

STRONG BOUNDS FOR WEIGHTED EMPIRICAL DISTRIBUTION FUNCTIONS BASED ON UNIFORM SPACINGS

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Let U_1, U_2, \dots be a sequence of independent rv's having the uniform distribution on $(0, 1)$. Let \hat{F}_n be the empirical distribution function based on the transformed uniform spacings $\mathbb{D}_{i,n} := G(nD_{i,n})$, $i = 1, 2, \dots, n$, where G is the $\exp(1)$ df and $D_{i,n}$ is the i th spacing based on U_1, U_2, \dots, U_{n-1} . In this paper a complete characterization is obtained for the a.s. behaviour of $\limsup_{n \rightarrow \infty} b_n V_{n,\nu}$ and $\limsup_{n \rightarrow \infty} b_n W_{n,\nu}$, where $\nu \in [0, \frac{1}{2}]$, $\{b_n\}_{n=1}^\infty$ is a sequence of norming constants,

$$V_{n,\nu} = \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} \quad \text{and} \quad W_{n,\nu} = \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}}.$$

It turns out that compared with the i.i.d. case only $W_{n,\nu}$ behaves differently. The results imply, e.g., laws of the iterated logarithm for $\log(n^{\nu-1}V_{n,\nu})$ and $\log(n^{\nu-1}W_{n,\nu})$. Of independent interest is the theorem on the lower-upper class behaviour of the maximal spacing, which gives the final solution for this problem and generalizes some recent results in the literature.

1. Introduction and main results. Let U_1, U_2, \dots be a sequence of independent random variables (rv's), each having the uniform distribution on $(0, 1)$. For $n = 2, 3, \dots$ we define the transformed uniform spacings by $\mathbb{D}_{i,n} := G(nD_{i,n})$, where

$$D_{i,n} := U_{i:n-1} - U_{i-1:n-1}, \quad i = 1, 2, \dots, n, \\ 0 := U_{0:n-1} \leq U_{1:n-1} \leq \dots \leq U_{n-1:n-1} \leq U_{n:n-1} := 1$$

are the order statistics of the first $n-1$ rv's in the given sequence and G is defined by $G(x) = 1 - e^{-x}$, $x \in (0, \infty)$. Note that for $t \in (0, 1 - e^{-n}]$,

$$F_n(t) := P(\mathbb{D}_{i,n} \leq t) = P(nU_{1:n-1} \leq -\log(1-t)) = 1 - \left(1 + \frac{\log(1-t)}{n}\right)^{n-1},$$

and for $t \in (0, 1)$,

$$F_n(t) \rightarrow t, \quad \text{as } n \rightarrow \infty.$$

Finally, we define the empirical df \hat{F}_n based on the $\mathbb{D}_{i,n}$ by

$$\hat{F}_n(t) = n^{-1} \sum_{i=1}^n 1_{(0,t]}(\mathbb{D}_{i,n}), \quad t \in (0, 1).$$

We will establish a complete characterization of the almost sure behaviour of $\limsup_{n \rightarrow \infty} b_n V_{n,\nu}$ and $\limsup_{n \rightarrow \infty} b_n W_{n,\nu}$, for $\nu \in [0, \frac{1}{2}]$ and a sequence of

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positive norming constants $\{b_n\}_{n=1}^\infty$, where

$$(1.1) \quad V_{n,\nu} = \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}}$$

and

$$(1.2) \quad W_{n,\nu} = \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}}.$$

The study of these rv's is motivated by similar studies for the i.i.d. case. In that case for dimension one Csáki (1974, 1975, 1982) investigated the value $\nu = \frac{1}{2}$, Shorack and Wellner (1978) $\nu = 0$ and Mason (1981) $\nu \in (0, \frac{1}{2})$. Furthermore, Mason (1982, $\nu = 0$) and Einmahl and Mason (1985, $\nu \in [0, \frac{1}{2}]$) obtained the results in the multivariate i.i.d. case. Although in the i.i.d. case the results are known also for $\nu \in (\frac{1}{2}, 1]$, the behaviour for these values of the weighted empirical distribution function based on uniform spacings is still an interesting open question.

To prove the result for $W_{n,\nu}$ we need the final solution for a problem on the a.s. behaviour of the maximal uniform spacing. This result, which is of independent interest and generalizes the results of Slud (1978), Devroye (1981) and Deheuvels (1982), will be stated as our first theorem. For its presentation we denote, as usual, the ordered $D_{i,n}$, $i = 1, 2, \dots, n$, by

$$D_{1:n} \leq D_{2:n} \leq \dots \leq D_{n:n}.$$

THEOREM 1.1. *Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive constants. Then we have*

$$(1.3) \quad \left[\sum a_n \log(a_n^{-1}) = \infty \right] \Rightarrow \left[P(nD_{n:n} \geq \log(a_n^{-1}) \text{ i.o.}) = 1 \right].$$

The proof of this theorem is deferred to Section 2.

Next let us present our characterizations concerning $V_{n,\nu}$ and $W_{n,\nu}$ and some corollaries. The proofs of these are deferred to Section 3. In both theorems $\{a_n\}_{n=1}^\infty$ is again a sequence of positive constants.

THEOREM 1.2.A. *For each $\nu \in [0, \frac{1}{2}]$ we have*

$$(1.4) \quad \left[\sum a_n = \infty \right] \Rightarrow \left[\limsup_{n \rightarrow \infty} a_n^{1-\nu} V_{n,\nu} = \infty \text{ a.s.} \right]$$

and

$$(1.5) \quad \left[\sum a_n < \infty, a_n \downarrow \text{ and } na_n \log n \rightarrow 0 \right] \Rightarrow \left[\lim_{n \rightarrow \infty} a_n^{1-\nu} V_{n,\nu} = 0 \text{ a.s.} \right].$$

COROLLARY 1.1.A. *For each $\nu \in [0, \frac{1}{2}]$,*

$$(1.6) \quad \limsup_{n \rightarrow \infty} \frac{\log(n^{\nu-1} V_{n,\nu})}{\log \log n} = 1 - \nu \text{ a.s.}$$

THEOREM 1.2.B. *For each $\nu \in [0, \frac{1}{2}]$ we have*

$$(1.7) \quad \left[\sum a_n \log(a_n^{-1}) = \infty \right] \Rightarrow \left[\limsup_{n \rightarrow \infty} a_n^{1-\nu} W_{n,\nu} = \infty \text{ a.s.} \right]$$

and

$$(1.8) \quad \left[\sum a_n \log(a_n^{-1}) < \infty \text{ and } a_n \downarrow 0 \right] \Rightarrow \left[\lim_{n \rightarrow \infty} a_n^{1-\nu} W_{n,\nu} = 0 \text{ a.s.} \right].$$

COROLLARY 1.1.B. *For each $\nu \in [0, \frac{1}{2}]$,*

$$(1.9) \quad \limsup_{n \rightarrow \infty} \frac{\log(n^{\nu-1} W_{n,\nu})}{\log \log n} = 2(1 - \nu) \text{ a.s.}$$

COROLLARY 1.2.

$$(1.10) \quad \left[\sum a_n \log(a_n^{-1}) < \infty \text{ and } a_n \downarrow 0 \right] \Rightarrow \left[P(nD_{n:n} \geq \log(a_n^{-1}) \text{ i.o.}) = 0 \right].$$

Corollary 1.2 is more or less the same as Theorem 4.1 in Devroye (1981). The only difference is that our monotonicity condition on the a_n is somewhat milder than his conditions.

Combination of Theorems 1.2.A and 1.2.B yields for

$$Z_{n,\nu} := \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{(t(1-t))^{1-\nu}}:$$

COROLLARY 1.3. *Theorem 1.2.B and Corollary 1.1.B hold with $W_{n,\nu}$ replaced by $Z_{n,\nu}$.*

Comparison of Theorems 1.2.A and 1.2.B with the aforementioned results for the i.i.d. case shows that the result for $V_{n,\nu}$ is exactly the same as that for the one-dimensional i.i.d. case. Surprisingly, the behaviour of $W_{n,\nu}$ is different: It coincides with the result for dimension two(!) in the i.i.d. case.

2. Proof of Theorem 1.1. Before we give the actual proof we need some notation and some lemmas. For $n \in \mathbb{N}$ define the stochastic interval $I_n = (I_{n1}, I_{n2}]$, which is one of the intervals determined by two successive order statistics of U_1, U_2, \dots, U_{n-1} , by

$$(2.1) \quad I_n = (U_{i-1:n-1}, U_{i:n-1}], \text{ if } U_{i-1:n-1} < U_n \leq U_{i:n-1}.$$

The length of I_n is denoted by S_n . Next we define

$$(2.2) \quad T_n = \begin{cases} S_n, & \text{if } I_n \not\subset I_k, \forall \frac{2}{3}n < k < n, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, we have

$$(2.3) \quad T_n \leq S_n \leq D_{n:n}.$$

For the proof of Theorem 1.1 we shall need the following extension of the Borel–Cantelli lemma.

LEMMA 2.1 [Kochen and Stone (1964)]. *If for a sequence of events A_1, A_2, \dots on some probability space*

$$(2.4) \quad \sum PA_n = \infty \quad \text{and} \quad \liminf_{N \rightarrow \infty} \frac{\sum \sum_{1 \leq n < m \leq N} PA_n A_m}{(\sum_{n=1}^N PA_n)^2} < \infty,$$

then $P(A_n \text{ i.o.}) > 0$.

The next three lemmas are distributional results for S_n .

LEMMA 2.2. *Let $c \in (0, 1)$. Then*

$$(2.5) \quad P(S_n \geq c) = (1 - c)^{n-1} (1 + (n - 1)c).$$

The proof of this lemma is easy, uses only elementary probability theory and will be omitted.

LEMMA 2.3. *Let $0 < a \leq b < 1$ and $m > n$. Then*

$$(2.6) \quad \begin{aligned} &P(S_m \geq a; S_n \geq b; I_m \subset I_n) \\ &= (1 - a)^{m-n-2} ((m - n - 2)a + 1)n(n - 1) \\ &\quad \times \left\{ \frac{(1 - a^2)}{n - 1} (1 - b)^{n-1} - \frac{2}{n} (1 - b)^n + \frac{1}{n + 1} (1 - b)^{n+1} \right\} \\ &\quad - 2(m - n - 1)a^2(1 - a)^{m-n-2} \\ &\quad \times \{n(1 - a)(1 - b)^{n-1} - (n - 1)(1 - b)^n\}. \end{aligned}$$

PROOF. Observe that, with $f_n(x) = -dP(S_n \geq x)/dx$,

$$(2.7) \quad \begin{aligned} &P(S_m \geq a; S_n \geq b; I_m \subset I_n) \\ &= \int_b^1 f_n(x) P(S_m \geq a; I_m \subset I_n | S_n = x) dx \\ &= \int_b^1 f_n(x) P(S_m \geq a | I_m \subset I_n; S_n = x) P(I_m \subset I_n | S_n = x) dx \\ &= \int_b^1 x f_n(x) P(S_m \geq a | I_m \subset I_n; S_n = x) dx. \end{aligned}$$

For the conditional probability in this last expression we have

$$(2.8) \quad \begin{aligned} &P(S_m \geq a | I_m \subset I_n; S_n = x) \\ &= P(S_m \geq a | I_m \subset I_n; S_n = x; U_m \leq U_n) \\ &= 2x^{-2} \int_a^x y P(S_m \geq a | I_m \subset I_n; S_n = x; U_m \leq U_n; U_n - I_{n1} = y) dy. \end{aligned}$$

Now define the rv Z by $Z = \#\{i \in \mathbb{N}: n < i < m \text{ and } U_i \in (I_{n1}, U_n]\}$, where for a set A , $\#A$ denotes its cardinality. Using Z we see that the conditional probability in the last expression in (2.8) is equal to

$$\begin{aligned}
 & \sum_{k=0}^{m-n-1} P(S_m \geq a | Z = k; I_m \subset I_n; S_n = x; U_m \leq U_n; U_n - I_{n1} = y) \\
 & \quad \times P(Z = k | I_m \subset I_n; S_n = x; U_m \leq U_n; U_n - I_{n1} = y) \\
 (2.9) \quad & = \sum_{k=0}^{m-n-1} P(S_{k+1} \geq a/y) \binom{m-n-1}{k} y^k (1-y)^{m-n-k-1} \\
 & = \sum_{k=0}^{m-n-1} (1-a/y)^k (1+ka/y) \binom{m-n-1}{k} y^k (1-y)^{m-n-k-1}.
 \end{aligned}$$

Substituting (2.9) in (2.8) and in turn (2.8) in (2.7) we obtain

$$\begin{aligned}
 P(S_m \geq a; S_n \geq b; I_m \subset I_n) &= 2n(n-1) \int_b^1 (1-x)^{n-2} \\
 (2.10) \quad & \times \int_a^x \sum_{k=0}^{m-n-1} \binom{m-n-1}{k} y^{k+1} (1-y)^{m-n-k-1} \\
 & \quad \times (1-a/y)^k (1+ka/y) dy dx.
 \end{aligned}$$

Now elementary analysis, including Newton's binomial formula, shows that the last expression in (2.10) is equal to the last expression in (2.6). \square

COROLLARY 2.1. *Let $a \in (0, 1)$ and $m > n$. Then*

$$(2.11) \quad P(I_m \subset I_n | S_m \geq a) = 2/(n+1).$$

PROOF. Note that

$$(2.12) \quad P(I_m \subset I_n | S_m \geq a) = P(S_m \geq a; S_n \geq a; I_m \subset I_n) / P(S_m \geq a)$$

and apply Lemma 2.3 to the numerator and Lemma 2.2 to the denominator. \square

LEMMA 2.4. *Let $a, b \in (0, 1)$ and $m > n$. Then*

$$(2.13) \quad P(S_m \geq a; S_n \geq b; I_m \not\subset I_n) \leq P(S_m \geq a) P(S_n \geq b).$$

PROOF. By similar ideas as in the proof of Lemma 2.3 it can be shown that for $x \leq 1-a$,

$$\begin{aligned}
 & P(S_m \geq a | S_n = x; I_m \not\subset I_n) \\
 (2.14) \quad & = \sum_{k=0}^{m-n-1} (1-x)^k x^{m-n-k-1} \binom{m-n-1}{k} (1-a/(1-x))^{n-2+k} \\
 & \quad \times (1 + (n-2+k)a/(1-x)).
 \end{aligned}$$

This last expression is decreasing in x . Hence, according to Lemma 4 in Lehmann (1966) it follows that

$$(2.15) \quad P(S_m \geq a; S_n \geq b | I_m \not\subset I_n) \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b | I_m \not\subset I_n).$$

Using (2.15) and Corollary 2.1 we easily see

$$\begin{aligned}
 & P(S_m \geq a; S_n \geq b; I_m \not\subset I_n) \\
 &= P(S_m \geq a; S_n \geq b | I_m \not\subset I_n) P(I_m \not\subset I_n) \\
 (2.16) \quad & \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b | I_m \not\subset I_n) P(I_m \not\subset I_n) \\
 & \leq P(S_m \geq a | I_m \not\subset I_n) P(S_n \geq b) \\
 &= P(S_m \geq a) P(S_n \geq b) P(I_m \not\subset I_n | S_m \geq a) / P(I_m \not\subset I_n) \\
 &= P(S_m \geq a) P(S_n \geq b). \quad \square
 \end{aligned}$$

Now we are prepared to give the proof of Theorem 1.1. First, we restrict ourselves without loss of generality to a_n with $n^{-4/3} \leq a_n \leq e^{-1}$, which implies $1 \leq \log(a_n^{-1}) \leq \frac{4}{3} \log n$. [For $n^{-4/3} \leq a_n$ we need that $P(nD_n: n \geq \frac{4}{3} \log n \text{ i.o.}) = 0$, which follows from our Corollary 1.2 or from Theorem 2.1 in Slud (1978).] Note also that on account of (2.3) and the Hewitt–Savage 0–1 law it suffices to show that

$$(2.17) \quad \left[\sum a_n \log(a_n^{-1}) = \infty \right] \Rightarrow \left[P(nT_n \geq \log(a_n^{-1}) \text{ i.o.}) > 0 \right]:$$

We shall prove (2.17) with the aid of Lemma 2.1. Hence we need to prove (2.4) with $A_n = [T_n \geq c_n]$, where $c_n = n^{-1} \log(a_n^{-1})$.

Let us first consider $P[S_n \geq c_n]$. By Lemma 2.2 it is easily seen that we have for large n ,

$$\begin{aligned}
 & P(S_n \geq c_n) \geq (1 - c_n)^n (n - 1) c_n \\
 (2.18) \quad & \geq a_n (\log(a_n^{-1})) \left(\frac{n - 1}{n} \right) (1 - nc_n^2) \geq \frac{1}{2} a_n \log(a_n^{-1}).
 \end{aligned}$$

Hence we have $\sum P(S_n \geq c_n) = \infty$. We also have

$$\begin{aligned}
 & P(T_n \geq c_n) = P(T_n \geq c_n; T_n = S_n) + P(T_n \geq c_n; T_n \neq S_n) \\
 (2.19) \quad &= P(S_n \geq c_n; T_n = S_n) \\
 &= P(T_n = S_n | S_n \geq c_n) P(S_n \geq c_n).
 \end{aligned}$$

So for establishing $\sum P(T_n \geq c_n) = \infty$ we only have to show that

$$\liminf_{n \rightarrow \infty} P(T_n = S_n | S_n \geq c_n) > 0.$$

But by Corollary 2.1 we have

$$\begin{aligned}
 & P(T_n \neq S_n | S_n \geq c_n) = P(\exists_{k: 2n/3 < k < n} I_n \subset I_k | S_n \geq c_n) \\
 & \leq \sum_{k=[2n/3]+1}^{n-1} P(I_n \subset I_k | S_n \geq c_n) \\
 (2.20) \quad &= \sum_{k=[2n/3]+1}^{n-1} 2/(k+1) \\
 & \leq 2 \int_{2n/3}^n x^{-1} dx \\
 &= 2 \log \frac{3}{2} < 1.
 \end{aligned}$$

Hence $P(T_n = S_n | S_n \geq c_n) \geq 1 - 2\log_2 \frac{3}{2} := \delta > 0$, which proves that the first condition in Lemma 2.1 is fulfilled.

Now it remains to show that the second condition in Lemma 2.1 is fulfilled. It is sufficient to prove that for large n and $m > n$,

$$(2.21) \quad P(T_m \geq c_m; T_n \geq c_n) \leq (2\delta^{-2})P(T_m \geq c_m)P(T_n \geq c_n),$$

with δ as above. In the proof of (2.21) we need that for large n and $m \geq \frac{3}{2}n$,

$$(2.22) \quad P(S_m \geq c_m; S_n \geq c_n; I_m \subset I_n) \leq P(S_m \geq c_m)P(S_n \geq c_n).$$

We shall now establish (2.22). We have to distinguish between the cases $c_m \leq c_n$ and $c_m > c_n$. The case $c_m > c_n$ is the easiest one, hence its proof will be omitted. So let us assume $c_m \leq c_n$. By an application of Lemma 2.3, using $mc_m \geq 1$, we have

$$(2.23) \quad \begin{aligned} P(S_m \geq c_m; S_n \geq c_n; I_m \subset I_n) &\leq 12mc_mnc_n^2(1-c)^{m-n-2}(1-c_n)^{n-1} \\ &\leq 13mc_ma_mnc_n^2a_n(1-c_m)^{-n}. \end{aligned}$$

Now elementary analysis shows, using $m \geq \frac{3}{2}n$ and $a_k \geq k^{-4/3}$, $k = n, m$, that for large n ,

$$(2.24) \quad c_n(1-c_m)^{-n} \leq n^{-1/18} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

But (2.23) and (2.24), combined with (2.18), imply (2.22).

Now we consider (2.21) for large n and $m \geq \frac{3}{2}n$. Then we have

$$(2.25) \quad \begin{aligned} P(T_m \geq c_m; T_n \geq c_n) &\leq P(S_m \geq c_m; S_n \geq c_n) \\ &\leq P(S_m \geq c_m; S_n \geq c_n; I_m \subset I_n) \\ &\quad + P(S_m \geq c_m; S_n \geq c_n; I_m \not\subset I_n) \\ &\leq 2P(S_m \geq c_m)P(S_n \geq c_n) \\ &\leq 2\delta^{-2}P(T_m \geq c_m)P(T_n \geq c_n), \end{aligned}$$

where for the third inequality also Lemma 2.4 is applied. Next, we consider (2.21) for large n and $n < m < \frac{3}{2}n$. Then we have

$$(2.26) \quad \begin{aligned} P(T_m \geq c_m; T_n \geq c_n) &= P(T_m \geq c_m; T_n \geq c_n; I_m \subset I_n) + P(T_m \geq c_m; T_n \geq c_n; I_m \not\subset I_n) \\ &\leq P(S_m \geq c_m; S_n \geq c_n; I_m \not\subset I_n) \\ &\leq P(S_m \geq c_m)P(S_n \geq c_n) \\ &\leq \delta^{-2}P(T_m \geq c_m)P(T_n \geq c_n), \end{aligned}$$

which completes the proof of (2.21) and hence the proof of Theorem 1.1. \square

3. Proofs of Theorems 1.2.A and 1.2.B. Before giving the actual proofs we need some lemmas. Let $E_1, E_2, \dots, E_n, \dots$ be a sequence of independent exponentially distributed rv's with parameter 1; let H_n be the df of $\sum_{i=1}^n E_i$ [the gamma $(n, 1)$ df], and let \hat{F}_n be the empirical df based on U_1, U_2, \dots, U_n from the original sequence of independent uniform $(0, 1)$ rv's. We define $\mathbb{D}_{i,n}^* := 1 - \mathbb{D}_{i,n}$ for $i = 1, 2, \dots, n$; let \hat{F}_n^* denote the empirical df based on these $\mathbb{D}_{i,n}^*$ and note

that

$$W_{n,\nu} = \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\nu}} = \sup_{0 < t < 1} \frac{n|\hat{F}_n^*(t) - t|}{t^{1-\nu}}.$$

The first lemma is well known [see, e.g., Pyke (1965), Beirlant, van der Meulen, Ruymgaart and van Zuijlen (1982) and Devroye (1981)].

LEMMA 3.1.

(i) *The random vector $(E_1/n\bar{E}_n, E_2/n\bar{E}_n, \dots, E_n/n\bar{E}_n)$ is distributed as $(D_{1,n}, D_{2,n}, \dots, D_{n,n})$ and is independent of \bar{E}_n , where $\bar{E}_n := n^{-1}\sum_{i=1}^n E_i$.*

(ii) *Uniform spacings are exchangeable and negative lower orthant dependent (NLOD), i.e.,*

$$P(D_{1,n} \leq x_1; D_{2,n} \leq x_2; \dots; D_{n,n} \leq x_n) \leq \prod_{i=1}^n P(D_{i,n} \leq x_i).$$

(iii) *For nonnegative numbers c_1, c_2, \dots, c_n we have*

$$P(D_{1,n} \geq c_1; D_{2,n} \geq c_2; \dots; D_{n,n} \geq c_n) = \begin{cases} \left(1 - \sum_{i=1}^n c_i\right)^{n-1}, & \text{for } \sum_{i=1}^n c_i < 1, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 3.2. *For each $\lambda > 0$ and each interval $A \subset [0, 1]$ we have*

$$(3.1) \quad P\left(\sup_{t \in A} n^{1/2}(\hat{F}_n(t) - t) \geq \lambda\right) \leq \frac{1}{H_n(n)} P\left(\sup_{t \in A} n^{1/2}(\hat{F}_n(t) - t) \geq \lambda\right),$$

$$(3.2) \quad P\left(\sup_{t \in A} n^{1/2}(t - \hat{F}_n(t)) \geq \lambda\right) \leq \frac{1}{1 - H_n(n)} P\left(\sup_{t \in A} n^{1/2}(t - \hat{F}_n(t)) \geq \lambda\right).$$

PROOF. With $\tilde{t} := G^{-1}(t)$ we have with the aid of Lemma 3.1

$$\begin{aligned} & P\left(\sup_{t \in A} n^{1/2}(\hat{F}_n(t) - t) \geq \lambda\right) \\ &= P\left(\sup_{t \in A} n^{1/2}\left(n^{-1} \sum_{i=1}^n 1_{(0, \tilde{t}]}(\mathbb{D}_{i,n}) - t\right) \geq \lambda\right) \\ &= P\left(\sup_{t \in A} n^{1/2}\left(n^{-1} \sum_{i=1}^n 1_{(0, \tilde{t}]}(E_i/\bar{E}_n) - t\right) \geq \lambda \mid \bar{E}_n \leq 1\right) \\ &= P\left(\sup_{t \in A} n^{1/2}\left(n^{-1} \sum_{i=1}^n 1_{(0, \tilde{t}]}(E_i/\bar{E}_n) - t\right) \geq \lambda; \bar{E}_n \leq 1\right) (P(\bar{E}_n \leq 1))^{-1} \\ &\leq P\left(\sup_{t \in A} n^{1/2}\left(n^{-1} \sum_{i=1}^n 1_{(0, \tilde{t}]}(E_i) - t\right) \geq \lambda; \bar{E}_n \leq 1\right) (H_n(n))^{-1} \\ &\leq (H_n(n))^{-1} P\left(\sup_{t \in A} n^{1/2}(\hat{F}_n(t) - t) \geq \lambda\right), \end{aligned}$$

which shows (3.1). The proof of (3.2) can be given in a similar way by

conditioning on the event $[\bar{E}_n \geq 1]$. Compare with Einmahl (1985), where a similar technique is used. \square

LEMMA 3.3. *There exist $c_1, c_2, c_3 \in (0, \infty)$ such that for every $\lambda > 0$ and every $\theta \in (0, 1)$,*

$$(3.3) \quad \max \left(P \left(\sup_{t \geq \theta} \frac{n^{1/2} |\hat{F}_n(t) - t|}{t^{1/2}} \geq \lambda \right), P \left(\sup_{t \geq \theta} \frac{n^{1/2} |\hat{F}_n^*(t) - t|}{t^{1/2}} \geq \lambda \right) \right) \\ \leq c_1 \left(\log \frac{2}{\theta} \right) \exp \left(-c_2 \lambda^2 \psi \left(c_3 \lambda (n\theta)^{-1/2} \right) \right),$$

where $\psi: [0, \infty] \rightarrow (0, \infty)$ satisfies $\psi(\sigma) \sim 2\sigma^{-1} \log \sigma$ as $\sigma \rightarrow \infty$.

PROOF. The given upper bound in the inequality of the lemma can be obtained as in Einmahl, Ruymgaart and Wellner (1984) from the well-known upper bounds for the right-hand sides of the inequalities in Lemma 3.2, taking into account that

$$P \left(\sup_{t \geq \theta} \frac{n^{1/2} |\hat{F}_n^*(t) - t|}{t^{1/2}} \geq \lambda \right) = P \left(\sup_{t \leq 1-\theta} \frac{n^{1/2} |\hat{F}_n(t) - t|}{(1-t)^{1/2}} \geq \lambda \right),$$

and that both $(H_n(n))^{-1}$ and $(1 - H_n(n))^{-1}$ are bounded above by some $M \in (0, \infty)$. \square

PROOF OF THEOREM 1.2.A. Throughout the proof let $\nu \in [0, \frac{1}{2}]$ be fixed. Suppose that $\Sigma a_n = \infty$. From Theorem 3.1 in Devroye (1982) we see that $P(\mathbb{D}_{1:n} \leq a_n \text{ i.o.}) = 1$ under the extra condition that $a_n/n \downarrow$. With the aid of $[\Sigma_{n=1}^\infty a_n = \infty] \Rightarrow [P(U_{1:n} \leq a_n \text{ i.o.}) = 1]$ and a suitable transformation it can be shown that this extra condition is superfluous.

Hence

$$(3.4) \quad P(\mathbb{D}_{1:n} < \varepsilon a_n \text{ i.o.}) = 1, \quad \text{for every } \varepsilon > 0.$$

The inequality $V_{n,\nu} \geq (2\mathbb{D}_{1:n}^{1-\nu})^{-1}$ and (3.4) easily lead to (1.4) if we let $\varepsilon \downarrow 0$.

Next we suppose that $\Sigma a_n < \infty$, $a_n \downarrow$ and $na_n \log n \rightarrow 0$ as $n \rightarrow \infty$. Define

$$(3.5) \quad V_n^\pm := \sup_{0 < t \leq \log n/n} \frac{\pm n(F_n(t) - t)}{t^{1-\nu}}$$

and let us start by showing that $\limsup_{n \rightarrow \infty} a_n^{1-\nu} V_n^+ \leq 1$ a.s., hence that $\lim_{n \rightarrow \infty} a_n^{1-\nu} V_n^+ = 0$ a.s. According to a version of the Borel–Cantelli lemma we need to show that $\Sigma PC_n < \infty$ and $PA_n \rightarrow 0$ as $n \rightarrow \infty$, where

$$A_n := \left\{ V_n^+ \geq \frac{1}{a_n^{1-\nu}} \right\} \quad \text{and} \quad C_n := A_n A_{n-1}^c.$$

Let $b_n := \log n/n$ and let $y_{k,n}$ be the solutions of the equations, $n = 1, 2, \dots$,

$$f_n(t) = k, \quad \text{for } k = 0, 1, \dots, k_n := \lfloor f_n(b_n) \rfloor + 1,$$

where

$$(3.6) \quad f_n(t) := nt + \left(\frac{t}{a_n} \right)^{1-\nu}, \quad t \in [0, \infty).$$

Note that

$$(3.7) \quad k_n = 1 + \left\lceil \log n + \left(\frac{\log n}{na_n} \right)^{1-\nu} \right\rceil.$$

Clearly, since $na_n \log n \rightarrow 0$ we have

$$(3.8) \quad k_n \leq 3 \left(\frac{\log n}{na_n} \right)^{1-\nu}, \quad \text{for large } n.$$

Moreover, let $x_k \equiv x_{k,n} := y_{k,n} \wedge b_n$. Observe that for large n , f_n is increasing in t , $f_{n-1} \leq f_n$ since $a_n \downarrow$, $b_{n-1} \geq b_n$, $x_{k_n} = b_n$ and $x_k \leq k^{1/(1-\nu)} a_n$. We have

$$\begin{aligned} C_n &= \left\{ V_n^+ \geq \frac{1}{a_n^{1-\nu}}; V_{n-1}^+ < \frac{1}{a_{n-1}^{1-\nu}} \right\} \\ &= \left\{ \sup_{0 < t \leq b_n} \frac{\hat{F}_n(t) - t}{t^{1-\nu}} \geq \frac{1}{na_n^{1-\nu}}; \sup_{0 < t \leq b_{n-1}} \frac{\hat{F}_{n-1}(t) - t}{t^{1-\nu}} < \frac{1}{(n-1)a_{n-1}^{1-\nu}} \right\} \\ (3.9) \quad &\subset \left\{ \exists_{t \in (0, b_n]} n\hat{F}_n(t) \geq f_n(t); \forall_{t \in (0, b_n]} (n-1)\hat{F}_{n-1}(t) < f_n(t) \right\} \\ &\subset \bigcup_{k=1}^{k_n} \left\{ \exists_{t \in (x_{k-1}, x_k]} n\hat{F}_n(t) = k; \forall_{t \in (x_{k-1}, x_k]} (n-1)\hat{F}_{n-1}(t) \leq k-1 \right\} \\ &\subset \bigcup_{k=1}^{n \wedge k_n} B_{n,k}, \end{aligned}$$

where

$$(3.10) \quad B_{n,k} := \{ \mathbb{D}_{k:n-1} > x_k, \mathbb{D}_{k:n} \leq x_k \}$$

and where $\mathbb{D}_{1:n} \leq \mathbb{D}_{2:n} \leq \dots \leq \mathbb{D}_{n:n}$ are the ordered $\mathbb{D}_{i,n}$, $i = 1, 2, \dots, n$. Define

$$A_{n,k} := \sum_{j=3}^n 1_{(0, z_k]}(D_{j,n}),$$

where

$$z_k \equiv z_{k,n} := G^{-1}(x_k)/n.$$

By conditioning on the event $\{U_{n-1} = U_{1:n-1}\}$ and using the independence of the order statistics and the ranks, we find for $n \geq 3$,

$$\begin{aligned} PB_{n,k} &= P\left(\sum_{j=1}^{n-1} 1_{(0, (n/(n-1))z_k]}(D_{j,n-1}) \leq k-1; \sum_{j=1}^n 1_{(0, z_k]}(D_{j,n}) \geq k\right) \\ &\leq P(D_{1,n} \leq z_k; D_{2,n} \leq z_k; A_{n,k} = k-2) \\ &\quad + P(D_{2,n} \leq z_k; A_{n,k} = k-1) + P(D_{1,n} \leq z_k; A_{n,k} = k-1). \end{aligned}$$

Because of the exchangeability and the negative lower orthant dependence of uniform spacings we have [by convention $\binom{n}{k} = 0$ for $k < 0$ or $k > n$]

$$P(D_{1,n} \leq z_k; D_{2,n} \leq z_k; A_{n,k} = k-2) \leq \binom{n-2}{k-2} (F_n(x_k))^k.$$

Moreover,

$$P(D_{2,n} \leq z_k; A_{n,k} = k-1) \leq \binom{n-2}{k-1} (F_n(x_k))^k$$

and

$$P(D_{1,n} \leq z_k; A_{n,k} = k-1) \leq \binom{n-2}{k-1} (F_n(x_k))^k,$$

so that we obtain $(F_n(x) \leq ex$ for $x \in [0, \frac{1}{2}]$)

$$\begin{aligned} PB_{n,k} &\leq 3 \binom{n-1}{k-1} (F_n(x_k))^k \\ &\leq 3 \binom{n-1}{k-1} (ex_k)^k \\ &\leq 3e^k \frac{n^{k-1}}{(k-1)!} k^{k/(1-\nu)} a_n^k \\ (3.11) \quad &\leq 3a_n e^k (na_n)^{k-1} \frac{k^{k/(1-\nu)+1}}{k!} \\ &\leq 3a_n e^{2k} (na_n)^{k-1} k^{k\nu/(1-\nu)+1} \\ &\leq 3a_n e^{2k} (na_n k_n^{\nu/(1-\nu)})^{k-1} k^{1/(1-\nu)}, \end{aligned}$$

where

$$na_n k_n^{\nu/(1-\nu)} \leq 3(na_n \log n)^{1/2}.$$

Since $na_n \log n \rightarrow 0$ as $n \rightarrow \infty$, it follows that $PC_n \leq M_\nu a_n$ for certain $M_\nu \in (0, \infty)$ (for large n) and hence $\sum PC_n < \infty$.

For the proof that $PA_n \rightarrow 0$ we note that for large n ,

$$\begin{aligned}
 PA_n &\leq P\left(\exists_{t \in (0, b_n]} n\hat{F}_n(t) \geq f_n(t)\right) \\
 &\leq P\left(\bigcup_{k=1}^{n \wedge k_n} \left\{ \sup_{t \leq x_k} n\hat{F}_n(t) \geq k \right\}\right) \\
 &\leq \sum_{k=1}^{n \wedge k_n} P(\mathbb{D}_{k: n} \leq x_k) \\
 (3.12) \quad &\leq \sum_{k=1}^{n \wedge k_n} \binom{n}{k} (F_n(x_k))^k \\
 &\leq \sum_{k=1}^{n \wedge k_n} \frac{n}{k} \binom{n-1}{k-1} (ex_k)^k \\
 &\leq \frac{1}{3} M_\nu n a_n \rightarrow 0, \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

which completes the proof of

$$(3.13) \quad \lim_{n \rightarrow \infty} a_n^{1-\nu} V_n^+ = 0 \quad \text{a.s.}$$

Remark that also

$$(3.14) \quad a_n^{1-\nu} V_n^- \leq (n a_n \log n)^{1-\nu} (\log n)^{2\nu-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, Lemma 3.3 immediately yields for large n ,

$$\begin{aligned}
 &P\left(\sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} \geq \frac{1}{a_n^{1-\nu}}\right) \\
 &\leq P\left(\sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1/2}} \geq \frac{1}{a_n^{1/2}}\right) \leq \frac{1}{n^2},
 \end{aligned}$$

so that because of the Borel–Cantelli lemma

$$(3.15) \quad \lim_{n \rightarrow \infty} a_n^{1-\nu} \sup_{\log n/n < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.}$$

Combination of the obtained partial results in (3.13)–(3.15) yields

$$(3.16) \quad \lim_{n \rightarrow \infty} a_n^{1-\nu} \sup_{0 < t < 1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.} \quad \square$$

PROOF OF THEOREM 1.2.B. We follow the lines of the proof of Theorem 1.2.A; however, the tools and the calculations will be different. Let $\nu \in [0, \frac{1}{2}]$ be fixed. Suppose that $\sum a_n \log(a_n^{-1}) = \infty$. Let $\mathbb{D}_{1,n}^* \leq \mathbb{D}_{2,n}^* \leq \dots \leq \mathbb{D}_{n,n}^*$ denote the ordered $\mathbb{D}_{i,n}^*$, $i = 1, 2, \dots, n$. From Theorem 1.1 we have

$$\begin{aligned} P(nD_{n:n} \geq \log(a_n^{-1}) \text{ i.o.}) &= P(\mathbb{D}_{n:n} \geq 1 - a_n \text{ i.o.}) \\ &= P(\mathbb{D}_{1:n}^* \leq a_n \text{ i.o.}) = 1, \end{aligned}$$

and hence also $P(\mathbb{D}_{1:n}^* \leq \varepsilon a_n \text{ i.o.}) = 1$ for every $\varepsilon > 0$. Now (1.7) follows by the argument used below (3.4) since $W_{n,\nu} \geq (2(\mathbb{D}_{1:n}^*)^{1-\nu})^{-1}$.

For the proof of (1.8) we suppose that $\sum a_n \log a_n^{-1} < \infty$ and $a_n \downarrow 0$ (which implies that $na_n \log n \rightarrow 0$ as $n \rightarrow \infty$ and $\sum a_n \log n < \infty$). Without loss of generality we restrict ourselves to sequences $\{a_n\}_{n=1}^\infty$ with $a_n \geq n^{-3/2}$. Define

$$(3.17) \quad W_n^\pm := \sup_{0 < t < \log n/n} \frac{\pm (\hat{F}_n^*(t) - t)}{t^{1-\nu}}.$$

To obtain $\lim_{n \rightarrow \infty} a_n^{1-\nu} W_n^+ = 0$ a.s. we need to show that $\sum PC_n^* < \infty$ and $PA_n^* \rightarrow 0$ as $n \rightarrow \infty$, where now

$$(3.18) \quad A_n^* := \left\{ W_n^+ \geq \frac{1}{a_n^{1-\nu}} \right\} \quad \text{and} \quad C_n^* := A_n^* A_{n-1}^{*c}.$$

Let b_n , $y_{k,n}$, $f_n(t)$, k_n and x_k be the quantities which are defined in the proof of Theorem 1.2.A. Note that

$$(3.19) \quad (2n^{3/2})^{-1} \leq x_k \leq k^{1/(1-\nu)} a_n,$$

and similarly as in (3.9) we have

$$\begin{aligned} (3.20) \quad C_n^* &= \left\{ W_n^+ \geq 1/a_n^{1-\nu}; W_{n-1}^+ < \frac{1}{a_n^{1-\nu}} \right\} \\ &\subset \bigcup_{k=1}^{n \wedge k_n} B_{n,k}^*, \end{aligned}$$

where now

$$(3.21) \quad B_{n,k}^* := \{\mathbb{D}_{k:n-1}^* > x_k; \mathbb{D}_{k:n}^* \leq x_k\}.$$

Letting

$$\begin{aligned} (3.22) \quad z_k^* &\equiv z_{k,n}^* := \frac{G^{-1}(1-x_k)}{n} = \frac{\log(x_k^{-1})}{n}, \\ u_k &:= \frac{n}{n-1} z_k^*, \\ I_n &:= \sum_{i=1}^n 1_I(D_{i,n}), \quad \text{for intervals } I, \end{aligned}$$

we obtain

$$\begin{aligned}
 PB_{n,k}^* &= P\left(\sum_{j=1}^{n-1} 1_{[1-x_k, 1]}(\mathbb{D}_{j, n-1}) \leq k-1; \sum_{j=1}^n 1_{[1-x_k, 1]}(\mathbb{D}_{j, n}) \geq k\right) \\
 (3.23) \quad &= P\left(\sum_{j=1}^{n-1} 1_{(0, 1-x_k)}(\mathbb{D}_{j, n-1}) \geq n-k; \sum_{j=1}^n 1_{(0, 1-x_k)}(\mathbb{D}_{j, n}) \leq n-k\right) \\
 &= P((0, u_k)_{n-1} \geq n-k; (0, z_k^*)_n \leq n-k) \\
 &= P_1 + P_2 + P_3,
 \end{aligned}$$

where

$$\begin{aligned}
 P_1 &:= P((0, u_k)_{n-1} \geq n-k; (0, z_k^*)_n \leq n-k; [z_k^*, u_k)_{n-1} = 0), \\
 P_2 &:= P((0, u_k)_{n-1} \geq n-k; (0, z_k^*)_n \leq n-k; [z_k^*, u_k)_{n-1} = 1), \\
 P_3 &:= P((0, u_k)_{n-1} \geq n-k; (0, z_k^*)_n \leq n-k; [z_k^*, u_k)_{n-1} \geq 2).
 \end{aligned}$$

Next, we define the rv's

$$\begin{aligned}
 J_{n,k} &:= \sum_{i=3}^n 1_{(0, z_k^*)}(D_{i,n}), & K_{n,k} &:= \sum_{i=4}^n 1_{(0, z_k^*)}(D_{i,n}), \\
 (3.24) \quad S_{n,k} &:= \sum_{i=3}^n 1_{[u_k, 1]}(D_{i,n}), & T_{n,k} &:= \sum_{i=4}^n 1_{[u_k, 1]}(D_{i,n}).
 \end{aligned}$$

We obtain, again by conditioning on the event $\{U_{n-1} = U_{1:n-1}\}$, with the aid of Lemma 3.1 for n sufficiently large,

$$\begin{aligned}
 P_1 &= P(J_{n,k} = n-k; D_{1,n} \geq z_k^*; D_{2,n} \geq z_k^*; S_{n,k} = k-2) \\
 &\leq \binom{n-2}{k-2} (1 - kz_k^*)^{n-1} \\
 (3.25) \quad &= \binom{n-2}{k-2} \left(1 - \frac{k \log(x_k^{-1})}{n}\right)^{n-1} \\
 &\leq \binom{n-2}{k-2} x_k^{k(n-1)/n} \\
 &\leq \binom{n-1}{k-1} (2x_k)^k.
 \end{aligned}$$

Moreover, observe that

$$\begin{aligned}
 P_2 &= P((0, z_k^*)_{n-1} \geq n-k-1; (0, z_k^*)_n \leq n-k; [z_k^*, u_k)_{n-1} = 1) \\
 &= P_{21} + P_{22} + P_{23},
 \end{aligned}$$

where

$$\begin{aligned}
 P_{21} &:= P((0, z_k^*)_{n-1} \geq n - k - 1; (0, z_k^*)_n \leq n - k; \\
 &\quad [z_k^*, u_k)_{n-1} = 1; D_{1,n} + D_{2,n} < z_k^*), \\
 P_{22} &:= P((0, z_k^*)_{n-1} \geq n - k - 1; (0, z_k^*)_n \leq n - k; \\
 &\quad [z_k^*, u_k)_{n-1} = 1; D_{1,n} + D_{2,n} \geq u_k), \\
 P_{23} &:= P((0, z_k^*)_{n-1} \geq n - k - 1; (0, z_k^*)_n \leq n - k; \\
 &\quad [z_k^*, u_k)_{n-1} = 1; D_{1,n} + D_{2,n} \in [z_k^*, u_k)).
 \end{aligned}$$

We obtain, similarly as in (3.25), with the aid of the mean-value theorem for n sufficiently large,

$$\begin{aligned}
 P_{21} &= (n-2)P(K_{n,k} = n - k - 2; D_{3,n} \in [z_k^*, u_k); \\
 &\quad D_{1,n} + D_{2,n} < z_k^*; T_{n,k} = k - 1) \\
 &\leq (n-2) \binom{n-3}{k-1} P(D_{1,n} \geq u_k; D_{2,n} \geq u_k; \dots; D_{k-1,n} \geq u_k; \\
 &\quad D_{k,n} \in [z_k^*, u_k)) \\
 &= (n-2) \binom{n-3}{k-1} \{ (1 - (k-1)u_k - z_k^*)^{n-1} - (1 - ku_k)^{n-1} \} \\
 &\leq (n-2) \binom{n-3}{k-1} z_k^* (1 - kz_k^*)^{n-2} \\
 &\leq (n-2) \binom{n-3}{k-1} \frac{\log(x_k^{-1})}{n} x_k^{k((n-2)/n)} \\
 &\leq \binom{n-1}{k-1} x_k^{k((n-2)/n)} \log(x_k^{-1}) \\
 &\leq \binom{n-1}{k-1} (2x_k)^k \log(x_k^{-1}), \\
 P_{22} &= 2(n-2)P(K_{n,k} = n - k - 1; D_{3,n} \in [z_k^*, u_k); D_{1,n} + D_{2,n} \geq u_k; \\
 &\quad D_{1,n} < z_k^*; D_{2,n} \geq z_k^*; T_{n,k} = k - 2) \\
 &\quad + (n-2)P(K_{n,k} = n - k; D_{3,n} \in [z_k^*, u_k); D_{1,n} \geq z_k^*; \\
 &\quad D_{2,n} \geq z_k^*; T_{n,k} = k - 3) \\
 &\leq 2(n-2) \binom{n-3}{k-2} P(D_{1,n} \geq z_k^*; D_{2,n} \geq z_k^*; \dots; D_{k-1,n} \geq z_k^*; \\
 &\quad D_{k,n} \in [z_k^*, u_k)) \\
 &\quad + (n-2) \binom{n-3}{k-3} P(D_{1,n} \geq z_k^*; D_{2,n} \geq z_k^*; \dots; D_{k-1,n} \geq z_k^*; \\
 &\quad D_{k,n} \in [z_k^*, u_k)) \\
 &\leq 3(n-2) \binom{n-1}{k-1} z_k^* (1 - kz_k^*)^{n-2} \\
 &\leq 3 \binom{n-1}{k-1} x_k^{k((n-2)/n)} \log(x_k^{-1}) \\
 &\leq 3 \binom{n-1}{k-1} (2x_k)^k \log(x_k^{-1})
 \end{aligned}$$

and

$$\begin{aligned}
 P_{23} &= 2P(J_{n,k} = n - k - 1; D_{1,n} + D_{2,n} \in [z_k^*, u_k); D_{1,n} < z_k^*; \\
 &\quad D_{2,n} \geq z_k^*; S_{n,k} = k - 1) \\
 &\leq 2 \binom{n-1}{k-1} P(D_{1,n} \geq u_k; D_{2,n} \geq u_k; \dots; D_{k-1,n} \geq u_k; D_{k,n} \geq z_k^*) \\
 &\leq 2 \binom{n-2}{k-1} (1 - kz_k^*)^{n-1} \\
 &\leq 2 \binom{n-1}{k-1} (2x_k)^k.
 \end{aligned}$$

Finally, for n sufficiently large,

$$\begin{aligned}
 P_3 &\leq P([z_k^*, u_k)_{n-1} \geq 2) \\
 &\leq n^2 P(D_{1,n-1} \in [z_k^*, u_k); D_{2,n-1} \in [z_k^*, u_k)) \\
 &= n^2 \left((1 - 2z_k^*)^{n-2} - (1 - z_k^* - u_k)^{n-2} \right. \\
 &\quad \left. - ((1 - u_k - z_k^*)^{n-2} - (1 - 2u_k)^{n-2}) \right) \\
 &\leq n^2 \left\{ \frac{n-2}{n-1} z_k^* (1 - 2z_k^*)^{n-3} - \frac{n-2}{n-1} z_k^* (1 - 2u_k)^{n-3} \right\} \\
 &\leq 2n^2 z_k^{*2} (1 - 2z_k^*)^{n-4} \\
 &\leq 2n^2 \left(\frac{\log x_k^{-1}}{n} \right)^2 x_k^{2((n-4)/n)} \\
 &\leq 2(2x_k)^2 (\log x_k^{-1})^2,
 \end{aligned}$$

so that

$$\begin{aligned}
 (3.26) \quad PB_{n,k} &\leq 7 \binom{n-1}{k-1} (2x_k)^k \log(x_k^{-1}) + 2(2x_k)^2 (\log(x_k^{-1}))^2 \\
 &\leq 11 \binom{n-1}{k-1} (2x_k)^k \log n + 5 \left(\frac{2 \log n}{n} \right)^2 (\log n)^2.
 \end{aligned}$$

As in (3.11) we obtain (since $a_n \geq n^{-3/2}$) for certain $M_\nu^* \in (0, \infty)$,

$$\begin{aligned}
 PC_n^* &\leq \sum_{k=1}^{k_n} 11 \log na_n e^{2k} (3(na_n \log n)^{1/2})^{k-1} k^{1/(1-\nu)} \\
 &\quad + 60 \left(\frac{\log n}{na_n} \right)^{1-\nu} \frac{(\log n)^4}{n^2} \\
 &\leq M_\nu^* (a_n \log n + (\log n)^5 n^{-3/2}),
 \end{aligned}$$

so that $\Sigma PC_n^* < \infty$. Moreover,

$$\begin{aligned}
 (3.27) \quad PA_n^* &\leq \sum_{k=1}^{n \wedge k_n} P(\mathbb{D}_{k:n}^* \leq x_k) \\
 &\leq \sum_{k=1}^{n \wedge k_n} \binom{n}{k} (1 - kz_k^*)^{n-1} \\
 &\leq \sum_{k=1}^{n \wedge k_n} \binom{n}{k} (2x_k)^k \rightarrow 0, \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

which completes the proof of

$$(3.28) \quad \lim_{n \rightarrow \infty} a_n^{1-\nu} W_n^+ = 0 \quad \text{a.s.}$$

Finally, from the reasoning in (3.13)–(3.16) with V_n^\pm replaced by W_n^\pm and \hat{F}_n replaced by \hat{F}_n^* it follows that

$$(3.29) \quad \lim_{n \rightarrow \infty} a_n^{1-\nu} \sup_{0 < t < 1} \frac{n|F_n^*(t) - t|}{t^{1-\nu}} = 0 \quad \text{a.s.},$$

which completes the proof of (1.8). \square

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