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ON A PROBLEM OF H. P. MCKEAN: INDEPENDENCE OF BROWNIAN HITTING TIMES AND PLACES[1]

BY LOREN D. PITT

University of Virginia

We show that for bounded domains $A \subseteq \mathbb{R}^N$ with $0 \in A$, if the exit time τ_A and exit place $X(\tau_A)$ are independent for a Brownian motion starting at 0, then A is essentially a ball centered at 0. Extensions are given when $X(t)$ is a Brownian motion with constant drift and when A is unbounded.

1. Introduction and statement of results. We denote the open ball of radius $r > 0$ in $\mathbb{R}^N$ and center at the origin 0 with B_r. The standard Brownian motion in $\mathbb{R}^N$ starting at 0 is written as $\{X(t): t \geq 0\}$, and the first exit time of $X(t)$ from B_r will be written as $t_r = \inf\{t > 0: X(t) \notin B_r\}$. For each r, the exit time t_r and the exit place $X(t_r)$ are independent random variables. To the best of our knowledge, it was H. P. McKean (private communication) who first raised the converse question: If $A \subset \mathbb{R}^N$ is a bounded domain containing the origin 0 and if the exit time $\tau_A = \inf\{t: X(t) \notin A\}$ and exit place $X(\tau_A)$ are independent, does it follow that for some $r_0 = 0$ we have $A = B_{r_0}$?

In this note we show that the answer is essentially yes. The qualifier "essentially" is necessary here because in dimensions $N \geq 2$, Brownian motion does not hit sets of zero Newtonian capacity (logarithmic capacity when $N = 2$). Thus if $A \subseteq B$ and $B - A$ has zero capacity the sets A and B cannot be distinguished by the Brownian motion $X(t)$. We avoid this difficulty in the following statement of our main result.

THEOREM 1. *The exit time τ_A and exit place $X(\tau_A)$ are independent iff there exists an $r_0 > 0$ with $A \subseteq B_{r_0}$ and $\tau_A = t_{r_0}$ a.s.*

COMMENTS. If τ_A and $X(\tau_A)$ are independent the theorem implies that with probability 1, $X(t)$ never hits $C = B_{r_0} - A$ on $(0, \infty)$. It follows that C is polar and has capacity 0; see Port and Stone [5], page 191. The set C is too small for the Brownian particle to notice its absence.

We give two proofs of Theorem 1. The first, a very simple proof based on exponential martingales, does not seem to generalize to arbitrary unbounded domains. The second proof was suggested by the referee and does generalize to arbitrary domains and even variable coefficient diffusions. This proof is essentially a large deviation argument.

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It is most remarkable that Theorem 1 remains true as stated when $X(t)$ is replaced by a Brownian motion with constant drift,

$$Y(t) \equiv X(t) + tv \quad \text{for fixed } v \in \mathbb{R}^N. \tag{1}$$

The surprising fact that $\tau(Y, A) = \inf\{t > 0: Y(t) \notin A\}$ and $Y(\tau(Y, A))$ are independent goes back to G. E. H. Reuter (unpublished) and has been extended by Kent [1], Stern [7] and Wendel [9]. A definitive version, and much more, is in the article of Pitman and Yor [4].

That Theorem 1 holds in this case is an immediate consequence of

THEOREM 2. *Let $Y(t)$ be given by* (1) *and let $A \subseteq \mathbb{R}^N$ be a domain containing* 0 *with $P\{\tau_A < \infty\} > 0$. Then $P\{\tau(Y, A) < \infty\} > 0$ and τ_A and $X(\tau_A)$ are independent conditionally on $\{\tau_A < \infty\}$ iff $\tau(Y, A)$ and $Y(\tau(Y, A))$ are independent conditionally on $\{\tau(Y, A) < \infty\}$.*

REMARK. The simple proofs that we give of Theorems 1 and 2 are almost algebraic in nature and shed little light on the general question of how robust these results may be. A more quantitative approach would be valuable. In particular we would like to know:

If $X(\tau_A)$ and τ_A are nearly independent, in some sense, is it possible to conclude that in some sense A is almost a ball?

2. Proofs.

FIRST PROOF OF THEOREM 1. We argue using the exponential martingales which McKean [2] popularized. Namely, for $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ and $X(t) = (X_1(t), \dots, X_N(t))$, let $\lambda \cdot X(t)$ denote the usual inner product. Then

$$M_\lambda(t) \equiv \exp\{\lambda \cdot X(t) - \tfrac{1}{2}|\lambda|^2 t\}, \qquad t \geq 0,$$

is a martingale with $M_\lambda(0) = 1$. Since A is assumed bounded, we note that $M_\lambda(t)$ is uniformly bounded for $t \leq \tau_A$. Hence, by the optional sampling theorem, we have $1 = EM_\lambda(0) = EM_\lambda(X(\tau_A))$ or

$$1 = E\exp\{\lambda \cdot X(\tau_A) - \tfrac{1}{2}|\lambda|^2 \tau_A\}.$$

By the assumed independence of τ_A and $X(\tau_A)$ we have

$$E\exp\{\lambda \cdot X(\tau_A)\} = \left[E\exp\{-\tfrac{1}{2}|\lambda|^2 \tau_A\}\right]^{-1}$$

and thus the moment generating function $E\exp\{\lambda \cdot X(\tau_A)\} = \psi(\lambda)$ is a radial function of λ. If $\mu(E) \equiv P\{X(\tau_A) \in E\}$ is the exit distribution, then $\psi(\lambda) = \int e^{\lambda \cdot x}\mu(dx)$ and we see μ is a rotation invariant measure.

To complete our argument, we let Γ equal the closed support of μ and we define $r_0 \equiv \inf\{|y|: y \in \Gamma\}$. Since Γ is closed and μ is rotation invariant, we see that $S_{r_0} \equiv \{x: |x| = r_0\} \subseteq \Gamma$. As a consequence, each point x in S_{r_0} is an accumulation point of the complement A^c of A. Thus $S_{r_0} \subseteq A^c$, since A^c is closed. Since $0 \in A$ and A is connected we see $A \subseteq B_{r_0}$ and thus $\tau_A \leq t_{r_0}$. By the definition of

r_0, we see for each $r < r_0$ that $\mu\{B_r\} = 0$ or, what is the same, that $|X(\tau_A)| > r$ a.s. Hence, $\tau_A > t_r$ a.s. Letting $r \uparrow r_0$ shows $\tau_A \geq t_{r_0}$ a.s. and completes the proof. □

SECOND PROOF OF THEOREM 1. Again let $\mu(dx)$ be the distribution of $X(\tau_A)$, let Γ be the closed support of μ and let $x_0 \in \Gamma$ be such that $|x_0| = r_0 \equiv \inf\{|x|: x \in \Gamma\}$. As seen before, it suffices to show that $\Gamma = S_{r_0} \equiv \{x: |x| = r_0\}$. Since A is bounded this will follow if we show

$$P\{|X(\tau_A)| > (1 + 4\delta)r_0\} = 0 \tag{2}$$

holds for each $\delta \in (0, \frac{1}{4})$.

Fixing δ and letting $B(t)$ be a one-dimensional Brownian motion, elementary arguments give

$$\begin{aligned} P\{\tau_A < t, |X(\tau_A)| > (1 + 4\delta)r_0\} &\leq P\Big\{\sup_{s<t}|X(s)| > (1 + 4\delta)r_0\Big\} \\ &\leq \text{const.}\, P\Big\{\sup_{s<t} B(s) > (1 + 3\delta)r_0\Big\} \\ &\leq \text{const.}\, t^{1/2} \exp\Big\{-\frac{1}{2t}((1 + 3\delta)r_0)^2\Big\}. \end{aligned} \tag{3}$$

From the independence assumption it follows that

$$P\{|X(\tau_A) > (1 + 4\delta)r_0\} = \frac{P\{\tau_A < t, |X(\tau_A)| > (1 + 4\delta)r_0\}}{P\{\tau_A < t\}} \tag{4}$$

and an upper bound on $P\{|X(\tau_A)| > (1 + 4\delta)r_0\}$ will follow from any lower bound on $P\{\tau_A < t\}$. For this let $H = \{y \notin A: |y - x_0| \leq \delta r_0\}$. By definition of x_0, $\mu(H) > 0$ and H has positive capacity. Let $\nu(dy)$ be the nonzero equilibrium measure of H and let L_H be the last exit time of $X(t)$ from H [either assume $N > 2$ or kill $X(t)$ at an exponential rate]. If $p_s(y)$ is the density of $X(s)$, then (see [5], page 61)

$$\begin{aligned} P\{\tau_A < t\} \geq P\{L_H < t\} &= \int \nu(dy) \int_0^t p_s(y)\, ds \\ &\geq \text{const.}\, t \inf\Big\{p_s(y): y \in H \text{ and } \frac{(1 + \delta)^2}{(1 + 2\delta)^2} t < s < t\Big\} \\ &\geq \text{const.}\, t^{1-N/2} \exp\Big\{-\frac{1}{2t}((1 + 2\delta)r_0)^2\Big\}. \end{aligned}$$

Combining this with (3) and (4) and letting $t \to 0$ gives the desired result (2). □

PROOF OF THEOREM 2. We will prove a more general theorem first: On the space $C = C(\mathbb{R}^N, [0, \infty))$, let $Z_t(\omega) = \omega(t)$, $\mathscr{F} = \sigma\{Z_s: 0 \leq s\}$ and $\mathscr{F}_t = \sigma\{Z_s: 0 \leq s \leq t\}$. Let P^0 be the law on $(C, \mathscr{F})$ of $\{X(t)\}$ and let P^v be the law of $\{Y(t)\}$. Let T be any $\mathscr{F}_{t+}$ stopping time, not necessarily finite but with

$P^0\{T < \infty\} > 0$. Then we will show:

> T and $X(t)$ are independent conditionally on $\{T < \infty\}$ with respect to P^0 iff they are independent conditionally with respect to P^v.

By the Cameron–Martin formula ([6], page 81 or [2], page 97), the laws P^0 and P^v are equivalent when restricted to each $\mathscr{F}_t$ and on $\mathscr{F}_t$ the Radon–Nikodym derivative is the exponential martingale

$$dP^v/dP^0 = M_v(t) = \exp\{v \cdot Z_t - \tfrac{1}{2}|v|^2 t\}.$$

More generally, for any stopping time T and any t, $M_v(T \wedge t)$ is the restriction of dP^v/dP^0 to $\mathscr{F}_{(T\wedge t)+}$. Since $M_v(T \wedge t) = M_v(T)$ on $\{T < t\}$ we see that on $(\{T < \infty\}, \mathscr{F}_{T+})$ the two laws are equivalent and for any nonnegative $\mathscr{F}_{T+}$ measurable function f,

$$\int_{\{T<\infty\}} f\, dP^v = \int_{\{T<\infty\}} f M_v(T)\, dP^0.$$

Thus we have, for the moment generating functions

$$\begin{aligned}\psi_v(\lambda, \alpha) &\equiv \int_{\{T<\infty\}} \exp\{\lambda \cdot Z(T) - \tfrac{1}{2}\alpha^2 T\}\, dP^v \\ &= \int_{\{T<\infty\}} \exp\{\lambda \cdot Z(T) - \tfrac{1}{2}\alpha^2 T\} M_v(T)\, dP^0 \\ &= \int_{\{T<\infty\}} \exp\{(\lambda + v) \cdot Z(T) - \tfrac{1}{2}(\alpha + |v|^2)T\}\, dP^0 \\ &\equiv \psi_0(\lambda + V, \alpha + |v|^2).\end{aligned}$$

If $\psi_v(\lambda, \alpha)$ is finite for all $\lambda \in \mathbb{R}^N$ and $\alpha \geq 0$ we know that T and $Z(T)$ are conditionally independent on $\{T < \infty\}$ w.r.t. P^v iff ψ_v has a factorization of the form $\psi_v(\lambda, \alpha) = F(\lambda)G(\alpha)$. But the identity $\psi_v(\lambda, \alpha) = \psi_0(\lambda + v, \alpha + |v|^2)$ shows this factorization is equivalent to a factorization of ψ_0, which in turn is equivalent to the independence of T and $Z(T)$ conditionally on $\{T < \infty\}$ w.r.t. P^0.

To complete the proof it only remains to show: If T and $Z(T)$ are independent conditionally on $\{T < \infty\}$ w.r.t. P^0 (resp., P^v), then for all λ,

$$(5) \qquad \int_{\{T<\infty\}} e^{\lambda \cdot X(T)}\, dP^0 < \infty \qquad \left(\text{resp., } \int_{\{T<\infty\}} e^{\lambda \cdot Z(T)}\, dP^v < \infty\right).$$

For this we fix a $t > 0$ with $P^0(T < t) > 0$. Then $P^v(T < t) > 0$. Defining $M = \max\{|X(s)|: s < t\}$ we have

$$\int_{\{T<t\}} e^{\lambda \cdot Z(T)}\, dP^0 \leq E^0 e^{|\lambda| M} < \infty$$

and since $P^v\{M > m + t|v|\} \le P^0\{M > m\}$ holds for all $m \ge 0$ we also have

$$\int_{\{T<t\}} e^{\lambda \cdot Z(T)}\, dP^v \le E^v e^{|\lambda| M} < \infty.$$

With the P^0 independence assumption we have

$$\infty \ge P^0\{T < \infty\} \int_{\{T<\infty\}} e^{\lambda \cdot Z(T)} \cdot 1_{\{T>t\}}\, dP^0$$

$$= \int_{\{T<\infty\}} e^{\lambda \cdot Z(T)}\, dP^v \cdot \int_{\{T<\infty\}} 1_{\{T<t\}}\, dP^0,$$

thereby showing the P^0 half of (5). Replacing P^0 with P^v gives the other half and completes the proof of Theorem 2. □

3. Extensions. All of the results and proofs given have significant extensions. The first proof of Theorem 1 does not require that A is bounded, but only that the martingales $\{M_\lambda(t \wedge \tau_A)\}$ each be uniformly integrable for some nonempty open set $\Lambda = \{\lambda\} \subseteq \mathbb{R}^N$. For example, this will be the case if A is contained in some half space $H_{\lambda,\delta} = \{x: x \cdot \lambda < \delta\}$ or if $Ee^{\alpha\tau_A} < \infty$ for some $\alpha > 0$.

A further extension is that τ_A may be replaced with a stopping time T. Thus one may prove: If $X(t)$ is Brownian motion, if T is a stopping time with $Ee^{\alpha T} < \infty$ for some $\alpha > 0$ and if T and $X(T)$ are independent, then the distribution μ of $X(T)$ is isotropic.

The second proof of Theorem 1 has other significant extensions. First, if the assumption that A is bounded is replaced with $P\{\tau_A < \infty\} > 0$ and the independence of $X(\tau_A)$ and τ_A is replaced with independence conditional on $\{\tau_A < \infty\}$, then the argument given shows that the closed support Γ of the distribution of $X(\tau_A)$ is contained in S_{r_0}. If $\Gamma \subsetneqq S_{r_0}$ and if $\{X(t)\}$ is transient, then $P\{\tau_A < \infty\} < 1$. Thus one may generalize Theorem 1 to

THEOREM 1′. *If $N \ge 3$ and if $A \subseteq R^N$ is a domain satisfying $P\{\tau_A < \infty\} = 1$, then τ_A and $X(\tau_A)$ are independent iff there is an $r_0 > 0$ with $\tau_A = t_{r_0}$ a.s.*

The basic idea in the second proof is very general and will work whenever there are good estimates for

$$P\{\tau_A < \varepsilon; x(\tau_A) \in B\} \quad \text{as } \varepsilon \to 0$$

with $B \subset \Gamma$ equal to the support of the distribution of $X(\tau_A)$.

This is standard large deviations problem, as may be seen by making the time change $t \to \sqrt{\varepsilon}\, t$. The simplest case is that of a regular diffusion without drift on a bounded domain with smooth boundary. This case is covered in essential detail in numerous places, e.g., [8], Section 6, and with the smooth boundary assumption the case with drift readily follows from the case without drift by estimation of the maximum deviation of the process with drift from that without. Omitting details we outline the results here.

Let $L = \frac{1}{2}\sum a_{ij}(x)\partial^2/\partial x_i\, \partial x_j + \sum b_j(x)\partial/\partial x_j$ be a uniformly elliptic diffusion generator on $\mathbb{R}^N$ with smooth bounded coefficient. Fix x_0 and let $X(t)$ and $X_\varepsilon(t)$ be the diffusion processes starting from x_0 with generators L and εL, $\varepsilon > 0$. If A contains x_0 and is bounded, τ_A and $\tau_{A,\varepsilon}$ will denote the exit times from A by $X(t)$ and $X_\varepsilon(t)$. Then for $B \subset \Gamma = \partial A$,

$$P\{\tau_A < \varepsilon;\, X(\tau_A) \in B\} = P\{\tau_{A,\varepsilon} < 1;\, X_\varepsilon(\tau_{A,\varepsilon}) \in B\}.$$

Write $a(x) = (a_{ij}(x))$ for the coefficient matrix and for x and y in $\bar{A}$ define

$$\phi_1(x, y) = \inf\left\{\int_0^1 \langle f'(t), a^{-1}(f(t))f'(t)\rangle\, dt\right\},$$

where the inf is taken over all smooth functions $f(t)$ with $f(0) = x$, $f(1) = y$ and $f(t) \in A$ for $0 < t < 1$.

For relatively open $B \subseteq \Gamma$ the large deviation principle gives

$$\lim_{\varepsilon \to 0} \varepsilon \log P\{\tau_A < \varepsilon;\, X(\tau_A) \in B\} = -\tfrac{1}{2} \inf_{x \in B} \phi_1(x_0, x).$$

But, if τ_A and $X(\tau_A)$ are independent this equals

$$\lim_{\varepsilon \to 0} \varepsilon[\log P\{\tau_A < \varepsilon\} + \log P\{X(\tau_A) \in B\}],$$

which does not depend on B. Hence $\inf\{\phi_1(x_0, x): x \in B\}$ does not depend on $B \subseteq \Gamma$ and Γ must be a sphere with respect to the distance $d(x, y) = \sqrt{\phi_1(x, y)}$. Namely, for some $r_0 > 0$, $\Gamma = \{x: \phi_1(x_0, x) = r_0^2\}$.

This is not sufficient for τ_A and $X(\tau_A)$ to be independent. Pinsky [3] investigated the case when L is the Laplacian on a Riemannian manifold (M, g) and has shown that if for all small balls $B(x_0, \varepsilon) = A$ and all x_0, τ_A and $X(\tau_A)$ are independent, then the metric g is an Einstein metric.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA 22903-3199