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TANAKA'S FORMULA AND RENORMALIZATION FOR INTERSECTIONS OF PLANAR BROWNIAN MOTION[1]

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We use a Tanaka-like formula to explain Varadhan's renormalization of the formally infinite measure of Brownian self intersections given by

$$\int_0^T \int_0^T \delta(W_t - W_s)\, ds\, dt.$$

1. Introduction. It is well known that a Brownian path in the plane must intersect itself. A measure of such self-intersections is formally given by

$$\int_B \int \delta(W_t - W_s)\, ds\, dt, \tag{1.1}$$

where δ is the "delta-function." Somewhat more precisely if

$$q_\varepsilon(x) = \frac{e^{-\varepsilon/2} e^{-|x|^2/2\varepsilon}}{2\pi\varepsilon} \tag{1.2}$$

then by (1.1) we mean the formal limit

$$\lim_{\varepsilon \to 0} \int_B \int q_\varepsilon(W_t - W_s)\, ds\, dt. \tag{1.3}$$

If $B \subseteq R_+^2$ is away from the diagonal we have shown (Rosen, 1983) that (1.1) is the value at $x = 0$ of the (continuous) local time $\alpha(x, B)$, of the random field: $(s, t) \to W_t - W_s$. The precise definitions will be recalled below.

When B intersects the diagonal, the limit (1.3) is infinite. Varadhan has shown that we can find a (nonrandom) constant $c(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon \to 0} \int_0^T \int_0^T q_\varepsilon(W_t - W_s)\, ds\, dt - c(\varepsilon)T \tag{1.4}$$

is a well-defined random variable (Varadhan, 1969). Although once described as "the most interesting property of the Wiener process which has been discovered in recent years" (Nelson, 1973), it has had little impact on probability, and in fact is unknown to most probabilists. [The situation is now improving. See the recent articles by Le Gall (1985), Yor (1985a, b), and Dynkin (1985).]

The purpose of this paper is to explain Varadhan's result in terms more familiar to probabilists. We will show how it follows naturally from a Tanaka formula.

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Let

$$K^{\varepsilon}(x)=\int_{\varepsilon}^{\infty}q_t(x)\,dt, \tag{1.5}$$

which is a C^{∞} function of x. If we apply Itô's formula to the nonanticipating function of t and x:

$$\int_0^t K^{\varepsilon}(x-W_s)\,ds \quad \text{on the interval } 0\le t\le T$$

and use the formula $(-\Delta+1)K^{\varepsilon}(x)=2q_{\varepsilon}(x)$ we obtain

$$\begin{aligned}
&\int_0^T K^{\varepsilon}(W_T-W_s)\,ds-0-\int_0^T K^{\varepsilon}(W_t-W_t)\,ds\\
&\quad-\int_{D_T}\int\nabla K^{\varepsilon}(W_t-W_s)\,ds\,dW_t\\
&\quad-\tfrac{1}{2}\int_{D_T}\int K^{\varepsilon}(W_t-W_s)\,ds\,dt\\
&\quad=-\int_{D_T}\int q^{\varepsilon}(W_t-W_s)\,ds\,dt,
\end{aligned}\tag{1.6}$$

where $D_T=\{(s,t)|0\le s\le t\le T\}$. We recognize the right-hand side as the integral in (1.3), so we must consider how the left-hand side behaves as $\varepsilon\to 0$. The second term is $TK^{\varepsilon}(0)\sim T/2\pi\ln(1/\varepsilon)$ as $\varepsilon\to 0$. We will show in Section 2 that the other terms on the left-hand side all have "nice" limits as $\varepsilon\to 0$. Dropping the second term, $TK^{\varepsilon}(0)$ is precisely Varadhan's renormalization.

To better appreciate the nature of the term which is being dropped, let

$$D_{T,\gamma}=\{(s,t)\in D_T|t-s\ge\gamma\}.$$

Apply Itô's formula to the nonanticipating functional of t and x

$$\int_0^{t-\gamma}K^{\varepsilon}(x-W_s)\,ds \quad \text{on the interval } \gamma\le t\le T$$

and then take the $\varepsilon\to 0$ limit, which we later justify, to find

$$\begin{aligned}
&\int_0^{T-\gamma}K(W_T-W_s)\,ds-0-\int_{\gamma}^{T}K(W_t-W_{t-\gamma})\,dt\\
&\quad-\int_{D_{T,\gamma}}\int\nabla K(W_t-W_s)\,ds\,dW_t\\
&\quad-\tfrac{1}{2}\int_{D_{T,\gamma}}\int K(W_t-W_s)\,ds\,dt=-\alpha(0,D_{T,\gamma}).
\end{aligned}\tag{1.7}$$

The last two terms on the left-hand side are "area terms" which behave nicely as $\gamma\to 0$, converging to the corresponding terms in (1.6). The first two terms are "boundary terms"; the first has a nice limit as $\gamma\to 0$ — while the second, which in the limit corresponds to the "information from the diagonal", is precisely the term which we drop in renormalization. This is as it should be — for this troublesome term corresponds to $t = s$ in (1.1) which is registering as a "spurious" intersection.

(1.7) in turn can be thought of as a special case of

$$\alpha(0, G) = \int_{\partial G} K(W_t - W_s)\, ds + \int_G \int \nabla K(W_t - W_s)\, ds\, dW_t + \tfrac{1}{2} \int_G \int K(W_t - W_s)\, ds\, dt. \tag{1.8}$$

This formula is certainly true if G is a finite union of rectangles in some $D_{T,\gamma}$ — but in this paper we do not attempt to establish a general version of (1.8).

All our formulas extend to $\alpha(x, D_{T,\gamma})$ and $\alpha^{\varepsilon}(x, D_T)$ where we replace $q_{\varepsilon}(y)$ by $q_{\varepsilon,x}(y) = q_{\varepsilon}(y - x)$: simply replace $K^{\varepsilon}(y)$ by $K_x^{\varepsilon}(y) = K^{\varepsilon}(y - x)$.

In the next section we will show that the integrals

$$\int_0^T K_x^{\varepsilon}(W_T - W_s)\, ds, \quad \int_{D_T} \int K_x^{\varepsilon}(W_t - W_s)\, ds\, dt, \quad \int_{D_T} \int \nabla K_x^{\varepsilon}(W_t - W_s)\, ds\, dW_t,$$

converge as $\varepsilon \to 0$ to random variables jointly continuous in x, T. Furthermore, we establish (1.7), defining its terms along the way, and show that its "nice" terms converge as $\gamma \to 0$ to the corresponding limits of (1.6) just described. This shows that dropping the second term in either (1.6) or (1.7) gives the same renormalization.

The idea of using Itô's formula to understand renormalization goes back to Symanzik (1969), albeit in a different form. Our contribution is to carry this out rigorously in a way which we feel is understandable.

The situation in three dimensions is more complicated. We refer to Symanzik (1969) and Westwater (1980). Speaking very roughly, the contribution from the line $y = x + \gamma$, the second integral in (1.7), is $O(K(\gamma)) = O(\ln(1/\gamma))$ in two dimensions, so that the area terms are $O(\int_{0^+} \ln(1/\gamma)\, d\gamma) < \infty$ — while in three dimensions $K(\gamma) = O(1/\gamma)$ so that even the area terms are infinite.

We refer the reader to recent work of Dynkin (1983) for a different attempt to put Varadhan's result, and the quantum field theory which gave rise to it, in a more general probabilistic setting, and to Fröhlich (1984) for a survey on recent progress in implementing Symanzik's ideas relating Brownian intersections and quantum field theory.

2. Let us introduce the notation

$$J_1(x, T, \varepsilon) = \int_0^T K^{\varepsilon}(x - W_s)\, ds, \tag{2.1}$$

$$J_2(x, T, \gamma, \varepsilon) = \int_{D_{T,\gamma}} \int K_x^{\varepsilon}(W_t - W_s)\, ds\, dt, \tag{2.2}$$

$$J_3(x, T, \gamma, \varepsilon) \doteq \int_{D_{T,\gamma}} \int \nabla K_x^{\varepsilon}(W_t - W_s)\, ds\, dW_t \doteq \int_\gamma^T \left(\int_0^{t-\gamma} \nabla K_x^{\varepsilon}(W_t - W_s)\, ds \right) dW_t. \tag{2.3}$$

These are well defined for $\varepsilon > 0$.

Lemma 1. *For any m even, $\alpha < \frac{1}{6}$, $i = 1,2,3$*

$$\mathbb{E}\big(J_i(x,T,\gamma,\varepsilon) - J_i(x',T',\gamma',\varepsilon')\big)^m \le c_m|(x,T,\gamma,\varepsilon) - (x',T',\gamma',\varepsilon')|^{\alpha m}, \tag{2.4}$$

uniformly on each

$$E_n = \{x, x', T, T' \in \{\|y\| \le n\}, \gamma, \gamma' \in [0,1], \varepsilon, \varepsilon' \in (0,1]\}.$$

Using this lemma with the multidimensional version of Kolmogorov's lemma (Meyer, 1980) we find

$$|J_i(x,T,\gamma,\varepsilon) - J_i(x',T',\gamma',\varepsilon')| \le c_\omega|(x,T,\gamma,\varepsilon) - (x',T',\gamma',\varepsilon')|^\alpha \tag{2.5}$$

for all *rational* points in E_n. In the case of J_1 and J_2, which are continuous in E_n, (2.5) must then hold for *all* points in E_n. From this we see that the $\varepsilon \to 0$ limit of J_1 and J_2 is jointly continuous—but $K^\varepsilon(x) \uparrow K(x)$ for all x so that by the monotone convergence theorem

$$\lim_{\varepsilon \to 0} J_i(\cdot, \varepsilon) = J_i(\cdot, 0), \qquad i = 1 \text{ or } 2. \tag{2.6}$$

The stochastic integral J_3 is in any event defined only almost surely—so if (2.5) holds for rational points in E_n, we can find a version of J_3 with (2.5) holding for all points in $\overline{E_n}$ (including $\varepsilon = 0$, $\varepsilon' = 0$).

In particular we can interchange the $\varepsilon \to 0$, $\gamma \to 0$ limits in all our J_i, which is precisely what is needed to justify the considerations of Section 1.

Proof of Lemma 1. We first consider the easiest case $J_1(x, T, \varepsilon)$. Of course, we have

$$\begin{aligned}\mathbb{E}\big(J_1(x,T,\varepsilon) - J_1(x',T',\varepsilon')\big)^m \le{}& c\mathbb{E}\big(J_1(x,T,\varepsilon) - J_1(x',T,\varepsilon)\big)^m \\ &+ c\mathbb{E}\big(J_1(x',T,\varepsilon) - J_1(x',T,\varepsilon')\big)^m \\ &+ c\mathbb{E}\big(J_1(x',T,\varepsilon') - J_1(x',T',\varepsilon')\big)^m.\end{aligned} \tag{2.7}$$

We use the identity

$$K^\varepsilon(x) = 2\int_{\mathbb{R}^2} e^{ip\cdot x}\frac{e^{-\varepsilon(|p|^2+1)/2}}{|p|^2+1}\,d^2p \tag{2.8}$$

to rewrite

$$J_1(x,T,\varepsilon) = 2\int_0^T\int_{\mathbb{R}^2} e^{ip\cdot(x-W_s)/2}\frac{e^{-\varepsilon(|p|^2+1)/2}}{|p|^2+1}\,d^2p\,ds. \tag{2.9}$$

Using this, the first term in (2.7) becomes

$$\begin{aligned} &\mathbb{E}\big(J_1(x,T,\varepsilon) - J_1(x',T,\varepsilon)\big)^m \\ &\quad = 2^m \int_{[0,T]^m} \int_{R^{2m}} \left[\prod_{l=1}^m \left(e^{ip^l\cdot x} - e^{ip^l \cdot x'}\right) \frac{e^{-\varepsilon(|p^l|^2+1)/2}}{|p^l|^2+1} \right] \\ &\qquad \times \mathbb{E}\left(\exp\left[i \sum_{l=1}^m p^l \cdot W_s \right] \right) ds\, dp. \end{aligned} \tag{2.10}$$

Let π denote a permutation of $\{1,2,\ldots,m\}$; and $\Delta(\pi) = \{(s_1,\ldots,s_m) | s_{\pi(l)} \le s_{\pi(l+1)}\}$. On $\Delta(\pi)$

$$\mathbb{E}\left(\exp\left[i \sum_{l=1}^m p^l \cdot W_{s_l} \right] \right) = \exp\left[- \sum_{l=1}^m |u^l|^2 \bar{s}_l / 2 \right], \tag{2.11}$$

where

$$u^l = \sum_{j \ge l} p^j, \qquad \bar{s}_l = s_{\pi(l)} - s_{\pi(l-1)}. \tag{2.12}$$

In (2.10) we first use the bounds, for any $\alpha < 1$

$$|e^{ip\cdot x} - e^{ip\cdot x'}| \le c_\alpha |p|^\alpha |x - x'|^\alpha, \tag{2.13}$$

then integrate $d\bar{s}$, using

$$\int_0^n e^{-v^2 s}\, ds < \frac{c}{1+v^2}, \tag{2.14}$$

to bound (2.10) by

$$\begin{aligned} &\mathbb{E}\big(J_1(x,T,\varepsilon) - J_1(x',T,\varepsilon)\big)^m \\ &\quad \le c|x-x'|^{\alpha m} \sum_\pi \int_{R^{2m}} \prod_{l=1}^m |p^l|^\alpha (1+|p^l|^2)^{-1} (1+|u^l|^2)^{-1}\, dp \\ &\quad \le c|x-x'|^{\alpha m}, \end{aligned} \tag{2.15}$$

since

$$\begin{aligned} &\int_{R^{2m}} \prod_{l=1}^m |p^l|^\alpha (1+|p^l|^2)^{-1} (1+|u^l|^2)^{-1}\, dp \\ &\quad \le \left(\int \prod_{l=1}^m \frac{|p^l|^{2\alpha}}{(1+|p^l|^2)^2}\, dp \right)^{1/2} \left(\int \prod_{l=1}^m (1+|u^l|^2)^{-2}\, dp \right)^{1/2} \\ &\quad < \infty \quad \text{if } \alpha < 1. \end{aligned} \tag{2.16}$$

[By (2.12) the u^l form a linear set of coordinates for R^{2m}.]

The second term in (2.7) is handled similarly. We use

$$\left| e^{-|p|^2 \varepsilon} - e^{-|p|^2 \varepsilon'} \right| \le c_\alpha \big(|p|^2 |\varepsilon - \varepsilon'| \big)^\alpha$$

in place of (2.13).

To handle the third term in (2.7) we return to (2.10) and bound

$$\begin{aligned}
&\int_{[T,T']^m\cap\Delta(\pi)}\exp\left[-\sum_{l=1}^m|u^l|^2\bar{s}_{l/2}\right]ds\\
&\quad\le|T-T'|^{m/q'}\left(\int_{[0,n]^m\cap\Delta(\pi)}\exp\left[-q\sum_{l=1}^m|u^l|^2\bar{s}_{l/2}\right]ds\right)^{1/q}\\
&\quad\le c|T-T'|^{m/q'}\prod_{l=1}^m\left(1+|u^l|^2\right)^{-1/q}\quad\text{if }\frac{1}{q}+\frac{1}{q'}=1.
\end{aligned}$$

From (2.16) we see that the third term will be bounded if $1/q>\frac{1}{2}$, i.e., $1/q'<\frac{1}{2}$.

This proves Lemma 1 for J_1.

Writing

$$J_2(x,T,\gamma,\varepsilon)=2\int_{D_{T,\gamma}}\int\int_{R^2}e^{ip\cdot(x-(W_t-W_s))}\frac{e^{-\varepsilon(|p|^2+1)/2}}{|p|^2+1}\,d^2p\,ds\,dt$$

and arguing as above, we see that it suffices to bound

$$\int_{[0,n]^{2m}}\int_{R^{2m}}\left[\prod_{l=1}^m\left(1+|p^l|^2\right)^{-1}\right]\mathbb{E}\left(\exp\left[i\sum_{l=1}^m p^l\cdot\left(W_{t_l}-W_{s_l}\right)\right]\right)dp\,ds\,dt. \tag{2.17}$$

Let

$$r_i=s_i,\qquad r_{i+m}=t_i.$$

Let now π be a permutation of $\{1,\dots,2m\}$ and let

$$\Delta(\gamma)=\left\{(r_1,\dots,r_{2m})|r_{\pi(l)}\le r_{\pi(l+1)}\right\}.$$

On each $\Delta(\pi)$

$$\mathbb{E}\left(\exp\left[i\sum_{l=1}^m p^l\cdot\left(W_{t_l}-W_{s_l}\right)\right]\right)=\exp\left[-\sum_{l=1}^{2m-1}|\bar{p}^l|^2\bar{r}_l\right],$$

where $\bar{r}_l=r_{\pi(l+1)}-r_{\pi(l)}$ and $\bar{p}^l$ is some linear combination of the p^j. Each p^i must appear in at least one $\bar{p}^l$—let v^i be the first $\bar{p}^l$ containing p^i. Then integrating out $ds\,dt$ and using (2.14) we see that (2.17) is bounded by

$$c\int_{R^{2m}}\prod_{l=1}^m\left(1+|p^l|^2\right)^{-1}\left(1+|v^l|^2\right)^{-1}dp<\infty,$$

using Cauchy–Schwarz as in (2.14) since the set $\{v^1,\dots,v^m\}$ forms a nonsingular set of coordinates.

To handle J_3 we use the well-known inequality [see Ikeda and Watanabe (1981), page 110]

$$\mathbb{E}\left(\left(\int f_t\,dW_t\right)^m\right)\le c\mathbb{E}\left(\left(\int f_t^2\,dt\right)^{m/2}\right)$$

to see that it suffices to bound

$$\int_{[0,n]^{m+m}/2}\int_{R^{2n}}\prod_{l=1}^{m}|p^l|(1+|p^l|^2)^{-1} \times \mathbb{E}\left(\exp\left[i\sum_{l=1}^{m}p^l\cdot\left(W_{t_{[l/2]}}-W_{s_l}\right)\right]\right)dp\,ds\,dt \tag{2.18}$$

and again we need only bound

$$\int_{R^{2m}}\prod_{l=1}^{m}|p^l|(1+|p^l|^2)^{-1}(1+|v^l|^2)^{-1}\,dp$$
$$\le\left(\int\prod_{l=1}^{m}\left(\frac{|p^l|}{1+|p^l|^2}\right)^3dp\right)^{1/3}\left(\int\prod_{l=1}^{m}(1+|v^l|^2)^{-3/2}\,dp\right)^{2/3}<\infty.$$

This completes the proof of Lemma 1. □

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