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RANDOM WALKS AND PERCOLATION ON TREES[1]

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There is a way to define an average number of branches per vertex for an arbitrary infinite locally finite tree. It equals the exponential of the Hausdorff dimension of the boundary in an appropriate metric. Its importance for probabilistic processes on a tree is shown in several ways, including random walk and percolation, where it provides points of phase transition.

1. Introduction. Consider a countable tree with a distinguished vertex called the root (Figure 1a). We may imagine the tree as growing away from the root, each vertex sending forth several branches (Figure 1b). There is a way to define the "average" number of branches per vertex, which we call the "branching number" of the tree. The definition is motivated by Hausdorff dimension; indeed, the logarithm of the branching number was first defined by Furstenberg [12], who called it the "dimension" of the tree. Furstenberg used this notion in a stunning manner to prove a result about the Hausdorff dimension of the intersection of certain Cantor sets. Essentially the same notion was considered independently by Holmes [15], whose work was flawed and was corrected by Hawkes [14]. Our aim is to show how fundamental the branching number is for probabilistic processes associated to a tree. In return, these processes give a precise and intuitive meaning to the sense in which the branching number is an average. Our results can also be used to calculate the Hausdorff dimension of random sets and to establish random covering theorems. Several known theorems that we need to use will be given proofs simpler than the original.

To explain our first result, suppose that we do a simple random walk on a given tree: start, say, at the root, and at each vertex, choose any of the edges coming out with equal probability (including the edge towards the root) and move to the other end. There are two possibilities: Either the walk is transient (i.e., the walk returns to the root only a finite number of times almost surely) or recurrent (i.e., it returns to the root infinitely often a.s.). If there is "usually" more than one branch heading away from the root, then we expect to increase our distance from the root most of the time, hence that the walk will be transient. To balance this, suppose, instead of the simple random walk, that we choose the edges going towards the root with greater probability than the branches heading away. Namely, fix $\lambda > 1$; if we are at a vertex with n branches out and one edge back, choose the edge back with probability

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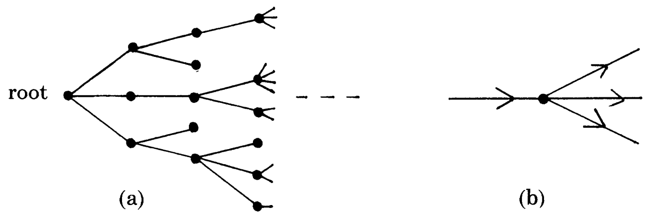

FIG. 1.

$\lambda/(\lambda+n)$ and choose each branch out with probability $1/(\lambda+n)$. At the root, we choose each branch with equal probability. Note that if $\lambda = n$, there is equal probability of moving forward or back; however, λ is a constant independent of which vertex we are at. Our result, then, is this: If λ is less than the branching number, the walk is transient, while if λ is greater than the branching number, the walk is recurrent. Coincidentally, this type of random walk has been used in studying random aggregation [18] and has proved useful in Monte Carlo simulations of self-avoiding walks [3]. More detailed information on this walk in special cases can be found in [19].

Our second result is somewhat less intuitive. Begin with a tree and fix $p \in [0,1]$. Remove each edge with probability $1-p$, independently of the other edges. Thus, if a given vertex originally has n branches heading away from the root, the expected number of branches remaining will be pn. We shall show that if p times the branching number is less than one, then the root will be connected to only finitely many vertices a.s., while if p times the branching number is greater than one, there is a positive probability that the root is connected to infinitely many vertices. In the language of percolation theory (see [17] for a review), we have found that the critical probability is the reciprocal of the branching number.

This percolation result should be compared with a basic result from the theory of branching processes. In such a process, any particle gives birth to k children ($0 \le k < \infty$) with probability p_k ($\sum p_k = 1$). We begin with a single particle; it gives birth; then its children give birth independently, each according to the same distribution $\{p_k\}$, and so on. The expected number of progeny per particle is $m = \sum kp_k$. A basic result [2, Theorem I.5.1] is that the process eventually becomes extinct a.s. iff $m \le 1$ (except when $p_1 = 1$). In conjunction with our percolation result, this strengthens the sense in which the branching number is an average, as does the following. We shall prove that for a branching process with $m > 1$, given the event that the process does not become extinct, the associated genealogical tree has branching number m a.s. This theorem, in different language, was established under the assumption that $\sum k(\log k)^2 p_k < \infty$ by Hawkes [14]. Our methods extend easily to the situation of a multitype branching process.

We arrive at our result on random walks through examination of a phase transition of electrical networks on trees: As a certain parameter is varied, the network passes from finite resistance to infinite resistance. This result in turn depends upon considerations of the energy of flows. The latter can be translated into one of the shortest descriptions of the branching number: Let Q be the directed adjacency matrix of a tree, where the edges are oriented so as to point away from the root. Then the supremum of the absolute value of the eigenvalues of Q acting on l^2 is equal to the square root of the branching number. These energy results are also crucial to the percolation calculation, providing a somewhat mysterious link between random walks and percolation.

Our study of percolation, including random resistive or capacitative networks, can be used to calculate the Hausdorff dimension of many random sets. We shall merely refer to [8] to show how this is done. We shall, however, give an interpretation of Hausdorff dimension via a random covering theorem and via a random interval walk. For simplicity, consider a closed subset E of $[0,1]$; euclidean space of any dimension would work as well. Choose an integer $r \geq 2$ and $p \in [0,1]$. Let $\{I_k\}$ be a list of all the r-adic intervals in $[0,1]$: $\{[0,1], [0, r^{-1}], [r^{-1}, 2r^{-1}], \dots, [(r-1)r^{-1}, 1], [0, r^{-2}], [r^{-2}, 2r^{-2}], \dots\}$. We "put down" each I_k with probability $1-p$ and ask whether we have covered E. More precisely, let $I_k(\omega_p)$ be I_k with probability $1-p$ and the empty set with probability p, these being mutually independent events in k. Let $F(\omega_p) = \bigcup_k I_k(\omega_p)$. Then Theorem 6.2 enables us to state that, for any r,

$$\dim E = \inf\{-\log p/\log r \colon 0 \leq p \leq 1,\ E \subseteq F(\omega_p) \text{ a.s.}\}.$$

In fact, we would get the same result if $F(\omega_p)$ were replaced by $G(\omega_p) = \limsup_k I_k(\omega_p)$. In addition, when $-\log p/\log r < \dim E$, Corollary 6.3 leads to the relation

$$\dim\left[E \setminus G(\omega_p)\right] = \dim E + \log p/\log r \quad \text{a.s.}$$

Most parts of these covering theorems were established in [14] by somewhat different methods. We shall see that $\dim G(\omega_p)^c = (\log p/\log r + 1) \vee 0$ a.s., whence

$$\dim E \cap G(\omega_p)^c = \left(\dim E + \dim G(\omega_p)^c - 1\right) \vee 0 \text{ a.s.}$$

To use Furstenberg's language [12], $G(\omega_p)^c$ is a random set which is almost surely transverse to any given closed set E. In fact, we will extend this to Borel, even analytic, sets E and place it in a more general setting in Section 7.

Now, if I_k has the form $[ar^{-l}, (a+1)r^{-l}]$, we say that l is the *order* of I_k. Given $\lambda > 0$, let us do a random walk on the r-adic intervals which intersect E (in at least one point). We start at $[0,1]$ and choose one of the r-adic intervals of order 1 which intersect E with equal probabilities. When we are at an r-adic interval I of order $l > 0$, we choose either the r-adic superinterval of order $l-1$ containing I with probability $\lambda/(\lambda+n)$ or one of the r-adic subintervals of order $l+1$ contained in I intersecting E, each with probability $1/(\lambda+n)$, where n is the number of such subintervals of I. Then Theorem 4.3 tells us that the walk converges (to a point of E) a.s. if $\lambda < r^{\dim E}$

but diverges a.s. if $\lambda > r^{\dim E}$. More precisely, if we are at the interval $J_n(\omega_\lambda)$ at time n, then $\liminf J_n(\omega_\lambda) = \limsup J_n(\omega_\lambda)$ is a point a.s. if $\lambda < r^{\dim E}$ while $\liminf J_n(\omega_\lambda) = \varnothing$ and $\limsup J_n(\omega_\lambda) = [0,1]$ a.s. if $\lambda > r^{\dim E}$. Therefore,

$$\dim E = \sup\{\log \lambda / \log r : J_n(\omega_\lambda) \text{ converges a.s.}\}.$$

We can think of $r^{\dim E}$ as an average number of subintervals of order $l+1$ intersecting E of a given interval of order l which intersects E.

2. Elementary results on the branching number. We shall use the term *tree* to mean a countable connected graph with a distinguished vertex called the *root* which has no loops or cycles and which is locally finite (i.e., each vertex belongs only to a finite number of edges). We denote the root by 0 and shall generally consider the tree as a directed graph, where edges go in the direction away from 0. If σ is a vertex, we write $|\sigma|$ for the number of edges on the shortest path from 0 to σ. Consistent with the orientation, we write the following: $\sigma \le \tau$ if σ is on the shortest (hence every) path from 0 to τ; $\sigma < \tau$ if $\sigma \le \tau$ and $\sigma \ne \tau$; $\sigma \to \tau$ if $\sigma \le \tau$ and $|\tau| = |\sigma| + 1$. If $\sigma \to \tau$, τ is said to be a *successor* of σ. If $\sigma \ne 0$, then $\overleftarrow{\sigma}$ denotes the (unique) vertex such that $\overleftarrow{\sigma} \to \sigma$. There is a (unique) one-to-one correspondence between vertices other than the root and edges such that a vertex belongs to its corresponding edge; an edge is said to *precede* its corresponding vertex. Because of this correspondence, we shall use the label of a vertex to denote its preceding edge as well. A *cutset* Π of a tree Γ is a finite set of vertices not including 0 such that for every vertex $\sigma \in \Gamma$, either $\sigma \le \tau$ for some $\tau \in \Pi$, $\tau \le \sigma$ for some $\tau \in \Pi$, or $\{\tau \in \Gamma: \sigma \le \tau\}$ is finite, and such that there is no pair $\sigma, \tau \in \Pi$ with $\sigma < \tau$. It will be handy later to denote the set $\{\tau \in \Gamma: \sigma \le \tau\}$ by Γ^σ. A special cutset is the sphere of radius n, $S_n = \{\sigma \in \Gamma: |\sigma| = n\}$. We write $|\Pi| = \min\{|\sigma|: \sigma \in \Pi\}$ and $M_n = \operatorname{card} S_n$. We will say that $\Pi_\alpha \to \infty$ when $|\Pi_\alpha| \to \infty$.

The simplest examples of trees are the n-trees, where Γ is said to be an n-tree if each vertex has exactly n successors. Figure 2 shows a 2-tree. Our definition of branching number will be such that the branching number of an n-tree is n. It will also be easily seen that in case the number of successors of every vertex lies in a fixed interval $[n_1, n_2]$, then so does the branching number.

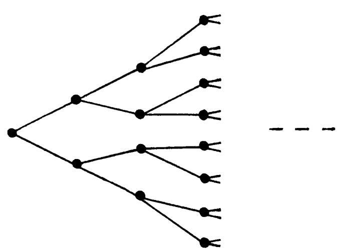

FIG. 2.

DEFINITION. The *branching number* of a tree Γ, denoted br Γ, is defined by

$$\operatorname{br}\Gamma = \inf\left\{\lambda > 0 \colon \liminf_{\Pi\to\infty} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = 0\right\}$$

$$= \sup\left\{\lambda > 0 \colon \liminf_{\Pi\to\infty} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = \infty\right\}$$

$$= \inf\left\{\lambda > 0 \colon \inf_{\Pi} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = 0\right\}.$$

The *growth* of Γ, denoted gr Γ, is defined by

$$\operatorname{gr}\Gamma = \inf\left\{\lambda > 0 \colon \liminf_{n\to\infty} \sum_{\sigma\in S_n} \lambda^{-|\sigma|} = 0\right\} = \liminf_{n\to\infty} M_n^{1/n}.$$

Note that neither of these numbers depends on the choice of root and that br $\Gamma \le$ gr Γ. In the case where all the vertices of Γ have degree at least 2 and V_n denotes the number of vertices of Γ within distance n of the root, we also have gr $\Gamma = \liminf_{n\to\infty} V_n^{1/n}$ since $M_n < V_n \le nM_n + 1$. It is not hard to see that if Γ is *spherically symmetric* (i.e., the degree of a vertex depends only on its distance from the root), then br Γ = gr Γ.

In case the vertices of Γ have uniformly bounded degree, it is easy to give the following interpretation of br Γ (see [12]). Let the number of successors of any vertex be at most r and label them arbitrarily with distinct numbers drawn from $\{0, 1, \dots, r-1\}$. Every infinite directed path from 0 then gives a string of integers in $[0, r-1]$, which we interpret as the base r expansion of a real number in $[0, 1]$. Let E be the set of all such real numbers. Then br $\Gamma = r^{\dim E}$.

EXAMPLE. Let Γ be a tree such that if $|\sigma|$ is even, then σ has 2 successors, while if $|\sigma|$ is odd, then σ has 3 successors (Figure 3). Then br $\Gamma = \sqrt{6}$.

EXAMPLE. If Γ_1 and Γ_2 are trees, let $\Gamma_1 \vee \Gamma_2$ denote a tree formed from disjoint copies of Γ_1 and Γ_2 whose roots are identified and taken as the root of $\Gamma_1 \vee \Gamma_2$ (see Figure 4). Then

$$\operatorname{br}(\Gamma_1 \vee \Gamma_2) = \operatorname{br}\Gamma_1 \vee \operatorname{br}\Gamma_2. \tag{2.1}$$

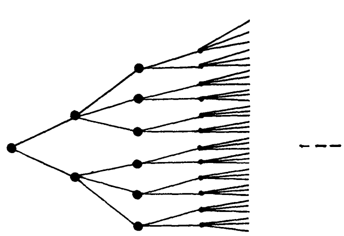

FIG. 3.

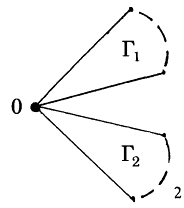

FIG. 4.

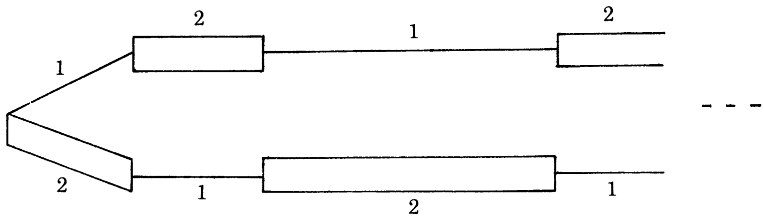

FIG. 5.

DEFINITION. A tree is called *quasispherical* if its branching number equals its growth.

EXAMPLE. Let $n_k \uparrow \infty$. Let Γ_1 (resp., Γ_2) be a tree such that $n_{2k} \le |\sigma| < n_{2k+1} \Rightarrow \sigma$ has one successor (resp., two successors), while $n_{2k-1} \le |\sigma| < n_{2k} \Rightarrow \sigma$ has two successors (resp., one successor). Then if n_k increases sufficiently rapidly, $\operatorname{br} \Gamma_1 = \operatorname{br} \Gamma_2 = 1$, whence $\operatorname{br}(\Gamma_1 \vee \Gamma_2) = 1$, while $\operatorname{gr}(\Gamma_1 \vee \Gamma_2) = 2$ (see Figure 5 for a schematic representation). Thus, $\Gamma_1 \vee \Gamma_2$ is *not* quasispherical.

EXAMPLE. Let Γ be a tree such that 0 has two successors; for every $\sigma \ne 0$, σ has either one or three successors; if σ has one successor and $\sigma \le \tau$, then τ has only one successor; and $M_n = 2^n$ (see Figure 6). Then $\operatorname{gr} \Gamma = 2$ and $\operatorname{br} \Gamma = 1$. Let Γ' be the subtree consisting of those $\sigma \in \Gamma$ such that $\overleftarrow{\sigma}$ has more than one successor. Then also $\operatorname{gr} \Gamma' = 2$ and $\operatorname{br} \Gamma' = 1$.

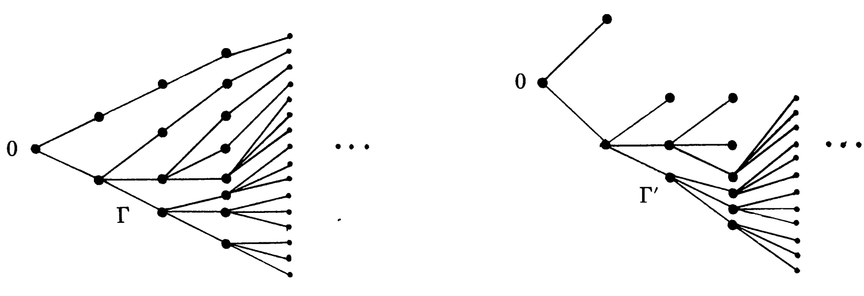

FIG. 6.

As Furstenberg showed [12], classical theorems on the relationship between Hausdorff dimension and measures have important analogues for trees.

DEFINITION. A *flow* on a tree Γ is a nonnegative function θ on the vertices of Γ such that for all $\sigma \in \Gamma$,

$$\theta(\sigma) = \sum_{\sigma \to \tau} \theta(\tau).$$

A flow θ such that $\theta(0) = 1$ is called a *unit flow*. The set of unit flows on Γ is denoted $U(\Gamma)$. For a flow θ with $\theta(0) \neq 0$, we define its *Lipschitz constant* to be

$$\operatorname{lip} \theta = \liminf_{|\sigma| \to \infty} \theta(\sigma)^{-1/|\sigma|}$$

and its *branching number* as

$$\operatorname{br} \theta = \exp \liminf_{\Pi \to \infty} \frac{-\sum_{\sigma \in \Pi} \theta(\sigma) \log \theta(\sigma)}{\sum_{\sigma \in \Pi} \theta(\sigma)|\sigma|},$$

where $0 \log 0$ is interpreted as 0.

Note that $\operatorname{lip} \theta \leq \operatorname{br} \theta$. We shall repeatedly use the fact that for any flow θ and any cutset Π,

$$\sum_{\sigma \in \Pi} \theta(\sigma) = \theta(0).$$

PROPOSITION 2.1 ([12]). *For any tree* Γ,

$$\operatorname{br} \Gamma = \sup_{\theta \in U(\Gamma)} \operatorname{lip} \theta = \sup_{\theta \in U(\Gamma)} \operatorname{br} \theta.$$

PROOF. It suffices to show that $\forall\, \theta \in U(\Gamma)$, $\operatorname{br} \theta \leq \operatorname{br} \Gamma$, and $\forall\, \lambda < \operatorname{br} \Gamma$ $\exists\, \theta \in U(\Gamma)$ $\operatorname{lip} \theta \geq \lambda$. Suppose first that $\theta \in U(\Gamma)$ and $\operatorname{br} \theta > \lambda$. Then for all large Π,

$$-\sum_{\sigma \in \Pi} \theta(\sigma) \log \theta(\sigma) \geq \log \lambda \sum_{\sigma \in \Pi} \theta(\sigma)|\sigma|. \tag{2.2}$$

From Jensen's inequality, we have

$$\begin{aligned}\sum_{\sigma \in \Pi} \lambda^{-|\sigma|} &= \sum_{\sigma \in \Pi} \theta(\sigma) \exp[-\log \theta(\sigma) - |\sigma| \log \lambda] \\ &\geq \exp \sum_{\sigma \in \Pi} \theta(\sigma)[-\log \theta(\sigma) - |\sigma| \log \lambda] \geq e^0 = 1\end{aligned}$$

by (2.2). Therefore $\operatorname{br} \Gamma \geq \lambda$, whereupon $\operatorname{br} \Gamma \geq \operatorname{br} \theta$.

Secondly, let $0 < \lambda < \operatorname{br} \Gamma$. Then $\inf_\Pi \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} > 0$. If we regard $\lambda^{-|\sigma|}$ as the capacity of the edge preceding σ, then it follows from the max-flow min-cut theorem (see, e.g., [10]) that there is a nonzero flow θ such that for all σ, $\theta(\sigma) \le \lambda^{-|\sigma|}$. Hence lip $\theta \ge \lambda$. □

In Section 4, we shall develop analogues of Frostman's classical theorem which relates capacity to dimension; see Theorem 4.3, the discussion preceding Proposition 4.5 and Proposition 4.5 itself.

3. Quasispherical and subperiodic trees. The calculation of branching numbers is considerably easier for quasispherical trees than for general trees. In this section, we shall give two conditions sufficient for quasisphericity. For $\sigma \in \Gamma$, let $M_n^\sigma = \operatorname{card}\{\tau \in \Gamma\colon \sigma \le \tau \text{ and } |\tau| = |\sigma| + n\}$.

DEFINITION. A tree Γ is called *subperiodic* if, for all $\sigma \ne 0$, there is an adjacency-preserving injection $T\colon \Gamma^\sigma \to \Gamma^{T(\sigma)}$ with $T(\sigma) \in S_1$.

PROPOSITION 3.1. *Let Γ be a subperiodic tree. Then Γ is quasispherical and there is a unit flow θ on Γ such that* lip θ = br θ = br Γ.

PROOF. In fact, we will show that $\lambda = \lim M_n^{1/n}$ exists and $\inf_\Pi \sum_{\sigma\in\Pi}\lambda^{-|\sigma|} = 1$. This immediately shows that Γ is quasispherical and, by the max-flow min-cut theorem, that there is a unit flow θ satisfying $\theta(\sigma) \le \lambda^{-|\sigma|}$.

Let $B_n = \max_{|\sigma|=1} M_n^\sigma$. The subperiodicity hypothesis entails that $B_{m+n} \le B_m B_n$ for $m, n \ge 0$, whence $\lambda = \lim B_n^{1/n}$ exists. It follows that $\lambda = \lim M_n^{1/n}$ as well. Also, $B_{nr} \le B_n^r$, whence $B_n \ge B_{nr}^{1/r} \to \lambda^n$ and so $M_n \ge \lambda^{n-1}$. Now if, contrary to our claim above, $\sum_{\sigma\in\Pi}\lambda^{-|\sigma|} < 1$ for some cutset Π, then let $G = \{\sigma \in \Gamma\colon (\exists\, \tau \in \Pi,\ \sigma < \tau) \text{ or } (\forall\, \tau \in \Pi,\ \sigma \nleq \tau \text{ and } \tau \nleq \sigma)\}$ and observe that

$$\Bigl(\sum_{\sigma\in G}\lambda^{-|\sigma|}\Bigr)\sum_{n\ge 0}\Bigl(\sum_{\sigma\in\Pi}\lambda^{-|\sigma|}\Bigr)^n < \infty.$$

Let F be the set of sequences $t = (\sigma_1, \sigma_2, \dots, \sigma_n, \sigma_0)$ with $n \ge 0$, $\sigma_k \in \Pi$ if $k \ge 1$, and $\sigma_0 \in G$; write $|t| = |\sigma_1| + \cdots + |\sigma_n| + |\sigma_0|$. The above inequality can be written as $\sum_{t\in F}\lambda^{-|t|} < \infty$. However, because Γ is subperiodic, there is an injection $S\colon \Gamma \to F$ such that $|S(\sigma)| = |\sigma|$ (cf. the proof of Proposition III.1 in [11]), whence

$$\sum_{t\in F}\lambda^{-|t|} \ge \sum_{\sigma\in\Gamma}\lambda^{-|\sigma|} = \sum_{n\ge 0}\lambda^{-n}M_n \ge \sum_{n\ge 0}\lambda^{-1} = \infty,$$

a contradiction. □

EXAMPLE. Let Γ be the tree whose vertices are the *self-avoiding walks* on $\mathbf{Z}^2$ from $(0,0)$, i.e., the finite paths starting at $(0,0)$, passing from (x,y) to any one of $(x \pm 1, y), (x, y \pm 1)$, and not passing through any vertex twice. Γ forms a subperiodic tree rooted at $\varnothing$ when each path is followed by all its possible extensions by one vertex. It is estimated that br Γ(= gr Γ) ≈ 2.64 [3].

Our next proposition shows that trees whose growth is fairly "regular" are quasispherical. This depends on a lemma due to Falconer. If $\sigma, \tau \in \Gamma$, we denote by $\sigma \wedge \tau$ the vertex farthest from 0 which is less than or equal to both σ and τ.

LEMMA 3.2 ([9]). *Let f and g be nonnegative functions on a tree Γ such that g is strictly positive and decreasing in the sense that $\sigma \le \tau \Rightarrow g(\sigma) \ge g(\tau)$. Let*

$$a_n = \sum_{\sigma, \tau \in S_n} g(\sigma \wedge \tau)^{-1} f(\sigma) f(\tau)$$

and

$$b_n = \sum_{\sigma \in S_n} f(\sigma).$$

If

$$\liminf_{n \to \infty} a_n b_n^{-2} < \infty, \tag{3.1}$$

then

$$\inf_{\Pi} \sum_{\sigma \in \Pi} g(\sigma) > 0. \tag{3.2}$$

PROOF. Let $0 < \varepsilon < \limsup_{n \to \infty} a_n^{-1} b_n^2$. We will show that $\sum_{\sigma \in \Pi} g(\sigma) > \varepsilon/4$ for all Π.

Given Π, choose n so that $\forall\ \sigma \in S_n$, $\exists\ \tau \in \Pi$, $\tau \le \sigma$ and $a_n^{-1} b_n^2 > \varepsilon$. Set $c_n = b_n/(2a_n)$ and

$$A = \left\{ \sigma \in S_n : \exists\ \tau \le \sigma,\ c_n \sum_{\tau \le \rho \in S_n} f(\rho) > g(\tau) \right\}.$$

If $\tau \le \sigma \in S_n$, then

$$\sum_{\rho \in S_n} g(\sigma \wedge \rho)^{-1} f(\rho) \ge \sum_{\tau \le \rho \in S_n} g(\sigma \wedge \rho)^{-1} f(\rho) \ge \sum_{\tau \le \rho \in S_n} g(\tau)^{-1} f(\rho).$$

If $\sigma \in A$, we can choose τ so that this latter sum is greater than c_n^{-1}, whence

$$\sum_{\sigma \in A} f(\sigma) < c_n \sum_{\substack{\sigma \in A \\ \rho \in S_n}} f(\sigma) g(\sigma \wedge \rho)^{-1} f(\rho) \le a_n c_n.$$

Also, if $|\tau| \le n$, then

$$c_n \sum_{\tau \le \sigma \in S_n \setminus A} f(\sigma) \le g(\tau).$$

Putting these last inequalities together, we obtain

$$\begin{aligned} \sum_{\tau \in \Pi} g(\tau) &\ge c_n \sum_{\tau \in \Pi} \sum_{\tau \le \sigma \in S_n \setminus A} f(\sigma) = c_n \sum_{\sigma \in S_n \setminus A} f(\sigma) \\ &= c_n \left(\sum_{\sigma \in S_n} f(\sigma) - \sum_{\sigma \in A} f(\sigma) \right) > c_n (b_n - a_n c_n) = \frac{b_n^2}{4a_n} > \frac{\varepsilon}{4}. \qquad \square \end{aligned}$$

PROPOSITION 3.3. *Let* Γ *be a tree such that for some* $\lambda \geq 1$,

$$\sup_{k, m \geq 0} M_k^{-1} \sum_{\sigma \in S_k} \left(\lambda^{-m} M_m^{\sigma}\right)^2 < \infty \tag{3.3}$$

and

$$\limsup_{n \to \infty} \lambda^{-n} M_n > 0. \tag{3.4}$$

Then $\lambda = \operatorname{br} \Gamma$ *and* Γ *is quasispherical.*

PROOF. From (3.3) with $k = 0$, we get that $\sup \lambda^{-m} M_m < \infty$, whence $\operatorname{gr} \Gamma \leq \lambda$. It remains to show that $\operatorname{br} \Gamma \geq \lambda$. To this end, we may as well assume that $\lambda > 1$. Choose $\lambda_1 \in]1, \lambda[$ and set $f(\sigma) = \lambda^{-|\sigma|}$, $g(\sigma) = \lambda_1^{-|\sigma|}$. If we establish (3.1), then (3.2) will entail that $\operatorname{br} \Gamma \geq \lambda_1$, which will complete the proof. Now

$$\begin{aligned}
a_n &= \sum_{\sigma, \tau \in S_n} g(\sigma \wedge \tau)^{-1} f(\sigma) f(\tau) = \sum_{\sigma, \tau \in S_n} \lambda_1^{|\sigma \wedge \tau|} \lambda^{-2n} \\
&= \sum_{|\rho| \leq n} \lambda_1^{|\rho|} \lambda^{-2n} \sum_{\substack{\sigma, \tau \in S_n \\ \sigma \wedge \tau = \rho}} 1 \\
&= \sum_{|\rho| \leq n} \lambda_1^{|\rho|} \lambda^{-2n} \left\{ \left(\sum_{\rho \leq \sigma \in S_n} 1 \right)^2 - \sum_{\rho \to \psi} \left(\sum_{\psi \leq \sigma \in S_n} 1 \right)^2 \right\} \\
&= \sum_{|\rho| \leq n} \lambda_1^{|\rho|} \lambda^{-2n} \left\{ \left(M_{n-|\rho|}^{\rho} \right)^2 - \sum_{\substack{\rho \to \psi \\ |\psi| \leq n}} \left(M_{n-|\psi|}^{\psi} \right)^2 \right\} \\
&= \sum_{0 < |\rho| \leq n} \left(\lambda_1^{|\rho|} - \lambda_1^{|\rho|-1} \right) \lambda^{-2n} \left(M_{n-|\rho|}^{\rho} \right)^2 + \lambda^{-2n} (M_n)^2 \\
&= \left(1 - \lambda_1^{-1}\right) \sum_{k=1}^{n} \left\{ \lambda^{-k} \left(\frac{\lambda_1}{\lambda} \right)^k \sum_{\rho \in S_k} \left(\lambda^{-n+k} M_{n-k}^{\rho} \right)^2 \right\} + \left(\lambda^{-n} M_n \right)^2.
\end{aligned}$$

Thus, a_n is uniformly bounded by (3.3). In conjunction with (3.4), this establishes (3.1). □

4. Electrical networks and random walks. Denumerable Markov chains correspond to random walks on directed graphs, where the edges are labelled with the transition probabilities. Reversible Markov chains correspond to random walks on undirected graphs, or electrical networks, where the edges are labelled with resistances (or conductances): The transition probability from σ to τ is the conductance from σ to τ divided by the sum of all conductances emanating from σ (see the superb [7] or [16, Chapter 9, Section 10]). We shall require transience criteria for reversible chains which stem from this correspondence.

Let Γ be a finite undirected graph with vertex set V and edge set $E \subseteq V \times V$. We shall regard E as a set of ordered pairs such that $(\sigma, \tau) \in E \Rightarrow (\tau, \sigma) \in E$. Let $l^2(V)$ be the usual complex Hilbert space on V and $l^2(E)$ be the space of complex-valued functions θ on E such that $\theta(\sigma, \tau) = -\theta(\tau, \sigma)$, with the inner product

$$(\theta, \theta') = \frac{1}{2} \sum_{(\sigma, \tau) \in E} \theta(\sigma, \tau) \overline{\theta'(\sigma, \tau)}.$$

We define the *coboundary operator* d: $l^2(V) \to l^2(E)$ by

$$(dP)(\sigma, \tau) = P(\tau) - P(\sigma).$$

Its adjoint is the *boundary operator*,

$$(d^*\theta)(\sigma) = -\sum_{(\sigma, \tau) \in E} \theta(\sigma, \tau).$$

When Γ is infinite, we will continue to use the above symbols when they are well defined.

Suppose now that Γ is a countable connected graph with conductances $C(\sigma, \tau) > 0$ assigned to edges satisfying

$$C(\sigma, \tau) = C(\tau, \sigma) \quad \text{and} \quad \forall\, \sigma \sum_{(\sigma, \tau) \in E} C(\sigma, \tau) < \infty.$$

Fix a vertex $0 \in \Gamma$. Let

$$\mathscr{F}(\Gamma) = \Big\{\theta\colon E \to \mathbf{R} \,|\, \theta C^{-1/2} \in l^2(E), \forall\, \sigma \sum_{(\sigma, \tau) \in E} |\theta(\sigma, \tau)| < \infty,$$
$$\text{and } \forall\, \sigma \neq 0,\ (d^*\theta)(\sigma) = 0\Big\}$$

be the space of *flows on* Γ *of finite energy*, where the *energy* of a function θ is defined as

$$E(\theta) = \|\theta C^{-1/2}\|_2.$$

The random walk on (Γ, C) has transition probabilities

$$p_{\sigma, \tau} = C(\sigma, \tau)/\alpha_\sigma,$$

where

$$\alpha_\sigma = \sum_{(\sigma, \tau) \in E} C(\sigma, \tau).$$

THEOREM 4.1 ([20]). *The random walk on* (Γ, C) *is transient iff* $\mathscr{F}(\Gamma) \neq \{0\}$.

PROOF. We may assume that Γ has no loops (so that $p_{\sigma, \sigma} = 0$) without affecting either condition. Let $\{\Gamma_n\}$ be a sequence of finite subgraphs of Γ containing 0 whose union is Γ. Let $P_n(\sigma)$ be the potential (the voltage) at σ when 0 is at unit potential and all $\tau \notin \Gamma_n$ are grounded. Let $\theta_n(\sigma, \tau)$ be the *current* flow, defined by Ohm's law $\theta_n(\sigma, \tau) = -C(\sigma, \tau) \cdot dP_n(\sigma, \tau)$. By [16, Proposition 9-131], the random walk is recurrent iff the current at 0 [$-d^*\theta_n(0)$] tends to 0 as $n \to \infty$. Now $d^*\theta_n = 0$ on $\Gamma_n \setminus \{0\}$ by Kirchhoff's law, whence

$$\begin{aligned} E(\theta_n) &= -(\theta_n, dP_n) = -(d^*\theta_n, P_n) \\ &= -d^*\theta_n(0) \cdot P_n(0) - \sum_{\sigma \notin \Gamma_n} d^*\theta_n(\sigma) \cdot P_n(\sigma) \\ &= -d^*\theta_n(0). \end{aligned}$$

That is, the energy equals the current at 0 when a unit potential is applied.

Now define θ_n to be the current flow when $\Gamma \setminus \Gamma_n$ is grounded and 0 is at a potential so that $-d^*\theta_n(0) = 1$. Then the walk is recurrent iff $E(\theta_n) \to \infty$. Furthermore, by Thompson's principle [7, page 63], θ_n is the unique function θ of minimum energy satisfying $-d^*\theta(0) = 1$ and $d^*\theta = 0$ on $\Gamma_n \setminus \{0\}$. Thus, if $\mathscr{F}(\Gamma) \neq \{0\}$, there exists $\theta \in \mathscr{F}(\Gamma)$ with finite energy and $-d^*\theta(0) = 1$, which implies that $E(\theta_n) \leq E(\theta) < \infty$ and the walk is transient. Conversely, if the walk is transient, then $E(\theta_n) \nrightarrow \infty$ and, by taking a subsequence if necessary, we have that $\theta_n \to \theta$ and $E(\theta) \leq \liminf E(\theta_n) = \lim E(\theta_n) = M < \infty$. Since $0 \leq P_n \leq M$, $|\theta_n(\sigma, \tau)| \leq MC(\sigma, \tau)$, whence $\forall\, \sigma\ \sum_{(\sigma,\tau) \in E} |\theta(\sigma, \tau) - \theta_n(\sigma, \tau)| \to 0$. Consequently, $0 \neq \theta \in \mathscr{F}(\Gamma)$. □

This theorem is referred to [20] as "Royden's criterion." Another criterion given in [20] weakens the conditions on θ: Random walk on (Γ, C) is transient iff $\exists\ \theta$: $E \to \mathbf{R}$ such that $\theta C^{-1/2} \in l^2(E)$, $\forall\ \sigma\ \sum_{(\sigma,\tau) \in E} |\theta(\sigma, \tau)| < \infty$, $\sum_{\sigma \in V} |d^*\theta(\sigma)| < \infty$, and $\sum_{\sigma \in V} d^*\theta(\sigma) \neq 0$. A simpler proof than the original may be given as follows. Given such a θ, add a vertex x to Γ with edges connecting x to every $\sigma \in \Gamma$ and with conductances $C(x, \sigma) = |d^*\theta(\sigma)|$. By Royden's criterion and the hypothesis, this new walk is transient. That is, a random walk beginning at 0 will visit 0 only a finite number of times a.s. and will never visit x with positive probability. Therefore, the original walk is transient.

When $\mathscr{F}(\Gamma) \neq \{0\}$, there is a unique $\theta \in \mathscr{F}(\Gamma)$ of minimum energy such that $-d^*\theta(0) = 1$; we call this θ the *unit current flow*. [That θ exists and is unique may be seen as follows: First, if θ_n is a minimizing sequence, then θ_n has a limit point θ with $E(\theta) \leq \liminf E(\theta_n)$. Second, if $\theta \neq \theta'$, then it is easily seen that $E(\frac{1}{2}(\theta + \theta')) < \frac{1}{2}(E(\theta) + E(\theta'))$.] The terminology is justified by noting that if $0 \in \Gamma_n$, Γ_n finite and increasing to Γ, θ_n the unit current flow when $\Gamma \setminus \Gamma_n$ is grounded and voltage is applied at 0, then $\theta_n \to \theta$. [Indeed, $E(\theta_n) \leq E(\theta)$, whence every limit point of θ_n has minimum energy, hence is θ.] The corresponding limit, $P = \lim P_n$, is called the *potential* and satisfies $\theta = -C \cdot dP$. Note that $P(0) = E(\theta)$. Recall the interpretation of $P(\sigma)/P(0)$ as the probability that the walk started at σ will (ever) visit 0. Also, the unit current flow $\theta(\sigma, \tau)$ is equal to the expected net number of crossings of the edge (σ, τ) when a random walk is started at 0 [a crossing of (σ, τ) is counted positively when the traversal is from σ to τ and negatively when from τ to σ] —see [7, page 52].

When Γ is a tree with conductances C, we shall write $\Phi(\sigma)$ for the resistance $C(\overleftarrow{\sigma}, \sigma)^{-1}$ of the edge preceding σ ($\sigma \neq 0$). It will also be convenient to write

$$E^{\Phi}(\theta) = \sum_{0 \neq \sigma \in \Gamma} \theta(\sigma)^2 \Phi(\sigma)$$

for the energy of a flow θ and

$$E^{\Phi}(\Gamma) = \min_{\theta \in U(\Gamma)} E^{\Phi}(\theta)$$

for the energy of unit current flow (or ∞ if current does not flow). Thus, Royden's criterion is that (Γ, Φ^{-1}) is transient iff $E^{\Phi}(\Gamma) < \infty$. In conjunction with the max-flow min-cut theorem, this leads to the following relative of the Nash–Williams criterion ([20], [22]).

COROLLARY 4.2. *Let Γ be a tree with resistances Φ. If (Γ, Φ^{-1}) is transient. Then*

$$\lim_{\Pi \to \infty} \sum_{\sigma \in \Pi} \Phi(\sigma)^{-1} = \infty. \tag{4.1}$$

Conversely, if there are positive numbers w_n such that $\sum_{n \geq 1} w_n < \infty$ and

$$\liminf_{\Pi \to \infty} \sum_{\sigma \in \Pi} w_{|\sigma|} \Phi(\sigma)^{-1} > 0, \tag{4.2}$$

then (Γ, Φ^{-1}) is transient.

PROOF. If (Γ, Φ^{-1}) is transient, then there is a unit flow θ of finite energy. The Cauchy–Schwarz inequality shows that

$$1 = \left(\sum_{\sigma \in \Pi} \theta(\sigma) \right)^2 \leq \left(\sum_{\sigma \in \Pi} \Phi(\sigma)^{-1} \right) \left(\sum_{\sigma \in \Pi} \theta(\sigma)^2 \Phi(\sigma) \right).$$

Since $\sum_{\sigma \in \Pi} \theta(\sigma)^2 \Phi(\sigma) \to 0$, we arrive at (4.1). On the other hand, if (4.2) is satisfied, then the max-flow min-cut theorem ensures the existence of a nonzero flow θ obeying $\theta(\sigma) \leq w_{|\sigma|} \Phi(\sigma)^{-1}$. This flow has finite energy:

$$\begin{aligned} \sum_{0 \neq \sigma \in \Gamma} \theta(\sigma)^2 \Phi(\sigma) &= \sum_{n \geq 1} \sum_{\sigma \in S_n} \theta(\sigma) \big(\theta(\sigma) \Phi(\sigma) \big) \leq \sum_{n \geq 1} w_n \sum_{\sigma \in S_n} \theta(\sigma) \\ &= \sum_{n \geq 1} w_n < \infty. \end{aligned}$$

□

Our canonical choice of resistances on a tree is $\Phi(\sigma) = \lambda^{|\sigma| - 1}$, where $\lambda > 0$ is arbitrary. The significance of this choice arises from the next theorem, whose intuitive basis was described in Section 1.

THEOREM 4.3. *Let Γ be a tree. When $\lambda < \operatorname{br} \Gamma$, the walk associated to λ is transient and when $\lambda > \operatorname{br} \Gamma$, the walk is recurrent. Correspondingly, current flows when $\lambda < \operatorname{br} \Gamma$ but not when $\lambda > \operatorname{br} \Gamma$.*

PROOF. If the walk associated to λ is transient, then (4.1) shows that $\lambda \le \operatorname{br} \Gamma$. On the other hand, if $\lambda < \operatorname{br} \Gamma$, choose $\lambda_0 \in]\lambda, \operatorname{br} \Gamma[$ and set $w_n = (\lambda \lambda_0^{-1})^n$. Then (4.2) holds, whence the walk is transient. $\square$

REMARK. At $\lambda = \operatorname{br} \Gamma$, the walk may be either recurrent or transient. For example, when Γ is an n-tree, the walk is recurrent. On the other hand, if Γ is a spherically symmetric tree such that $M_n = 3^k$ for $2^k \le n < 2^{k+1}$, then the walk is transient.

From Theorem 4.3, we may deduce a formula for the branching number similar to that for the growth. Let

$$M_n(\theta) = \left(\sum_{\sigma \in S_n} \theta(\sigma)^2 \right)^{-1};$$

notice that if θ_n is a unit flow which is constant on S_n, then $M_n(\theta_n) = M_n$.

COROLLARY 4.4. *For any tree* Γ,

$$\operatorname{br} \Gamma = \sup_{\theta \in U(\Gamma)} \liminf_{n \to \infty} M_n(\theta)^{1/n}.$$

PROOF. This is an immediate consequence of the fact that $\liminf M_n(\theta)^{1/n}$ is the radius of convergence of the series

$$\sum_{0 \ne \sigma \in \Gamma} \theta(\sigma)^2 \lambda^{|\sigma|}. \qquad \square$$

It is interesting to note that when $\lambda > \operatorname{br} \Gamma$, the walk may not be ergodic. Indeed, [16, Proposition 9-131] states that (Γ, Φ^{-1}) is ergodic iff $\sum_{\sigma \in \Gamma} \Phi(\sigma)^{-1} < \infty$. For the canonical Φ, this amounts to the requirement that $\sum M_n \lambda^{-n} < \infty$. Thus, if $\lambda > \limsup M_n^{1/n}$, the walk is ergodic, while if $\operatorname{br} \Gamma < \lambda < \limsup M_n^{1/n}$, the walk is null recurrent.

The above considerations of energy are illuminated further by an abstract potential theory on the (Martin) *boundary*, $\partial \Gamma$, of Γ [4]: This is the space of infinite directed paths beginning at 0. (The boundary bears a clear resemblance to the set E described in Section 2 following the definition of branching number.) For distinct points s and t of $\partial \Gamma$, we let $s \wedge t$ denote the point of Γ farthest from 0 which is common to both s and t. Natural kernels $K(s,t)$ on $\partial \Gamma \times \partial \Gamma$ are those of the form

$$K(s,t) = \begin{cases} \Psi(s \wedge t) & \text{if } s \ne t, \\ \lim\limits_{\substack{\sigma \in s \\ |\sigma| \to \infty}} \Psi(\sigma) =: \Psi(s) & \text{if } s = t, \end{cases} \tag{4.3}$$

where Ψ is any nondecreasing nonnegative function on Γ, for example, $\Psi(\sigma) = \lambda^{|\sigma|}$, $\lambda \ge 1$. The *potential* of a finite positive Borel measure μ on $\partial \Gamma$ is defined to be the function $\int K(s,t)\, d\mu(t)$ and the *energy* is the number

$\int\int K(s,t)\,d\mu(s)\,d\mu(t)$. Finite Borel measures μ on $\partial\Gamma$ are in 1–1 correspondence with flows θ on Γ through the relation

$$\theta(\sigma) = \mu(\{s : \sigma \in s\}). \tag{4.4}$$

This correspondence permits the following expressions for the potential and energy with respect to the kernel (4.3). Let

$$\Phi(\sigma) = \Psi(\sigma) - \Psi(\overleftarrow{\sigma}), \qquad \sigma \neq 0,$$

and denote by $\vec{\sigma}$ the unique successor of σ in s if $\sigma \in s$. We shall interpret $\infty \cdot 0$ as 0. Then

$$\begin{aligned}\int_{\partial\Gamma} K(s,t)\,d\mu(t) &= \sum_{\sigma \in s} \int_{s \wedge t = \sigma} \Psi(\sigma)\,d\mu(t) + \Psi(s)\mu(\{s\})\\ &= \sum_{\sigma \in s} \Psi(\sigma)\big[\theta(\sigma) - \theta(\vec{\sigma})\big] + \Psi(s)\mu(\{s\})\\ &= \sum_{0 < \sigma \in s} \Phi(\sigma)\theta(\sigma) + \Psi(0)\theta(0)\end{aligned}$$

[the last step requires some care when $\mu(\{s\}) = 0$; in this case, if we write $s \cap S_n = \{\sigma_n\}$, then we use the fact that $\sum_{1 \le k \le n} \Phi(\sigma_k)(\theta(\sigma_k) - \theta(\sigma_{n+1})) \to \sum_{0 < \sigma \in s} \Phi(\sigma)\theta(\sigma)$ by the monotone convergence theorem] and so

$$\begin{aligned}\int\int K(s,t)\,d\mu(s)\,d\mu(t) &= \sum_{0 \neq \sigma \in \Gamma} \Phi(\sigma)\theta(\sigma) \int_{\sigma \in s} d\mu(s) + \Psi(0)\theta(0)\mu(\partial\Gamma)\\ &= E^{\Phi}(\theta) + \Psi(0)\theta(0)^2.\end{aligned}$$

Note that given θ and Φ, we can define μ by (4.4) and Ψ by

$$\Psi(\sigma) = \sum_{0 < \tau \le \sigma} \Phi(\tau), \tag{4.5}$$

whereupon the above calculations are again valid. Since $\Psi(0) = 0$, the energy of μ is equal to the energy of θ.

When $E^{\Phi}(\Gamma) < \infty$, we let θ_{Φ} be the unit flow of minimum energy and μ_{Φ} the boundary measure associated to it by (4.4). It is evident that μ_{Φ} is the distribution of current outflow on $\partial\Gamma$. In probabilistic terms, μ_{Φ} is harmonic measure for the random walk on (Γ, Φ^{-1}), i.e., the "hitting" distribution on $\partial\Gamma$. This is because $\theta_{\Phi}(\sigma)$ is the expected number of net crossings of the edge preceding σ. Since Γ is a tree, each path to ∞ crosses this edge either zero or one times net, whence $\theta_{\Phi}(\sigma)$ is simply the probability of hitting $\{s\colon \sigma \in s\}$. Because of these facts, it is reasonable to regard the electrical network on Γ as being "grounded at infinity." Further justification stems from the following observation.

PROPOSITION 4.5. *Let Γ be a tree with resistances Φ, unit current flow θ_{Φ}, potential P^{Φ}, and harmonic measure μ_{Φ}. Let μ be any probability measure on*

$\partial\Gamma$ whose unit flow θ determined by (4.4) *has finite energy. Then*

$$\lim_{\substack{\sigma \in s \\ |\sigma| \to \infty}} P^{\Phi}(\sigma) = 0 \quad \mu\text{-a.e.}\,[s]. \tag{4.6}$$

The potential of μ_Φ with respect to the kernel given by (4.3) *and* (4.5) *is equal to $E^\Phi(\Gamma)$ μ-a.e.*

PROOF. Write $(\theta_1, \theta_2)_n = \sum_{|\sigma| \le n} \theta_1(\sigma)\overline{\theta_2(\sigma)}$. Then

$$I_n := \left(\theta\Phi^{1/2}, \theta_\Phi \Phi^{1/2}\right)_n = -(\theta, dP^\Phi)_n = -(d^*\theta, P^\Phi)_n$$

$$= -d^*\theta(0) \cdot P^\Phi(0) - \sum_{\sigma \in S_n} \theta(\sigma) P^\Phi(\sigma) = E^\Phi(\Gamma) - \int_{\partial\Gamma} F_n \, d\mu,$$

where $F_n(s) = P^\Phi(\sigma)$ if $\sigma \in s$, $|\sigma| = n$. Since θ_Φ is the point of the affine hyperplane $-d^*\theta' = 1$ closest to 0 when $\mathscr{F}(\Gamma)$ is considered as a vector space with inner product $\langle \theta_1, \theta_2 \rangle := (\theta_1\Phi^{1/2}, \theta_2\Phi^{1/2})$, it follows that

$$0 = \langle \theta_\Phi - \theta, \theta_\Phi \rangle = E^\Phi(\Gamma) - \left(\theta\Phi^{1/2}, \theta_\Phi\Phi^{1/2}\right).$$

Thus, $I_n \to E^\Phi(\Gamma)$, whence $\int_{\partial\Gamma} F_n \, d\mu \to 0$. Let $F(s) = \lim_{\sigma \in s} P^\Phi(\sigma)$. Since $F_n \downarrow F$, we have $\int_{\partial\Gamma} F \, d\mu = 0$, which is to say $F = 0$ μ-a.e.

In light of our previous calculation, the potential of μ_Φ at s is $\sum_{0 < \sigma \in s} \Phi(\sigma)\theta_\Phi(\sigma)$. This is the same as the potential drop from 0 to s, which, by (4.6), is $E^\Phi(\Gamma)$ μ-a.e. □

Theorem 4.3 is equivalent to a bound on the eigenvalues of a certain operator Q associated to Γ. Namely, given a complex-valued function f on the vertices of Γ, define

$$(Qf)(\sigma) = \sum_{\sigma \to \tau} f(\tau).$$

In the usual basis of $l^2(\Gamma)$, the matrix of Q is the directed adjacency matrix of Γ. [It is perhaps of interest that the graph Laplacian $\Delta = +d^*d$ can be expressed in terms of Q: $\Delta = Q + Q^* - QQ^* = I - (Q - I)(Q - I)^*$.] The operator Q is bounded on $l^2(\Gamma)$ iff Γ is of uniformly bounded degree. In any case, Q is always closed and it is natural to examine its spectrum, in particular, to see whether it enables us to calculate $\operatorname{br}\Gamma$. Note first that $\operatorname{br}\Gamma$ is not simply a function of the spectral radius of Q: If Γ is a tree such that "most" vertices have only one successor while Γ still contains arbitrarily large but finite portions of a 2-tree, then $\operatorname{br}\Gamma = 1$ while $\operatorname{rad}(Q) = \sqrt{2}$, which is the spectral radius of the operator associated to an entire 2-tree. It transpires, instead, that the *point spectral radius*,

$$\operatorname{rad}_p(Q) = \sup\{|\lambda| : \exists\, f \in l^2(\Gamma) \setminus \{0\}, Qf = \lambda f\},$$

determines the branching number.

THEOREM 4.6. *For any tree* Γ, $\operatorname{rad}_p(Q) = (\operatorname{br}\Gamma)^{1/2}$.

PROOF. We claim that it is enough to consider nonnegative eigenvalues and eigenfunctions in computing $\operatorname{rad}_p(Q)$. Assuming this, note that nonnegative λ-eigenfunctions f are in 1–1 correspondence with flows θ via $\theta(\sigma) = \lambda^{-|\sigma|}f(\sigma)$. Since $\|f\|_2^2 = \sum_{\sigma\in\Gamma}\theta(\sigma)^2(\lambda^2)^{|\sigma|}$, we see that $f \in l^2(\Gamma)$ iff θ has finite energy for the resistances $\Phi(\sigma) = (\lambda^2)^{|\sigma|-1}$. The conclusion ensues.

In order to establish our claim, first observe that the point spectrum is a disc centered at the origin (this uses an artifice similar to the above correspondence of eigenfunctions and flows). Next, given $\lambda > 0$ and a λ-eigenfunction $f \in l^2(\Gamma)\setminus\{0\}$, we may assume that $f(0) \neq 0$. Define

$$g(\sigma) = |f(\sigma)| \prod_{0\le\tau<\sigma} a_\tau,$$

where

$$a_\tau = \begin{cases} |Qf(\tau)|/(Q|f|)(\tau) & \text{if } (Q|f|)(\tau) \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then $g \geq 0$, $g \in l^2(\Gamma)\setminus\{0\}$, and $Qg = \lambda g$. □

5. Periodic trees. Explicit calculations of energy or current flow are generally impossible, even for a tree with the canonical resistances. Essentially the only exception we know of is the case of periodic trees, which are trees whose structure can be defined from a finite piece of information. We shall see that their calculation reduces to solving a finite system of slightly nonlinear equations, which in turn permits a convergent approximation scheme. Before defining and "solving" periodic trees, we shall present a general relationship which enables current to be calculated from the energies of subtrees.

Given $\sigma \in \Gamma$, we regard $\Gamma^\sigma = \{\tau \in \Gamma\colon \sigma \le \tau\}$ as a tree with root σ. If resistances Φ on Γ are given, then we use the resistances $\Phi|\Gamma^\sigma$ induced on Γ^σ. We claim that

$$E^\Phi(\Gamma) = \sum_{0<\tau\le\sigma}\theta_\Phi(\tau)\Phi(\tau) + \theta_\Phi(\sigma)E^{\Phi|\Gamma^\sigma}(\Gamma^\sigma) \tag{5.1}$$

when $E^{\Phi|\Gamma^\sigma}(\Gamma^\sigma) < \infty$. Indeed,

$$E^\Phi(\Gamma) - \sum_{0<\tau\le\sigma}\theta_\Phi(\tau)\Phi(\tau) = P^\Phi(\sigma)$$

is the potential drop from σ to ∞. Therefore $\theta_\Phi(\sigma)P^\Phi(\sigma)$ is the energy in Γ^σ of the unit current flow in Γ, which equals $\theta_\Phi(\sigma)^2E^{\Phi|\Gamma^\sigma}(\Gamma^\sigma)$ when normalization is accounted for.

When $|\sigma| = 1$, (5.1) yields

$$\theta_\Phi(\sigma) = \left[\Phi(\sigma) + E^{\Phi|\Gamma^\sigma}(\Gamma^\sigma)\right]^{-1}E^\Phi(\Gamma). \tag{5.2}$$

Since $\sum_{|\sigma|=1}\theta_\Phi(\sigma) = 1$, we may also deduce that

$$(5.3)\qquad E^\Phi(\Gamma) = \left\{\sum_{|\sigma|=1}\left[\Phi(\sigma) + E^{\Phi|\Gamma^\sigma}(\Gamma^\sigma)\right]^{-1}\right\}^{-1}$$

In case $\Phi(\sigma) = \lambda^{|\sigma|-1}$, we will use the notations θ_λ and E^λ for θ_Φ and E^Φ, respectively. Equations (5.2) and (5.3) become

$$(5.4)\qquad \theta_\lambda(\sigma) = \left[1 + \lambda E^\lambda(\Gamma^\sigma)\right]^{-1}E^\lambda(\Gamma), \qquad |\sigma| = 1,$$

$$(5.5)\qquad E^\lambda(\Gamma) = \left\{\sum_{|\sigma|=1}\left[1 + \lambda E^\lambda(\Gamma^\sigma)\right]^{-1}\right\}^{-1}$$

We are now prepared to find the current flow and energy of periodic trees.

DEFINITION. A tree Γ is called *periodic* if, for all $\sigma \neq 0$, there is an adjacency-preserving bijection $T\colon \Gamma^\sigma \to \Gamma^{T(\sigma)}$ with $T(\sigma) \in S_1$.

Periodic trees can be constructed as follows. Let G be a finite directed graph with at least one cycle. The *directed cover* Γ of G is the set of finite (directed) paths in G; Γ forms a tree rooted at the empty set when each path is followed by all its possible extensions by one vertex (see Figure 7). Clearly Γ is periodic and every periodic tree is isomorphic to a directed cover of a finite digraph.

A similar notion is that of the *universal cover* of a connected undirected graph H. This is the set of all finite paths in H beginning at some fixed vertex $x \in H$ which never backtrack (i.e., no traversed edge is immediately traversed in the opposite direction). We assume that H is finite and has at least one cycle. The universal cover of H can be obtained from the directed cover of a certain digraph G in the following manner. If $H = (V, E)$, let $G = (E, F)$, where $F = \{((u,v),(v,w))\colon (u,v) \in E,\ (v,w) \in E,\ w \neq u \text{ if } v \neq u\}$. [As in Section 4, we are assuming that $E \subseteq V \times V$ and $(u,v) \in E \Rightarrow (v,u) \in E$. This has to be modified if we wish to allow graphs H with multiple edges.] If Γ is the directed cover of G and $|\sigma| = 1$, then Γ^σ is isomorphic (as an unrooted tree) to the universal cover of H.

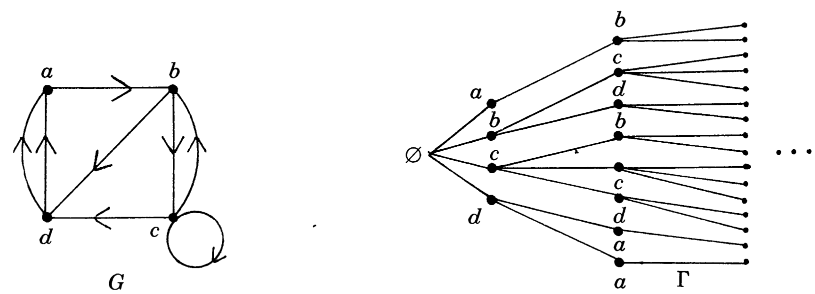

FIG. 7.

Our first task is to compute the branching number of the directed cover of a digraph G. Let A be the directed adjacency matrix of G, i.e., A_{uv} is the number of edges going from u to v. By the Perron–Frobenius theorem, the spectral radius of A is equal to its largest positive eigenvalue, λ_*. This coincides with br Γ. For if $\mathbf{1}$ denotes a column vector all of whose entries are 1, then M_{n+1} is the number of paths in G of length n, and this is $\mathbf{1}^T A^n \mathbf{1}$. Consequently, $M_n^{1/n} \to \lambda_*$; this is br Γ in view of Proposition 3.1.

We may identify S_1 with the vertices of G. For $v \in G$, we set

$$J_\lambda(v) = \left[1 + \lambda E^\lambda(\Gamma^v)\right]^{-1}.$$

If we regard J_λ as a column vector, then (5.5) applied to the tree Γ^v leads to

$$(5.6) \qquad (AJ_\lambda)(v) = \lambda \frac{J_\lambda(v)}{1 - J_\lambda(v)}.$$

Solution of this system of equations is facilitated by the nonlinear operator

$$B_\lambda(K) := \lambda^{-1} A\tilde{K},$$

where

$$\tilde{K}(v) = \frac{K(v)}{1 + K(v)}.$$

We shall term the digraph G *minimal* if there do *not* exist disjoint subsets V_1 and V_2 of the vertices of G whose induced directed adjacency matrices both have spectral radius λ_* and such that if $v_1 \in V_1$ and $v_2 \in V_2$, then there is no directed path in G from v_1 to v_2 nor from v_2 to v_1.

THEOREM 5.1. *For $\lambda < \lambda_*$, the energy $E^\lambda(\Gamma)$ is given by*

$$E^\lambda(\Gamma) = \left(\mathbf{1}^T J_\lambda\right)^{-1}$$

and J_λ is the unique solution of

$$(5.7) \qquad (AJ)(v) = \lambda \frac{J(v)}{1 - J(v)}, \qquad J \geq 0,$$

which has the maximum number of nonzero entries. Furthermore, if we choose any vector $K \geq \lambda^{-1} A\mathbf{1}$, then $B_\lambda^n K \to K_\lambda$, with $J_\lambda = \tilde{K}_\lambda$. Let π: $\Gamma \setminus \{0\} \to G$ be the natural projection mapping paths to their endpoints. Unit current flow is given by

$$\theta_\lambda(\sigma) = \lambda^{-|\sigma|+1}\left(\mathbf{1}^T J_\lambda\right)^{-1} J_\lambda(\pi(\sigma)) \prod_{0 < \tau < \sigma} \left[1 - J_\lambda(\pi(\tau))\right], \qquad \sigma \neq 0.$$

As $\lambda \uparrow \lambda_*$, $J_\lambda \to 0$. Suppose now that G is minimal. Then as $\lambda \uparrow \lambda_*$,

$$J_\lambda \cdot E^\lambda(\Gamma) \to J_*,$$

where J_* is the unique nonnegative λ_*-eigenvector of A satisfying $\mathbf{1}^T J_* = 1$.

Also, the limiting flow

$$\theta_*(\sigma) := \lim_{\lambda \uparrow \lambda_*} \theta_\lambda(\sigma) = \lambda_*^{-|\sigma|+1} J_*(\pi(\sigma))$$

exists and satisfies $\operatorname{lip} \theta_* = \operatorname{br} \Gamma$.

PROOF. Solutions to (5.7) are related to fixed points of B_λ,

$$B_\lambda K = K, \qquad K \geq 0,$$

via $J = \tilde{K}$. Define K_λ by $\tilde{K}_\lambda = J_\lambda$. Let G_λ be the subgraph induced by G on the set of vertices v such that $J_\lambda(v) = 0$. In view of (5.6), we may write

$$A = \begin{bmatrix} A_\lambda & 0 \\ * & * \end{bmatrix},$$

where A_λ is the directed adjacency matrix of G_λ. Since no current flows in $\pi^{-1}(G_\lambda)$, the directed cover of G_λ must have infinite λ-energy, whence $\lambda \geq \mu$, where μ is the spectral radius of A_λ. It follows that all solutions of (5.7) are 0 on G_λ: If J satisfies (5.7) and $J' = J|G_\lambda \neq 0$, then

$$(A_\lambda J')(v) = \lambda \frac{J'(v)}{1 - J'(v)} > \lambda J'(v),$$

whence $A_\lambda^n J' > \lambda^n J'$ and, by the spectral radius formula, $\mu > \lambda$, a contradiction. Therefore, if J is a solution of (5.7) with the maximum number of nonzero entries, J is nonzero on $G \setminus G_\lambda$, as is J_λ. If $J \neq J_\lambda$, then there are two distinct nonnegative fixed points of B_λ positive on $G \setminus G_\lambda$, call them K and K'. We may suppose that K is not greater than or equal to K' (else we switch notation). Then there is a number $t \in]0,1[$ such that $K \geq tK'$ and t is maximal with this property. Now $\widetilde{tK'} \geq t' \cdot \tilde{K}'$ for some $t' > t$, since $t < 1$. Consequently,

$$K = B_\lambda K \geq B_\lambda(tK') \geq t' B_\lambda K' = t'K',$$

which violates our choice of t. This shows that J_λ is the unique solution to (5.7) with the maximum number of nonzero entries.

We turn now to the iterative solution of (5.7). Choose $K \geq \lambda^{-1} A\mathbf{1}$. Then $B_\lambda K \leq \lambda^{-1} A\mathbf{1} \leq K$, which leads to $K \geq B_\lambda K \geq B_\lambda^2 K \geq \cdots$ by monotonicity of B_λ. In particular, $B_\lambda^n K$ converges to some fixed point K'. Now $K \geq \lambda^{-1} A\mathbf{1} \geq \lambda^{-1} A J_\lambda = K_\lambda$, whence $B_\lambda^n K \geq K_\lambda$ and so $K' \geq K_\lambda$. As K_λ is the unique fixed point with the maximum number of nonzero entries, this means that $K' = K_\lambda$.

To determine the current flow, we use (5.4) and (5.5) applied to the appropriate tree and normalized for nonunit current flow:

$$\begin{aligned} \theta_\lambda(\sigma) &= \left[1 + \lambda E^\lambda(\Gamma^\sigma)\right]^{-1} E^\lambda(\Gamma^{\overleftarrow{\sigma}}) \theta_\lambda(\overleftarrow{\sigma}) \\ &= \begin{cases} \lambda^{-1} J_\lambda(\pi(\sigma)) J_\lambda(\pi(\overleftarrow{\sigma}))^{-1} \left[1 - J_\lambda(\pi(\overleftarrow{\sigma}))\right] \theta_\lambda(\overleftarrow{\sigma}) & \text{if } \overleftarrow{\sigma} \neq 0, \\ J_\lambda(\pi(\sigma)) \left(\mathbf{1}^T J_\lambda\right)^{-1} & \text{if } \overleftarrow{\sigma} = 0. \end{cases} \end{aligned}$$

Iteration of this formula enables us to compute $\theta_\lambda(\sigma)$ as stated above.

Suppose now that J is a limit point of J_λ. Since $0 \le J_\lambda < 1$, we have $0 \le J \le 1$ and J satisfies (5.7) for λ_*. It was shown earlier in the proof (when considering A_λ) that this is impossible unless $J = 0$. Thus, $J_\lambda \to 0$.

Consider next any limit point, J, of $J_\lambda \cdot E^\lambda(\Gamma)$ as $\lambda \uparrow \lambda_*$. Since $E^\lambda(\Gamma) = (\mathbf{1}^T J_\lambda)^{-1}$, it follows that $0 \le J \le 1$ and $\mathbf{1}^T J = 1$. Furthermore, if we multiply (5.6) by $E^\lambda(\Gamma)$ and take limits, we obtain $AJ = \lambda_* J$. If G is minimal, then J_* as described is unique [5], whence $J = J_*$. This demonstrates the existence of $\lim J_\lambda E^\lambda(\Gamma)$. Finally, substitution of this result into the formula for θ_λ gives θ_*. It follows from the form of θ_* that $\operatorname{lip}\theta_* = \lambda_* = \operatorname{br}\Gamma$. □

6. Percolation. Let Γ be a countable graph and fix $p \in [0,1]$. We remove each edge with probability $1-p$ independently of the other edges. The random graph which is left will be denoted $\Gamma(\omega_p)$, where ω_p is a point in an underlying probability space Ω_p. For any vertex $\sigma \in \Gamma$, let $\Gamma_\sigma(\omega_p)$ denote the connected component of σ in $\Gamma(\omega_p)$. By the zero–one law, the probability that $\Gamma_\sigma(\omega_p)$ is infinite for some $\sigma \in \Gamma$ is either 0 or 1. This probability is increasing in the parameter p, whence we define the *critical probability*,

$$p_c = p_c(\Gamma) = \sup\{p\colon \mathbf{P}[\exists\, \sigma \in \Gamma, \operatorname{card}\Gamma_\sigma(\omega_p) = \infty] = 0\}.$$

One of the main interests of percolation theory is the calculation of p_c. This problem is usually phrased in a slightly different manner. Namely, when we are concerned with a *connected* countable graph Γ and a fixed vertex $0 \in \Gamma$, we ask for

$$\sup\{p\colon \mathbf{P}[\operatorname{card}\Gamma_0(\omega_p) = \infty] = 0\}.$$

It is not hard to see that this equals $p_c(\Gamma)$ by virtue of the connectivity and countability of Γ. Thus, as we have defined it, $p_c(\Gamma)$ is equal to the infimum of the critical probabilities of the connected components of Γ.

It transpires that the critical probability of the random graph $\Gamma(\omega_p)$ bears a simple relation to $p_c(\Gamma)$.

Proposition 6.1. *For any countable graph* Γ,

$$p_c(\Gamma(\omega_p)) = (p_c(\Gamma)/p) \wedge 1 \quad a.s.$$

Proof. It suffices to establish the equation for $p > p_c(\Gamma)$. Fix such a p and consider any $q \in [0,1]$. Edge removal with parameter p followed by independent edge removal with parameter q is equivalent to edge removal with parameter pq: Abusing notation, we may write, for fixed ω_p,

$$\exists\, \sigma \in \Gamma(\omega_p), \operatorname{card}\Gamma(\omega_p)_\sigma(\omega_q) = \infty \quad \Leftrightarrow \quad \exists\, \sigma \in \Gamma, \operatorname{card}\Gamma_\sigma(\omega_p) \cap \Gamma_\sigma(\omega_q) = \infty$$

and, on $\Omega_p \times \Omega_q$ and Ω_{pq},

$$\mathbf{P}\big[\exists\, \sigma \in \Gamma, \operatorname{card} \Gamma_\sigma(\omega_p) \cap \Gamma_\sigma(\omega_q) = \infty\big] = \mathbf{P}\big[\exists\, \sigma \in \Gamma, \operatorname{card} \Gamma_\sigma(\omega_{pq}) = \infty\big].$$

The result follows from Fubini's theorem in conjunction with this pair of observations. □

As a rule, $p_c(\Gamma)$ is extremely difficult to calculate. For example, when Γ is the square lattice $\mathbf{Z}^2$, it was proved only a few years ago that $p_c = \frac{1}{2}$; the critical probability for the cubic lattice $\mathbf{Z}^3$ is still unknown [17]. When Γ is an n-ary tree, it has long been understood that $p_c = 1/n$. This is easily proved with the aid of some elementary theory of branching processes [2, Theorem I.5.1]. Our principal result extends this calculation to all trees.

THEOREM 6.2. *For any tree* Γ,

$$p_c(\Gamma) = \frac{1}{\operatorname{br} \Gamma}.$$

PROOF. Given a cutset Π and $\lambda > 0$, let

$$Z_\Pi^\lambda = \sum_{\sigma \in \Pi} \lambda^{-|\sigma|}. \tag{6.1}$$

Also write $\Pi(\omega_p) = \Pi \cap \Gamma_0(\omega_p)$ and

$$Z_{\Pi(\omega_p)}^\lambda = \sum_{\sigma \in \Pi(\omega_p)} \lambda^{-|\sigma|} = \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} \mathbf{1}_{\Gamma_0(\omega_p)}(\sigma).$$

Then

$$\mathbf{E}\big[Z_{\Pi(\omega_p)}^1\big] = \sum_{\sigma \in \Pi} \mathbf{P}\big[\sigma \in \Gamma_0(\omega_p)\big] = \sum_{\sigma \in \Pi} p^{|\sigma|} = Z_\Pi^{p^{-1}}. \tag{6.2}$$

If $p^{-1} > \operatorname{br} \Gamma$, i.e., $p < (\operatorname{br} \Gamma)^{-1}$, then there is a sequence $\Pi_n \to \infty$ such that $Z_{\Pi_n}^{p^{-1}} \to 0$. From Fatou's lemma applied to (6.2), we obtain

$$\liminf_{n \to \infty} Z_{\Pi_n(\omega_p)}^1 = 0 \quad \text{a.s.}$$

This is the same as $\Gamma_0(\omega_p)$ being finite a.s.

It remains to show that if $p^{-1} < \operatorname{br} \Gamma$, then $\Gamma_0(\omega_p)$ is infinite with positive probability. For any unit flow θ, set

$$X_n^\theta(\omega_p) = \sum_{\sigma \in S_n(\omega_p)} \theta(\sigma) p^{-|\sigma|}.$$

(This may be considered as a normalized version of $Z_{S_n(\omega_p)}^1$.) Let $\mathscr{F}_n$ be the σ-field of subsets of Ω_p generated by the events $\{\omega_p \in \Omega_p\colon \text{edge } \sigma \in \Gamma(\omega_p)\}$, $0 < |\sigma| \le n$. Then $\{X_n^\theta\}$ is adapted to $\{\mathscr{F}_n\}$ and forms a martingale, $\mathbf{E}[X_{n+1}^\theta | \mathscr{F}_n] = X_n^\theta$. Since $\{X_n^\theta\}$ is nonnegative, the martingale converges a.s. to some random variable X^θ. Now

$$\begin{aligned}
\mathbf{E}\left[\left(X_n^\theta\right)^2\right] &= \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-2n}\mathbf{P}\left[\sigma,\tau\in\Gamma_0(\omega_p)\right] \\
&= \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-2n}\left(p^{|\sigma\wedge\tau|}p^{|\sigma|-|\sigma\wedge\tau|}p^{|\tau|-|\sigma\wedge\tau|}\right) \\
&= \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-|\sigma\wedge\tau|} \\
&= \sum_{|\psi|\le n} p^{-|\psi|} \sum_{\substack{\sigma\wedge\tau=\psi\\ \sigma,\tau\in S_n}} \theta(\sigma)\theta(\tau) \\
&= \sum_{|\psi|\le n} p^{-|\psi|}\left[\theta(\psi)^2 - \sum_{\substack{\psi\to\sigma\\ |\sigma|\le n}}\theta(\sigma)^2\right] \\
&= \sum_{0<|\psi|\le n} \theta(\psi)^2\left[p^{-|\psi|} - p^{-|\overleftarrow{\psi}|}\right] + 1 \\
&= (1-p)\sum_{0<|\psi|\le n} \theta(\psi)^2 p^{-|\psi|} + 1.
\end{aligned}$$

If $p^{-1} < \operatorname{br}\Gamma$, then we may choose θ to be of finite p^{-1}-energy. In this case, $\sup \mathbf{E}[(X_n^\theta)^2] < \infty$, whence $X_n^\theta \to X^\theta$ in L^2-norm [23, Proposition IV-2-7] and so $\mathbf{E}[X^\theta] = \mathbf{E}[X_n^\theta] = 1$. Since $X^\theta \neq 0$ with positive probability, $\Gamma_0(\omega_p)$ is infinite with positive probability. $\square$

COROLLARY 6.3. *If* Γ *is a tree and* $p > (\operatorname{br}\Gamma)^{-1}$, *then*

$$\sup_{\sigma\in\Gamma} \operatorname{br}\Gamma_\sigma(\omega_p) = \sup_{\sigma\in\Gamma}\operatorname{br}\Gamma^\sigma(\omega_p) = p\operatorname{br}\Gamma \quad a.s.,$$

where $\Gamma^\sigma(\omega_p) = \Gamma^\sigma \cap \Gamma_\sigma(\omega_p)$ *and where the branching number of a finite tree is deemed to be zero. Therefore,* $\operatorname{ess\,sup}\operatorname{br}\Gamma_0(\omega_p) = p\operatorname{br}\Gamma$.

This is an immediate consequence of Theorem 6.2 and Proposition 6.1. The final conclusion can be strengthened when Γ is an n-tree: Given that card $\Gamma_0(\omega_p) = \infty$,

$$\operatorname{br}\Gamma_0(\omega_p) = p\operatorname{br}\Gamma \quad \text{a.s.}$$

for $p > (\operatorname{br}\Gamma)^{-1} = 1/n$. In fact, we may state the following extension of the usual growth law for branching processes. (For background, see [2, pages 1–10].)

PROPOSITION 6.4. *Let the probability of k offspring of a given particle be* p_k $(0 \le k < \infty)$ *in a Galton–Watson process. If* $m = \sum_{k\ge 0} kp_k > 1$, *then, given the event that the process does not become extinct, the associated genealogical tree has branching number m a.s. and is quasispherical a.s.*

PROOF. If the Galton–Watson process is followed by independent edge removal with parameter q, the resulting connected component of 0 has the same law as a Galton–Watson process with mean mq. The latter process becomes extinct a.s. iff $mq \leq 1$. Therefore, the original process leads to a tree with critical probability at least $1/m$ a.s., i.e., with branching number at most m a.s. Let $\lambda \leq m$ be such that a tree with branching number at least λ is produced with positive probability, π, and E the corresponding event. Then

$$1 - \pi = \mathbf{P}[E^c] = \sum_{k=0}^{\infty} \mathbf{P}[E^c | \operatorname{card} S_1 = k] p_k = \sum_{k=0}^{\infty} \mathbf{P}[E^c]^k p_k = \sum_{k=0}^{\infty} p_k (1-\pi)^k$$

It follows that π is the probability of nonextinction and that the branching number is a constant a.s. given nonextinction. Finally, that constant is m, since when $mq > 1$, Fubini's theorem shows that the (original) process has positive probability of producing a tree with critical probability at most q.

It is immediate from [2, Theorem I.6.1] that the growth of any nonextinct tree is a.s. at most m. Since the growth is at least the branching number, the nonextinct trees are a.s. quasispherical. □

Completely analogous reasoning leads to the following generalization of Proposition 6.4 to multitype branching processes. (For the relevant definitions and theorems, see [13, pages 34–42] and [2, page 192].)

PROPOSITION 6.5. *Let m be the maximum eigenvalue of the mean matrix of a supercritical positive regular nonsingular multitype branching process. Given the event of nonextinction, the genealogical tree has branching number m a.s. and is quasispherical a.s.*

Another critical value of p often considered in percolation theory is

$$p_T = \sup\{p \colon \mathbf{E}[\operatorname{card} \Gamma_0(\omega_p)] < \infty\},$$

where "card" refers to the cardinality of, say, the vertex set. For certain homogeneous graphs, $p_T = p_c$ ([1, 21]). In the case of trees, p_T is very easily calculated:

$$\mathbf{E}[\operatorname{card} \Gamma_0(\omega_p)] = \sum_{0 \neq \sigma \in \Gamma} \mathbf{P}[\sigma \in \Gamma_0(\omega_p)] = \sum_{0 \neq \sigma \in \Gamma} p^{|\sigma|} = \sum_{n \geq 1} M_n p^n,$$

whence

$$p_T^{-1} = \limsup_{n \to \infty} M_n^{1/n}.$$

(It is curious that p_T^{-1} also separates ergodicity from nonergodicity of the canonical random walk.) In particular, $p_T = p_c$ iff Γ is quasispherical and $\lim M_n^{1/n}$ exists. Examples of such trees Γ include subperiodic trees and almost any nonextinct tree produced by a Galton–Watson process (in light of Proposition 6.4 and [2, Theorem I.6.1]) or by a positive regular nonsingular multitype branching process.

The above considerations suggest investigation of the following critical probability for an arbitrary connected countable graph Γ. If $0 \in \Gamma$, define

$$p_{\mathrm{cut}}(\Gamma) = \sup\Big\{p\colon \inf_{\Pi} \mathbf{E}\big[\operatorname{card} \Pi \cap \Gamma_0(\omega_p)\big] = 0\Big\}$$

$$= \sup\Big\{p\colon \inf_{\Pi} \sum_{v \in \Pi} \mathbf{P}\big[v \in \Gamma_0(\omega_p)\big] = 0\Big\},$$

where the infimum is taken over all cutsets Π (i.e., collections Π of vertices such that each path emanating from 0 and containing infinitely many distinct vertices must include a vertex from Π). Then $p_{\mathrm{cut}}(\Gamma) = (\operatorname{br} \Gamma)^{-1}$ virtually by definition when Γ is a tree. In general, we have

$$p_T(\Gamma) \leq p_{\mathrm{cut}}(\Gamma) \leq p_c(\Gamma).$$

Therefore, $p_{\mathrm{cut}}(\Gamma) = p_c(\Gamma)$ when Γ is any tree or when Γ is a euclidean cubic lattice ([1], [21]). It is quite possible that this relation holds for all connected graphs.

Random resistive or capacitative networks will be the last topic from percolation theory that we consider. The kind we shall examine have proved useful in the study of random fractals ([8], [9]). Consider a tree Γ, to each edge of which is assigned a nonnegative random variable A_σ. Let

$$C(\sigma) = \prod_{0 < \tau \leq \sigma} A_\tau;$$

this will be the conductance or the capacity of the edge preceding σ, depending on the problem considered. We are interested in the probability that the associated electrical network admits current flow or that the associated capacitative network admits flow [i.e., that there is a nonzero flow θ bounded by C: $\forall\ \sigma\ \theta(\sigma) \leq C(\sigma)$].

THEOREM 6.6. *Suppose that $\{A_\sigma\}$ are independent, each having mean p. If $p \operatorname{br} \Gamma < 1$, then a.s. the electrical network has infinite resistance and the capacitative network admits no flow. If $p \operatorname{br} \Gamma > 1$, $\forall\ \sigma$, $\mathbf{E}[A_\sigma^2] \leq p$, and for some $\lambda \in]1, p \operatorname{br} \Gamma[$, $\forall\ \sigma$, $A_\sigma \leq \lambda$ a.s., then with positive probability, the electrical network has finite resistance and the capacitative network admits flow.*

REMARK 1. The case where A_σ takes only the values 0 or 1 reduces to ordinary ("Bernoulli") percolation. More precisely, when A_σ takes only one nonzero value, $\lambda^{-1} \leq 1$, the above result for capacitative networks is equivalent to Theorem 6.2 and Corollary 6.3 by the max-flow min-cut theorem.

REMARK 2. In view of Section 4, the above result for electrical networks may be interpreted as determining the character of a random walk in a random environment.

REMARK 3. If $A_\sigma > 0$ a.s., then by the zero–one law, the final conclusions hold not only with positive probability, but a.s.

PROOF OF THEOREM 6.6. Define the random variables $\zeta_\Pi = \sum_{\sigma \in \Pi} C(\sigma)$. Then

$$\mathbf{E}[\zeta_\Pi] = \sum_{\sigma \in \Pi} \prod_{0<\tau\le\sigma} \mathbf{E}[A_\tau] = \sum_{\sigma\in\Pi} p^{|\sigma|} = Z_\Pi^{p^{-1}},$$

in the notation of (6.1). Hence, if $p^{-1} > \operatorname{br}\Gamma$, there is a sequence $\Pi_n \to \infty$ such that $Z_{\Pi_n}^{p^{-1}} \to 0$, whereupon $\liminf_{n\to\infty} \zeta_{\Pi_n} = 0$ a.s. By the max-flow min-cut theorem, the capacitative network admits no flow a.s. and, by Corollary 4.2, the electrical network has infinite resistance a.s.

For the converse, if $p^{-1}\lambda < \operatorname{br}\Gamma$ and $\lambda > 1$, set

$$\xi_n^\theta = \sum_{\sigma \in S_n} \theta(\sigma) p^{-|\sigma|} C(\sigma),$$

where θ is a unit flow of finite $p^{-1}\lambda$-energy. Then $(\xi_n^\theta, \mathscr{F}_n)$ is a martingale, where $\mathscr{F}_n$ is the σ-field generated by $\{A_\sigma: |\sigma| \le n\}$, and thus converges a.s. to some random variable ξ^θ. Furthermore,

$$\begin{aligned}
\mathbf{E}\Big[\big(\xi_n^\theta\big)^2\Big] &= \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-2n}\mathbf{E}\Bigg[\prod_{0<\psi\le\sigma\wedge\tau} A_\psi^2\Bigg] p^{2n-2|\sigma\wedge\tau|}\\
&\le \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-|\sigma\wedge\tau|}\\
&= (1-p)\sum_{0<|\sigma|\le n} \theta(\sigma)^2 p^{-|\sigma|} + 1
\end{aligned}$$

if $\mathbf{E}[A_\sigma^2] \le p$. As this is bounded, $\xi_n^\theta \to \xi^\theta$ in L^2, hence $\xi^\theta \ne 0$ with positive probability. This, however, is not enough to ensure a nonzero flow bounded by C; we need a lower bound for all cutset sums. We use Lemma 3.2 for this purpose.

Let $\Gamma(\omega_p) = \{\sigma \in \Gamma: C(\sigma)(\omega_p) \ne 0\}$ and set $f(\sigma) = \theta(\sigma)p^{-|\sigma|}C(\sigma)$ and $g(\sigma) = \lambda^{-|\sigma|}C(\sigma)$, where λ is as in the hypotheses. Define

$$Y_n = \sum_{\sigma,\tau\in S_n\cap\Gamma(\omega_p)} g(\sigma\wedge\tau)^{-1} f(\sigma) f(\tau) = \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)p^{-2n}\lambda^{|\sigma\wedge\tau|} \prod_{\substack{0<\psi\le\sigma\\ \text{or}\\ 0<\psi\le\tau}} A_\psi.$$

Then

$$\mathbf{E}[Y_n] = \sum_{\sigma,\tau\in S_n} \theta(\sigma)\theta(\tau)(p^{-1}\lambda)^{|\sigma\wedge\tau|} = (1-p\lambda^{-1}) \sum_{0<|\sigma|\le n} \theta(\sigma)^2 (p^{-1}\lambda)^{|\sigma|} + 1,$$

which is bounded in n. Therefore, $\liminf_{n\to\infty} Y_n < \infty$ a.s. [In fact, one may show that $(Y_n, \mathscr{F}_n)$ is a submartingale and hence converges to a finite limit a.s.] We have already shown that $\sum_{\sigma\in S_n\cap\Gamma(\omega_p)} f(\sigma) = \xi_n^\theta$ has a nonzero limit with positive probability. Lemma 3.2 ensures that $\inf_\Pi \sum_{\sigma\in\Pi\cap\Gamma(\omega_p)} g(\sigma) > 0$ with positive probability. The proof is completed with the aid of Corollary 4.2. □

7. Boundary sets. Very little effort is needed to extend several of our results to subsets of the boundary of a tree. Given a subset E of the boundary of Γ, we denote the set of *covers* of E by

$$\mathscr{C}(E) = \{\Pi \subseteq \Gamma\colon \forall\, s \in E,\ s \cap \Pi \neq \varnothing\}.$$

We define the *dimension* of E by

$$\dim E = \log \inf\left\{\lambda > 0\colon \liminf_{\mathscr{C}(E) \ni \Pi \to \infty} \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} = 0\right\}.$$

Thus, $\dim \partial\Gamma = \log \operatorname{br} \Gamma$. It is easy to see that this is the usual Hausdorff dimension when $\partial\Gamma$ is considered as a compact space with metric $d(s,t) = e^{-|s \wedge t|}$. For $\lambda > 0$, the function

$$I^{\lambda}(E) = \inf_{\Pi \in \mathscr{C}(E)} \sum_{\sigma \in \Pi} \lambda^{-|\sigma|}$$

is a capacity on $\partial\Gamma$ [6, Chapitre VI, Corollaire 19]. Since $\dim E = \inf\{\log \lambda\colon I^{\lambda}(E) = 0\}$, it follows that

$$\dim E = \sup\{\dim K\colon K \subseteq E,\ K \text{ compact}\}$$

when E is Borel or, more generally, analytic. Of course, compact subsets of $\partial\Gamma$ correspond to boundaries of subtrees of Γ rooted at 0. From Section 4, we may conclude that, when E is analytic,

$$\dim E = \sup\left\{\log \lambda\colon E \text{ carries a Borel probability measure } \mu \text{ such that } \iint \lambda^{|s \wedge t|}\, d\mu(s)\, d\mu(t) < \infty\right\}.$$

A slight modification of the first half of the proof of Theorem 6.2 shows that for *any* set $E \subseteq \partial\Gamma$,

$$\dim\big(E \cap \partial\Gamma_0(\omega_p)\big) \le (\dim E + \log p) \vee 0 \quad \text{a.s.}$$

and when $\dim E + \log p < 0$, $E \cap \partial\Gamma_0(\omega_p) = \varnothing$ a.s. Hence, when E is analytic,

$$\operatorname{ess\,sup} \dim\big(E \cap \partial\Gamma_0(\omega_p)\big) = (\dim E + \log p) \vee 0.$$

Let us write

$$\partial\Gamma(\omega_p) = \big\{s \in \partial\Gamma\colon \operatorname{card}\big(s \setminus \Gamma(\omega_p)\big) < \infty\big\};$$

this is essentially the set of boundary points of the infinite components of $\Gamma(\omega_p)$. Corollary 6.3 implies that if E is analytic, then

$$\begin{aligned}\dim\big(E \cap \partial\Gamma(\omega_p)\big) &= (\dim E + \log p) \vee 0 \\ &= \big(\dim E + \dim \partial\Gamma(\omega_p) - \dim \partial\Gamma\big) \vee 0 \quad \text{a.s.}\end{aligned}$$

In the language of [12], $\partial\Gamma(\omega_p)$ is a.s. transverse to any given analytic set $E \subseteq \partial\Gamma$.

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