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ON LOG-CONCAVE AND LOG-CONVEX INFINITELY DIVISIBLE SEQUENCES AND DENSITIES

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We consider nonnegative infinitely divisible random variables whose Lévy measures are either absolutely continuous or supported by the integers. Necessary conditions are found ensuring that such distributions are log-concave or log-convex.

1. Introduction. Log-concavity and log-convexity of functions and sequences in probability have been of interest to several authors, e.g., Karlin (1968). Ibragimov (1956) calls a distribution strongly unimodal if its convolution with any unimodal distribution is unimodal. He proves that the set of strongly unimodal probability densities is equal to the set of log-concave densities. An equivalent result for log-concave discrete probability distributions has been proved by Keilson and Gerber (1971). Much work has been done on the unimodality of infinitely divisible distributions [cf. Yamazato (1978) and Sato and Yamazato (1978)], but little on strong unimodality. The study of log-concave functions and sequences is thus a relatively unknown field in probability, with important applications in the fields of statistics and optimization. Log-convexity is of interest in the study of reliability and of infinitely divisible random variables. Steutel (1970) proves that all log-convex discrete probability distributions are infinitely divisible. The absolutely continuous analogue is also proved in Steutel (1970).

In this note we consider distributions of nonnegative infinitely divisible random variables whose Lévy measures are either absolutely continuous or supported by the integers. We prove that for such distributions to be log-concave (log-convex), it is necessary that their Lévy measures be log-concave (log-convex). Our results in the discrete case contain an analogue of Yamazato's (1982) concavity result (it also provides an alternative proof of this result), and an analogue to the convexity result for renewal sequences in de Bruijn and Erdös (1953).

2. Discrete distributions. In this section we consider infinitely divisible discrete probability distributions $(p_n)_{n=0}^{\infty}$ on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. All sequences considered here will be real-valued and indexed by $\mathbb{N}_0$: they are denoted by (a_n), (p_n) etc. A sequence (a_n) is log-concave if (a_n) is nonnegative and $(\log(a_n))$ is concave, or equivalently if $a_n \geq 0$ and

$$a_n^2 \geq a_{n+1}a_{n-1}, \qquad n = 1, 2, 3, \dots . \tag{1}$$

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If the sequence satisfies (1) with strict inequality, then the sequence is said to be strictly log-concave. Similarly, (a_n) is log-convex if $a_n \geq 0$ and the sequence satisfies

$$a_n^2 \leq a_{n+1}a_{n-1}, \qquad n = 1,2,3,\ldots. \tag{2}$$

(a_n) is said to be strictly log-convex if (2) is satisfied with strict inequality.

A probability distribution (p_n) on $\mathbb{N}_0$ with $p_0 > 0$ is infinitely divisible if and only if it satisfies

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \qquad n = 0,1,2,\ldots, \tag{3}$$

with nonnegative r_k and, necessarily, $\sum_{k=0}^{\infty} r_k/(k+1) < \infty$ [cf. Steutel (1970)]. All log-convex distributions are infinitely divisible. This is easily proved by induction since

$$r_n p_n p_0 = p_n p_{n+1} + \sum_{k=0}^{n-1} r_k(p_{n+1}p_{n-k-1} - p_{n-k}p_n)$$

is positive if (p_n) is strictly log-convex and noting that any log-convex sequence can be written as a limit of strictly log-convex sequences. Not all log-concave distributions are infinitely divisible since

$$r_1 = p_0^{-2}(2p_2p_0 - p_1^2)$$

is not necessarily nonnegative when (p_n) is log-concave.

The proofs of the main theorems in this section rely on two equations derived from (3). Though easily verified using (3), the equations were rather hard to find. Because of their importance we state them in a lemma.

LEMMA 1. *Let* (p_n) *and* (r_n) *be related by* (3) *and let* $p_{-1} := 0$. *Then*

$$\begin{aligned} &m(m+2)(p_{m+1}^2 - p_m p_{m+2}) \\ &\quad = p_{m+1}(r_0 p_m - p_{m+1}) \\ &\qquad + \sum_{l=0}^{m}\sum_{k=0}^{l}(p_{m-1}p_{m-k-1} - p_{m-k}p_{m-l-1})(r_{k+1}r_l - r_{l+1}r_k), \end{aligned} \tag{4}$$

$$\begin{aligned} &r_{m+1}(m+2)(p_{m+1}p_{m+3} - p_{m+2}^2) \\ &\quad = p_{m+1}(r_{m+2}p_{m+2} - r_{m+1}p_{m+3}) \\ &\qquad + \sum_{k=0}^{m}(p_{m-k}p_{m+2} - p_{m+1}p_{m-k+1})(r_{m+2}r_k - r_{k+1}r_{m+1}). \end{aligned} \tag{5}$$

Relation (4) is a discrete analogue of equation (10) in Yamazato (1982), whereas (5) is an analogue of formula (7) used by Bruijn and Erdös (1953). We shall need the following lemma.

LEMMA 2. *Let* (p_n) *and* (r_n) *be related by* (3) *with* $p_0 > 0$. *Then*

(i) *if* (p_n) *is strictly log-concave for* $n = 1, 2, \ldots, m$, *then* $r_0 p_m - p_{m+1} > 0$;
(ii) *if* (r_n) *is strictly log-convex and* $r_0^2 - r_1 < 0$, *then* $r_{m+2}p_{m+2} - r_{m+1}p_{m+3} > 0$.

PROOF. If (p_n) is strictly log-concave, then (p_{n+1}/p_n) is decreasing, so $r_0 = p_1/p_0 > p_{m+1}/p_m$.

If (r_n) is strictly log-convex, then (r_{n+1}/r_n) is increasing. Hence,

$$(m+3)p_{m+3} < p_{m+2}r_0 + (m+2)p_{m+2} \max_{1 \le k \le m+2} \left\{ \frac{r_k}{r_{k-1}} \right\}$$

$$< p_{m+2} \frac{r_{m+2}}{r_{m+1}} + (m+2)p_{m+2} \frac{r_{m+2}}{r_{m+1}}. \qquad \square$$

THEOREM 1. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \qquad n = 0, 1, 2, \ldots,$$

with $r_k \ge 0$, $p_0 > 0$ *and let* (r_n) *be log-concave. Then*

(p_n) *is log-concave if and only if* $r_0^2 - r_1 \ge 0$.

PROOF. Suppose that (r_n) is strictly log-concave and $r_0^2 - r_1 > 0$, then (r_n) is positive and hence (p_n) is positive. Observe that

$$2(p_1^2 - p_0 p_2) = p_0^2(r_0^2 - r_1). \tag{6}$$

By using (6), Lemma 2(i) and applying induction to (4), we see that (p_n) is strictly log-concave. The proof is completed by noting that any log-concave sequence can be written as a limit of strictly log-concave sequences. $\square$

THEOREM 2. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \qquad n = 0, 1, 2, \ldots,$$

with nonnegative r_k, $p_0 > 0$ *and let* (r_n) *be log-convex. Then*

(p_n) *is log-convex if and only if* $r_0^2 - r_1 \le 0$.

PROOF. As in Theorem 1, except that Lemma 2(ii) is used and induction is applied to (5). □

It is curious to note the difference in (4) and (5). We were not able to find an equation of the form (4) to prove Theorem 2 or one of the form (5) to prove Theorem 1.

REMARK 1. The assumption that (p_n) is a probability distribution is not used in the proofs of Theorems 1 and 2. These theorems are thus true for arbitrary nonnegative sequences related by (3).

3. Absolutely continuous distributions. In this section infinitely divisible probability distributions F on $\mathbb{R}_+$ with absolutely continuous Lévy measures are considered. We obtain two results on the log-concavity and log-convexity of the densities of F, which are analogues to those obtained in Section 2. The result on log-concave densities is proved in Yamazato (1982). We here propose a proof based on applying a limiting argument to Theorem 1. This proof can easily be adapted to log-convex densities, thus giving the absolutely continuous analogue of Theorem 2.

A function f on $\mathbb{R}$ is log-concave (log-convex) on an interval I if f is positive on I and $\log(f)$ is concave (convex) on I. f is said to be log-concave (log-convex) if $I = \{x|f > 0\}$ is an interval and f is log-concave (log-convex) on I. As in the discrete case, f is strictly log-concave (strictly log-convex) if $\log(f)$ is strictly concave (strictly convex).

A probability distribution F on $(0, \infty)$ is infinitely divisible if and only if there exists a nondecreasing measure H such that

$$\int_0^x u\,dF(u) = \int_0^x F(x-u)\,dH(u), \tag{7}$$

$$\int_1^\infty u^{-1}\,dH(u) < \infty, \tag{8}$$

where H and F determine each other uniquely [cf. Steutel (1970)]. If F and H have densities f and h, then

$$xf(x) = \int_0^x h(x-u)f(u)\,du. \tag{9}$$

Without loss of generality we assume that $\inf\{x|f(x) > 0\} = 0$. It is shown in Steutel (1970) that all absolutely continuous distributions with log-convex densities are infinitely divisible. As in the discrete case, not all distributions having log-concave densities are infinitely divisible, e.g., $f(x) = c\exp(-x^2)$ for $x \in (0, \infty)$.

We begin with a lemma.

LEMMA 3. *Let f and h be continuous and related by* (9). *Suppose h is monotone on $(0, \varepsilon)$ for some $\varepsilon > 0$ and $0 < f(0+) < \infty$. Then $h(0+) = 1$.*

PROOF. Suppose h is nonincreasing on $(0, \varepsilon)$ and $0 < f(0+) < \infty$. Then $h(0+) > 0$. From (9) it follows that for $0 < x < \varepsilon$,

$$h(0+) \geq xf(x) \Big/ \int_0^x f(u)\, du,$$

$$h(x) \leq xf(x) \Big/ \int_0^x f(u)\, du.$$

Letting $x \to 0$ the right-hand sides tend to one, so $h(0+) = 1$. Similarly, if h is nondecreasing. □

THEOREM 3 (Yamazato). *Let F be an infinitely divisible distribution function on $(0, \infty)$ with an absolutely continuous Lévy measure H. Let f and h be the densities of F and H, respectively, and assume that h is log-concave. Then*

$$f \text{ is log-concave if and only if } h(0+) \geq 1.$$

PROOF. Suppose h is log-concave and $h(0+) > 1$; then h must be continuous on I. Define $(r_n(k))$ by

$$r_n(k) = h\left(\frac{n+1}{k}\right), \qquad n = 0, 1, 2, \ldots,$$

and any $k \in \mathbb{N}_0$. Then $(r_n(k))$ is log-concave, and since $h(0+) > 1$ we have $(r_0(k))^2 > r_1(k)$, for large k. By (8) and the continuity of h we see that $\sum r_n(k)/(n+1) < \infty$. For fixed k define $(p_n(k))$ by

$$(n+1)p_{n+1}(k) = \sum_{l=0}^{n} p_{n-l}(k) r_l(k), \qquad n = 0, 1, 2, \ldots,$$

$$p_0(k) = k \exp\left(-\sum_{n=0}^{\infty} r_n(k)/(n+1)\right) > 0, \tag{10}$$

with $\sum p_n(k) = k$. By Theorem 1 and Remark 1 the sequence $(p_n(k))$ is log-concave. Let

$$p_k(x) = \sum_{\substack{n \geq 0 \\ n \leq kx}} k^{-1} p_n(k),$$

$$H_k(x) = \sum_{\substack{n \geq 0 \\ n \leq kx}} k^{-1} r_n(k).$$

From (10) it follows that

$$\int_{[0,\, x+k^{-1}]} u\, dP_k(u) = \int_{[0,\, x]} P_k(x-u)\, dH_k(u),$$

$$\frac{n+1}{k} p_{n+1}(k) = \int_{[0,\, n/k]} h\left(\frac{n+1}{k} - u\right) dP_k(u). \tag{11}$$

By Helly's first theorem [cf. Feller (1971)] there is a subsequence $(P_{k(s)})$ converging weakly to some distribution function, P say, as $s \to \infty$. Hence, since

$H_k \to H$, by Helly's second theorem

$$\int_{[0,x]} u\,dP(u) = \int_{[0,x]} P(x-u)\,dH(u).$$

Since H uniquely determines F in (7) we must have $F = P$. Let

$$f_k(x) = (p_{n+1}(k))^{kx-n}(p_n(k))^{n+1-xk}, \qquad x \in \left[\frac{n}{k}\,\frac{n+1}{k}\right). \tag{12}$$

Then f_k is a log-concave function of x. Let $n \to \infty$ and $k \to \infty$ in such a way that $k^{-1}(n+1) \to x$. Then it follows from (9), (11) and (12) that

$$xp(x) := \lim_{\substack{k\to\infty \\ n\to\infty \\ k^{-1}(n+1)\to x}} \frac{n+1}{k} f_k\left(\frac{n+1}{k}\right) = \int_{[0,x)} h(x-u)\,dF(u) = xf(x) \quad \text{a.e.}$$

Since log-concavity is preserved under convergence, F has a log-concave density p. Any log-concave function with $h(0+) \geq 1$ can be written as a limit of log-concave functions with $h_k(0+) > 1$, completing this part of the proof.

Conversely, if f and h are log-concave, then $h(0+) = 1$ by Lemma 3 if $0 < f(0+) < \infty$. If f is log-concave, then $f(0+)$ cannot be infinite. If $f(0+) = 0$, then f is nondecreasing on $(0, \varepsilon)$ and

$$xf(x) \leq f(x)\int_0^x h(u)\,du.$$

Letting $x \to 0$ yields $h(0+) \geq 1$. □

The proof of Theorem 3 can easily be adapted to log-convex densities by using Theorem 2 instead of Theorem 1. We then obtain

THEOREM 4. *Let F be an infinitely divisible distribution function with an absolutely continuous Lévy measure H. Let f and h be the densities of F and H, respectively, and assume that h is log-convex. Then*

$$f \text{ is log-convex if and only if } h(0+) \leq 1.$$

4. Applications and counterexamples. In this section we define a class of infinitely divisible distributions in terms of their Lévy measures and determine under what conditions a distribution in this class is log-concave or log-convex. An application of this result shows that the reverse statements of our main theorems do not hold. Finally, we characterize the log-convex discrete stable distributions.

Let I_d denote the class of distributions having Lévy measures (r_n) of the form

$$r_n = (n+1)\left\{\int_0^b y^n\,dm(y) + \int_c^a y^n\,dy\right\}, \qquad n = 0,1,2,\ldots, \tag{13}$$

with $b \le 1$, $c \le a \le 1$, m bounded by Lebesgue measure and

$$\int_a^b dm(y) < b - a, \quad \text{if } b > a,$$

$$\int_0^c dm(y) < c, \qquad \text{if } c > 0.$$

The proof of Theorem 2 in Yamazato (1982) can be adapted to prove the following theorem if Theorem 3 in Hansen and Steutel (1987) is used in the same fashion as Lemma 4.1 in Yamazato (1982).

THEOREM 5. *Let* (p_n) *and* (r_n) *be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_k p_{n-k}, \qquad n = 0, 1, 2, \ldots,$$

with nonnegative r_k *and* $p_0 > 0$. *Let* $(p_n) \in I_d$. *Then*

(i) *if* $c = 0$ *and* $a \ge b$, *then* (p_n) *is log-concave*;
if $c = 0$ *and* $a < b$, *then* (p_n) *is not log-concave*;
(ii) *if* $c \ge 0$ *and* $a \ge c \ge b$, *then* (p_n) *is log-convex*;
if $c \ge 0$ *and* $a \ge b > c$, *then* (p_n) *is not log-convex*;
if $c \ge 0$ *and* $b > a > c$, *then* (p_n) *is not log-convex*;
if $c \ge 0$ *and* $b < a = c$, *then* (p_n) *is log-convex*.

REMARK 2. The absolutely continuous analogue of Theorem 5 can be obtained by applying the same type of limiting argument as in the proof of Theorem 3.

REMARK 3. Let m in (13) be Lebesgue measure on (d, b), and 0 otherwise. Then $r_n = b^n - d^n + a^n$ and $r_n^2 - r_{n+1}r_{n-1} < 0$ for large n if $a > b > d > 0$, whereas (p_n) is log-concave by Theorem 5(i). Similarly, (r_n) is asymptotically log-concave if $0 = d < b < c < a$, whereas (p_n) is log-convex by Theorem 5(ii). Hence, the reverse statements of Theorems 1 and 2 do not hold.

A discrete analogue of an absolutely continuous stable distribution was proposed in Steutel and van Harn (1979). They proved that a distribution (p_n) is discrete stable with exponent γ if and only if its generating function is of the form

$$P(z) = \exp\bigl(-\lambda(1-z)^{\gamma}\bigr), \qquad \gamma \in (0,1], \lambda \ge 0.$$

Taking generating functions on both sides of (3) and comparing with the Taylor series expansion of $-\lambda(1-z)^{\gamma}$, one sees that (r_n) is strictly log-convex and that $r_0^2 - r_1 < 0$ if and only if $\gamma < 1 - r_0$. Applying Theorem 2 to these observations gives

THEOREM 6. *Let* (p_n) *be discrete stable with exponent* γ. *Then*

(p_n) *is strictly log-convex if and only if* $\lambda < \gamma^{-1} - 1$.

The Lévy density h of an absolutely continuous stable distribution on $(0, \infty)$ is of the form $cx^{-\gamma}$, hence h is log-convex and $h(0+) = \infty$. Applying Theorem 4 we have, rather unexpectedly, that there are no log-convex stable densities on $(0, \infty)$.

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