $f: Y \to R$ be measurable. Then the composite function $f \circ g$: $X \to R$ is measurable, so there exists a (D-)quotient rcp $\nu': R \times \mathbf{S} \to [0, 1]$. Define $\nu: R \times \mathbf{T} \to [0, 1]$ by $\nu(t, G) = \nu'(t, g^{-1}(G))$. It is easily verified that ν is a (D-)quotient rcp for $(Y, \mathbf{T}, \rho)$ and $f: Y \to R$. □

LEMMA 4. *Let $(X, \mathbf{S}, \mu)$ have the (D-)quotient rcpp, let $X_0 \in \mathbf{S}$ with $\mu(X_0) > 0$, let $\mathbf{S}_0$ be $\mathbf{S}$ restricted to subsets of X_0, and let λ on $(X_0, \mathbf{S}_0)$* be given by $\mu/\mu(X_0)$. Then $(X_0, \mathbf{S}_0, \lambda)$ *has the (D-)quotient rcpp.*

PROOF. Fix measurable $f: X_0 \to R$. Let $g: X \to R$ be a measurable extension of f. Let $\nu': R \times \mathbf{S} \to [0, 1]$ be a (D-)quotient rcp for g. Define $\nu: R \times \mathbf{S}_0 \to [0, 1]$ as follows:

$$\text{(5)} \qquad \begin{aligned} &\text{if} \quad \nu'(t, X_0) = 0, \quad \text{let} \quad \nu(t, F) = \lambda(F) \\ &\text{if} \quad \nu'(t, X_0) > 0, \quad \text{let} \quad \nu(t, F) = \nu'(t, F)/\nu'(t, X_0). \end{aligned}$$

ν is clearly a (D-)kernel. It remains to verify (2).

Let μ_R, λ_R be the measures induced on R from μ, λ by g, f, respectively. Then, for any $E \in \mathbf{B}$,

$$\begin{aligned} \int_E \nu'(t, X_0)\mu_R(dt) &= \mu(X_0 \cap g^{-1}(E)) \\ &= \mu(f^{-1}(E)) = \lambda(f^{-1}(E)) \cdot \mu(X_0) = \lambda_R(E) \cdot \mu(X_0). \end{aligned}$$

Hence $(d\lambda_R/d\mu_R)(t) = (\nu'(t, X_0))/\mu(X_0)$. Hence for any $F \in \mathbf{S}_0$, $E \in \mathbf{B}$, we have

$$\begin{aligned} &\int_E \nu(t, F)\lambda_R(dt) \\ &\quad = \int_E \nu(t, F) \frac{\nu'(t, X_0)}{\mu(X_0)} \mu_R(dt) = \int_E \frac{\nu'(t, F)}{\mu(X_0)} \mu_R(dt) \quad \text{from (5)}, \\ &\quad = \frac{\mu(F \cap g^{-1}(E))}{\mu(X_0)} = \frac{\mu(F \cap f^{-1}(E))}{\mu(X_0)} = \lambda(F \cap f^{-1}(E)). \end{aligned}$$

Thus, ν is a (D-)quotient rcp for λ. □

We now prove Theorem 10 by contradiction. Suppose $(X, \mathbf{S}, \mu)$ is complete, has the quotient rcpp, but is not discrete. Then there exists $X_0 \in \mathbf{S}$ with $\mu(X_0) > 0$, such that $(X_0, \mathbf{S}_0, \lambda)$ is nonatomic, where $\lambda = \mu/\mu(X_0)$. λ remains complete, and by Lemma 4 retains the quotient rcpp.

Let **B** be the Borel field on [0, 1], and let ([0, 1], **B**, ρ) be Lebesgue measure. Since λ is nonatomic, there exists a measurable function g: $(X_0, \mathbf{S}_0) \to$ ([0, 1], **B**) such that ρ is the measure induced from λ.

Next, g remains measurable for $\mathbf{B}^*$, the completion of **B** with respect to ρ, on [0, 1]. For, $g^{-1}(\mathbf{B}) \subset \mathbf{S}_0$ yields $g^{-1}(\mathbf{B}^*) \subset \mathbf{S}_0^* = \mathbf{S}_0$, since $\mathbf{S}_0$ is complete.

Now $(X_0, \mathbf{S}_0, \lambda)$ has the quotient rcpp. Hence ([0, 1], $\mathbf{B}^*$, ρ) has the quotient rcpp by Lemma 3. But this contradicts Lemma 2. □

Putting these results together yields:

THEOREM 11. *Let* $(X, \mathbf{S}, \mu)$ *be complete*; *then the following properties are logically equivalent*:

(i) *discrete,*
(ii) *product rcpp,*
(iii) *subfield rcpp and perfect,*
(iv) *quotient rcpp.*

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are given (without assuming completeness) by Theorems 9, 2, 4 and 7; (iv) $\Rightarrow$ (i) is Theorem 10. □

11. Complete spaces and the *D*-quotient regular conditional probability property. Theorem 10 is not altogether satisfying from the viewpoint of probability theory. The distinction between quotient and *D*-quotient rcpp's is slight, and one would like to see equally strong conclusions flowing from the latter. This can be accomplished by introducing a plausible set-theoretic assumption.

ASSUMPTION L. There is no countably-additive extension of Lebesgue measure to the class of all subsets of [0, 1].

The set-theoretic situation is this. If assumption L is false, the continuum must contain a weakly inaccessible cardinal. Hence L is implied by the continuum hypothesis (Drake, 1974, page 178, Example 3; Birkhoff, 1948). Thus, L is certainly consistent with the axioms of set theory, and may be implied by them. (There are some versions of set theory, involving a weakened axiom of choice, in which L may be consistently denied (Solovay, 1970).)

LEMMA 5. *Under assumption L, Lebesgue measure completed on the unit square does not have the D-quotient rcpp.*

PROOF. Let ([0, 1], **B**, ρ) be Lebesgue measure on the Borel field of the unit interval, so that the space in question is $([0, 1]^2, (\mathbf{B} \times \mathbf{B})^*, \rho \times \rho)$. Consider the map f: $[0, 1]^2 \to R$ given by $f(x, y) = x$. Note that the induced measure $(\rho \times \rho)_R = \rho$, Lebesgue measure on [0, 1]. Suppose ν: $R \times (\mathbf{B} \times \mathbf{B})^* \to [0, 1]$ were a *D*-quotient rcp.

From Lemma 1 we conclude that ρ-almost surely,

$$1 = \nu(t, f^{-1}(t)) = \nu(t, \{t\} \times [0, 1]), t \in [0, 1].$$

Next, let $B_1, B_2, \cdots$ enumerate the intervals with rational endpoints in [0, 1]. Then, for all $E \in \mathbf{B}$, $\rho(B_n)\rho(E) = (\rho \times \rho)(([0, 1] \times B_n) \cap f^{-1}(E)) = \int_E \nu(t, [0, 1] \times B_n)\rho(dt)$, which implies $\nu(t, [0, 1] \times B_n) = \rho(B_n)$, for ρ-almost all t. Combined with the argument above, this yields $\nu(t, \{t\} \times B_n) = \rho(B_n)$, ρ-almost surely, for all n. Thus, $\nu(t, \cdot)$ is (one-dimensional) Lebesgue measure concentrated on the line with x-coordinate t in the unit square. Let t_0 be such a t.

Now $(\rho \times \rho)(\{t_0\} \times [0, 1]) = 0$, hence the completed σ-field $(\mathbf{B} \times \mathbf{B})^*$ includes *all* subsets of $\{t_0\} \times [0, 1]$, that is, all sets of the form $\{t_0\} \times F$, $F \subset [0, 1]$. $\nu(t_0, \{t_0\} \times F)$ must be defined for all F, countably additive, and $= \rho(F)$ when F is a Borel set, violating assumption L. □

(A related but more complicated example may be found in Pachl, 1978, page 158; see the note, Ramachandran, 1981).

THEOREM 12. *Under assumption L, if $(X, \mathbf{S}, \mu)$ is complete and has the D-quotient rcpp, then it is discrete.*

PROOF. Follows exactly the lines of Theorem 10, with Lemmas 3 and 4 in their D-quotient versions. $(X_0, \mathbf{S}_0, \lambda)$ is mapped onto Lebesgue measure on the unit square instead of the unit interval. This is permissible since there is a Borel isomorphism between Lebesgue measure on the unit interval and on the unit square (e.g., Ramachandran, 1979, Part II, page 25). The conclusion is that Lebesgue measure completed on the square has the D-quotient rcpp, contradicting Lemma 5. □

Summarizing:

THEOREM 13. *Under assumption L, if $(X, \mathbf{S}, \mu)$ is complete, the following properties are logically equivalent*:

(i) *discrete,*
(ii) *product rcpp,*
(iii) *subfield rcpp and perfect,*
(iv) *quotient rcpp,*
(v) *D-quotient rcpp.*

PROOF. Theorems 11, 12, and 2. □

12. Open questions. A number of unresolved questions concern the relations among the various rcp concepts.

Is there a space having quotient rcpp but not product rcpp? D-quotient rcpp but not quotient? One of quotient rcpp or subfield rcpp, but not the other?

The subfield rcpp seems to be the most popular in the literature, but we have found it the least useful. Is there a nonperfect subfield rcpp space?

In the last part, is assumption L really needed for Theorems 12 and 13?

Finally, is there a simple *n.* and *s.* condition for any of the rcpp's, not requiring any special assumptions such as completeness or countable separability?

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