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SOME LIMIT THEOREMS FOR VOTER MODEL OCCUPATION TIMES[1]

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Let η_t be the (basic) voter model on $\mathbb{Z}^d$. We consider the occupation time functionals $\int_0^t f(\eta_s)\,ds$ for certain functions f and initial distributions. The first result is a pointwise ergodic theorem in the case $d = 2$, extending the work of Andjel and Kipnis. The second result is a central limit type theorem for $f(\eta) = \eta(0)$ and initial distributions: (i) δ_η, for a class of states η, $d \geq 2$, and (ii) ν_θ, the extremal invariant measures, $d \geq 3$.

1. Introduction. Occupation time functionals have been studied for several infinite particle systems. A sampling of this work is independent random walk systems ([9] and [20]–[22]), branching Brownian motion and random walk systems ([10]), the contact process ([12]–[14]), the voter model ([4] and [8]) and the simple exclusion process ([1]). The objective of this paper is to extend some of the work in [1] and [8] on pointwise ergodic theorems and "central limit" type theorems for the voter model. We begin by defining our process.

Let $X = \{0,1\}^{\mathbb{Z}^d}$, endowed with the usual product topology. The (basic) voter model η_t is the X-valued Markov process, which has flip rates at each site $x \in \mathbb{Z}^d$ and time $t \geq 0$,

$$\eta_t(x) \to 1 - \eta_t(x), \quad \text{at rate } (2d)^{-1}\#\{y\colon |x-y| = 1,\ \eta_t(x) \neq \eta_t(y)\}.$$

A complete description of η_t can be found in [18]. For each $0 \leq \theta \leq 1$ let μ_θ denote the Bernoulli product measure on X with density θ, $\mu_\theta\{\eta(x) = 1\} = \theta$ for all $x \in \mathbb{Z}^d$. For each probability measure μ on X let P_μ denote the law of η_t with initial measure μ, and let $\mathscr{I}$ be the set of such measures that are invariant for η_t, i.e., $\mathscr{I} = \{\mu\colon P_\mu(\eta_t \in \cdot) = \mu\}$. $\mathscr{I}_e$ will denote the set of extreme points of $\mathscr{I}$. Let δ_η be the point mass at η, and write P_η for P_{δ_η}. Finally, $\Rightarrow$ will denote weak convergence.

The fundamental result concerning the ergodic behavior of η_t is (see [7] and [15])

Theorem 0. *For* $0 \leq \theta \leq 1$,

$$P_{\mu_\theta}(\eta_t \in \cdot) \Rightarrow \nu_\theta, \quad \text{as } t \to \infty. \tag{1.1}$$

For $d \leq 2$, $\nu_\theta = (1-\theta)\mu_0 + \theta_{\mu_1}$ *and* $\mathscr{I} = \{\nu_\theta,\ 0 \leq \theta \leq 1\}$. *For* $d \geq 3$, $\mathscr{I}_e = \{\nu_\theta,\ 0 \leq \theta \leq 1\}$.

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For $d \geq 3$ the measures ν_θ are not product measures; their macroscopic structure is studied in [5]. For $d = 1, 2$ Theorem 0 indicates that clustering occurs; see [2] and [6] for $d = 1$, and [3] and [11] for $d = 2$.

Holley and Liggett [15] have given explicit necessary and sufficient conditions for a measure μ to be in the "domain of attraction" of a ν_θ, i.e., for

$$P_\mu(\eta_t \in \cdot) \Rightarrow \nu_\theta, \quad \text{as } t \to \infty.$$

For fixed $\eta \in X$ and $\mu = \delta_\eta$, this condition is

$$\lim_{t \to \infty} \sum_{y \in \mathbb{Z}^d} p_t(x, y)\eta(y) = \theta, \qquad \forall\, x \in \mathbb{Z}^d, \tag{1.2}$$

where p_t is the transition function of a rate 1 simple symmetric random walk on $\mathbb{Z}^d$. As noted in [18], pages 68–69, it is enough to require (1.2) just for $x = 0$.

We are interested here in the asymptotic behavior of the occupation time functionals

$$\int_0^t f(\eta_s)\, ds,$$

for certain functions f on X and various initial measures for η_t, especially μ_θ, ν_θ and δ_η. A "strong law" type result, or pointwise ergodic theorem, was proved in [1].

THEOREM 1. *Suppose $d \geq 3$, $f \in C(X)$ and η satisfies* (1.2). *Then*

$$\frac{1}{t}\int_0^t f(\eta_s)\, ds \to \int f(\zeta)\, d\nu_\theta(\zeta), \quad \text{as } t \to \infty,\ P_\eta\ a.s. \tag{1.3}$$

This type of pointwise ergodic theorem has also been proved for the contact process (see [12]–[14]) and the simple exclusion process (see [1]). As is pointed out in [1], general considerations imply that the set of η for which (1.3) must hold has ν_θ measure 1, but this fact alone fails to identify a single η for which (1.3) is true.

The case $d = 1$ is properly excluded from Theorem 1, since results of [8] indicate that for $f(\eta) = \eta(0)$, and at least some η satisfying (1.2), $t^{-1}\int_0^t f(\eta_s)\, ds$ converges weakly to a nondegenerate random variable. But there is no reason to suspect the case $d = 2$ should be excluded, and this is our first result. Let $\mathbf{0}$ and $\mathbf{1}$ denote the elements of X that are identically 0 and 1.

THEOREM 2. *Suppose $d = 2$, $f \in C(X)$ and η satisfies* (1.2). *Then*

$$\frac{1}{t}\int_0^t f(\eta_s)\, ds \to (1 - \theta)f(\mathbf{0}) + \theta f(\mathbf{1}), \quad \text{as } t \to \infty,\ P_\eta\ a.s. \tag{1.4}$$

The right-hand sides of (1.3) and (1.4) are the same for $d = 2$ by Theorem 0.

"Central limit" type behavior for the occupation time functionals was considered in [8] for $f(\eta) = \eta(0)$ and initial distribution μ_θ. The result is

Theorem 3. *For the voter model with initial distribution* μ_θ, $0 < \theta < 1$, *there are constants* $0 < \sigma_d^2 < \infty$ *such that as* $t \to \infty$,

$$\sigma^2(t) = \operatorname{var}\left(\int_0^t \eta_s(0)\,ds\right) \sim \sigma_d^2\theta(1-\theta)\begin{cases} t^2, & d = 1,\\ t^2/\log t, & d = 2,\\ t^{3/2}, & d = 3,\\ t\log t\,, & d = 4,\\ t, & d = 5,\end{cases}$$

and

$$\left(\int_0^t \eta_s(0)\,ds - t\theta\right)\Big/\sigma(t) \Rightarrow Z,$$

where Z *is nondegenerate, and normal if and only if* $d \geq 2$.

[$f(t) \sim g(t)$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 1$.] The reasons for considering μ_θ for the initial distribution are twofold. Product measure is a natural initial distribution given the dynamics of the model, representing "complete independence" of the voters at time 0. Moreover, the duality equation used to study the voter model (see the next section) is readily analyzed in this case, and is far less tractable in others. For $d \geq 3$ it seems that the most natural choice for an initial distribution is ν_θ, which makes η_t a stationary process. We have been able to show that in this case, and for some δ_η, central limit type behavior can still be obtained. Our result is

Theorem 4. *If* η_t *has initial distribution*

(i) ν_θ, $d \geq 3$,

or

(ii) δ_η, $d \geq 2$, *where* η *satisfies* (1.2) *uniformly in* x,

then as $t \to \infty$,

$$\left(\int_0^t \left(\eta_s(0) - E_\mu\eta_s(0)\right)ds\right)\Big/\left(\operatorname{var}_\mu\int_0^t \eta_s(0)\,ds\right)^{1/2} \Rightarrow Z, \tag{1.5}$$

where Z *is standard normal.*

The extra assumption of uniformity in (1.2) for initial distributions δ_η is probably not needed, but our proof requires something a little stronger than (1.2). The remainder of the paper is organized as follows. In Section 2 we extend a key lemma of [5] to cover initial distributions δ_η and ν_θ. Theorem 2 is proved in Section 3 and Theorem 4 is proved in Section 4.

2. The key lemma. To analyze the voter model, one needs a duality equation, which connects the voter model with a coalescing random walk system.

We define, for $n \geq 1$, $x_i \in \mathbb{Z}^d$, $s_i \geq 0$ and $t \geq \max_i s_i$:

$\xi_t((x_1, s_1), \dots, (x_n, s_n))$ = the system of coalescing rate 1 simple symmetric random walks on $\mathbb{Z}^d$, starting at $x_i \in \mathbb{Z}^d$, the walk at x_i frozen until time s_i, such that two walks coalesce only after both are unfrozen.

The system ξ_t is constructed on a "percolation substructure"; see [13] for details of the construction. For $\eta \in X$ and $A \subset \mathbb{Z}^d$ we will write $A \subset \eta$ for $A \subset \{x: \eta(x) = 1\}$.

The duality equation for η_t and ξ_t (see [13]) can be written as

$$(2.1)\quad P_\eta\big(\eta_{t-s_i}(x_i) = 1, 1 \leq i \leq n\big) = P\big(\xi_t((x_1, s_1), \dots, (x_n, s_n)) \subset \eta\big).$$

We will also need to start η_t in μ_θ and ν_θ, in which case we have

$$\begin{aligned} P_\mu\big(\eta_{t-s_i}(x_i) = 1, 1 \leq i \leq n\big) &= E\theta^{\#\xi_t((x_1, s_1), \dots, (x_n, s_n))}, & \mu &= \mu_\theta, \\ &= E\theta^{N_\infty((x_1, s_1), \dots, (x_n, s_n))}, & \mu &= \nu_\theta, \end{aligned}$$

where $N_\infty(\cdot) = \lim_{t\to\infty} \#\xi_t(\cdot)$.

As in [5] and [8] our results depend on obtaining good moment and cumulant estimates. If η_t has initial distribution μ let $S_m^\mu(t)$ be the mth cumulant of $\int_0^t \eta_s(0)\, ds$, formally

$$\log E_\mu\left(\exp\left(\lambda \int_0^t \eta_s(0)\, ds\right)\right) = \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} S_m^\mu(t).$$

We will need the so-called Ursell functions (see [16], [17] and [19]). Given random variables $Y_1, \dots, Y_m$, not necessarily distinct, denote the mth-order Ursell function u_m,

$$u_m(Y_1, \dots, Y_m) = \sum_{s=1}^{m} (-1)^{s-1}(s-1)! \sum_{\pi = (\pi_1, \dots, \pi_s)} \rho(\pi_1) \cdots \rho(\pi_s),$$

the second sum over partitions π of $\{1, 2, \dots, m\}$, and $\rho(\pi_i) = E\big(\prod_{j\in\pi_i} Y_j\big)$. We will need a combinatorial result from [16], [17] and [19]. Let (π', π'') be a nontrivial partition of $\{1, 2, \dots, m\}$. Then one can write

$$(2.2)\quad u_m(Y_1, \dots, Y_m) = \sum \pm \big[\rho(\pi_1 \cup \pi_2) - \rho(\pi_1)\rho(\pi_2)\big]\rho(\pi_3) \cdots,$$

where the sum is over all partitions $\pi = (\pi_1, \pi_2, \pi_3, \dots)$ such that $\pi_1 \subset \pi'$, $\pi_2 \subset \pi''$. We will also use the much simpler combinatorial fact that

$$(2.3)\quad S_m^\mu(t) = \int_0^t ds_1 \cdots \int_0^t ds_m\, u_m^\mu\big(\eta_{t-s_1}(x_1), \dots, \eta_{t-s_m}(x_m)\big).$$

The main technical result we need is the

KEY LEMMA. *For $m = 2, 3, \dots$ there exist finite constants K_m such that if $x_i \in \mathbb{Z}^d$, $s_i \geq 0$, $t \geq \max_i s_i$,*

$$\begin{aligned} &u_m^\mu\big(\eta_{t-s_1}(x_1), \dots, \eta_{t-s_m}(x_m)\big) \\ (2.4)\quad &\leq K_m P\big(\#\xi_t((x_1, s_1), \dots, (x_m, s_m)) = 1\big), \quad \mu = \mu_\theta \text{ or } \delta_\eta, \\ &\leq K_m P\big(N_\infty((x_1, s_1), \dots, (x_m, s_m)) = 1\big), \quad \mu = \nu_\theta. \end{aligned}$$

The case $\mu = \mu_\theta$ is in [8], based on Proposition 2 of [5]. The estimate is useful because it is shown in [8] [equations (4.3) and (4.4)] that if

$$(2.5)\quad g_m(t) = \sup_{x_i} \int_0^t ds_1 \cdots \int_0^t ds_m\, P(\#\xi_t((x_1, s_1), \dots, (x_m, s_m)) = 1),$$

then

$$(2.6)\quad \begin{aligned} g_m(t) &= O\big(t \cdot (t/\log t)^{m-1}\big), & d &= 2,\\ &= O\big(t \cdot t^{(m-1)/2}\big), & d &= 3,\\ &= O\big(t \cdot (\log t)^{m-1}\big), & d &= 4,\\ &= O(t), & d &\ge 5. \end{aligned}$$

We will omit the straightforward modifications of the proof of this fact needed to prove (for $d \ge 3$)

$$(2.7)\quad \sup_{x_i} \int_0^t ds_1 \cdots \int_0^t ds_m\, P(N_\infty((x_1, s_1), \dots, (x_m, s_m)) = 1) = O(g_m(t)).$$

PROOF OF THE KEY LEMMA. Proposition 2 of [5] was proved assuming η_t had initial distribution μ_θ. The key lemma asserts, in effect, that it holds for two other choices of initial measures, namely δ_η and ν_θ. Here are the details. Fix m, the s_i and the x_i.

Construct m independent random walks $X_t(i)$, $1 \le i \le m$, $X_t(i)$ frozen at its starting point x_i until time s_i. Let $\tilde{P}$ and $\tilde{E}$ denote the probability law and expectation operator for these walks. For each nontrivial partition $\pi = (\pi_1, \dots, \pi_s)$ of $\{1, 2, \dots, m\}$, let $\hat{X}_t^\pi$ denote the process such that only those walks with index from the same π_α interact. This interaction is that if $X_\cdot(i)$ and $X_\cdot(j)$ collide at some time $s \ge \max(s_i, s_j)$, then $X_\cdot(i)$ survives if and only if $i < j$. Given π, let $X_t^{\pi_\alpha}$ denote the positions of the set of surviving particles with indices from π_α at time t.

Now consider

$$\begin{aligned} &u_m^\mu(\eta_{t-s_1}(x_1), \dots, \eta_{t-s_m}(x_m))\\ &\quad = \sum_{s=1}^m (-1)^{s-1}(s-1)! \sum_{\pi = (\pi_1, \dots, \pi_s)} E_\mu\Big(\prod_{i\in\pi_1} \eta_{t-s_i}(x_i)\Big) \cdots E_\mu\Big(\prod_{i\in\pi_s} \eta_{t-s_i}(x_i)\Big). \end{aligned}$$

We will prove that

$$(2.8)\quad u_m^\mu(\eta_{t-s_1}(x_1), \dots, \eta_{t-s_m}(x_m)) = \tilde{E}(\tilde{\Sigma})$$

where

$$(2.9)\quad \tilde{\Sigma} = \sum_{s=1}^m (-1)^{s-1}(s-1)! \sum_{\pi = (\pi_1, \dots, \pi_s)} \rho(\pi_1) \cdots \rho(\pi_s)$$

and

$$(2.10)\quad \begin{aligned} \rho(\pi_\alpha) &= 1(X_t^{\pi_\alpha} \subset \eta), & &\text{if } \mu = \delta_\eta,\\ &= \theta^{\tilde{N}_\infty(\pi_\alpha)}, & &\text{if } \mu = \nu_\theta, \end{aligned}$$

$\tilde{N}_\infty(\pi_\alpha) = \lim_{t\to\infty} \# X_t^{\pi_\alpha}$.

Once this is done, the purely combinatorial argument used to prove (2.2) applies to show that if (π', π'') is a nontrivial partition of $\{1, 2, \dots, m\}$, then

$$\tilde{\Sigma} = \sum \pm \left[\rho(\pi_1 \cup \pi_2) - \rho(\pi_1)\rho(\pi_2)\right]\rho(\pi_3)\cdots, \tag{2.9'}$$

the sum over partition $(\pi_1, \pi_2, \dots)$ with $\pi_1 \subset \pi'$, $\pi_2 \subset \pi''$. Now if $\#X_t^{\{1,2,\dots,m\}} > 1$ [or $\tilde{N}_\infty(\{1, 2, \dots, m\}) > 1$], then there is some nontrivial partition (π', π'') such that the walks $X_\cdot^i$, $\in \pi'$ and the walks $X_\cdot^j$, $\in \pi''$, do not meet by time t (or never meet). In either case, if $\pi_1 \subset \pi'$ and $\pi_2 \subset \pi''$ and ρ is given by (2.10), then

$$\rho(\pi_1 \cup \pi_2) = \rho(\pi_1)\rho(\pi_2).$$

Consequently, $\tilde{\Sigma} = 0$ on $\{\#X_t^{\{1,2,\dots,m\}} > 1\}$ [or $\tilde{N}_\infty(\{1, 2, \dots, m\}) > 1$]. Taking expectation yields (2.4) with

$$K_m = \sum_{s=1}^{m} (s-1)!\,\#\{\pi = (\pi_1, \dots, \pi_s)\}.$$

All that remains is to prove (2.9) and (2.10).

Suppose $\mu = \delta_\eta$. Letting $\xi_t(\pi_\alpha) = \xi_t((x_i, s_i), \dots, (x_{i_k}, s_{i_k}))$ if $\pi_\alpha = \{i_1, \dots, i_k\}$, duality implies

$$E_\eta\Big(\prod_{j\in\pi_\alpha} \eta_{t-s_j}(x_j)\Big) = P(\xi_t(\pi_\alpha) \subset \eta)$$

$$= \tilde{P}(X_t^{\pi_\alpha} \subset \eta).$$

Thus

$$u_m^\eta\big(\eta_{t-s_1}(x_1), \dots, \eta_{t-s_m}(x_m)\big)$$

$$= \sum_{s=1}^{m} (-1)^{s-1}(s-1)! \sum_{\pi=(\pi_1,\dots,\pi_s)} \tilde{P}(X_t^{\pi_1} \subset \eta)\cdots\tilde{P}(X_t^{\pi_s} \subset \eta)$$

$$= \sum_{s=1}^{m} (-1)^{s-1}(s-1)! \sum_{\pi=(\pi_1,\dots,\pi_s)} \tilde{P}(X_t^{\pi_1} \subset \eta, \dots, X_t^{\pi_s} \subset \eta),$$

since for a given $\pi = (\pi_1, \dots, \pi_s)$, $X_t^{\pi_1}, \dots, X_t^{\pi_s}$ are independent. This is the $\mu = \delta_\eta$ case of (2.9) and (2.10). Now suppose $\mu = \nu_\theta$. Then duality implies

$$E_{\nu_\theta}\Big(\prod_{j\in\pi_\alpha} \eta_{t-s_j}(x_j)\Big) = \int d\nu_\theta(\xi)\, P(\xi_t(\pi_\alpha) \subset \zeta)$$

$$= \sum_{A\subset \mathbf{Z}^d} P(\xi_t(\pi_\alpha) = A) E\theta^{N_\infty(A)}$$

$$= E\theta^{N_\infty((x_i, s_i), \dots, (x_{i_k}, s_{i_k}))}$$

$$= \tilde{E}\theta^{\tilde{N}_\infty(\pi_\alpha)}.$$

Thus

$$u_m^{\nu_\theta}\big(\eta_{t-s_1}(x_1),\dots,\eta_{t-s_m}(x_m)\big)$$
$$= \sum_{s=1}^{m}(-1)^{s-1}(s-1)! \sum_{\pi=(\pi_1,\dots,\pi_s)} \tilde{E}\theta^{\tilde{N}_\infty(\pi_1)}\cdots\tilde{E}\theta^{\tilde{N}_\infty(\pi_s)}$$
$$= \sum_{s=1}^{m}(-1)^{s-1}(s-1)! \sum_{\pi=(\pi_1,\dots,\pi_s)} \tilde{E}\big[\theta^{\tilde{N}_\infty(\pi_1)}\cdots\theta^{\tilde{N}_\infty(\pi_s)}\big],$$

since for a given $\pi=(\pi_1,\dots,\pi_s)$, we must have $\tilde{N}_\infty(\pi_1),\dots,\tilde{N}_\infty(\pi_s)$ independent. This completes the proof. □

3. Proof of Theorem 2. We will proceed by first showing that (1.4) is valid for the special case $f(\eta)=\eta(0)$, and then arguing that (due to the clustering) this case suffices. For the first step, we note that

$$\begin{aligned} E_\eta\int_0^t \eta_s(0)\,ds &= \int_0^t P(\xi_{t-s}(0,0)\in\eta)\,ds \\ &= \int_0^t \sum_{y\in\mathbb{Z}^2} p_s(0,y)\eta(y)\,ds \\ &\sim t\theta, \end{aligned} \tag{3.1}$$

as $t\to\infty$ by (1.2). Furthermore, the key lemma (with $m=4$) and (2.5) and (2.6) imply that

$$\begin{aligned} E_\eta\left[\int_0^t\big(\eta_s(0)-E_\eta(\eta_s(0))\big)\,ds\right]^4 &= S_4(t)+3S_2(t) \\ &= O(t^4/\log^2 t), \end{aligned}$$

as $t\to\infty$. The proof of

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\eta_s(0)\,ds=\theta,\quad P_\eta \text{ a.s.,} \tag{3.2}$$

is now standard. By Chebyshev, for any $\varepsilon>0$,

$$P_\eta\left(\left|\frac{1}{t}\int_0^t\big(\eta_s(0)-E_\eta(\eta_s(0))\big)\,ds\right|>\varepsilon\right)=O\left(\frac{1}{\log^2 t}\right),$$

as $t\to\infty$. By Borel–Cantelli, for any $r>1$, this estimate and (3.1) give

$$\lim_{n\to\infty}\frac{1}{r^n}\int_0^{r^n}\eta_s(0)\,ds\to\theta,\quad P_\eta \text{ a.s.}$$

By considering $r^n\le t\le r^{n+1}$, we obtain

$$\frac{\theta}{r}\le\liminf_{t\to\infty}\frac{1}{t}\int_0^t\eta_s(0)\,ds\le\limsup_{t\to\infty}\frac{1}{t}\int_0^t\eta_s(0)\,ds\le r\theta,\quad P_\eta \text{ a.s.}$$

Let $r\downarrow 1$ to complete the proof of (3.2).

For the next step we note that it suffices, by standard arguments, to assume f is of the form

$$f(\eta)=\prod_{x\in A}\eta(x),\quad \text{finite } A\subset\mathbb{Z}^2.$$

Due to the clustering that occurs one suspects that $\eta_s(x) = \eta_s(0)$ "most of the time." This is indeed correct, as we will prove

$$(3.3) \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t 1(\eta_s(0) \neq \eta_s(x))\, ds = 0, \quad \text{a.s. } P_\eta,\ x \in \mathbf{Z}^2.$$

To prove (3.3), we will first establish

$$(3.4) \qquad E_\eta\left(\int_0^t 1(\eta_s(0) \neq \eta_s(x))\, ds\right)^2 = O(t^2/\log^2 t),$$

as $t \to \infty$; with this estimate (3.3) is proved the same way (3.2) was proved.

Consider

$$E_\eta\left(\int_0^t \eta_s(0)(1 - \eta_s(x))\, ds\right)^2$$
$$= \left\{\int_{0\le s\le u\le t} + \int_{0\le u\le s\le t}\right\} P_\eta(\eta_s(0) = \eta_u(0) = 1,\ \eta_u(x) = \eta_s(x) = 0)\, du\, ds.$$

By duality, the first integral is

$$(3.5) \qquad \int_{0\le s\le u\le t} P(\xi_t(0, t-s) \in \eta,\ \xi_t(x, t-s) \notin \eta,$$
$$\xi_t(0, t-u) \in \eta,\ \xi_t(x, t-u) \notin \eta)\, du\, ds.$$

A little thought shows that (3.5) is bounded above by

$$\int_{0\le s\le u\le t} P(\#\xi_t((0, t-s), (x, t-s)) = 2,\ \#\xi_t((0, t-u), (x, t-u)) = 2)\, du\, ds$$
$$= \int_{0\le s\le u\le t} P(\#\xi_{t-s}((0, t-u), (x, t-u)) = 2)$$
$$\times P(\#\xi_t((0, t-s), (x, t-s)) = 2)\, du\, ds$$
$$= \int_{0\le s\le u\le t} P(\tau_0(x) > 2(u-s)) P(\tau_0(x) > 2s)\, du\, ds,$$

where $\tau_0(x)$ is the first hitting time of 0 for a rate 1 simple symmetric random walk on $\mathbf{Z}^2$ starting at x. The well-known estimate (see [23], for example)

$$P_x(\tau_0 > u) \sim \frac{\pi}{\log u}, \quad \text{as } u \to \infty,$$

used in the last integral produces an expression which is

$$O(t^2/\log t), \quad \text{as } t \to \infty.$$

We conclude (3.4) holds. □

Remark. Since only the $m = 4$ case of the key lemma was used here, one can avoid the general combinatorial argument that produces (2.2), and instead verify "by hand" the $m = 4$ case. However, the full strength of the key lemma is needed in the next section.

4. Proof of Theorem 4. Theorem 3 (for $d \geq 2$) was proved in [8] by first establishing the variance estimates and then showing

$$\lim_{t \to \infty} \frac{S_m^{\mu}(t)}{\sigma^m(t)} = 0, \quad \text{for all } m \geq 3,\ \mu = \mu_\theta.$$

Our strategy here is exactly the same. Since the key lemma and equations (2.5)–(2.7) imply the cumulants $S_m^{\mu}(t)$ for $\mu = \delta_\eta$ and ν_θ are of the same order as for $\mu = \mu_\theta$, it is only necessary to prove

$$\liminf_{t \to \infty} \operatorname{var}_\mu\left(\int_0^t \eta_s(0)\, ds\right) \Big/ \sigma^2(t) > 0, \qquad \mu = \delta_\eta, \nu_\theta, \tag{4.1}$$

to establish (1.5). For the point masses δ_η this is where we use the extra assumption of uniformity in (1.2).

For the initial distribution μ write $\mu_t(x)$ for $E_\mu \eta_t(x)$. Let $X_t(u)$, $X_t(s)$ be two independent random walks, both starting at the origin, frozen until their starting times u and s. Finally, let $\tau(u,s) = \inf\{t \geq \max(u,s)\colon X_t(u) = X_t(s)\}$. Then

$$\begin{aligned}\operatorname{var}_\mu\left(\int_0^t \eta_s(0)\, ds\right) &= E_\mu\left(\int_0^t (\eta_{t-s}(0) - \mu_{t-s}(0))\, ds\right)^2 \\ &= 2\int_{0 \leq s \leq u \leq t} \left[P_\mu(\eta_{t-s}(0) = \eta_{t-u}(0) = 1)\right. \\ &\qquad \left. - \mu_{t-s}(0)\mu_{t-u}(0)\right] du\, ds.\end{aligned} \tag{4.2}$$

Letting η_0 have distribution μ, independent of the random walks $X_t(u)$, $X_t(s)$, the preceding integrand can be written as

$$\begin{aligned}&P(X_t(u) \in \eta_0, \tau(u,s) \leq t) + P(X_t(u) \in \eta_0, X_t(s) \in \eta_0, \tau(u,s) > t) \\ &\qquad - \mu_{t-s}(0)\mu_{t-u}(0) \\ &\quad = P(X_t(u) \in \eta_0, X_t(s) \in \eta_0) - \mu_{t-s}(0)\mu_{t-u}(0) \\ &\qquad + P(X_t(u) \in \eta_0, X_t(s) \notin \eta_0, \tau(u,s) \leq t) \\ &\quad \geq P(X_t(u) \in \eta_0, X_t(s) \notin \eta_0, \tau(u,s) \leq t),\end{aligned}$$

for $\mu = \delta_\eta$ and ν_θ. The last inequality follows because

$$\begin{aligned}&P(X_t(u) \in \eta_0, X_t(s) \in \eta_0) \\ &\quad = \sum_{x,y} p_{t-u}(0,x)p_{t-s}(0,y)\mu(\{\eta\colon \eta(x) = \eta(y) = 1\}) \\ &\quad \geq \sum_{x,y} p_{t-u}(0,x)p_{t-s}(0,y)\mu(\{\eta\colon \eta(x) = 1\})\mu(\{\eta\colon \eta(y) = 1\}) \\ &\qquad\qquad\qquad \left(\text{both } \mu = \delta_\eta \text{ and } \mu = \nu_\theta \text{ are positively correlated}\right) \\ &\quad = P(X_t(u) \in \eta_0)P(X_t(s) \in \eta_0) \\ &\quad = \mu_{t-u}(0)\mu_{t-s}(0).\end{aligned}$$

We have established that the integrand of the right-hand side of (4.2) is at least as large as

$$P(X_t(u) \in \eta_0,\ X_t(s) \notin \eta_0,\ \tau(u,s) \le t),$$

which can be written as

$$\sum_x \int_u^t P(\tau(u,s) \in dv,\ X_t(u) = x) P(X_{t-v}^x \in \eta_0,\ Y_{t-v}^x \notin \eta_0), \tag{4.3}$$

where X_t^x, Y_t^x are two independent random walks starting at x.

Now suppose that $\mu = \delta_\eta$, so $\eta_0 \equiv \eta$. Then

$$\begin{aligned} P(X_{t-v}^x \in \eta_0,\ Y_{t-v}^x \notin \eta_0) &= P(X_{t-v}^x \in \eta)(1 - P(Y_{t-v}^x \in \eta_0)) \\ &= \theta(1-\theta) + \varepsilon(x, t-v), \end{aligned}$$

where $\lim_{t\to\infty} \sup_x |\varepsilon(x,t)| = 0$. This is the assumption that (1.2) holds uniformly in x. If $\mu = \nu_\theta$, then

$$\begin{aligned} P(X_{t-v}^x \in \eta_0,\ Y_{t-v}^x \notin \eta_0) &= P(X_{t-v}^0 \in \eta_0,\ Y_{t-v}^0 \notin \eta_0) \\ &= \sum_{y,z} p_{t-v}(0,y) p_{t-v}(0,z) \nu_\theta\{\eta: \eta(y) = 1, \eta(z) = 0\} \\ &= \theta(1-\theta) + \varepsilon(t-v), \end{aligned}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$. This last fact follows from elementary properties of ν_θ; see [13] or [18] for instance. In either case, $\mu = \delta_\eta$ or ν_θ, (4.3) can be written as

$$\int_u^t P(\tau(u,s) \in dv)[\theta(1-\theta) + \bar{\varepsilon}(t-v)],$$

for some $\bar{\varepsilon}$, $|\bar{\varepsilon}(t)| \to 0$ as $t \to \infty$. Consequently,

$$\operatorname{var}_\mu\left(\int_0^t \eta_s(0)\,ds\right) \ge 2\int_{0\le s\le u\le v\le t} du\,ds\,P(\tau(u,s) \in dv)[\theta(1-\theta) + \bar{\varepsilon}(t-v)],$$

for $\mu = \delta_\eta$ and ν_θ, η satisfying (1.2) uniformly in x.

We will omit the details checking that this integral is

$$[\theta(1-\theta) + o(1)]2\int_{0\le s\le u\le t} du\,ds\,P(\tau(u,s) \le t),$$

and now remark that (see the calculation in [8]) this expression equals

$$(1 + o(1))\operatorname{var}_{\mu_\theta}\left(\int_0^t \eta_s(0)\,ds\right) = (1 + o(1))\sigma^2(t),$$

as $t \to \infty$. Thus we have established (4.1) (with the lim inf at least 1). This completes the proof. □

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