The Annals of Probability
1988, Vol. 16, No. 4, 1742–1766

RANDOM CELL COMPLEXES AND GENERALISED SETS

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The concepts of random cell complexes, generalised sets and their mean sets in R^d are introduced. Under stationarity conditions various relations between associated geometrical quantities are derived.

Introduction. Cell complexes have been studied in topology as well as in differential geometry and algebraic geometry. In the literature on stochastic geometry one can find them until now only in the special form of random convex tessellations of the plane and the space or of random mosaics generated by stationary (Poissonian) hyperplane processes in R^d. Thereby, methods of classical convex and integral geometry in the sense of Hadwiger (1957) and Santaló (1976) are combined with stochastic invariance properties of the random sets. Because of the failing geometrical background, similar results for arbitrary stationary random tessellations of R^d have not been known even in the case of convex cells.

The present paper is based on a new global-analytical approach to related problems of geometry developed by the author. As a particular result, one gets an answer to a question of Blaschke formulated at the end of the last edition of his *Integral Geometry* in 1955. Blaschke found relations between certain euclidean invariants (the additive extensions of Minkowski's quermassintegrals) of two- and three-dimensional euclidean polyhedra and those of the underlying cells. (Euler's polyhedron theorem may be considered to be a special case.) In this connection, he asked whether it is possible to obtain similar results for topological polyhedra with smooth cells. This problem was solved in Zähle (1987b) in the more general context of topological cell complexes in R^d whose cells are piecewise C^1-smooth and satisfy a certain second order rectifiability condition. It will be shown in a later paper that the whole theory and the stochastic applications may be extended to arbitrary cells of bounded curvatures in the sense of Zähle (1987c).

There are at least three reasons for considering nonconvex (random) cells or, more generally, cells with singularities:

1. It is mathematically interesting and tractable; the tools are universal and clear. Many formulas known from the convex case remain valid for general polyhedra or tessellations and may be completed by new relationships.
2. There exist practical examples, where nonconvex models seem to be more adapted. For instance, amorphous structures in physical chemistry may be

Received October 1986; revised August 1987.
AMS 1980 *subject classifications*. Primary 60D05; secondary 60G55, 58A25.
Key words and phrases. Random cell complexes, stationary point processes, i-skeletons, typical j-cells, currents, generalised sets, mean sets, curvature and direction properties, polyhedron theorems, random tessellations.

described by means of random cell complexes, where the atoms are the vertices and the atomic bonds arising from certain potential principles determine the edges. Since, in general, the atoms of a ring do not lie in a plane, the faces of the cell complex cannot be assumed to be linear. Furthermore, the example of cracks or breaches in materials shows that singularities may appear.

3. (Random) complexes of "piecewise smooth" cells may be used for approximating certain (random) fractal sets via current metrics.

The basic idea in the general approach consists in replacing the sets X under consideration, i.e., the i-cells and the j-skeletons, by their so-called unit normal cycle C_X. This is a continuous linear functional on the space of differential $(d-1)$-forms on $R^d \times S^{d-1}$ associated with integration over the corresponding generalised unit normal bundle

$$\operatorname{Nor} X = \{(x, n) \in R^d \times S^{d-1}: x \in X, n \text{ normal to } X \text{ at } x\}.$$

In this way geometrical quantities as volume, direction and curvature (measures) and topological characteristics as connectivity numbers may be obtained by the values of C_X at special differential forms. Simultaneously, from the standpoint of stochastics, the treatment of random linear functionals (and random measures) is more convenient than that of random geometric sets. For example, this approach leads to a new concept of the mean set of a random set or of a Gaussian random set.

The paper is divided into the following parts:

Section 1: Random cell complexes of the kind mentioned are introduced.

Section 2: Contains a summary of the geometrical background, namely, the definition of unit normal bundle, associated current and curvature measures, and the formulation of the basic polyhedron relation (6). The concept of a generalised set as an arbitrary unit normal cycle is introduced in Section 2.4.

Section 3: Here the main results of the paper are presented: The random sets determined by a random cell complex are identified with the associated random currents (unit normal cycles) and their mean sets are defined as generalised sets. Under the assumption of stationarity, the corresponding random curvature measures (including m-volume and direction measures as margins) have a simple structure and the polyhedron formula (6) gets a nice stochastic version (Theorem 3.3.6). This allows computation of the mean geometrical quantities of the random i-skeletons by those of the typical j-cells and the mean numbers of j-cells per unit volume, $j \leq i$, and vice versa (Corollary 3.3.7).

The special case of stationary random tessellations of R^d is treated separately and illustrated by a complete system of mean value relations for the practically interesting three-dimensional case. [Known formulas for the convex model may be derived and the results of Weiss and Zähle (1987) are extended.] Finally, the general relationships are applied to random tessellations generated by stationary weakly flat hypersurface processes. In particular, the resulting statements complete the corresponding knowledge about Poissonian hyperplane processes and extend it to the weakly flat case.

1. Random cell complexes.

DEFINITIONS. For $i = 0, \dots, d$ let $\mathscr{M}_i$ be the space of connected compact i-dimensional submanifolds m_i of R^d with boundary and positive reach [cf. Federer (1959)]. Here by definition $m_i = m_i \cup \partial m_i$.

A *p-dimensional PR-cell complex* in R^d is a $(p+1)$-tuple $M = (M_0, \dots, M_p)$, where the M_i are locally finite families from $\mathscr{M}_i$ (the i-cells) satisfying the following incidence relations:

1. The intersection of two i-cells from M_i is empty or a j-cell from M_j, $j = 0, \dots, i-1$.
2. Any $(i-1)$-cell from M_{i-1} is contained in the boundary of some i-cell from M_i.
3. The boundary of any i-cell from M_i is the (finite) union of some $(i-1)$-cells from M_{i-1}.

As usual, the corresponding union sets $\bigcup M_i$ are denoted by $|M_i|$, $i = 0, \dots, p$, and said to be the *i-skeletons* of the cell complex M. The set M_p is called a *p-dimensional* (*topological*) *PR-polyhedron* in R^d.

We now omit the smoothness condition on the i-cells and let $\mathscr{U}_i$ be the space of i-dimensional compact submanifolds with or without boundary which are representable as PR-polyhedra. Any $(p+1)$-tuple $U = (U_0, \dots, U_p)$ of locally finite families U_i from $\mathscr{U}_i$ satisfying the corresponding relations 1′–3′ is said to be a *p-dimensional* U_{PR}*-cell complex* in R^d. $|U_i|$, $i = 0, \dots, p-1$, denotes the *i-skeleton* and $|U_p|$ the U_{PR}*-polyhedron* associated with U. [Notice that the i-cells of U, i.e., the elements of U_i need not be (simply) connected.]

For a stochastic model it is appropriate to describe the families of i-cells by point processes: Let $\mathscr{N}_i$ be the space of locally finite simple counting measures on $[\mathscr{U}_i, \mathfrak{U}_i]$. Here $\mathfrak{U}_i$ is the Borel σ-algebra with respect to a suitable topology on $\mathscr{U}_i$ which will be introduced in the Remark of Section 2 for geometrical reasons. The set $A(\varphi_i)$ of atoms (counts) of $\varphi_i \in \mathscr{N}_i$ will correspond to the family U_i of i-cells. For abbreviation we identify φ_i with $A(\varphi_i)$ and write $|\varphi_i| := |A(\varphi_i)|$, $u_i \dot{\in} \varphi_i$, iff $u_i \in A(\varphi_i)$. On $\mathscr{N}_i$ the usual σ-algebra $\mathfrak{N}_i$ is given. Then the basic space of a p-dimensional random U_{PR}-cell complex may be introduced as

$$\mathscr{N}^{(p)} = \Big\{\eta = \big(\eta_0, \dots, \eta_p\big) \colon \eta_i \in \mathscr{N}_i,$$
$$A(\eta) := \big(A(\eta_0), \dots, A(\eta_p)\big) \text{ is a } U_{PR}\text{-cell complex}\Big\}.$$

(It can be shown that $\mathscr{N}^{(p)} \in \mathfrak{N}_0 \otimes \cdots \otimes \mathfrak{N}_p$, i.e., this subspace is product measurable.) Let $\mathfrak{N}^{(p)} := (\mathfrak{N}_0 \otimes \cdots \otimes \mathfrak{N}_p) \cap \mathscr{N}^{(p)}$ be the corresponding σ-algebra.

In this notation a *p-dimensional random* U_{PR}*-cell complex* is a measurable mapping ξ from a basic probability space into $[\mathscr{N}^{(p)}, \mathfrak{N}^{(p)}]$. We also write $\xi = (\xi_0, \dots, \xi_p)$ as a random vector. $|\xi_i|$ is the associated *random i-skeleton* (the *random p-polyhedron* if $i = p$).

2. Normal bundles and currents.

2.1. *Generalised unit normal bundles and measurabilities.* The concept of generalised unit normal bundles introduced in Zähle (1987a) is appropriate for describing geometrical properties of cell complexes and more general sets. Let U_{PR} be the class of locally finite unions of sets with positive reach whose finite intersections also possess positive reach. The aforementioned i-dimensional topological polyhedra are all contained in U_{PR}. The unit normal bundle of $X \in U_{PR}$ is determined by a certain index function which is due to Schneider (1978) in the special case of finite unions of compact convex sets. Let $x \in R^d$, $n \in S^{d-1}$ and b be the closed unit ball. The *index* of X at x with respect to the direction n is given by

$$(1) \qquad i_X(x,n) := 1 - \lim_{r \to 0+} \lim_{s \to 0+} \chi(X \cap ([x + (r+s)n] + rb)).$$

(Here χ denotes the Euler characteristic.)

The existence of the limit is proved in Zähle (1987c). This index function agrees with the variant introduced in Zähle (1987a) under more restrictive assumptions. (All related results remain valid in the general case.)

In this notation the *generalised unit normal bundle* is given by

$$\operatorname{Nor} X := \{(x,n) \in R^d \times S^{d-1} : i_X(x,n) \neq 0\}.$$

The *normal cone* of X at x is then defined by

$$N_x X := \{cn : (x,n) \in \operatorname{Nor} X,\ c \geq 0\}.$$

[More details can be found in Zähle (1987a). In particular, for classical sets as convex bodies and smooth submanifolds or, more generally, for sets of positive reach, the definition of $N_x X$ coincides with the classical variant of the convex cone dual to $T_x X$. Notice that $\operatorname{Nor} X$ is a $(d-1)$-dimensional locally rectifiable subset of $R^d \times S^{d-1}$. The closure $\overline{\operatorname{Nor} X}$ differs from $\operatorname{Nor} X$ by a set of Hausdorff measure zero.]

This concept was used in Zähle (1987a) for defining currents associated with $\operatorname{Nor} X$ instead of X. It enables us to study curvature properties by current methods. At the same time this is the key for introducing generalised sets in the sense of distribution theory.

From the standpoint of stochastics, the current approach via unit normal bundles is also useful. In particular, open measurability problems become clear by introducing an *appropriate topology in* U_{PR}. Namely, let τ be the hit topology on the space $\mathscr{F}$ of closed subsets of $R^d \times S^{d-1}$ in the sense of Matheron (1975) and $f: U_{PR} \to \mathscr{F}$ be the mapping with $f(X) = \overline{\operatorname{Nor} X}$. Then, by definition, $\tau(U_{PR})$ is the weakest topology in U_{PR} with respect to which f is τ-continuous. (Notice that it contains the trace on U_{PR} of the hit topology on the space of closed subsets of R^d.)

REMARK 2.1.1. We now are able to complete the definition of a random cell complex (cf. Section 1): The topology generating the σ-algebra $\mathfrak{U}_i$ is the trace of $\tau(U_{PR})$ on $\mathscr{U}_i$. Further, let $\mathscr{P}_i$ be the space of i-dimensional topological

U_{PR}-polyhedra as introduced in Section 1 and $\mathfrak{P}_i$ the Borel σ-algebra with respect to the trace of $\tau(U_{PR})$ on $\mathscr{P}_i$.

DEFINITION. A *random i-dimensional U_{PR}-polyhedron* is a random element in $[\mathscr{P}_i, \mathfrak{P}_i]$.

Then the following measurability property holds.

PROPOSITION 2.1.2. *For any p-dimensional random cell complex $\xi = (\xi_0, \dots, \xi_p)$, the i-skeletons $|\xi_i|$ are random i-polyhedra.*

The proof is analogous to that given in Zähle [(1986), Section 1.3.1].

2.2. *Currents and curvatures associated with unit normal bundles.* Currents are continuous linear functionals on the space of differential forms of fixed order provided with the Schwartz topology, i.e., special Schwartz distributions. For details cf. Federer (1969, 1978). (We will use adopted notation.) For our polyhedral model we will investigate locally rectifiable currents associated with the unit normal bundles of the cells and the skeletons. Thereby the sets are represented by integrals of differential $(d-1)$-forms over the corresponding normal bundles. (This enables us to study geometrical i-volume i-direction and curvature properties of sets with singularities by means of advanced calculus.)

NOTATION. $\wedge_{d-1}$ is the space of $(d-1)$-vectors on $R^d \times R^d$. $\wedge^{d-1}$ is the space of $(d-1)$-covectors on $R^d \times R^d$. $\mathscr{E}^{d-1}$ is the space of differential $(d-1)$-forms (of class ∞) on $R^d \times R^d$. $\mathscr{D}^{d-1}$ $[= \mathscr{D}(R^d \times R^d, \wedge^{d-1})]$ is the subspace of $(d-1)$-forms with compact support. $\mathscr{H}^i$ is the i-dimensional Hausdorff measure in euclidean space.

More details about the following concepts and statements can be found in Zähle (1987a).

For $X \in U_{PR}$, $\operatorname{Nor} X$ is a locally Hausdorff $(d-1)$-rectifiable subset of $R^d \times R^d$. In order to define a current associated with $\operatorname{Nor} X$ we introduce an appropriate *simple $(d-1)$-vector field orienting* $\operatorname{Nor} X$: Let $e_1, \dots, e_d$ be an orthonormal basis in R^d with dual basis $e_1', \dots, e_d'$ and let π_i: $R^d \times R^d \to R^d$, $i = 0, 1$, be the coordinate mappings

$$\pi_i(x, n) = \begin{cases} x, & i = 0, \\ n, & i = 1. \end{cases}$$

At $\mathscr{H}^{d-1}$-almost all $(x, n) \in \operatorname{Nor} X$ the tangent cone $T_{(x,n)} \operatorname{Nor} X$ is a $(d-1)$-dimensional vector space. For these (x, n), let

$$a_X(x, n) = a_1(x, n) \wedge \cdots \wedge a_{d-1}(x, n)$$

be the simple $(d-1)$-vector in $R^d \times R^d$ associated with $T_{(x,n)} \operatorname{Nor} X$ such that

$|a_X(x,n)| = 1$ and

$\operatorname{sign}\langle(\pi_0 + t\pi_1)a_1(x,n) \wedge \cdots \wedge (\pi_0 + t\pi_1)a_{d-1}(x,n) \wedge n, e_1' \wedge \cdots \wedge e_d'\rangle = 1.$

The symbol $\langle \cdot , \cdot \rangle$ denotes the natural bilinear pairing of m-vectors and m-covectors. (The last expression has been shown to be independent of t for sufficiently small $t > 0$. It corresponds to the sign of the volume element of infinitesimal parallel sets of X for a suitable parametrization.)

Now define differential forms $\varphi_k \in \mathscr{E}^{d-1}$, $k = 0, \dots, d-1$, which do not depend on the first argument by means of

$$\langle v, \varphi_k(n)\rangle = \frac{1}{(d-k)\alpha(d-k)} \times \sum_{\substack{\varepsilon_i = 0,1 \\ \Sigma\varepsilon_i = d-1-k}} \langle \pi_{\varepsilon_1} v_1 \wedge \cdots \wedge \pi_{\varepsilon_{d-1}} v_{d-1} \wedge n, e_1' \wedge \cdots \wedge e_d'\rangle, \tag{2}$$

if $v = v_1 \wedge \cdots \wedge v_{d-1} \in \Lambda_{d-1}$, where $\alpha(k)$ denotes the volume of the k-dimensional unit ball. The restriction of φ_k to S^{d-1} is said to be *kth symmetric curvature form* over R^d.

For $X \in U_{PR}$ with Nor X oriented as before, let

$$C_X = (\mathscr{H}^{d-1} \llcorner \operatorname{Nor} X) \wedge i_X a_X$$

be the *locally rectifiable* $(d-1)$*-dimensional current* on $R^d \times R^d$ (or on $R^d \times S^{d-1}$) defined by

$$C_X(\psi) = \int_{\operatorname{Nor} X} \langle i_X(x,n) a_X(x,n), \psi(x,n)\rangle \mathscr{H}^{d-1}(d(x,n)),$$

$\psi \in \mathscr{D}^{d-1}$. Here and in the following let B be an arbitrary bounded Borel subset of $R^d \times S^{d-1}$. By definition, $C_X \llcorner B$ is the current

$$(C_X \llcorner B)(\psi) = \int_{\operatorname{Nor} X} 1_B(x,n) \langle i_X(x,n) a_X(x,n), \psi(x,n)\rangle \times \mathscr{H}^{d-1}(d(x,n)). \tag{3}$$

Recall that general $(d-1)$-dimensional currents are defined for $\psi \in \mathscr{D}^{d-1}$, but $(C_X \llcorner B)(\psi)$ also makes sense for $\psi \in \mathscr{E}^{d-1}$. [There is no difference, since

$$(C_X \llcorner B)(\psi) = (C_X \llcorner B)(f\psi)$$

for any C_∞-function f: $R^d \times R^d \to R^1$ with compact support such that $f(x,n) = 1$ if $(x,n) \in B$.] In particular, the values

$$C_k(X,B) := (C_X \llcorner B)(\varphi_k), \qquad k = 0, \dots, d-1, \tag{4}$$

for the symmetric curvature forms φ_k regarded as functions in B are Radon measures, the signed *curvature measures* of X. Their relationship to classical versions of convex and differential geometry is established in Zähle (1987a). This paper also contains an interpretation as integrals of generalised mean curvatures which does not use current concepts. Note that in the special case when X is a convex polyhedron, $C_X(\varphi_k)$ equals, up to a constant, the sum of the exterior angles at the k-faces weighted by the corresponding k-face areas. For $X, Y, X \cap Y, X \cup Y \in U_{PR}$, one obtains the additivity property

$$C_{X \cup Y} \llcorner B = C_X \llcorner B + C_Y \llcorner B - C_{X \cap Y} \llcorner B. \tag{5}$$

2.3. *Curvature measures for cell complexes.* Now return to the special case when the sets X as before are selected from a cell complex. In this section we will only consider the curvature measures $C_k(X, B^*)$ for *directional symmetric* bounded Borel sets $B^* \subset R^d \times S^{d-1}$, i.e., when

$$(x, n) \in B^* \quad \text{iff } (x, -n) \in B^*.$$

(Note that the basic set X itself need not be symmetric.) In this case the following fundamental relations hold [for a proof, see Zähle (1987b)]:

Let $U = (U_0, \ldots, U_p)$ be a p-dimensional U_{PR}-cell complex. Then for $i = 0, \ldots, p$,

$$C_k(|U_i|, B^*) = \sum_{j=k}^{i} (-1)^{j-k} \sum_{u_j \in U_j} C_k(u_j, B^*). \tag{6}$$

According to this formula the curvature measures of a U_{PR}-polyhedron may be computed by those of the underlying cells. This enables us to establish relations between the cell curvature measures provided that some of the polyhedron curvature measures are known, e.g., in the case of tessellations of R^d or S^{d-1}.

In particular, any i-cell u_i of U may be considered as a special polyhedron. Therefore,

$$\left(1 - (-1)^{i-k}\right) C_k(u_i, B^*) = \sum_{j=k}^{i-1} (-1)^{j-k} \sum_{u_j \in U_j} C_k(u_j, B^*).$$

By applying (6) to the boundary polyhedron ∂u_i, we find $C_k(\partial u_i, B^*)$ for the right-hand side. Hence,

$$\left(1 - (-1)^{i-k}\right) C_k(u_i, B^*) = C_k(\partial u_i, B^*). \tag{7}$$

As a special case, one gets

$$C_{i-1}\left(u_i, R^d \times S^{d-1}\right) = \tfrac{1}{2} \mathscr{H}^{d-1}(\partial u_i). \tag{8}$$

Further, let $N_i(u_j)$ be the *number of i-cells* of the cell complex which are *adjacent to the j-cell u_j*. Then for $i = 0, \ldots, p-1$,

$$\sum_{j=k}^{i-1} (-1)^{j-k} \sum_{u_j \in U_j} N_i(u_j) C_k(u_j, B^*) = \left(1 - (-1)^{i-k}\right) \sum_{u_i \in U_i} C_k(u_i, B^*). \tag{9}$$

In particular, if the cell complex is finite and all cells are simply connected, one obtains

$$\sum_{j=0}^{i-1} (-1)^j \sum_{u_j \in U_j} N_i(u_j) = \left(1 - (-1)^p\right) a_i, \tag{10}$$

where a_i is the number of i-cells of U.

2.4. *Generalised sets.* Recall that to any $X \in U_{PR}$ there corresponds the locally rectifiable $(d-1)$-dimensional current C_X (or the current family $C_X \llcorner B$ for bounded Borel sets B in $R^d \times S^{d-1}$). It can be shown that for any differential form $\varphi \in \mathscr{D}^{d-2}$,

$$\partial C_X(\varphi) := C_X(d\varphi) = 0,$$

i.e., the boundary current ∂C_X vanishes. Therefore, C_X may also be called the *unit normal cycle* of X. This leads us to the following definition: A *generalised set* in R^d is a generalised normal cycle, i.e., an arbitrary $(d-1)$-dimensional current G on $R^d \times S^{d-1}$ which is a cycle ($\partial G = 0$). According to this definition, curvature properties of a generalised set G may be treated in the form of the values

$$G(f\varphi_k), \qquad k = 0, \dots, d-1,$$

for arbitrary C_∞-functions f: $R^d \times S^{d-1} \to R^1$ with compact support.

3. Currents and mean values for random cell complexes.

3.1. *Random currents associated with random cell complexes.* A *random i-dimensional current* in R^n is a random Schwartz distribution in R^n of type $\wedge^i$.

THEOREM 3.1.1. *Let $\xi = (\xi_0, \dots, \xi_p)$ be a random U_{PR}-cell complex in R^d. Then for any bounded Borel set $B \subset R^d \times S^{d-1}$, $C_{|\xi_i|} \llcorner B$ is a random $(d-1)$-dimensional current in $R^d \times R^d$ (associated with the i-skeleton), $i = 0, \dots, p$. Hence, for arbitrary $\psi \in \mathscr{E}^{d-1}$, $(C_{|\xi_i|} \llcorner B)(\psi)$ as a function in B is a random signed Radon measure.*

PROOF. In view of Proposition 2.1.2, it suffices to show that for any $\psi \in \mathscr{D}^{d-1}$ the mapping

$$h(P) := (C_P \llcorner B)(\psi)$$

from the space of i-dimensional polyhedra into R^1 is measurable. Recall that

$$(C_P \llcorner B)(\psi) = \int_{\operatorname{Nor} P} 1_B(x,n) i_p(x,n) \langle a_P(x,n), \psi(x,n) \rangle \mathscr{H}^{d-1}(d(x,n))$$

and that $\operatorname{Nor} P$ in this formula may be replaced by its closure $\overline{\operatorname{Nor} P}$. $\overline{\operatorname{Nor} P}$ is a locally $\mathscr{H}^{d-1}$-rectifiable subset of $R^d \times R^d$ and $a_P(x,n)$ is the simple $(d-1)$-vector field associated with $T_{(x,n)}\overline{\operatorname{Nor} P}$ in the way mentioned already. Using the theorem and the remark in the Appendix and continuity of the mapping, one obtains that $\langle a_P(x,n), \psi(x,n) \rangle$, regarded as a function of (P,x,n) on the set

$$E := \left\{ (P,x,n) \colon P \in \mathscr{P}_i, (x,n) \in \overline{\operatorname{Nor} P} \right\} \subset \mathscr{P}_i \times R^d \times R^d,$$

is measurable with respect to the corresponding product σ-algebra on $\mathscr{P}_i \times R^d \times R^d$ restricted to E.

According to the definition of the index function, the mentioned theorem and continuity of the Euler characteristic, $i_P(x,n)$ possesses the same property.

Therefore, it suffices to prove that for any bounded Borel set $D \subset R^d \times S^{d-1}$ and any $W \in \mathfrak{P}_i$,

$$\int_{\overline{\mathrm{Nor}\, P}} 1_W(P) 1_D(x,n) \mathscr{H}^{d-1}(d(x,n))$$

is measurable in P. (The general case follows from approximating the positive and negative parts of the expression under the integral by linear combinations of such product functions.) Finally, the measurability of $\mathscr{H}^{d-1}(\overline{\mathrm{Nor}\, P} \cap D)$ in P results from Zähle (1982). The second assertion follows immediately from the definition of random signed Radon measures [cf. Zähle (1986)]. □

REMARK. The measurability property of the associated currents may be obviously extended to arbitrary random U_{PR}-sets being not necessarily polyhedra, i.e., to random elements in $[U_{PR}, \mathfrak{U}_{PR}]$. Here $\mathfrak{U}_{PR}$ is the Borel σ-algebra with respect to the topology $\tau(U_{PR})$ defined in Section 2.1.

3.2. *Mean sets of random (generalised) sets.* Since we identify the sets with their associated normal cycles the *mean set* $\mathbf{E}\mathscr{G}$ of a random generalised set $\mathscr{G}$, i.e., of a random $(d-1)$-dimensional current in $R^d \times S^{d-1}$, is defined in a natural way: $\mathbf{E}\mathscr{G}$ is determined if the expectation of the random variables $\mathscr{G}(\psi)$ for $\psi \in \mathscr{D}^{d-1}$ exist and if $\mathbf{E}\mathscr{G}(\psi) := \mathbf{E}(\mathscr{G}(\psi))$ regarded as a function of ψ is continuous. Then $\mathbf{E}\mathscr{G}$ is a generalised set. A random U_{PR}-set Γ is said to be *locally integrable* if for any bounded Borel set $B \subset R^d \times S^{d-1}$,

$$\mathbf{E}\int_{\overline{\mathrm{Nor}\, \Gamma}} 1_B(x,n) |i_\Gamma(x,n)| \mathscr{H}^{d-1}(d(x,n)) < \infty.$$

PROPOSITION 3.2.1. *Any locally integrable random U_{PR}-set Γ possesses a mean set $\mathbf{E}\Gamma = \mathbf{E}C_\Gamma$ which is representable by integration, i.e., there exist a Radon measure $\|\mathbf{E}\Gamma\|$ and a locally $\|\mathbf{E}\Gamma\|$-integrable $(d-1)$-vector field $\overrightarrow{\mathbf{E}\Gamma}$ on $R^d \times S^{d-1}$ with $|\overrightarrow{\mathbf{E}\Gamma}(x,n)| = 1$ such that for $\psi \in \mathscr{D}^{d-1}$,*

$$\mathbf{E}\Gamma(\psi) = \int \langle \overrightarrow{\mathbf{E}\Gamma}(x,n), \psi(x,n) \rangle \|\mathbf{E}\Gamma\|(d(x,n)).$$

Hence, for arbitrary $\psi \in \mathscr{E}^{d-1}$,

$$\mathbf{E}(\Gamma \llcorner B)(\psi) = [(\mathbf{E}\Gamma) \llcorner B](\psi)$$

is a signed Radon measure in B, the intensity measure of $(\Gamma \llcorner B)(\psi)$.

PROOF. Let $\psi \in \mathscr{D}^{d-1}$. Then the function $\|\psi(x,n)\|$ has compact support and so is bounded. Therefore,

$$\mathbf{E}|\Gamma(\psi)| \leq \mathbf{E}\int_{\overline{\mathrm{Nor}\, \Gamma}} \|\psi(x,n)\| |i_\Gamma(x,n)| \mathscr{H}^{d-1}(dx) < \infty$$

by the integrability condition. Thus, $\mathbf{E}(\Gamma(\psi))$ exists. Further, for any compact

$K \subset R^d \times S^{d-1}$ and any natural l,

$$|\mathbf{E}\Gamma(\psi)| \le \nu_K^l(\psi)\mathbf{E}\int_{\mathrm{Nor}\,\Gamma} 1_K(x,n)|i_\Gamma(x,n)|\mathscr{H}^{d-1}(d(x,n))$$

for the seminorms ν_K^l used for the concept of Schwartz distribution, i.e., $\mathbf{E}\Gamma$ is a current.

In order to establish the integral representation of $\mathbf{E}\Gamma$, consider an arbitrary C^∞-function f: $R^d \times S^{d-1} \to [0, +\infty)$ with compact support. Then

$$\begin{aligned}\|\mathbf{E}\Gamma\|(f) &:= \sup\{\mathbf{E}\Gamma(\psi)\colon \psi \in \mathscr{D}^{d-1}, \|\psi\| \le f\}\\ &\le \mathbf{E}\int_{\mathrm{Nor}\,\Gamma} f(x,n)|i_\Gamma(x,n)|\mathscr{H}^{d-1}(d(x,n)) < \infty\end{aligned}$$

and the assertion follows from Federer [(1969), Section 4.1.5]. □

DEFINITION. A random U_{PR}-cell complex $\xi = (\xi_0, \dots, \xi_p)$ is said to be *integrable* if the random i-skeletons $|\xi_i|$, $i = 0, \dots, p$, are locally integrable. Then the *curvature intensity measures*

$$\left[\left(\mathbf{E}C_{|\xi_i|}\right)\llcorner B\right](\varphi_k) = \mathbf{E}C_k(|\xi_i|, B)$$

will be denoted by $\overline{C}_k^i(B)$ in the sequel. They provide special information about the mean sets $\mathbf{E}|\xi_i| = \mathbf{E}C_{|\xi_i|}$.

3.3. *Stationary random cell complexes.* For arbitrary random U_{PR}-cell complexes $\xi = (\xi_0, \dots, \xi_p)$, the polyhedron theorems of Section 2.3 yield relations for the curvature measures of the random sets. The statements for the corresponding mean sets become very simple if the random cell complex is stationary, i.e., if its probability distribution is invariant under the translations of R^d. Here the Palm theory for stationary random point processes and measures may be used.

We first introduce *total currents* associated with the point processes ξ_i, $i = 0, \dots, p$, by means of

$$C_{\xi_i}(\psi) := \sum_{u_i \dot{\in} \xi_i} C_{u_i}(\psi),$$

in particular, *total curvature measures* by

$$C_k(\xi_i, B) := \sum_{u_i \dot{\in} \xi_i} C_k(u_i, B).$$

The corresponding measurability property may be proved similarly as before.

DEFINITION. ξ is said to be *totally integrable* if for any bounded Borel set $B \subset R^d \times S^{d-1}$ and $i = 0, \dots, p$,

$$\mathbf{E}\sum_{u_i \dot{\in} \xi_i} \int_{\mathrm{Nor}\,u_i} 1_B(x,n)|i_{u_i}(x,n)|\mathscr{H}^{d-1}(d(x,n)) < \infty.$$

For such processes the mean sets $\mathbf{E}C_{\xi_i}$ and the *total curvature intensity measures*

$$\bar{C}^i_{k,t}(B) := \mathbf{E}C_k(\xi_i, B)$$

make sense. (The proof is analogous to that of Proposition 3.2.1.)

PROPOSITION 3.3.1. *Any totally integrable U_{PR}-cell complex is integrable.*

PROOF. By the definition of the index function and additivity of the Euler characteristic, we have

$$\left|i_{|U_i|}(x, n)\right| \le \sum_{j=1}^{i} \sum_{u_j \in U_j} \left|i_{u_j}(x, n)\right|.$$

Hence, for any i,

$$\begin{aligned}
&\mathbf{E}\int_{\overline{\operatorname{Nor}|\xi_i|}} 1_B(x, n)\left|i_{|\xi_i|}(x, n)\right|\mathscr{H}^{d-1}(d(x, n)) \\
&\quad \le \sum_{j=1}^{i} \mathbf{E} \sum_{u_j \dot{\in} \xi_j} \int_{\overline{\operatorname{Nor}|\xi_i|}} 1_B(x, n)\left|i_{u_j}(x, n)\right|\mathscr{H}^{d-1}(d(x, n)) \\
&\quad \le \sum_{j=1}^{i} \mathbf{E} \sum_{u_j \dot{\in} \xi_j} \int_{\overline{\operatorname{Nor} u_j}} 1_B(x, n)\left|i_{u_j}(x, n)\right|\mathscr{H}^{d-1}(d(x, n)) < \infty
\end{aligned}$$

by assumption, i.e., the integrability condition is fulfilled. □

By stationarity we have the following *factorization property* of the curvature intensity measures with respect to the Lebesgue measure $\mathscr{L}^d$ [cf. Zähle (1986)].

PROPOSITION 3.3.2. (i) *If ξ is integrable and stationary, then*

$$\bar{C}^i_k(B) = \left(\mathscr{L}^d \times D^i_k\right)(B)$$

for uniquely determined signed Radon measures D^i_k on S^{d-1}.

(ii) *If ξ is totally integrable and stationary, then*

$$\bar{C}^i_{k,t}(B) = \left(\mathscr{L}^d \times D^i_{k,t}\right)(B)$$

for uniquely determined signed Radon measures $D^i_{k,t}$ on S^{d-1}.

The constants

$$c^i_k := D^i_k(S^{d-1}), \qquad c^i_{k,t} := D^i_{k,t}(S^{d-1})$$

are said to be the corresponding *curvature* (*total curvature*) *intensities*. More particularly, c^i_i coincides with the *i-volume intensity* of the random i-skeleton. (It is called ith curvature intensity only for completeness.)

c^i_0 may be interpreted as the *mean Euler number* of the i-skeleton (in a limit sense if the process is ergodic). The measures D^i_k, $D^i_{k,t}$ may be considered as *direction measures weighted by curvatures.* (They correspond to the well-known notion of *surface area measures* of convex bodies.)

There exist the following calculation methods: Let K_1 and ∂K_1^+ be the d-dimensional variants of a unit cube and its upper right boundary, respectively.

PROPOSITION 3.3.3. *If ξ is integrable and stationary, then*

$$D_k^i(\Omega) = \mathbf{E}\left[C_k(|\xi_i|, K_1 \times \Omega) - C_k(|\xi_i|, \partial K_1^+ \times \Omega)\right]$$

for any Borel set $\Omega \subset S^{d-1}$.

The proof is similar to that of Theorem 3.2.4 in Zähle (1986). This formula provides a method for determining the measures D_k^i, in particular the curvature intensities c_k^i, by observation of the skeleton $|\xi_i|$ inside the window K_1.

Further, for totally integrable, stationary ξ let Q^i be the *shape distribution* of the typical cell of ξ_i (in the Palm sense) and N^i the *mean number of i-cells* per unit volume. (Here the shape of a cell is considered as the euclidean equivalence class determined by this cell; cf. Zähle [(1986), Section 3.2].) Denote

$$\mathbf{D}_k^i(\Omega) := \int C_k(u_i, R^d \times \Omega) Q^i(du_i);$$

in particular,

$$\mathbf{C}_k^i := \int C_k(u_i, R^d \times S^{d-1}) Q^i(du_i).$$

This is the mean value of the total kth curvature of the typical i-cell of ξ. In particular, $\mathbf{C}_i^i$ coincides with the mean i-volume of the typical i-cell and $2\mathbf{C}_{i-1}^i$ with the mean $(i-1)$-volume of its boundary [cf. (8)]. $\mathbf{C}_0^i$ corresponds to the mean Euler number of the typical i-cell.

PROPOSITION 3.3.4. *If ξ is totally integrable and stationary, then*

$$D_{k,t}^i(\Omega) = N^i \mathbf{D}_k^i(\Omega);$$

in particular,

$$c_{k,t}^i = N^i \mathbf{C}_k^i.$$

The proof is similar to that of Zähle [(1986), Section 3.2.3].

Next, we will introduce total curvature measures of the cell complex with multiplicities. (For the corresponding currents, analogous concepts may be considered.)

DEFINITION. The random cell complex $\xi = (\xi_0, \dots, \xi_p)$ is said to be *multiply integrable* if for any $0 \le i \le j \le p$,

$$\mathbf{E} \sum_{u_i \dot{\in} \xi_i} N_j(u_i) \int_{\overline{\mathrm{Nor}\, u_i}} 1_B(x,n) \left| i_{u_i}(x,n) \right| \mathscr{H}^{d-1}(d(x,n)) < \infty.$$

Clearly, the condition is fulfilled for all i, j if it holds for $j = i + 1$ and all i. Furthermore, multiple integrability implies total integrability of ξ. For such processes we define the *multiple total curvature intensity measures* as

$$\bar{C}^{ij}_{k,t}(B) := \mathbf{E} \sum_{u_i \dot{\in} \xi_i} N_j(u_i) C_k(u_i, B), \qquad i \leq j,$$

$$\bar{C}^{ij}_{k,t}(B) := \mathbf{E} \sum_{u_i \dot{\in} \xi_i} \sum_{\substack{u_j \dot{\in} \xi_j \\ u_j \subset \partial u_i}} C_k(u_i, B), \qquad i > j.$$

A simple combinatorial argument shows that

$$\bar{C}^{ij}_{k,t}(B) = \bar{C}^{ji}_{k,t}(B). \tag{11}$$

Similarly one obtains the following.

PROPOSITION 3.3.5. *If ξ is multiply integrable and stationary, then*

(i) $$\bar{C}^{ij}_{k,t}(B) = \left(\mathscr{L}^d \times D^{ij}_{k,t}\right)(B),$$

(ii) $$D^{ij}_{k,t}(\Omega) = N^i \mathbf{D}^{ij}_k(\Omega)$$

for uniquely determined signed Radon measures $D^{ij}_{k,t}$ and $\mathbf{D}^{ij}_k$ on S^{d-1}.

We also write

$$c^{ij}_{k,t} := D^{ij}_{k,t}(S^{d-1}) \quad \text{and} \quad \mathbf{C}^{ij}_k := \mathbf{D}^{ij}_k(S^{d-1}).$$

Thus,

$$c^{ij}_{k,t} = N^i \mathbf{C}^{ij}_k. \tag{12}$$

In the case $i \leq j$, $\mathbf{C}^{ij}_k$ may be interpreted as the mean value of the total kth curvature of the typical i-cell multiplied by the number of adjacent j-cells. For $i > j$, it coincides with the mean value of the sum of the total kth curvatures of all j-cells lying in the boundary of the typical i-cell. In view of (11),

$$N^i \mathbf{D}^{ij}_k(\Omega) = N^j \mathbf{D}^{ji}_k(\Omega). \tag{13}$$

In particular, if all i- and j-cells are *simply connected*, then

$$\mathbf{N}^{ij} := \mathbf{C}^{ij}_0$$

coincides with the mean number of j-cells which are adjacent to the typical i-cell and

$$N^i \mathbf{N}^{ij} = N^j \mathbf{N}^{ji}. \tag{14}$$

These formulas together with the polyhedron theorems (6) and (9) yield the following *main results*. (Let Ω^* be an arbitrary centrally symmetric Borel subset of S^{d-1}.)

THEOREM 3.3.6. *For any totally integrable stationary random cell complex* $\xi = (\xi_0, \dots, \xi_p)$, *we have*

$$D_k^i(\Omega^*) = \sum_{j=k}^{i} (-1)^{j-k} N^j \mathbf{D}_k^j(\Omega^*).$$

In particular,

$$c_k^i = \sum_{j=k}^{i} (-1)^{j-k} N^j \mathbf{C}_k^j.$$

This means that the weighted direction measures of the i-skeleton $|\xi_i|$ may be computed by those of the typical j-cells and the mean numbers of j-cells, $j = 0, \dots, i$, $i = 0, \dots, p$. In particular, a relation between the curvature intensities of $|\xi_i|$ and the mean total curvatures of the typical j-cells together with the mean numbers of j-cells ($j = 0, \dots, i$) is established.

The following *inversion formulas* result.

COROLLARY 3.3.7. *Under the conditions of Theorem* 3.3.6 *for* $i \geq k + 1$, *we have*

$$\mathbf{D}_k^i(\Omega^*) = (-1)^{i-k} (N^i)^{-1} \big(D_k^i(\Omega^*) - D_k^{i-1}(\Omega^*)\big).$$

In particular,

$$\mathbf{C}_k^i = (-1)^{i-k} (N^i)^{-1} \big(c_k^i - c_k^{i-1}\big).$$

If all cells are simply connected, then

$$N^i = (-1)^i \big(c_0^i - c_0^{i-1}\big).$$

THEOREM 3.3.8. *For any multiply integrable stationary random cell complex* $\xi = (\xi_0, \dots, \xi_p)$, *we have*

$$\sum_{j=k}^{i-1} (-1)^{j-k} N^j \mathbf{D}_k^{ji}(\Omega^*) = \big(1 - (-1)^{i-k}\big) N^i \mathbf{D}_k^i(\Omega^*).$$

In particular,

$$\sum_{j=k}^{i-1} (-1)^{j-k} N^j \mathbf{C}_k^{ji} = \big(1 - (-1)^{i-k}\big) N^i \mathbf{C}_k^i.$$

REMARK. If $k = i - 1$, the statement reads

(15) $$N^{i-1}\mathbf{D}_{i-1}^{i-1i}(\Omega^*) = 2N^i\mathbf{D}_{i-1}^i(\Omega^*).$$

In view of (7), this relation may be directly derived by Palm methods. Here the structural conditions on the $(i-1)$-skeleton may be weakened to rectifiability, since only $(i-1)$-volume measures are involved.

COROLLARY 3.3.9. *If ξ is as before and all j-cells $(j = 0,\dots,i)$ are simply connected, then*

$$\sum_{j=0}^{i-1}(-1)^j N^j \mathbf{N}^{ji} = \left(1-(-1)^i\right)N^i.$$

Theorem 3.3.8 immediately leads to the following result.

THEOREM 3.3.10. *Suppose ξ is as in Theorem 3.3.8. If $\mathbf{D}_k^{ij}(\Omega^*) = N^{ji}\mathbf{D}_k^j(\Omega^*)$, $j = 0,\dots,i-1$, then*

$$\sum_{j=k}^{i-1}(-1)^{j-k}\mathbf{N}^{ji}N^j\mathbf{D}_k^j(\Omega^*) = \left(1-(-1)^{i-k}\right)N^i\mathbf{D}_k^i(\Omega^*).$$

In particular, if $\mathbf{C}_k^{ji} = N^{ji}\mathbf{C}_k^j$, $j = 0,\dots,i-1$, then

$$\sum_{j=k}^{i-1}(-1)^{j-k}\mathbf{N}^{ji}N^j\mathbf{C}_k^j = \left(1-(-1)^{i-k}\right)N^i\mathbf{C}_k^i.$$

REMARK. The conditions of this theorem are fulfilled if ξ is totally integrable and the number of i-cells adjacent to the typical j-cell is independent of the kth weighted direction measure (or the total kth curvature, respectively) of the typical j-cell, $j = 0,\dots,i$. The statement presents a pseudorecurrence relation for the numbers $N^j\mathbf{D}_k^j(\Omega^*)$ (or $N^j\mathbf{C}_k^j$) provided that the $\mathbf{N}^{ji}$ are given. "Pseudo" means that the right-hand sides vanish if $i-k$ is even.

3.4. *Intersections.* The intersection formulas concerning curvature intensities of motion invariant U_{PR}-processes proved in Zähle (1986) remain valid for motion invariant U_{PR}-cell complexes. (The only difference for the proof is to infer the corresponding measurability conditions as in Section 3.1 of this paper.)

3.5. *Random tessellations.* We now will examine the special case of *stationary random mosaics* of R^d, i.e., of stationary random cell complexes $\xi = (\xi_0,\dots,\xi_d)$ such that ξ_d determines a tessellation of R^d: $|\xi_d| = R^d$. In this case, the formulas of Section 3.3 may be completed by the relations

(16) $$c_d^d = N^d\mathbf{C}_d^d = 1.$$

(Since this is a statement for volume measures, ξ_d might be replaced by any Borel tessellation of R^d.) Further, for $k < d$, $D_k^d(\Omega^*) \equiv 0$. Therefore, Theorem 3.3.6 yields the following relation between the cell measures provided that ξ is totally integrable:

$$\sum_{j=k}^{d} (-1)^{j-k} N^j \mathbf{D}_k^j(\Omega^*) \equiv 0, \qquad k < d. \tag{17}$$

In particular, if the cells are simply connected, then

$$\sum_{j=0}^{d} (-1)^j N^j = 0. \tag{18}$$

Further, since for any $(d-1)$-cell $u_{d-1} \dot{\in} \xi_{d-1}$, $N_d(u_{d-1}) = 2$, one gets $N^{d-1d} = 2$. Moreover,

$$\mathbf{D}_k^{d-1d}(\Omega^*) = 2\mathbf{D}_k^{d-1}(\Omega^*), \qquad k = 0, \dots, d-1, \tag{19}$$

provided that ξ is totally integrable. [For the special case of mosaics with simply connected cells of positive reach and $\Omega^* = S^{d-1}$, many of these formulas were proved in Weiss and Zähle (1987).]

As an example, we consider stationary random tessellations in R^3 with simply connected cells which were examined in the literature for the special case of convex cells. [Most general related results are formulated in Mecke (1984a). For further references, cf. Weiss and Zähle (1987).]

Here we introduce the following special notation (under the corresponding integrability conditions):

$S := c_2^2$	surface area intensity
$C := c_1^2$	curvature intensity
$\chi_2 := c_0^2$	mean Euler number of the side skeleton
$L := c_1^1$	edge length intensity
$\chi_1 := c_0^1$	mean Euler number of the edge skeleton
$\chi_0 := c_0^0 = N^0$	mean number of vertices
$\mathbf{V} := \mathbf{C}_3^3$	mean volume of the typical (3-) cell
$\mathbf{S} := 2\mathbf{C}_2^3$	mean surface area of the typical cell
$\mathbf{C} := \mathbf{C}_1^3$	mean total curvature of the typical cell
$\mathbf{A} := \mathbf{C}_2^2$	mean surface area of the typical side
$\mathbf{B} := 2\mathbf{C}_1^2$	mean boundary length of the typical side
$\mathbf{L} := \mathbf{C}_1^1$	mean length of the typical edge

We are also interested in the remaining characteristics:

N_3	mean number of cells
N_2	mean number of sides
N_1	mean number of edges
N^{ij}	mean number of j-cells adjacent to the typical i-cell
$\mathbf{L}^{1j} := \mathbf{C}_1^{1j}$, $j = 2, 3$	mean multiple length of the typical edge
$\mathbf{L}^{31} := \mathbf{C}_1^{31}$	mean total edge length of the typical cell

The parameter tuple $(S, C, \chi_2, L, \chi_1, \chi_0)$ associated with the i-skeletons, $i = 0, 1, 2$, is in one-to-one correspondence to the j-cell parameters $(\mathbf{V}, \mathbf{S}, \mathbf{C}, \mathbf{A}, \mathbf{B}, \mathbf{L})$.

Namely, Theorem 3.3.6, Corollary 3.3.7, (16), (17) and (18) yield

$$\mathbf{V} = \frac{1}{\chi_2}, \qquad \mathbf{S} = \frac{2S}{\chi_2}, \qquad \mathbf{C} = -\frac{C}{\chi_2},$$

$$\mathbf{A} = \frac{S}{\chi_2 - \chi_1}, \qquad \mathbf{B} = 2\frac{L - C}{\chi_2 - \chi_1}, \qquad \mathbf{L} = \frac{L}{\chi_0 - \chi_1}.$$

The inversion formulas read

$$S = \frac{1}{2}\frac{\mathbf{S}}{\mathbf{V}}, \qquad C = -\frac{\mathbf{C}}{\mathbf{V}}, \qquad \chi_2 = \frac{1}{\mathbf{V}},$$

$$L = \frac{1}{\mathbf{V}}\left(\frac{1}{4}\mathbf{B}\frac{\mathbf{S}}{\mathbf{A}} - \mathbf{C}\right), \qquad \chi_1 = \frac{1}{\mathbf{V}}\left(1 - \frac{1}{2}\frac{\mathbf{S}}{\mathbf{A}}\right),$$

$$\chi_0 = \frac{1}{\mathbf{V}}\left(1 - \frac{1}{2}\frac{\mathbf{S}}{\mathbf{A}} + \frac{1}{4}\frac{\mathbf{B}}{\mathbf{L}}\frac{\mathbf{S}}{\mathbf{A}} - \frac{\mathbf{C}}{\mathbf{L}}\right).$$

The parameters χ_0, χ_1, χ_2 may be replaced by N^0, N^1, N^2 by use of the relations

$$\chi_0 = N^0, \qquad \chi_2 = N^0 - N^1 + N^2, \qquad \chi_2 - \chi_1 = N^2, \qquad \chi_0 - \chi_1 = N^1.$$

Then one obtains the inversion formulas

$$N^1 = \frac{1}{\mathbf{V}}\left(\frac{1}{4}\frac{\mathbf{B}}{\mathbf{L}}\frac{\mathbf{S}}{\mathbf{A}} - \frac{\mathbf{C}}{\mathbf{L}}\right), \qquad N^2 = \frac{1}{2}\frac{1}{\mathbf{V}}\frac{\mathbf{S}}{\mathbf{A}}.$$

Note that $N^3 = N^0 - N^1 + N^2 = \chi_2 = 1/\mathbf{V}$. Further, the multiple lengths $\mathbf{L}^{ij}$ may be also expressed by the preceding parameters:

In view of (15) and (13),

$$\mathbf{L}^{13} = \mathbf{L}^{12} = \frac{N^2}{N^1}\mathbf{B} \quad \text{and} \quad \mathbf{L}^{31} = \frac{N^2}{N^3}\mathbf{B},$$

i.e.,

$$\mathbf{L}^{13} = \mathbf{L}^{12} = 2\frac{L - C}{\chi_0 - \chi_1} \quad \text{and} \quad \mathbf{L}^{31} = 2\frac{L - C}{\chi_2}.$$

Finally, Corollary 3.3.9 and (13) imply the dual topological mean value relations

$$\mathbf{N}^{01} = 2\frac{\chi_0 - \chi_1}{\chi_0}, \qquad \mathbf{N}^{32} = 2\frac{\chi_2 - \chi_1}{\chi_2},$$

$$\mathbf{N}^{02} = \frac{T - 2\chi_1}{\chi_0}, \qquad \mathbf{N}^{31} = \frac{T - 2\chi_1}{\chi_2},$$

$$\mathbf{N}^{03} = \frac{T}{\chi_0}, \qquad \mathbf{N}^{30} = \frac{T}{\chi_2},$$

$$\mathbf{N}^{12} = \frac{T - 2\chi_1}{\chi_0 - \chi_1}, \qquad \mathbf{N}^{21} = \frac{T - 2\chi_1}{\chi_2 - \chi_1},$$

$$\mathbf{N}^{13} = \frac{T - 2\chi_1}{\chi_0 - \chi_1}, \qquad \mathbf{N}^{20} = \frac{T - 2\chi_1}{\chi_2 - \chi_1}.$$

Here the new parameter $T = N^0\mathbf{N}^{03} = c^{03}_{0,t}$ is introduced. Hence, the tuple $(S, C, \chi_2, L, \chi_1, \chi_0, T)$ or, equivalently, $(\mathbf{V}, \bar{\mathbf{S}}, \mathbf{C}, \mathbf{N}^{30}, \mathbf{A}, \mathbf{B}, \mathbf{L})$ provides all information about the intensities and mean cell functionals under consideration. [Another parameter representation is given in Mecke (1984a) for the special case of convex tessellations. It may also be derived from the preceding formulas.]

REMARK. According to the general results obtained before these formulas remain valid if the mean values are replaced by the corresponding weighted direction measures $D^i_k(\Omega^*)$, $\mathbf{D}^i_k(\Omega^*)$, $D^{ij}_k(\Omega^*)$ and $\mathbf{D}^{ij}_k(\Omega^*)$, respectively. For simplicity we have considered here only the case that Ω^* coincides with the unit sphere.

3.6. *Random tessellations generated by stationary hypersurface processes.* There exist many papers on stationary random hyperplane processes in R^d and the associated random tessellation, mainly in the Poissonian case [cf. Miles (1974), Santaló (1976), Matheron (1975) and their references on related materials]. Here we will consider random tessellations generated by stationary hypersurface processes whose section properties are the same as in the hyperplane case. More precisely, it is assumed that for the stationary totally integrable random tessellation $\xi = (\xi_0, \dots, \xi_d)$ there exists a family $\eta_{d-1} = \{H_1, H_2, \dots\}$ of random hypersurfaces without boundary such that $|\xi_{d-1}| = \bigcup_{l=1}^{\infty} H_l$ and the intersection of $d - i$ of these hypersurfaces is empty or an i-dimensional submanifold without boundary. η_{d-1} may also be interpreted as a point process on the space of U_{PR}-hypersurfaces. The intersection process arising from all $(d-i)$-fold intersections will be denoted by η_i. Then $|\eta_i| = |\xi_i|$, $i = 0, \dots, d-1$. Let $M_i(u_j)$ be the number of i-dimensional submanifolds from η_i which are adjacent to the j-cell u_j from ξ_j, $j < i$. In this case

$$(20) \qquad M_i(u_j) = \binom{d-j}{d-i} \quad \text{and} \quad N_i(u_j) = \binom{d-j}{d-i} 2^{i-j}.$$

For $i - k$ *odd* it follows from (7) that for any $v_i \dot{\in} \eta_i$, $C_k(v_i, B^*) = 0$. Hence, by means of (6) one gets

$$0 = \sum_{v_i \dot{\in} \eta_i} C_k(v_i, B^*) = \sum_{v_i \dot{\in} \eta_i} \sum_{j=k}^{i} (-1)^{j-k} \sum_{\substack{u_j \dot{\in} \xi_j \\ u_j \subset v_i}} C_k(u_j, B^*)$$

$$= \sum_{j=k}^{i} (-1)^{j-k} \sum_{u_j \dot{\in} \xi_j} M_i(u_j) C_k(u_j, B^*),$$

i.e.,

$$\sum_{j=k}^{i} (-1)^{j-k} \sum_{u_j \dot{\in} \xi_j} \binom{d-j}{d-i} C_k(u_j, B^*) = 0.$$

Taking the expectation and using Proposition 3.3.4, one obtains

$$\sum_{j=k}^{i-1} (-1)^{j-k} \binom{d-j}{d-i} N^j \mathbf{D}_k^j(\Omega^*) = N^i \mathbf{D}_k^i(\Omega^*).$$

(It is easy to verify that the last equations are equivalent to the relations

$$\sum_{j=k}^{i-1} (-1)^{j-k} \binom{d-j}{d-i} 2^{i-j} N^j \mathbf{D}_k^j(\Omega^*) = 2N^i \mathbf{D}_k^i(\Omega^*),$$

which also follow from Theorem 3.3.10.)

We now will assume that the condition also holds for $i-k$ even, i.e.,

$$\mathbf{E} \sum_{v_i \in \eta_i} C_k(v_i, B^*) = 0, \qquad k = 0, \dots, i-1,\ i = 0, \dots, d-1.$$

Such hypersurface processes are said to be *weakly flat*. (In the three-dimensional case this means that the total Gaussian curvature intensity measure vanishes.) Then the preceding equations hold for all $k \le i-1$, are recurrent in $N^i \mathbf{D}_k^i(\Omega^*)$ for varying i and have the solution

$$N^i \mathbf{D}_k^i(\Omega^*) = \binom{d-k}{d-i} N^k \mathbf{D}_k^k(\Omega^*). \tag{21}$$

In the special case $\Omega^* = S^{d-1}$ and $k = 0$, (21) reads

$$N^j = \binom{d}{j} N^0. \tag{22}$$

(Note that all results concerning the case $\Omega^* = S^{d-1}$ may be derived under the weaker assumption that the total kth curvature intensities of the process η_i, $i-k$ even, are equal to zero.) Thus, (21), (22), Theorem 3.3.6 and (13) yield that all related values may be expressed by the proper *k-direction measures*

$$\vartheta_k(\Omega) := D_k^k(\Omega) = N^k \mathbf{D}_k^k(\Omega)$$

or the *k-volume intensities*

$$\nu_k := c_k^k = D_k^k(S^{d-1}),$$

respectively.

THEOREM 3.6.1. *If ξ is a totally integrable stationary random tessellation generated by a weakly flat hypersurface process in the above sense, then*

(i)
$$D_k^i(\Omega^*) = \vartheta_k(\Omega^*) \sum_{j=0}^{i-k} (-1)^j \binom{d-k}{j};$$

in particular,

$$c_k^i = \nu_k \sum_{j=0}^{i-k} (-1)^j \binom{d-k}{j}.$$

(ii)
$$\mathbf{D}_k^i(\Omega^*) = \binom{i}{k} \vartheta_k(\Omega^*) \left(\binom{d}{k} \nu_0 \right)^{-1};$$

in particular,

$$(*)\qquad \mathbf{C}_k^i = \binom{i}{k}\nu_k\left(\binom{d}{k}\nu_0\right)^{-1},$$

$$N^i = \binom{d}{i}\nu_0 = \binom{d}{i}N^0.$$

$$\text{(iii)}\qquad \mathbf{D}_k^{ij}(\Omega^*) = \binom{i}{i-j}\binom{j}{k}2^{i-j}\vartheta_k(\Omega^*)\left(\binom{d}{k}\nu_0\right)^{-1}, \qquad j < i;$$

in particular,

$$\mathbf{C}_k^{ij} = \binom{i}{i-j}\binom{i}{k}2^{i-j}\nu_k\left(\binom{d}{k}\nu_0\right)^{-1}.$$

This theorem implies the following special formulas:

$$(23)\qquad c_0^i = N^0\sum_{j=0}^{i}(-1)^j\binom{d}{j} \quad \text{(mean Euler number of the } i\text{-skeleton)},$$

$$(24)\qquad \mathbf{C}_k^i = \binom{i}{k}\mathbf{C}_k^k \quad \text{(mean total } k\text{th curvature of the typical } i\text{-cell)},$$

$$(25)\qquad N^i = \binom{d}{i}N^0 \quad \text{(mean number of } i\text{-cells)},$$

$$(26)\qquad \mathbf{N}^{ij} = \binom{i}{i-j}2^{i-j} \quad \text{(mean number of } j\text{-cells of the boundary of the typical } i\text{-cell)}.$$

Recall that $\mathbf{C}_k^k$ means the k-volume and $2\mathbf{C}_{k-1}^k$ the $(k-1)$-volume of the boundary. [For the special case of hyperplane processes, relations (25) and (26) are the central result of Mecke (1984b).]

Suppose now in addition that the *dth factorial moment measure* of the hypersurface process η_{d-1} is a *product measure*. We will prove that in this case,

$$(27)\qquad \nu_k = ((d-k)!)^{-1}\int\cdots\int|n_1\wedge\cdots\wedge n_{d-k}| \times\vartheta_{d-1}(dn_1)\cdots\vartheta_{d-1}(dn_{d-k}).$$

If η_{d-1} is *motion invariant,* this leads to

$$(28)\qquad \nu_k = (\nu_{d-1})^{d-k}\left(\frac{\Gamma(d/2)}{\Gamma((d+1)/2)}\right)^{d-k}\frac{\Gamma(d/2)}{\Gamma((k+1)/2)},$$

where Γ is the Euler function. [For the special case of motion invariant Poissonian hyperplane processes, relation $(*)$ with substitution (28) is proved in Matheron [(1975), Section 6.3] for $i = d$ and in Miles (1974) a modified expression for general i is given.]

In order to prove (27) note that the direction measure ϑ_{d-1} is related to the intensity measure λ of the hypersurface process η_{d-1} as follows: By definition of

$\vartheta_{d-1} = D_{d-1}^{d-1}$ for any Borel set $A \subset R^d$ with $\mathscr{L}^d(A) = 1$,

$$\vartheta_{d-1}(\Omega) = \mathbf{E}\tfrac{1}{2}\int_{|\eta_{d-1}| \cap A} \left(\delta_{n_x(|\eta_{d-1}|)}(\Omega) + \delta_{-n_x(|\eta_{d-1}|)}(\Omega)\right)\mathscr{H}^{d-1}(dx),$$

where $n_x(|\eta_{d-1}|)$ denotes the a.e. uniquely determined unit normal of $|\eta_{d-1}|$ at x lying in a fixed semisphere and δ_n is the Dirac measure concentrated on n. By additivity, the last expression equals

$$\tfrac{1}{2}\mathbf{E} \sum_{H \dot{\in} \eta_{d-1}} \int_{H \cap A} \left(\delta_{n_x(H)}(\Omega) + \delta_{-n_x(H)}(\Omega)\right)\mathscr{H}^{d-1}(dx).$$

Hence,

$$\vartheta_{d-1}(\Omega) = \tfrac{1}{2}\int\int_{H \cap A} \left(\delta_{n_x(H)}(\Omega) + \delta_{-n_x(H)}(\Omega)\right)\mathscr{H}^{d-1}(dx)\lambda(dH).$$

Since the dth factorial moment measure of η_{d-1} is a product measure, in view of stationarity and Fubini,

$$\begin{aligned}
\nu_k &= \frac{1}{(d-k)!}\int \cdots \int \mathscr{H}^k(A \cap H_1 \cap \cdots \cap H_{d-k})\lambda(dH_1)\cdots\lambda(dH_{d-k}) \\
&= \frac{1}{(d-k)!}\int \cdots \int 1_{A \cap H_1}(y) 1_{H_2}(y) \cdots 1_{H_{d-k}}(y) \int 1_A(x_2 + y)\mathscr{L}^d(dx_1) \\
&\qquad \cdots \int 1_A(x_{d-k} + y)\mathscr{L}^d(dx_{d-k})\mathscr{H}^k(dy)\lambda(dH_1)\cdots\lambda(dH_{d-k}) \\
&= \frac{1}{(d-k)!}\int \cdots \int 1_{A \cap H_1}(y) 1_{A \cap H_2}(x_2 + y) \cdots 1_{A \cap H_{d-k}}(x_{d-k} + y)\mathscr{H}^k(dy) \\
&\qquad \times\lambda(dH_1)\cdots\lambda(dH_{d-k})\mathscr{L}^d(dx_2)\cdots\mathscr{L}^d(dx_{d-k}) \\
&= \frac{1}{(d-k)!}\int \cdots \int\Big[\int \cdots \int \mathscr{H}^k((A \cap H_1) \cap (A \cap H_2 - x_2) \\
&\qquad \cap \cdots \cap (A \cap H_{d-k} - x_{d-k})) \\
&\qquad \times\mathscr{L}^d(dx_2)\cdots\mathscr{L}^d(dx_{d-k})\Big]\lambda(dH_1)\cdots\lambda(dH_{d-k}).
\end{aligned}$$

The translative kinematic formula for Hausdorff measures implies that the integral in the brackets equals

$$\begin{aligned}
&\int \cdots \int 1_{A \cap H_1}(z_1) \cdots 1_{A \cap H_{d-k}}(z_{d-k}) \\
&\qquad \times |n_{z_1}(H_1) \wedge \cdots \wedge n_{z_{d-k}}(H_{d-k})|\mathscr{H}^{d-1}(dz_1)\cdots\mathscr{H}^{d-1}(dz_{d-k}).
\end{aligned}$$

Therefore,

$$\nu_k = \frac{1}{(d-k)!}\int \cdots \int |n_1 \wedge \cdots \wedge n_{d-k}|\vartheta_{d-1}(dn_1)\cdots\vartheta_{d-1}(dn_{d-k}).$$

3.7. *Mean cells and projection bodies.* Here we will study further the mean cells of the random tessellation in the last example. Recall that we only consider the values of the mean cells (in the sense of Section 3.2) on differential forms of the kind $1_{\Omega^*}(n)\varphi_k(n)$, which determine the weighted direction measures $\mathbf{D}_k^i(\Omega^*)$. In our special case these measures may be calculated by means of the surface area measures of the associated *Steiner compact set.*

In the case of stationary hyperplane processes, the last concept can be found in Matheron (1975), Wieacker (1986) introduced it for arbitrary stationary rectifiable hypersurface processes in form of the *projection body* Π. The support function $h(\Pi, \cdot)$ of this centrally symmetric compact convex set is defined by the direction measure ϑ_{d-1},

$$h(\Pi, v) = \int |v \cdot n| \vartheta_{d-1}(dn).$$

According to Matheron [(1975), Section 4.5],

$$\begin{aligned} C_k(\Pi) &:= C_k(\Pi, R^d \times S^{d-1}) \\ &= \frac{1}{k!} \int \cdots \int |n_1 \wedge \cdots \wedge n_k| \vartheta_{d-1}(dn_1) \cdots \vartheta_{d-1}(dn_k), \end{aligned}$$

$k = 1, \dots, d$. [Matheron's $W_{d-k}(\cdot)$ coincides up to a constant with our $C_k(\cdot)$.] Therefore, we infer from (27) that

$$\nu_k = C_{d-k}(\Pi), \qquad k = 0, \dots, d-1.$$

COROLLARY 3.7.1. *For any totally integrable stationary random tessellation generated by a weakly flat hypersurface process whose dth factorial moment measure is a product measure, we have*

$$\mathbf{C}_k^i = \binom{i}{k} C_{d-k}(\Pi) \left(\binom{d}{k} C_d(\Pi) \right)^{-1}, \qquad 0 \le k \le i \le d.$$

REMARK. An analogous relation can be proved for the weighted direction measures $\mathbf{D}_k^i$. (Corollary 3.7.1 for the special case of hyperplane processes and $i = d$ is proved in Matheron [(1975), Section 6.3] by other methods.)

Planar case. For $d = 2$ and $i = 2$ the statement may be sharpened: By the Palm relation,

$$\mathbf{D}_1^2(\Omega) = (N^2)^{-1} D_1^1(\Omega) = (\nu_0)^{-1} \vartheta_1(\Omega).$$

Recall that $2D_1^2(\Omega)$ is the mean length of the boundary points of the typical cell with exterior unit normal in Ω. Since the measure $\mathbf{D}_1^2$ is even and not concentrated on a single pair $(-n_0, n_0)$, there exists an (up to translations) unique centrally symmetric convex cell P with

$$2\mathbf{D}_1^2(\Omega) = \int_{\partial P} 1_\Omega(n_x(P)) \mathscr{H}^1(dx), \tag{29}$$

where $n_x(P)$ denotes the a.e. uniquely determined unit normal to P at x

(Minkowski problem from the theory of convex bodies). In particular, for $\Omega = S^1$ one gets

$$\mathbf{B} := 2\mathbf{C}_1^2 = \mathscr{H}^1(\partial P) \quad \textit{mean boundary length}. \tag{30}$$

It is easy to see that the support function of P takes the form

$$h(P, v) = \int |v \cdot n| \mathbf{D}_1^2(dn). \tag{31}$$

Therefore, we have in the preceding notation,

$$P = \nu_0^{-1}\Pi = \left(\mathscr{L}^2(\Pi)\right)^{-1}\Pi. \tag{32}$$

[In the special case of convex cells, the support function (31) coincides with the mean support function of the typical cell, i.e., P is the "mean" set of the typical cell as defined in the theory of random convex sets.] Furthermore, since $\nu_0 = \mathscr{L}^2(\Pi) = \mathscr{L}^2(\nu_0 P) = \nu_0^2 \mathscr{L}^2(P)$, we have $\mathscr{L}^2(P) = \nu_0^{-1} = (N^2)^{-1} = \mathbf{C}_2^2$, i.e.,

$$\mathbf{C}_2^2 = \mathscr{L}^2(P) \quad \textit{mean area}. \tag{33}$$

Formulas (30) and (33) state that in the general case the mean boundary length and the mean area of the typical cell of the tessellation agree with those of the convex cell P given by (32).

APPENDIX

Measurability of tangent bundles. In the proof of Theorem 3.1.1, the measurability of the tangent bundle of Nor X as a closed subset of R^{2d} was used. A more general version of this assertion may be considered as a basic measurability property for translative stochastic geometry in euclidean space: In choosing appropriate spaces and mappings, many integral-geometric formulas may be reduced to Federer's coarea theorem and there the tangent bundles of the sets under consideration play a fundamental role.

Let $[\mathscr{X}_m, \mathfrak{X}_m]$ be the measurable space of (locally) $\mathscr{H}^m$-rectifiable closed subsets of R^n as introduced in Zähle (1982), based on concepts of Federer. [Each element of $\mathscr{X}_m$ possesses a locally finite (Hausdorff) $\mathscr{H}^m$-measure and, except for a $\mathscr{H}^m$-zero set, it is contained in the countable union of m-dimensional C_1-submanifolds.) For general $X \in \mathscr{X}_m$, the classical concept of tangent bundle is not appropriate for integral geometry. For example, one easily obtains $X \in \mathscr{X}_m$ which are dense in R^n. Therefore, Federer [(1969), Section 3.2.16] introduced the concept of approximate tangent cones $\tan^m[\mathscr{H}^m LX, x]$ to X at $x \in R^n$. They are related to the tangent spaces of the underlying C_1-submanifolds. The definition also holds for arbitrary X from the space $\mathscr{F}_m$ of $\mathscr{H}^m$-measurable closed subsets of R^n with locally finite measure. $\mathfrak{F}_m$ denotes the Borel σ-algebra with respect to the usual topology on $\mathscr{F}_m$ generated as for $\mathscr{X}_m$ by the two families of sets $\{X \in \mathscr{F}_m \colon X \cap K = \varnothing\}$, K compact, and $\{X \in \mathscr{F}_m \colon X \cap G \neq \varnothing\}$, G open. Note that $\mathfrak{F}_m$ is already generated by either family.

Proposition. *Let $K, K_1, K_2, \dots$ be a sequence of open cones in R^d with $\overline{K}_i \subset K$ and $\bigcup_{i=1}^{\infty} K_i = K$. Then*

$$\{X \in \mathscr{F}_m : 0 \in X, \tan^m[\mathscr{H}^m L X, 0] \cap K \neq \varnothing\}$$
$$= \bigcup_{i=1}^{\infty} \left\{X \in \mathscr{F}_m : 0 \in X, \limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K_i \cap rb) > 0\right\}.$$

Proof. Let $X \in \mathscr{F}_m$ and $0 \in X$. According to the arguments in Federer [(1969), Section 3.2.16] $v \in \tan^m[\mathscr{H}^m L X, 0]$ iff $\limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K(v) \cap rb) > 0$ for any open cone $K(v)$ containing v. Hence, it follows for $v \in \tan^m[\mathscr{H}^m \llcorner X, 0] \cap K_i$ that

$$\limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K_i \cap rb) > 0.$$

Conversely, suppose that

$$\limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K_i \cap rb) > 0.$$

If $\tan^m[\mathscr{H}^m \llcorner X, 0] \cap K = \varnothing$, then for any $v \in K$ there exists an open cone $K(v)$ containing v such that $\limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K(v) \cap rb) = 0$. Since $X \cap \overline{K}_i \cap b$ is compact it can be covered by finitely many of the cones $K(v)$, $v \in K$. Therefore, the additivity of $\mathscr{H}^m \llcorner X$ leads to

$$\lim_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K_i \cap rb) = 0,$$

which contradicts the preceding condition. Hence,

$$\tan^m[\mathscr{H}^m \llcorner X, 0] \cap K \neq \varnothing. \qquad \square$$

Let $\mathscr{C}$ be the space of closed cones of R^n provided with the Borel σ-algebra generated by the sets $\{C \in \mathscr{C} : C \cap K \neq \varnothing\}$ for arbitrary open cones K.

Theorem. *The mapping $f: \mathscr{F}_m \times R^n \to \mathscr{C}$ with $f(X, x) = \tan^m[\mathscr{H}^m L X, x]$ is measurable with respect to the product Borel σ-algebra.*

Proof. Since the mapping $(X, x) \to X - x$ is continuous and

$$\tan^m[\mathscr{H}^m \llcorner X, x] = \tan^m[\mathscr{H}^m \llcorner (X - x), 0],$$

it suffices to show that the mapping $X \to \tan^m[\mathscr{H}^m \llcorner X, 0]$ from $\mathscr{F}_m$ into $\mathscr{C}$ is measurable. In view of the definition of the σ-algebra on $\mathscr{C}$ and the proposition, this is fulfilled if the sets

$$\{X \in \mathscr{F}_m : 0 \in X\} \quad \text{and} \quad \left\{X \in \mathscr{F}_m : \limsup_{r \to 0+} r^{-m}\mathscr{H}^m(X \cap K_i \cap rb) > 0\right\},$$

$i = 1, 2, \dots,$ are elements of the σ-algebra $\mathfrak{F}_m$. (Note that for any K there exist suitable K_i.) The first measurability follows directly from the definition of $\mathfrak{F}_m$ and the second one from Zähle [(1982), Section 2.1.4]. $\square$

REMARK. Note that for the purposes of integral geometry $[\mathscr{F}_m, \mathfrak{F}_m]$ has to be replaced by $[\mathscr{X}_m, \mathfrak{X}_m]$. Recall that $\mathscr{X}_m \in \mathfrak{F}_m$ (cf. Zähle [(1982), Section 2.2.1]).

Because of the structure of the normal bundle in Theorem 3.1.1, $\overline{\text{Nor}\, X}$ is an element of $\mathscr{X}_{d-1}$ in R^{2d} such that $T_{(x,n)}\overline{\text{Nor}\, X} = \tan^{d-1}[\mathscr{H}^{d-1} \llcorner \overline{\text{Nor}\, X}, (x, n)]$ for $\mathscr{H}^{d-1}$-almost all $(x, n) \in \overline{\text{Nor}\, X}$.

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