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CONDITIONAL EXITS FOR SMALL NOISE DIFFUSIONS WITH CHARACTERISTIC BOUNDARY

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The small noise exit problem of Wentzell and Freidlin is of particular interest for regions whose boundary consists of trajectories of the underlying (unperturbed) dynamical system. This is called the case of characteristic boundary. One fruitful approach to attacking this problem involves conditioning the probability measure so that exit to the boundary occurs more quickly. In a previous paper, this approach was applied to some simplified examples, revealing some previously unanticipated phenomena for the characteristic boundary exit problem. In this paper we develop certain aspects of this approach more generally. In particular we present stochastic differential equations which give an asymptotically correct description of this conditioned process by using a carefully chosen system of coordinates near the boundary.

1. Introduction. The past decade has seen extensive study of the effects of adding small random perturbations to a (deterministic) dynamical system

$$\dot{x}(t) = \mathbf{b}(x(t)); \qquad x(0) = x_0. \tag{1.1}$$

The work of Wentzell and Freidlin [14] developed a rich mathematical theory for the study of "small noise diffusion" perturbations:

$$dx^\varepsilon(t) = \mathbf{b}(x^\varepsilon(t))\,dt + \varepsilon^{1/2}\sigma(x^\varepsilon(t))\,dw(t); \qquad x^\varepsilon(0) = x_0. \tag{1.2}$$

Although other types of perturbations may be more reasonable in applied contexts, (1.2) is very appealing mathematically. It yields a Markov diffusion process $x^\varepsilon(t)$ and thus makes a tremendous arsenal of mathematical theory available: Itô calculus and stochastic differential equations, elliptic and parabolic partial differential equations, the large deviations theory of Wentzell and Freidlin and more.

One of the most interesting problems in this area has been the exit problem. Suppose D is a bounded region, with smooth boundary ∂D, which is attracted by (1.1) to a unique stable critical point $0 \in D$. Let

$$\tau_D^\varepsilon = \inf\{t > 0\colon x^\varepsilon(t) \in \partial D\} \tag{1.3}$$

be the first exit time of x^ε from D. The exit problem concerns the asymptotic behavior, as $\varepsilon \downarrow 0$, of the exit time τ_D^ε and exit position $x^\varepsilon(\tau_D^\varepsilon)$.

The exit problem was first studied under the assumption that (1.1) enters D nontangentially:

$$\langle \mathbf{b}(y), \mathbf{n}(y)\rangle < 0, \quad \text{all } y \in \partial D,$$

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where $\mathbf{n}(y)$ is the unit outward normal. Recently there has been increasing interest in the case of characteristic boundary:

$$\langle \mathbf{b}(y), \mathbf{n}(y) \rangle = 0, \quad \text{all } y \in \partial D. \tag{1.4}$$

Several papers have addressed aspects of this version of the problem: [2, 3, 7, 8, 9, 10, 17, 18]. Large deviations results for the distributions of τ_D^ε and $x^\varepsilon(\tau_D^\varepsilon)$ generalize to the case (1.4); see [9]. But, as was pointed out there, these large deviations results are often inconclusive in the characteristic boundary case.

In [8] we presented an approach to the asymptotic behavior of the exit distribution

$$\mu^\varepsilon(x_0, A) = P_{x_0}[x^\varepsilon(\tau_D^\varepsilon) \in A], \qquad A \subseteq \partial D,$$

which revealed some previously unexpected phenomena. Taking D to be the unit disk, we assumed x^ε could be expressed as a decoupled system in terms of its radial and angular components. Specifically with $\rho = 1 - r$ denoting the distance to ∂D we assumed

$$\begin{aligned} d\rho^\varepsilon(t) &= B\rho^\varepsilon(t)\, dt + \varepsilon^{1/2}\, d\beta_1(t), \\ d\theta^\varepsilon(t) &= b(\theta^\varepsilon(t))\, dt + \varepsilon^{1/2}\, d\beta_2(t). \end{aligned} \tag{1.5}$$

Based on this we described the possible cycling or precession of μ^ε around ∂D as a periodic function of $\log(\varepsilon^{1/2})$. We explained how μ^ε could have a limit μ^0 giving positive probability to repelling critical points of (1.1) on ∂D, or that μ^0 could have positive density on a section of ∂D between two critical points. In particular, the boundary invariance conjecture made in Section 6 of [7] is false in general.

Our purpose in the present paper is to look somewhat more carefully at the conditional exit approach of [8]. In Section 2 we will exhibit the main features of this approach in a more general setting. We will describe a coordinate system (ρ, θ) near ∂D in which (1.2) looks something like (1.5). The motive for this is to allow us to appeal to the explicit conditioning calculations carried out in Section 3. However the change of coordinates will introduce additional $\mathscr{O}(\varepsilon)$ drift terms. We will show in Section 4 that these terms are negligible for our asymptotic calculations. This issue was noted in [8] but not addressed there; it was avoided by assuming (1.5). However we must deal with such $\mathscr{O}(\varepsilon)$ additions to the drift if we want to discuss (1.2) in more generality.

Some other issues involved in developing the conditional exit approach in general are brought into focus, but not resolved here. One of these is the existence of the system of coordinates mentioned above. We will see that this boils down to certain regularity conditions on the "boundary quasipotential":

$$W(x) = V(x, \partial D),$$

V being the variational distance function of Freidlin and Wentzell [14]. We will simply assume these properties for our discussion here.

Our analysis is aimed at providing an asymptotically correct characterization of the process x^ε when considered with respect to the probability $\bar{P}$ obtained by conditioning on exit. This culminates in Section 5 where the conditional exit kernel Q^ε of Section 3 is described in terms of stochastic differential equations for the conditioned process. However we do not carry the analysis of these equations to the point necessary for resolving the characteristic boundary exit problem in general. Instead, in Section 6, we describe a class of examples which, by the analysis of this paper, reduce to the simplified situation considered in [8]. This shows that the various phenomena described in [8] do indeed occur for examples which are properly of the form (1.2).

We finish this introduction by describing our assumptions and notation. We limit our discussion to two-dimensional systems, though all our arguments generalize to higher dimensions. The two-dimensional case is simpler since ∂D can be identified with $\mathbb{R}$ modulo some period S. (In higher dimensions ∂D is a manifold without a single chart parameterization.) For $A \subseteq \mathbb{R}^2$, $\bar{A}$ denotes its closure and ∂A its boundary. $D \subseteq \mathbb{R}^2$ is assumed to be bounded, open and simply connected.

We will take smooth to mean C^3 in the following. ∂D is assumed smooth, as are the coefficient functions $\mathbf{b}\colon \bar{D} \to \mathbb{R}^2$ and $\sigma\colon \bar{D} \to \mathbb{R}^2 \otimes \mathbb{R}^2$. σ is nonsingular everywhere in $\bar{D}$ and $a(x)$ denotes $\sigma(x)\sigma(x)^T$.

We assume $0 \in D$ and for every $x_0 \in D$ the solution of (1.1) satisfies $x(t) \in D$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$. (1.4) will be a consequence of our hypotheses of W-regularity in Section 2.

The $w(t)$ in (1.2) is of course a two-dimensional Brownian motion. For the bulk of the paper $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_s\}$ will be as described at the end of Section 2. The various other probability measures that occur below $(P^{\triangleright}, P^{\circ}, \dots)$ are all defined on this same $(\Omega, \mathscr{F})$. Diagram (5.2) illustrates how these are related to each other.

2. Conditioning on exit and the natural coordinates. We have two objectives in this section. The first is to discuss the general idea of conditioning on exit in the exit problem, elaborating on Section 2 of [8]. The second is to discuss a system of coordinates (ρ, θ) in which this approach can be reduced to explicit calculations.

Consider a nested pair of slightly smaller (smooth) domains $0 \in C \subseteq G \subseteq D$. (We will choose them more carefully below.) Define the following pair of stopping times:

$$\tau_G^\varepsilon = \inf\{t \geq 0\colon x^\varepsilon(t) \in \partial G\} \quad \text{and} \quad \tau_C^\varepsilon = \inf\{t \geq \tau_G^\varepsilon\colon x^\varepsilon(t) \in \partial C\}. \tag{2.1}$$

Notice that if $x^\varepsilon(0) \in \partial G$, then $\tau_G^\varepsilon = 0$ and τ_C^ε is the first hitting time of ∂C. Otherwise τ_C^ε is the first time *after* τ_G^ε at which ∂C is hit. $0 \leq \tau_G^\varepsilon < \tau_C^\varepsilon$ in general. In addition to the exit measure μ^ε from D consider the distribution of the first exit position from G:

$$\nu^\varepsilon(x_0, B) = P_{x_0}[x^\varepsilon(\tau_G^\varepsilon) \in B]; \qquad B \subseteq \partial G,\ x_0 \in G.$$

We will be able to choose G so that the exit problem from G is a nontangential one:

$$\langle \mathbf{b}(y), \mathbf{n}(y) \rangle < 0, \quad \text{all } y \in \partial G. \tag{2.2}$$

The basic idea of the conditional exit approach is to represent μ^ε in terms of ν^ε and the behavior of x^ε subject to the *conditional* probability that $\tau_D^\varepsilon < \tau_C^\varepsilon$. This is a very rare event, but conditioning on it will allow us to see much more clearly how the transition from ∂G to ∂D occurs, when it eventually does. To make this conditioning argument precise, define two probability kernels:

$$\begin{aligned} R^\varepsilon(x, A) &= P_x\left[x^\varepsilon(\tau_C^\varepsilon) \in A | \tau_C^\varepsilon < \tau_D^\varepsilon\right], \qquad x \in \partial G,\ A \subseteq \partial C, \\ Q^\varepsilon(x, A) &= P_x\left[x^\varepsilon(\tau_D^\varepsilon) \in A | \tau_D^\varepsilon < \tau_C^\varepsilon\right], \qquad x \in \partial G,\ A \subseteq \partial D. \end{aligned} \tag{2.3}$$

Then $E_x[R^\varepsilon(x^\varepsilon(\tau_G^\varepsilon), A)]$ for $x \in \partial C$ and $A \subseteq \partial C$ is a transition probability on ∂C. It can be shown that this has the Feller property so that an invariant probability measure π^ε on ∂C exists for it:

$$\pi^\varepsilon(A) = E_{\pi^\varepsilon}\left[R^\varepsilon(x^\varepsilon(\tau_G^\varepsilon), A)\right] \quad \text{all (measurable) } A \subseteq \partial C. \tag{2.4}$$

Let

$$h_\varepsilon(y) = P_y\left[\tau_D^\varepsilon < \tau_C^\varepsilon\right], \qquad y \in \partial G,$$

be the probability of the event we want to condition on. Consider $x_0 \in \overline{C}$. If we use the strong Markov property to calculate $\mu^\varepsilon(x_0, A)$ by conditioning on the two intermediate stopping times $0 < \tau_G^\varepsilon < \tau_C^\varepsilon \wedge \tau_D^\varepsilon \le \tau_D^\varepsilon$, we find that

$$\begin{aligned} \mu^\varepsilon(x_0, A) = E_{x_0}\Big[&(1 - h_\varepsilon(x^\varepsilon(\tau_G^\varepsilon)))R^\varepsilon(x^\varepsilon(\tau_G^\varepsilon), \mu^\varepsilon(\cdot, A)) \\ &+ h_\varepsilon(x^\varepsilon(\tau_G^\varepsilon))Q^\varepsilon(x^\varepsilon(\tau_G^\varepsilon), A)\Big]. \end{aligned} \tag{2.5}$$

[We use the convenient notations

$$R(x, \phi(\cdot)) = \int \phi(y) R(x, dy) \quad \text{and} \quad R(\mu, A) = \int R(x, A)\mu(dx),$$

where ϕ is a measurable function and μ is a probability measure.]

We will see in Theorem 5.3 that if C and G are chosen as in (2.19) below, then the y and ε dependencies of h_ε can be separated asymptotically:

$$h_\varepsilon(y) = \gamma^\varepsilon \cdot (g(y) + o(1)), \tag{2.6}$$

where g is a positive continuous function on ∂G, γ^ε is independent of $y \in \partial G$ and positive (though asymptotically small as $\varepsilon \downarrow 0$) and $o(1) \to 0$ uniformly over ∂G as $\varepsilon \downarrow 0$. Note that there is really a free constant in (2.6); for any $c > 0$ we can rewrite (2.6) as

$$\gamma_c^\varepsilon \cdot (g_c(y) + o(1)/c),$$

where $\gamma_c^\varepsilon = c\gamma^\varepsilon$ and $g_c(x) = g(x)/c$. Using this,

$$1 - h_\varepsilon(y) = (1 - \gamma_c^\varepsilon) + \gamma_c^\varepsilon[1 - g_c(y) - o(1)/c].$$

According to the basic exponential leveling result of [5] (see also [12]),

$$\mu^\varepsilon(y, A) = \mu^\varepsilon(0, A) + o(1), \tag{2.7}$$

where the $o(1)$ is uniform over $y \in \partial G$. Using these facts we can write

$$\begin{aligned}E_{x_0}&[(1-h_\varepsilon(x^\varepsilon(\tau_G^\varepsilon)))R^\varepsilon(x^\varepsilon(\tau_G^\varepsilon),\mu^\varepsilon(\cdot,A))]\\&=(1-\gamma_c^\varepsilon)E_{x_0}[R^\varepsilon(x^\varepsilon(\tau_G^\varepsilon),\mu^\varepsilon(\cdot,A))]\\&\quad+\gamma_c^\varepsilon E_{x_0}[1-g_c(x^\varepsilon(\tau_G^\varepsilon))](\mu^\varepsilon(0,A)+o(1))+\gamma_c^\varepsilon\cdot o(1)/c.\end{aligned}$$

Make these substitutions in (2.5) and then integrate the initial point x_0 over ∂G with respect to the invariant measure π^ε from (2.4). This yields

$$\begin{aligned}\mu^\varepsilon(\pi^\varepsilon,A)&=(1-\gamma_c^\varepsilon)\mu^\varepsilon(\pi^\varepsilon,A)+\gamma_c^\varepsilon E_{\pi^\varepsilon}[g_c(x^\varepsilon(\tau_G^\varepsilon))Q^\varepsilon(x^\varepsilon(\tau_G^\varepsilon),A)]\\&\quad+\gamma_c^\varepsilon E_{\pi^\varepsilon}[1-g_c(x^\varepsilon(\tau_G^\varepsilon))](\mu^\varepsilon(0,A)+o(1))+\gamma_c^\varepsilon o(1)/c.\end{aligned}$$

We can choose c to eliminate the term with $1-g_c$:

$$c=c^\varepsilon=E_{\pi^\varepsilon}[g(x^\varepsilon(\tau_G^\varepsilon))];\quad\text{that is,}\quad E_{\pi^\varepsilon}[1-g_c(x^\varepsilon(\tau_G^\varepsilon))]=0.$$

With this choice we can solve for $\mu^\varepsilon(\pi^\varepsilon, A)$:

$$\mu^\varepsilon(\pi^\varepsilon,A)=\frac{1}{c^\varepsilon}\{E_{\pi^\varepsilon}[g(x^\varepsilon(\tau_G^\varepsilon))Q^\varepsilon(x^\varepsilon(\tau_G^\varepsilon),A)]+o(1)\}.$$

Although c^ε is ε-dependent, it obeys the same positive lower bound as $g(\cdot)$, so that $o(1)/c^\varepsilon = o(1)$. Thus in terms of the *averaged* exit distributions

$$\begin{aligned}\nu^\varepsilon(A)&=P_{\pi^\varepsilon}[x^\varepsilon(\tau_G^\varepsilon)\in A]=\nu^\varepsilon(\pi^\varepsilon,A),\qquad A\subseteq\partial G,\\\mu^\varepsilon(A)&=P_{\pi^\varepsilon}[x^\varepsilon(\tau_D^\varepsilon)\in A]=\mu^\varepsilon(\pi^\varepsilon,A),\qquad A\subseteq\partial D,\end{aligned}$$

we have, for all measurable $A \subseteq \partial D$,

$$\mu^\varepsilon(A)=Q^\varepsilon\left(\frac{g}{c^\varepsilon}\nu^\varepsilon,A\right)+o(1)=\int_{\partial G}Q^\varepsilon(y,A)\frac{g(y)}{c^\varepsilon}\nu^\varepsilon(dy)+o(1).\tag{2.8}$$

$1/c^\varepsilon$ is simply the normalizing constant which makes $g(\cdot)$ a probability density with respect to ν^ε:

$$c^\varepsilon=\int_{\partial G}g(x)\,d\nu^\varepsilon.$$

The exponential leveling result cited in (2.7) implies that, for any $x_0 \in D$,

$$|\mu^\varepsilon(x_0,A)-\mu^\varepsilon(A)|\to 0$$

and similarly for ν^ε. Thus for any $x_0 \in D$ and measurable $A \subseteq \partial D$, $\mu^\varepsilon(x_0, A)$ and $\mu^\varepsilon(A)$ have the same limiting behavior as $\varepsilon \downarrow 0$.

Equation (2.8) is the fundamental relationship on which the conditional approach to the exit problem is based. It expresses the exit measure μ^ε for the characteristic boundary problem in terms of the exit measure ν^ε and the conditional exit probability Q^ε. Since G has a nontangential boundary (2.2), we have a good understanding of how ν^ε behaves from studies such as [6] of the nontangential exit problem. In most cases ν^ε will converge to some limiting measure on ∂G:

$$\nu^\varepsilon \Rightarrow \nu^0.$$

In this event

$$c^\varepsilon \to c^0 = \int_{\partial G} g(x)\, d\nu^0$$

and so the asymptotic behavior of μ^ε is given by

$$\mu^\varepsilon(A) \sim Q^\varepsilon\left(\frac{g}{c^0}\nu^\varepsilon, A\right). \tag{2.9}$$

We concentrate in this paper on finding an asymptotically correct representation of Q^ε in terms of a system of stochastic differential equations. Our approach is to choose a new system of coordinates (ρ, θ) in which (1.2) takes a form somewhat like (1.5). In particular we want:

1. $\rho^\varepsilon = \rho(x^\varepsilon)$ to be (essentially) independent of $\theta^\varepsilon = \theta(x^\varepsilon)$;
2. ∂D, ∂C and ∂G to be level sets of ρ, so that the conditioning event $\tau_D^\varepsilon < \tau_C^\varepsilon$ is also independent of θ^ε;
3. the ρ^ε equation to be (essentially) linear, so that explicit calculations as in [8] can be used to study the conditional probability asymptotically as $\varepsilon \downarrow 0$.

We will now identify smooth functions $\rho = \rho(x)$ and $\theta = \theta(x)$, defined in $D \setminus C$, that satisfy these criteria.

Itô's formula tells us that $\rho^\varepsilon(t) = \rho(x^\varepsilon(t))$ will satisfy

$$d\rho^\varepsilon(t) = [\langle \nabla\rho, \mathbf{b}\rangle + \varepsilon\alpha_1]\, dt + \varepsilon^{1/2}\langle \nabla\rho, \sigma\, dw(t)\rangle,$$

where

$$\alpha_1(x) = \frac{1}{2}\sum a_{ij}(x)\frac{\partial^2\rho}{\partial x_i\, \partial x_j}(x).$$

[The functions appearing in the coefficients are all evaluated at $x^\varepsilon(t)$.] Making the random time change

$$ds = \langle \nabla\rho, a\nabla\rho\rangle\, dt$$

we can write this as

$$d\rho^\varepsilon(s) = \left[\frac{\langle \nabla\rho, \mathbf{b}\rangle}{\langle \nabla\rho, a\nabla\rho\rangle} + \mathscr{O}(\varepsilon)\right] ds + \varepsilon^{1/2}\, d\beta_1(s), \tag{2.10}$$

where $\beta_1(s)$ is a Brownian motion on the s time scale. (We would also like $\langle \nabla\rho, a\nabla\rho\rangle$ to have a positive lower bound to avoid singularities in the time change.) For (2.10) to be (essentially) linear we would like the drift in (2.10), neglecting the $\mathscr{O}(\varepsilon)$, to simplify to ρ itself. Thus we want $\rho(x)$ to satisfy

$$\rho = \frac{\langle \nabla\rho, \mathbf{b}\rangle}{\langle \nabla\rho, a\nabla\rho\rangle} \quad \text{in } D \setminus C, \tag{2.11}$$

with $\rho = 0$ on ∂D, $\rho > 0$ inside D and $\langle \nabla\rho, a\nabla\rho\rangle$ bounded below away from 0.

The function ρ which we seek is not as mysterious as it might seem. Indeed (2.11) implies that

$$W = \rho^2$$

would be a smooth function which is 0 on ∂D, positive inside D and satisfying

$$\tfrac{1}{2}\langle \nabla W, A\nabla W\rangle = \langle \nabla W, \mathbf{b}\rangle$$

or

$$H(x, -\nabla W(x)) = 0, \tag{2.12}$$

where H is the Hamiltonian

$$H(x,p) = \tfrac{1}{2}\langle p, a(x)p\rangle + \langle p, \mathbf{b}(x)\rangle.$$

Readers familiar with the large deviations theory associated with the exit problem might recognize (2.12) as a Hamilton–Jacobi equation, similar to that for the usual quasipotential function. Here it follows (similar to Theorem 3.1, Chapter 4 of [14]) that W is the *boundary quasipotential*:

$$\begin{aligned} W(x) &= V(x, \partial D) \\ &= \inf\{S_{0T}(\varphi): 0 < T < \infty,\ \varphi \in C[0,T],\ \varphi(0) = x,\ \varphi(T) \in \partial D\}, \end{aligned} \tag{2.13}$$

where $S_{0T}(\varphi) = \int_0^T L(\varphi, \dot\varphi)$ is the usual Wentzell–Freidlin action functional. Thus the existence problem of the desired function ρ becomes that of establishing various regularity properties of $W(x) = V(x, \partial D)$. It is not our purpose here to address this issue but to proceed with the analysis of Q^ε assuming that these properties do hold. Specifically we assume the following.

HYPOTHESES OF W-REGULARITY. There exists a boundary strip

$$D_0 = \{x \in \overline{D}: \operatorname{dist}(x, \partial D) < \delta\},$$

some $\delta > 0$, in which the boundary quasipotential $W(x) = V(x, \partial D)$ is smooth, satisfying (2.12), and that $\rho(x) = \sqrt{W(x)}$ is also smooth (on ∂D in particular). In addition,

$$\langle \nabla\rho, a\nabla\rho\rangle = \langle \nabla W, a\nabla W\rangle / 4W$$

has a positive lower bound in D_0.

We will be able to verify these hypotheses using an explicit expression for W in the examples of Section 6 below.

We should point out that the above hypotheses can *only* be satisfied for cases of characteristic boundary, (1.4). Indeed, since $\rho = 0$ on ∂D but $\|\nabla\rho\| \neq 0$, $\nabla\rho$ is a nonzero multiple of $\mathbf{n}$ on ∂D. From (2.11) we have that

$$\langle \nabla\rho, \mathbf{b}\rangle = \rho\langle \nabla\rho, a\nabla\rho\rangle = 0 \quad \text{on } \partial D,$$

which implies (1.4). In fact, with ρ playing the role of distance to ∂D, (2.11) also implies the first order degeneracy condition (2.9) of [7] and $b(s) > 0$ in (2.6) of [17]. Thus some condition along these lines is necessary for our hypotheses to be satisfied.

Next we need to finish our choice of the coordinate system by selecting θ. Since ρ is a coordinate in the normal direction, θ should be a tangential coordinate. First parameterize ∂D by θ in some nice way (arclength with respect to some reference point, say). Since ∂D is one-dimensional, we can think of $\theta \in \mathbb{R}$ with periodic identification (mod S where S is the arclength of ∂D). We want to extend this parameterization of ∂D to a function $\theta(x)$ from D_0 into $\mathbb{R}$ (mod S) such that

$$\langle \nabla\rho, a\nabla\theta\rangle = 0 \quad \text{in } D_0. \tag{2.14}$$

This is the same as saying that $\theta(\cdot)$ should be constant along the solutions of

$$\dot{x} = a(x)\nabla\rho(x); \qquad x(0) \in \partial D.$$

Our hypotheses above can be shown to imply that D_0 is simply covered by these paths, so that $\theta(x(t)) = \theta(x(0))$ determines a C^2 extension of $\theta(x)$ to D_0 satisfying (2.14), with $\|\nabla\theta\| \neq 0$ in D_0 as well. Notice that (2.14) implies

$$\left(\frac{\partial(\rho,\theta)}{\partial x}\right) a(x) \left(\frac{\partial(\rho,\theta)}{\partial x}\right)^T = \begin{bmatrix} \sigma_1^2(x) & 0 \\ 0 & \sigma_2^2(x) \end{bmatrix}, \tag{2.15}$$

where

$$\sigma_1 = \langle \nabla\rho, a\nabla\rho\rangle^{1/2}, \qquad \sigma_2 = \langle \nabla\theta, a\nabla\theta\rangle^{1/2}$$

are both smooth positive functions in D_0. In particular $x \mapsto (\rho, \theta)$ has nonvanishing Jacobian in D_0. Thus (ρ, θ) does provide a suitable coordinate system in D_0. We will use x and (ρ, θ) interchangeably in referring to points in D_0. For instance,

$$\sigma_i(x) = \sigma_i(\rho, \theta), \qquad i = 1, 2,$$

and $x^\varepsilon(t) = (\rho^\varepsilon(t), \theta^\varepsilon(t))$.

The significance of (2.14) is that the off-diagonal elements in (2.15) are 0. We can define a new two-dimensional Brownian motion by

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix} \frac{\partial(\rho,\theta)}{\partial x} \sigma(x^\varepsilon)\, dw(t).$$

The original system (1.2) can now be expressed as follows in our new coordinate system:

$$\begin{aligned} d\rho^\varepsilon(t) &= \left[\langle \nabla\rho, \mathbf{b}\rangle + \varepsilon\alpha_1\right] dt + \varepsilon^{1/2}\sigma_1\, dw_1(t), \\ d\theta^\varepsilon(t) &= \left[\langle \nabla\theta, \mathbf{b}\rangle + \varepsilon\alpha_2\right] dt + \varepsilon^{1/2}\sigma_2\, dw_2(t). \end{aligned} \tag{2.16}$$

The function α_1 was defined previously. Its counterpart in the θ^ε equation is

$$\alpha_2 = \frac{1}{2}\sum_{i,j} a_{ij}\frac{\partial^2\theta}{\partial x_i\, \partial x_j},$$

which is also continuous in D_0.

The random time change mentioned above is

$$ds = \sigma_1^2(\rho^\varepsilon, \theta^\varepsilon)\, dt.$$

On this time scale,

$$\beta_i(s) = \int_0^t \alpha_1(\rho^\varepsilon, \theta^\varepsilon)\, dw_i(t) \tag{2.17}$$

defines a two-dimensional Brownian motion, $(\beta_1(s), \beta_2(s))$. Equation (2.11) says that $\langle \nabla\rho, \mathbf{b}\rangle = \rho\sigma_1^2$. Thus in our new coordinates and on the new time scale, (1.2) becomes

$$\begin{aligned} d\rho^\varepsilon(s) &= \left[\rho^\varepsilon + \varepsilon \frac{\alpha_1}{\sigma_1^2}(\rho^\varepsilon, \theta^\varepsilon)\right] ds + \varepsilon^{1/2}\, d\beta_1(s), \\ d\theta^\varepsilon(s) &= \left[\frac{\langle \nabla\theta, \mathbf{b}\rangle}{\sigma_1^2} + \varepsilon \frac{\alpha_2}{\sigma_1^2}\right] ds + \varepsilon^{1/2} \frac{\sigma_2}{\sigma_1}\, d\beta_2(s). \end{aligned} \tag{2.18}$$

We have $\partial D = \{\rho = 0\}$ by definition of ρ. We also want ∂C and ∂G to be level sets of ρ, so we take

$$C = D \setminus \{\rho \le R\} \quad \text{and} \quad G = D \setminus \{\rho \le \rho_0\}, \tag{2.19}$$

where $0 < \rho_0 < R$ are chosen so that $\{x: 0 \le \rho(x) \le R\} \subseteq D_0$. Thus

$$\partial C = \{\rho = R\} \quad \text{and} \quad \partial G = \{\rho = \rho_0\},$$

and will inherit smoothness from ρ. We observe that the outward normal $\mathbf{n}$ on ∂G will be $\mathbf{n} = -\nabla\rho/\|\nabla\rho\|$, so that by (2.11),

$$\langle \mathbf{b}, \mathbf{n}\rangle = -\frac{\langle \nabla\rho, \mathbf{b}\rangle}{\|\nabla\rho\|} = -\rho_0 \frac{\langle \nabla\rho, a\nabla\rho\rangle}{\|\nabla\rho\|} < 0$$

on ∂G. Thus (2.2) is indeed satisfied for this choice of G.

Any $x^\varepsilon(0) = x_0 \in \partial G$ corresponds to an initial ρ value of $\rho^\varepsilon(0) = \rho_0$. The stopping times on the s-time scale which correspond to τ_D^ε and τ_C^ε, respectively, are

$$\sigma_0^\varepsilon = \inf\{s > 0: \rho^\varepsilon(s) = 0\} \quad \text{and} \quad \sigma_R^\varepsilon = \inf\{s > 0: \rho^\varepsilon(s) = R\}. \tag{2.20}$$

We will denote their infimum by

$$\sigma_{0,R}^\varepsilon = \sigma_0^\varepsilon \wedge \sigma_R^\varepsilon.$$

For a path with $\sigma_0^\varepsilon < \sigma_R^\varepsilon$ we recover τ_D^ε by

$$\tau_D^\varepsilon = \int_0^{\sigma_0^\varepsilon} \frac{1}{\sigma_1^2(\rho^\varepsilon, \theta^\varepsilon)}\, ds. \tag{2.21}$$

The coefficient functions in (2.18) are only well-defined for (ρ, θ) values corresponding to points in D_0. In particular for $s > \sigma_{0,R}^\varepsilon$, (2.18) and even the construction of $\beta_i(s)$ breaks down. However in the following sections we will want to consider (2.18) for all $s < \infty$. We will accommodate this with two conventions. First we consider all the coefficient functions appearing in (2.18) to be extended to all $\rho \in \mathbb{R}$, $\theta \in \mathbb{R}$ by defining

$$\begin{aligned} \sigma_i(\rho,\theta) &= \sigma_i([\rho]_{0,R},\theta), \qquad \alpha_i(\rho,\theta) = \alpha_i([\rho]_{0,R},\theta), \\ \langle \nabla\theta, \mathbf{b}\rangle(\rho,\theta) &= \langle \nabla\theta, \mathbf{b}\rangle([\rho]_{0,R},\theta), \end{aligned} \tag{2.22}$$

where

$$[\rho]_{0,R} = (\rho \wedge R) \vee 0.$$

In this way the coefficient functions in (2.18) are defined for all values of ρ and θ, S-periodic in θ, bounded and Lipshitz continuous with σ_i uniformly positive. Second, instead of considering the $\beta_i(s)$ to be constructed from the original $w(t)$ of (1.2), we assume that we are given a priori a probability space $(\Omega, \mathscr{F}, P)$ with filtration $\{\mathscr{F}_s\colon 0 \le s < \infty\}$ and an adapted two-dimensional Brownian motion $(\beta_1(s), \beta_2(s))$. With these conventions, given any initial values (ρ_0, θ_0), (2.18) has a unique solution pair $(\rho^\varepsilon(s), \theta^\varepsilon(s))$ defined for all $0 \le s < \infty$. Although these conventions affect the distribution of $(\rho^\varepsilon, \theta^\varepsilon)$ for $s > \sigma^\varepsilon_{0,R}$, the distribution for $s \le \sigma^\varepsilon_{0,R}$ is unaffected. This is fine for our purposes, since our goal is to study

$$\begin{aligned} Q^\varepsilon(x_0, A) &= P\big[(\rho^\varepsilon(\sigma_0^\varepsilon), \theta^\varepsilon(\sigma_0^\varepsilon)) \in A | \sigma_0^\varepsilon < \sigma_R^\varepsilon\big] \\ &= P\big[(0, \theta^\varepsilon(\sigma_0^\varepsilon)) \in A | \sigma_0^\varepsilon < \sigma_R^\varepsilon\big], \end{aligned} \tag{2.23}$$

where

$$(\rho^\varepsilon(0), \theta^\varepsilon(0)) = (\rho_0, \theta_0) \quad \text{corresponds to } x_0 \in \partial G.$$

Throughout the remainder of this paper P, $(\Omega, \mathscr{F})$, $\{\mathscr{F}_s\}$ and $\beta_i(s)$ will refer to those just discussed.

3. Detailed conditional analysis of a simplified process. In this section we will make two simplifications which allow some explicit calculations to be used for the analysis. We will see that these simplifications do not affect the asymptotic behavior that we are interested in.

The first simplification is to drop the $\mathscr{O}(\varepsilon)\,ds$ terms from (2.18). Theorem 4.3 will justify the omission of these terms. The idea will be to use a Girsanov transformation, changing the P and $\beta_i(s)$ of Section 2 to a probability measure $P^\triangleright$ and associated Brownian pair $\beta_i^\triangleright(s)$. The construction of $P^\triangleright$ and $\beta_i^\triangleright$ will be discussed at the end of Section 4. For purposes of this section we consider $P^\triangleright$ to be any probability measure on $(\Omega, \mathscr{F})$ and $(\beta_i^\triangleright(s), \beta_2^\triangleright(s))$ to be any $\{\mathscr{F}_s\}$ adapted two-dimensional Brownian motion with respect to $P^\triangleright$. In particular we allow them to be ε-dependent, though we do not include that dependence in the notation. Thus the equations that we consider in this section are

$$\begin{aligned} d\rho^\varepsilon(s) &= \rho^\varepsilon(s)\,ds + \varepsilon^{1/2}\,d\beta_1^\triangleright(s), \qquad \rho^\varepsilon(0) = \rho_0, \\ d\theta^\varepsilon(s) &= \frac{\langle \nabla\theta, \mathbf{b}\rangle}{\sigma_1^2}\,ds + \varepsilon^{1/2}\frac{\sigma_2}{\sigma_1}\,d\beta_2^\triangleright(s), \qquad \theta^\varepsilon(0) = \theta_0. \end{aligned} \tag{3.1}$$

We want to understand the asymptotic ($\varepsilon \downarrow 0$) behavior of (3.1) when considered with respect to the conditional probability

$$\bar{P}^{\triangleright}[\cdot] = P^{\triangleright}[\cdot | \sigma_0^\varepsilon < \sigma_R^\varepsilon].$$

The stopping times remain as defined in (2.20).

The second simplification is to condition only on $\{\sigma_0^\varepsilon < \infty\}$ rather than the more restrictive event $\{\sigma_0^\varepsilon < \sigma_R^\varepsilon\}$. Proposition 3.2 will imply that this simplification does not affect the asymptotic behavior either. Following this we want to consider (3.1) with respect to

$$P^\circ[\cdot] = P^{\triangleright}[\cdot | \sigma_0^\varepsilon < \infty].$$

Define

$$h_\varepsilon(y) = P_y^{\triangleright}[\sigma_0^\varepsilon < \infty]. \tag{3.2}$$

We can calculate h_ε explicitly by solving

$$\frac{\varepsilon}{2} h_\varepsilon''(y) + y h_\varepsilon'(y) = 0; \qquad h_\varepsilon(0) = 1, \qquad h_\varepsilon(+\infty) = 0.$$

The change of variables $x = \varepsilon^{-1/2} y$ eliminates ε from the equation:

$$\tfrac{1}{2} h''(x) + x h'(x) = 0.$$

The standard complementary error function

$$h(x) = \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \tag{3.3}$$

gives the desired solution:

$$h_\varepsilon(y) = h(\varepsilon^{-1/2} y). \tag{3.4}$$

We can perform the conditioning on $\sigma_0^\varepsilon < \infty$ rather explicitly in terms of h_ε (this is the h-transform operation of Doob):

$$P^\circ[A] = P^{\triangleright}[A; \sigma_0^\varepsilon < \infty] / h_\varepsilon(\rho_0) = E^{\triangleright}[\zeta_\infty; A],$$

where

$$\zeta_\infty = \frac{dP^\circ}{dP^{\triangleright}} = \frac{1_{\sigma_0^\varepsilon < \infty}}{h_\varepsilon(\rho_0)}. \tag{3.5}$$

Notice that

$$\zeta_s = E^{\triangleright}[\zeta_\infty | \mathscr{F}_s] = \frac{h_\varepsilon(\rho^\varepsilon(s \wedge \sigma_0^\varepsilon))}{h_\varepsilon(\rho_0)} \tag{3.6}$$

is a $P^{\triangleright}$-martingale. Applying Itô's lemma we have, for finite $s \le \sigma_0^\varepsilon$,

$$d \log(h_\varepsilon(\rho^\varepsilon(s))) = -\frac{\varepsilon}{2} \left[\frac{h_\varepsilon'}{h_\varepsilon}(\rho^\varepsilon(s)) \right]^2 ds + \varepsilon^{1/2} \frac{h_\varepsilon'}{h_\varepsilon}\, d\beta_1^{\triangleright}(s).$$

Thus we can write ζ_s as an exponential $P^{\triangleright}$-martingale:

$$\zeta_s = \exp\left[\varepsilon^{1/2}\int_0^{s\wedge\sigma_0^\varepsilon}\frac{h'_\varepsilon}{h_\varepsilon}\,d\beta^{\triangleright} - \frac{\varepsilon}{2}\int_0^{s\wedge\sigma_0^\varepsilon}\left(\frac{h'_\varepsilon}{h_\varepsilon}\right)^2 ds\right]. \tag{3.7}$$

Since we know $E^{\triangleright}[\zeta_\infty] = 1$, our next result follows from the usual Girsanov theorem.

PROPOSITION 3.1. *Define* $\beta^\circ(s) = (\beta_1^\circ(s), \beta_2^\circ(s))$, *where* $\beta_2^\circ(s) = \beta_2^{\triangleright}(s)$ *is unchanged from* (3.1) *and*

$$\beta_1^\circ(s) = \beta_1^{\triangleright}(s) - \varepsilon^{1/2}\int_0^{s\wedge\sigma_0^\varepsilon}\frac{h'_\varepsilon}{h_\varepsilon}(\rho^\varepsilon(s))\,ds.$$

Then β° *is a two-dimensional Brownian motion with respect to* P°. (3.1) *is described by*

$$\begin{aligned} d\rho^\varepsilon(s) &= \left[\rho^\varepsilon(s) + \varepsilon\frac{h'_\varepsilon}{h_\varepsilon}(\rho^\varepsilon(s))1_{s\le\sigma_0^\varepsilon}\right]ds + \varepsilon^{1/2}\,d\beta_1^\circ(s), \\ d\theta^\varepsilon(s) &= \frac{\langle\nabla\theta,\mathbf{b}\rangle}{\alpha_1^2}\,ds + \varepsilon^{1/2}\frac{\sigma_2}{\sigma_1}\,d\beta_1{}^\circ(s), \end{aligned} \tag{3.8}$$

with respect to P°.

The effect of conditioning on $\sigma_0^\varepsilon < \infty$ is therefore to replace the drift coefficient in the first equation of (3.1) with the *conditional drift*,

$$b_\varepsilon^\circ(y) = y + \varepsilon\frac{h'_\varepsilon}{h_\varepsilon}(y)$$

for $s \le \sigma_0^\varepsilon$. The θ^ε equation remains unchanged. Thus for $s \le \sigma_0^\varepsilon$, (3.8) is

$$\begin{aligned} d\rho^\varepsilon(s) &= b_\varepsilon^\circ(\rho^\varepsilon)\,ds + \varepsilon^{1/2}\,d\beta_1^\circ(s), \\ d\theta^\varepsilon(s) &= \frac{\langle\nabla\theta,\mathbf{b}\rangle}{\sigma_1^2}\,ds + \varepsilon^{1/2}\frac{\sigma_2}{\sigma_1}\,d\beta_2^\circ(s). \end{aligned} \tag{3.9}$$

To see the structure of b_ε° more clearly, define d^* by

$$x + \frac{h'(s)}{h(x)} = -x - d^*(x),$$

that is,

$$d^*(x) = -\frac{h'(x)}{h(x)} - 2x = \frac{e^{-x^2}}{\int_x^\infty e^{-s^2}\,ds} - 2x.$$

Then

$$b_\varepsilon^\circ(y) = y + \varepsilon\frac{h'_\varepsilon(y)}{h'_\varepsilon(y)} = y + \varepsilon^{1/2}\frac{h'(\varepsilon^{-1/2}y)}{h(\varepsilon^{-1/2}y)} \tag{3.10}$$
$$= -\big(y + \varepsilon^{1/2}d^*(\varepsilon^{-1/2}y)\big).$$

The inequalities [1] (7.1.13) tell us that

$$x + \sqrt{x^2 + 4/\pi} \le \frac{-h'(x)}{h(x)} \le x + \sqrt{x^2 + 2}\,.$$

This means that

$$0 \le \sqrt{x^2 + 4/\pi} - x \le d^*(x) \le \sqrt{x^2 + 2} - x,$$

and so

$$0 \le \sqrt{y^2 + 4\varepsilon/\pi} - y \le \varepsilon^{1/2}d^*(\varepsilon^{-1/2}y) \le \sqrt{y^2 + 2\varepsilon} - y. \tag{3.11}$$

Notice that $\varepsilon^{1/2}d^*(\varepsilon^{-1/2}y)$ is $\mathscr{O}(\varepsilon)$ uniformly on any compact subset of $(0,\infty)$, but *not* at 0. Indeed, $\varepsilon^{1/2}d^*(0) = \varepsilon^{1/2}2/\sqrt{\pi}$, which is only $\mathscr{O}(\varepsilon^{1/2})$. The leftmost inequality in (3.11) gives us a lower bound:

$$b_\varepsilon^\circ(y) \le -y \quad \text{all } y \in [0,\infty). \tag{3.12}$$

Using this in (3.10) yields

$$-\varepsilon\frac{h'_\varepsilon}{h_\varepsilon} \le -2b_\varepsilon^\circ. \tag{3.13}$$

We also have the following uniform approximations:

$$b_\varepsilon^\circ(y) = -y + \mathscr{O}(\varepsilon) \quad \text{on compacts } \subseteq (0,\infty), \tag{3.14}$$

$$b_\varepsilon^\circ(y) = -y + \mathscr{O}(\varepsilon^{1/2}) \quad \text{on compacts } \subseteq [0,\infty). \tag{3.15}$$

According to (3.10), b_ε° is the reversal of the ρ^ε drift from (3.1) plus a boundary layer correction given by the d^* term. Theorem 4.3 below will say that $\mathscr{O}(\varepsilon)$ perturbations to the drift are asymptotically negligible. The d^* term is $\mathscr{O}(\varepsilon)$ away from the boundary, but only $\mathscr{O}(\varepsilon^{1/2})$ in the boundary layer and *not* negligible. Indeed it turns out that the distribution of σ_0^ε for the diffusion

$$dy^\varepsilon(s) = -y^\varepsilon(s)\,ds + \varepsilon^{1/2}\,d\beta_1^\circ(s); \qquad y^\varepsilon(0) = \rho_0,$$

is described asymptotically by

$$\sigma_0^\varepsilon - \log\big(\rho_0\varepsilon^{-1/2}\big) \Rightarrow \mathscr{K},$$

where $\mathscr{K}$ is a random variable with density $(2/\sqrt{\pi})\exp(-s - e^{-2s}) = (1/\sqrt{\pi})e^s q(s)$. Thus dropping the $\varepsilon^{1/2}d^*(\varepsilon^{-1/2}y)$ from b_ε^0 produces a different asymptotic limit law for σ_0^ε than that given by Proposition 3.3 below.

Next we want to establish the asymptotic equivalence of $\overline{P}^\triangleright$ and P°. Since $\{\sigma_0^\varepsilon < \sigma_R^\varepsilon\} \subseteq \{\sigma_0^\varepsilon < \infty\} \cup \{\sigma_R^\varepsilon = \infty\}$ and $P^\triangleright[\sigma_R^\varepsilon = \infty] = 0$, we have that $\overline{P}^\triangleright \ll P^\circ$

and

$$\frac{d\bar{P}^{\triangleright}}{dP^{\circ}} = \frac{1_{\sigma_0^\varepsilon < \sigma_R^\varepsilon}}{P^{\circ}[\sigma_0^\varepsilon < \sigma_R^\varepsilon]}.$$

Using the strong Markov property,

$$\begin{aligned} 1 - P^{\circ}[\sigma_0^\varepsilon < \sigma_R^\varepsilon] &= P^{\circ}[\sigma_R^\varepsilon < \sigma_0^\varepsilon] \\ &= \frac{P^{\triangleright}[\sigma_R^\varepsilon < \sigma_0^\varepsilon < \infty]}{P^{\triangleright}[\sigma_0^\varepsilon < \infty]} \\ &\le \frac{P_R^{\triangleright}[\sigma_0^\varepsilon < \infty]}{P_{\rho_0}^{\triangleright}[\sigma_0^\varepsilon < \infty]} \\ &= \frac{h(\varepsilon^{-1/2}R)}{h(\varepsilon^{-1/2}\rho_0)}. \end{aligned}$$

Now, it is clear from the formula $e^{x^2}h(x) = \int_0^\infty e^{-v^2 - vx}\,dv$ that $e^{x^2}h(x)$ is decreasing. This implies that

$$0 \le \frac{h(\varepsilon^{-1/2}R)}{h(\varepsilon^{-1/2}\rho_0)} \le e^{-(R^2-\rho_0^2)/\varepsilon}.$$

Hence

$$P^{\circ}[\sigma_0^\varepsilon < \sigma_R^\varepsilon] \to 1. \tag{3.16}$$

It follows that for any $r \ge 1$,

$$E^{\circ}\left[\left|1 - \frac{d\bar{P}^{\triangleright}}{dP^{\circ}}\right|^r\right] \to 0.$$

This proves the following result which will justify our assertion that conditioning on $\{\sigma_0^\varepsilon < \infty\}$ instead of $\{\sigma_0^\varepsilon < \sigma_R^\varepsilon\}$ does not affect the $\varepsilon \downarrow 0$ asymptotics.

PROPOSITION 3.2. *$\bar{P}^{\triangleright} \ll P^{\circ}$ and, for any $r \ge 1$,*

$$E^{\circ}\left[\left|1 - \frac{d\bar{P}^{\triangleright}}{dP^{\circ}}\right|^r\right] \to 0$$

as $\varepsilon \downarrow 0$.

We turn now to the conditional distribution of σ_0^ε.

PROPOSITION 3.3. *The distribution of σ_0^ε with respect to either $\bar{P}^{\triangleright}$ or P° is such that*

$$\sigma_0^\varepsilon - \log(\varepsilon^{-1/2}\rho_0) \Rightarrow \mathscr{E} \quad \text{as } \varepsilon \downarrow 0,$$

where $\mathscr{E}$ *is a random variable with density given by*

$$q(s) = 2e^{-(2s+e^{-2s})}.$$

This was proven in [8], where Proposition 3.1 is the case of P° and Proposition 3.2 implies that of $\overline{P}^{\triangleright}$. (Note that B in [8] has the value 1 here and our ρ_0 had the value $\frac{1}{4}$ there; so $y_0 = \varepsilon^{-1/2}\rho_0$ in the propositions there.) Also note that the assertion for $\overline{P}^{\triangleright}$ follows from that for P^0 using Proposition 3.2 above.

Next we want to establish some bounds on exponential moments of σ_0^ε and $\sigma_{0,R}^\varepsilon$. For this purpose we are interested in positive solutions to

$$\frac{\varepsilon}{2}u_\varepsilon'' + yu_\varepsilon' + \lambda u_\varepsilon = 0. \tag{3.17}$$

The change of variable $x = \varepsilon^{-1/2}y$, $u_\varepsilon(y) = u(x)$ eliminates ε from the equation, yielding

$$\tfrac{1}{2}u'' + xu' + \lambda u = 0. \tag{3.18}$$

The transformation

$$u(x) = e^{-x^2/2}w(\sqrt{2}\,x) \tag{3.19}$$

now puts the equation in the form

$$w''(x) - \left(\tfrac{1}{4}x^2 + a\right)w(x) = 0, \qquad a = \tfrac{1}{2} - \lambda. \tag{3.20}$$

The solutions of this are parabolic cylinder functions, as described in Chapter 19 of [1]. For $\lambda < 1$ the standard solution $U(a, x)$ has a closed form expression given in [1] (19.5.3). Taking

$$w_1(x) = 2^{(\lambda-1)/2}\Gamma(1-\lambda)U(a,x)$$

leads to the following solution of (3.18):

$$u_1(x) = \int_0^\infty s^{-\lambda}e^{-(x+s)^2}\,ds. \tag{3.21}$$

We note that u_1 is everywhere positive. We will need its value at 0:

$$\begin{aligned} u_1(0) &= \int_0^\infty s^{-\lambda}e^{-s^2}\,ds = \frac{1}{2}\int_0^\infty v^{-(\lambda+1)/2}e^{-v}\,dv \\ &= \frac{1}{2}\Gamma\left(\frac{1-\lambda}{2}\right). \end{aligned} \tag{3.22}$$

The second standard solution of (3.20) is

$$w_2(x) = V(a,x).$$

From [1] (19.3.6),

$$w_2(0) = \frac{2^{(1/2)a+(1/4)}\sin\left(\pi\left(\frac{3}{4} - \frac{1}{2}a\right)\right)}{\Gamma\left(\frac{3}{4} - \frac{1}{2}a\right)} > 0$$

for $|\lambda| < 1$. The asymptotic formula [1] (19.8.1),

$$w_2(x) \sim \sqrt{\frac{2}{\pi}}\, e^{x^2/4} x^{a-(1/2)}\big(1 + \mathscr{O}(x^{-2})\big) \quad \text{as } x \to +\infty, \tag{3.23}$$

implies that $w_2(x) > 0$ for all large x. Thus if $w_2(x) < 0$ for some $x > 0$, then w_2 would have at least two distinct positive zeros and the Sturm comparison theorem [4] would imply that w_1 also has a positive zero, which it does not. Thus $w_2(x) \geq 0$ for all $x \geq 0$. If $w_2(x^*) = 0$ for some $x^* > 0$, then $w_2'(x^*) = 0$ as well, implying $w_2 \equiv 0$ which is also false. Thus $w_2(x) > 0$ for all $x \geq 0$, provided $|\lambda| < 1$. The solution of (3.18) related to w_2 by (3.19) will be denoted u_2. It follows from (3.23) that

$$u_2(x) \sim \sqrt{\frac{2}{\pi}}\, \big(\sqrt{2}\, x\big)^{-\lambda} \quad \text{as } x \to +\infty. \tag{3.24}$$

With these facts we proceed to establish the desired exponential moments.

PROPOSITION 3.4. *For any* $0 < \lambda < 1$,

$$\limsup_{\varepsilon \downarrow 0} E^{\triangleright}\big[e^{\lambda \sigma_{0,R}^{\varepsilon}}\big] < \infty.$$

PROOF. Let $u_\varepsilon(y) = u_2(\varepsilon^{-1/2} y)$. By (3.17), $e^{\lambda s} u_\varepsilon(\rho^\varepsilon(s))$ is a nonnegative martingale. Using Fatou's lemma,

$$\begin{aligned} &E^{\triangleright}\big[e^{\lambda \sigma_{0,R}^{\varepsilon}} \cdot u_\varepsilon\big(\rho^\varepsilon(\sigma_{0,R}^{\varepsilon})\big)\big] \\ &\quad \leq \liminf_{T \to \infty} E^{\triangleright}\big[e^{\lambda(\sigma_{0,R}^{\varepsilon} \wedge T)} u_\varepsilon\big(\rho^\varepsilon(\sigma_{0,R}^{\varepsilon} \wedge T)\big)\big] = u_\varepsilon(\rho_0). \end{aligned}$$

Therefore, since $\rho^\varepsilon(\sigma_{0,R}^{\varepsilon}) = 0$ or R,

$$E^{\triangleright}\big[e^{\lambda \sigma_{0,R}^{\varepsilon}}\big] \leq \frac{u_\varepsilon(\rho_0)}{u_\varepsilon(0) \wedge u_\varepsilon(R)}.$$

Now $u_\varepsilon(0) = w_2(0) > 0$ and is independent of ε. By (3.24), $u_\varepsilon(R) = u_2(\varepsilon^{-1/2} R) \to 0$ as $\varepsilon \downarrow 0$ and so $u_\varepsilon(0) \wedge u_\varepsilon(R) = u_\varepsilon(R)$ for all sufficiently small ε. Thus

$$\begin{aligned} E^{\triangleright}\big[e^{\lambda \sigma_{0,R}^{\varepsilon}}\big] &\leq \frac{u_\varepsilon(\rho_0)}{u_\varepsilon(R)} \\ &= \frac{u_2(\varepsilon^{-1/2}\rho_0)}{u_2(\varepsilon^{-1/2} R)} \\ &\to \left(\frac{\rho_0}{R}\right)^{-\lambda} \quad \text{as } \varepsilon \downarrow 0, \end{aligned}$$

the last line following from (3.24). This proves the proposition. □

COROLLARY 3.5. *For any constant* $c > 0$,

$$E^{\rhd}[e^{\varepsilon c \sigma_{0,R}^{\varepsilon}}] \to 1 \quad \textit{and} \quad E^{\rhd}[\varepsilon \sigma_{0,R}^{\varepsilon}] \to 0 \quad \textit{as } \varepsilon \downarrow 0.$$

PROOF. Let $K = \limsup E^{\rhd}[e^{\sigma_{0,R}^{\varepsilon}/2}]$. Using the inequality $E^{\rhd}[e^{\varepsilon c \sigma}] \le E^{\rhd}[e^{p\varepsilon c\sigma}]^{1/p}$, $1 \le p < \infty$, with $1/p = 2\varepsilon c$ gives

$$\begin{aligned} 1 &\le E^{\rhd}[e^{\varepsilon c \sigma_{0,R}^{\varepsilon}}] \le E^{\rhd}[e^{\sigma_{0,R}^{\varepsilon}/2}]^{2\varepsilon c} \\ &\le (K+1)^{2\varepsilon c} \qquad [\text{for all sufficiently small } \varepsilon] \\ &\to 1. \end{aligned}$$

This proves the first assertion. For the second,

$$0 \le E^{\rhd}[\varepsilon \sigma_{0,R}^{\varepsilon}] \le E^{\rhd}[e^{\varepsilon \sigma_{0,R}^{\varepsilon}} - 1] \to 0. \qquad \square$$

PROPOSITION 3.6. *For any constant* $c > 0$,

$$\limsup_{\varepsilon \downarrow 0} E^{\circ}\Big[e^{c\sigma_0^{\varepsilon}/\log(\varepsilon^{-1/2})}\Big] < \infty.$$

PROOF. Let $\lambda = c/\log(\varepsilon^{-1/2})$. Then $\lambda \to 0$ as $\varepsilon \downarrow 0$. In particular $\lambda < 1$ for all ε sufficiently small. Take $u_\varepsilon(y) = u_1(\varepsilon^{-1/2}y)$. Using Fatou's lemma on the martingale $e^{\lambda s}u_\varepsilon(\rho^\varepsilon(s))$,

$$\begin{aligned} &E^{\rhd}\Big[e^{\lambda\sigma_0^\varepsilon}u_\varepsilon(\rho^\varepsilon(\sigma_0^\varepsilon)); \sigma_0^\varepsilon < \infty\Big] \\ &\quad \le \liminf_{T\to\infty} E^{\rhd}\Big[e^{\lambda(\sigma_0^\varepsilon \wedge T)}u_\varepsilon(\rho^\varepsilon(\sigma_0^\varepsilon \wedge T))\Big] = u_\varepsilon(\rho_0). \end{aligned}$$

Therefore

$$E^{\rhd}\Big[e^{\lambda\sigma_0^\varepsilon}; \sigma_0^\varepsilon < \infty\Big] \le u_\varepsilon(\rho_0)/u_\varepsilon(0),$$

and so

$$E^{\circ}[e^{\lambda\sigma_0^\varepsilon}] \le \frac{1}{u_\varepsilon(0)}\frac{u_\varepsilon(\rho_0)}{h_\varepsilon(\rho_0)}. \tag{3.25}$$

By (3.22), $u_\varepsilon(0) = u_1(0) = \frac{1}{2}\Gamma((1-\lambda)/2) \to \Gamma(\frac{1}{2})/2 > 0$ as $\varepsilon \downarrow 0$. For the other term in (3.25), check that

$$\begin{aligned} \frac{u_1(x)}{h(x)} &= \frac{\sqrt{\pi}}{2}\frac{\int_0^\infty s^{-\lambda}e^{-s^2-2sx}\,ds}{\int_0^\infty e^{-s^2-2sx}\,ds} \\ &= \frac{\sqrt{\pi}}{2}x^\lambda\left[\frac{\int_0^\infty v^{-\lambda}e^{-(v/x)^2}e^{-2v}\,dv}{\int_0^\infty e^{-(v/x)^2}e^{-2v}\,dv}\right]. \end{aligned}$$

As $x \to \infty$ and $\lambda = c/\log(\varepsilon^{-1/2}) \to 0$ (simultaneously) the ratio of integrals in the last expression above converges to 1. With $x_0 = \varepsilon^{-1/2}\rho_0$ in particular we have

$$\frac{u_\varepsilon(\rho_0)}{h_\varepsilon(\rho_0)} = \frac{u_1(x_0)}{h(x_0)} \sim \frac{\sqrt{\pi}}{2} x_0^\lambda$$
$$= \frac{\sqrt{\pi}}{2}\left(\varepsilon^{-1/2}\rho_0\right)^{c/\log(\varepsilon^{-1/2})} \to \frac{\sqrt{\pi}}{2} e^c \quad \text{as } \varepsilon \downarrow 0.$$

Using this in (3.25) proves the proposition. □

COROLLARY 3.7. *For any constant c, $E^\circ[e^{\varepsilon c \sigma_0^\varepsilon}] \to 1$ and $E^\circ[\varepsilon \sigma_0^\varepsilon] \to 0$ as $\varepsilon \downarrow 0$.*

PROOF. This is the same as Corollary 3.5, using $1/p = \varepsilon \log(\varepsilon^{-1/2})$. □

The last issue that we want to consider in this section is the behavior of integrals such as

$$\int_0^{\sigma_0^\varepsilon} f(\rho^\varepsilon, \theta^\varepsilon)\, ds, \tag{3.26}$$

when considered with respect to P°. It follows from (3.9) and (3.14) that on bounded time intervals $(\rho^\varepsilon, \theta^\varepsilon)$ converges to (ρ^*, θ^*), where

$$\dot{\rho}^*(s) = -\rho^*(s), \qquad \rho^*(0) = \rho_0,$$
$$\dot{\theta}^*(s) = \frac{\langle \nabla\theta, \mathbf{b}\rangle}{\sigma_1^2}(\theta^*(s)), \qquad \theta^*(0) = \theta_0.$$

To be precise, it follows that for any $T < \infty$,

$$\sup_{[0,T]} \|(\rho^\varepsilon(\cdot), \theta^\varepsilon(\cdot)) - (\rho^*(\cdot), \theta^*(\cdot))\| \to 0 \quad \text{in } P^\circ\text{-probability}. \tag{3.27}$$

Because $\sigma_0^\varepsilon \to \infty$, we anticipate that (3.26) approaches

$$\int_0^\infty f(\rho^*, \theta^*)\, ds. \tag{3.28}$$

Since $\rho^*(s) = e^{-s}\rho_0$, (3.28) will be finite if $f(\rho, \theta) = \mathscr{O}(\rho)$ as $\rho \to 0$, uniformly in θ.

PROPOSITION 3.8. *Suppose $f(\rho, \theta)$ is a continuous function of $\rho > 0$ and $\theta \in \mathbb{R}$, periodic in θ, with $f(\rho, \theta) = \mathscr{O}(\rho)$ as $\rho \to 0$, uniformly in θ. Then*

$$\int_0^{\sigma_0^\varepsilon} f(\rho^\varepsilon, \theta^\varepsilon)\, ds \to \int_0^\infty f(\rho^*, \theta^*)\, ds$$

in P°-probability as $\varepsilon \downarrow 0$.

PROOF. Consider the stopping time $\sigma_\delta^\varepsilon = \inf\{s > 0: \rho^\varepsilon(s) \le \delta\}$ for any $0 < \delta < \rho_0$. It follows from (3.27) that, in probability,

$$\sigma_\delta^\varepsilon \to s_\delta = \log(\rho_0/\delta) \quad \text{and} \quad \int_0^{\sigma_\delta^\varepsilon} f(\rho^\varepsilon, \theta^\varepsilon)\, ds \to \int_0^{s_\delta} f(\rho^*, \theta^*)\, ds.$$

From our hypotheses,

$$\int_{s_\delta}^{\infty} f(\rho^*, \theta^*)\, ds$$

can be made arbitrarily small by choosing δ sufficiently small. We need to show that, for any given $\delta' > 0$, we can make

$$\limsup_{\varepsilon \downarrow 0} P^\circ\left[\left|\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} f(\rho^\varepsilon, \theta^\varepsilon)\, ds\right| > \delta'\right] \tag{3.29}$$

arbitrarily small by choosing δ sufficiently small.

The hypotheses imply that for some constant B and all $0 < \rho \le R$,

$$|f(\rho, \theta)| \le B\rho.$$

Therefore

$$P^\circ\left[\left|\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} f(\rho^\varepsilon, \theta^\varepsilon)\, ds\right| > \delta'\right] \le P^\circ[\sigma_R^\varepsilon < \sigma_0^\varepsilon] + P^\circ\left[B\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} \rho^\varepsilon\, ds > \delta'\right]. \tag{3.30}$$

Because of (3.16) we can concentrate on the last term. Using (3.12) and (3.9), we have

$$\begin{aligned}
\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} \rho^\varepsilon\, ds &\le -\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} b_\varepsilon^\circ(\rho^\varepsilon)\, ds \\
&= -\left[\rho^\varepsilon(\sigma_0^\varepsilon) - \rho^\varepsilon(\sigma_\delta^\varepsilon) + \varepsilon^{1/2}\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} d\beta_1^\circ(s)\right] \qquad (3.31)\\
&= \delta - \varepsilon^{1/2}\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} d\beta_1^\circ(s).
\end{aligned}$$

Since we know that σ_0^ε has finite second moment, $E^\circ[\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} d\beta_1^\circ] = 0$ and so, independently of ε,

$$E_\delta^\circ\left[\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} \rho^\varepsilon(s)\, ds\right] \le \delta.$$

Consequently,

$$P^\circ\left[B\int_{\sigma_\delta^\varepsilon}^{\sigma_0^\varepsilon} \rho^\varepsilon\, ds > \delta'\right] \le \frac{B}{\delta'}\delta,$$

which $\to 0$ as $\delta \to 0$ uniformly in ε. This establishes what we wanted for (3.29) and completes the proof. □

We comment that because of (3.16) we do not need f to be defined for all $\rho > 0$. Instead f could be a continuous function on $D \setminus C$ expressed in our (ρ,θ) coordinates and so defined only for $0 < \rho < R$. Then we could replace $\int_0^{\sigma_0^\varepsilon}$ with $\int_0^{\sigma_0^\varepsilon, R}$ in the proposition, with the same conclusion. This is the setting for the following revision, which we will use in Section 5. The $h'_\varepsilon/h_\varepsilon$ essentially provides the needed $\mathcal{O}(\rho)$ factor as $\rho \to 0$.

PROPOSITION 3.9. *Suppose $\phi\colon D \setminus C \to \mathbb{R}$ is a bounded continuous function. Then*

$$\varepsilon \int_0^{\sigma_{0,R}^{\varepsilon}} \phi(\rho^{\varepsilon}, \theta^{\varepsilon}) \frac{h'_{\varepsilon}}{h_{\varepsilon}}(\rho^{\varepsilon})\, ds \to -2\int_0^{\infty} \phi(\rho^*, \theta^*)\rho^*\, ds$$

in P°-probability as $\varepsilon \downarrow 0$.

PROOF. From (3.10) and (3.11), we know that

$$\varepsilon \frac{h'_{\varepsilon}}{h_{\varepsilon}}(\rho) \to -2\rho \quad \text{uniformly on compacts.}$$

We repeat the reasoning of the preceding proof:

$$\varepsilon \int_0^{\sigma_{0,R}^{\varepsilon} \wedge \sigma_{\delta}^{\varepsilon}} \phi(\rho^{\varepsilon}, \theta^{\varepsilon}) \frac{h'_{\varepsilon}}{h_{\varepsilon}}(\rho^{\varepsilon})\, ds \to -2\int_0^{s_{\delta}} \phi(\rho^*, \theta^*)\rho^*\, ds$$

in P°-probability. If $|\phi| \le B$, then in place of the second term of (3.30) we get

$$P^{\circ}\left[B \int_{\sigma_{\delta}^{\varepsilon}}^{\sigma_0^{\varepsilon}} -\varepsilon \frac{h'_{\varepsilon}}{h_{\varepsilon}}(\rho^{\varepsilon})\, ds > \delta' \right].$$

By (3.13),

$$\int_{\sigma_{\delta}^{\varepsilon}}^{\sigma_0^{\varepsilon}} -\varepsilon \frac{h'_{\varepsilon}}{h_{\varepsilon}}(\rho^{\varepsilon})\, ds \le -2\int_{\sigma_{\delta}^{\varepsilon}}^{\sigma_0^{\varepsilon}} b_{\varepsilon}^{\circ}(\rho^{\varepsilon})\, ds,$$

from which point we proceed as in (3.31). □

4. Asymptotically negligible perturbations. The situation described by Proposition 3.2 will occur several times in the remainder of our discussion. In this section we focus on that notion, establishing some notation and properties.

DEFINITION. Suppose P_1 and P_2 are both probability measures on $(\Omega, \mathscr{F})$, depending on the parameter $\varepsilon > 0$. If for all sufficiently small ε, $P_2 \ll P_1$ and, for every $r \ge 1$,

$$E_1\left[\left| \frac{dP_2}{dP_1} - 1 \right|^r \right] \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

then we say P_2 is asymptotically replaceable by P_1 and write

$$P_2 \overset{\approx 1}{\leftarrow} P_1.$$

Thus Proposition 3.2 simply says $\bar{P}^{\triangleright} \overset{\approx 1}{\leftarrow} P^{\circ}$. The following lemmas describe some simple properties of this notion. The first is a transitive property.

LEMMA 4.1. *Suppose that P_1, P_2 and P_3 are probability measures on $(\Omega, \mathscr{F})$ with $P_3 \overset{\approx 1}{\leftarrow} P_2$ and $P_2 \overset{\approx 1}{\leftarrow} P_1$. Then $P_3 \overset{\approx 1}{\leftarrow} P_1$.*

PROOF. Write

$$\zeta_1 = \frac{dP_2}{dP_1} \quad \text{and} \quad \zeta_2 = \frac{dP_3}{dP_2}.$$

Then

$$\frac{dP_3}{dP_1} = \zeta_2 \zeta_1.$$

Since

$$\zeta_2\zeta_1 - 1 = (\zeta_2 - 1)\zeta_1 + (\zeta_1 - 1)$$

and we know that $E_1[|\zeta_1 - 1|^r] \to 0$, it is enough to show that

$$E_1\big[|(\zeta_2 - 1)\zeta_1|^r\big] \to 0. \tag{4.1}$$

To see this,

$$\begin{aligned} E_1\big[|(\zeta_2 - 1)\zeta_1|^r\big] &= E_1\big[|\zeta_2 - 1|^r(\zeta_1)^{1/2}(\zeta_1)^{r-1/2}\big] \\ &\le E_1\big[|\zeta_1 - 1|^{2r}\zeta_1\big]^{1/2} \cdot E_1\big[(\zeta_1)^{2r-1}\big]^{1/2} \\ &= E_2\big[|\zeta_1 - 1|^{2r}\big]^{1/2} \cdot E_1\big[(\zeta_1)^{2r-1}\big]^{1/2}. \end{aligned}$$

This implies (4.1) because $E_2[|\zeta_1 - 1|^{2r}] \to 0$ and $E_1[(\zeta_1)^{2r-1}] \to 1$. □

The next lemma says that asymptotic replacements preserve exponential moment results such as Corollaries 3.5 and 3.7.

LEMMA 4.2. *Suppose $P_2 \overset{\simeq 1}{\leftarrow} P_1$ are probability measures and σ^ε is a stopping time such that as $\varepsilon \downarrow 0$,*

$$E_1[e^{\varepsilon c \sigma^\varepsilon}] \to 1 \quad \textit{for every } c > 0.$$

Then

$$E_2[e^{\varepsilon c \sigma^\varepsilon}] \to 1 \quad \textit{for every } c > 0.$$

PROOF.

$$\begin{aligned} E_2[e^{\varepsilon c \sigma^\varepsilon}] - E_1[e^{\varepsilon c \sigma^\varepsilon}] &= E_1\left[e^{\varepsilon c \sigma^\varepsilon}\left(\frac{dP_2}{dP_1} - 1\right)\right] \\ &\le E_1[e^{\varepsilon 2 c \sigma^\varepsilon}]^{1/2} E_1\left[\left|\frac{dP_2}{dP_1} - 1\right|^2\right]^{1/2} \to 0. \qquad \square \end{aligned}$$

Next we want to consider conditions which imply $P_2 \overset{\simeq 1}{\leftarrow} P_1$ for situations where dP_2/dP_1 is a Girsanov density. Suppose that for each $\varepsilon > 0$, we have a probability measure P_1 on $(\Omega, \mathcal{F})$, an $\{\mathcal{F}_s\}$-adapted Brownian motion $\beta^1(s)$, a progressively measurable process $\psi(s)$ satisfying a uniform bound in s and ε:

$$|\psi(s)| \le K \quad \text{a.s.}, \tag{4.2}$$

and a stopping time σ^ε with the property that, for each constant $c > 0$,

$$E_1[e^{\varepsilon c \sigma^\varepsilon}] \to 1 \quad \text{as } \varepsilon \downarrow 0. \tag{4.3}$$

Let

$$\zeta_t(\psi) = \exp\left(\varepsilon^{1/2} \int_0^{t\wedge\sigma^\varepsilon} \psi(s)\, d\beta^1(s) - \frac{\varepsilon}{2} \int_0^{t\wedge\sigma^\varepsilon} \psi(s)^2\, ds\right) \tag{4.4}$$

be the exponential martingale which would occur as the Girsanov density associated with adding the $\mathscr{O}(\varepsilon)$ perturbation $\varepsilon\psi(s)1_{s\le\sigma_0^\varepsilon}\, ds$ to an equation such as (3.1). We begin by recalling the following result (see [16], Theorem 6.1 and the Note following its proof):

THE NOVIKOV CONDITION. Suppose $\phi(s)$ is progressively measurable and σ is a stopping time (finite a.s.) for which

$$E_1\left[\exp\left(\tfrac{1}{2}\int_0^\sigma \phi(s)^2\, ds\right)\right] < \infty. \tag{4.5}$$

Then

$$\zeta_t = \exp\left(\int_0^{t\wedge\sigma} \phi(s)\, d\beta^1(s) - \tfrac{1}{2}\int_0^{t\wedge\sigma} \phi(s)^2\, ds\right)$$

is a uniformly integrable martingale. In particular $E_1[\zeta_\sigma] = 1$.

In our case $\phi(s) = \varepsilon^{1/2}\psi(s)$, and (4.5) is satisfied because

$$E_1\left[\exp\left(\frac{\varepsilon}{2}\int_0^{\sigma^\varepsilon} \psi(s)^2\, ds\right)\right] \le E_1\left[e^{\varepsilon K^2 \sigma^\varepsilon/2}\right] < \infty$$

for all sufficiently small ε, by our hypothesis (4.3) above. Thus, for all sufficiently small ε, $\psi_t(\psi)$ is a uniformly integrable martingale and

$$E_1[\zeta_{\sigma^\varepsilon}(\psi)] = 1. \tag{4.6}$$

Thus $\zeta_{\sigma^\varepsilon}(\psi)$ *is* the density of a probability measure P_2 on $(\Omega, \mathscr{F})$ with $dP_2/dP_1 = \zeta_{\sigma^\varepsilon}(\psi)$. We can now prove our theorem.

THEOREM 4.3. *With the above hypotheses*, (4.4) *and* (4.3) *in particular*, $P_2[A] = \int_A \zeta_{\sigma^\varepsilon}(\psi)\, dP_1$ *defines a probability measure on* $(\Omega, \mathscr{F})$ *for which*

$$P_2 \overset{\approx 1}{\leftarrow} P_1.$$

PROOF. It follows from (4.3) that $E_1[\varepsilon\sigma^\varepsilon] \to 0$. Therefore

$$\left|\frac{\varepsilon}{2}\int_0^{\sigma^\varepsilon} \psi(s)^2\, ds\right| \le \varepsilon \frac{K^2}{2} \sigma^\varepsilon \to 0$$

in probability and

$$E_1\left[\left(\varepsilon^{1/2}\int_0^{\sigma^\varepsilon} \psi\, dw\right)^2\right] = E_1\left[\varepsilon\int_0^{\sigma^\varepsilon} \psi(s)^2\, ds\right] \le E_1[\varepsilon K^2 \sigma^\varepsilon] \to 0.$$

These facts imply that

$$\zeta_{\sigma^\varepsilon}(\psi) \to 1$$

in probability. It suffices to establish uniform integrability of $(\zeta_{\sigma^\varepsilon}(\psi) - 1)^r$. For this it is sufficient to show that $E_1[\zeta_{\sigma^\varepsilon}(\psi)^r]$ remains bounded as $\varepsilon \downarrow 0$, for each $r \geq 1$. Now one may check that

$$\zeta_{\sigma^\varepsilon}(\psi)^r = \zeta_{\sigma^\varepsilon}(2r\psi)^{1/2} \cdot \exp\left(\frac{\varepsilon}{2}\int_0^{\sigma^\varepsilon}(2r^2 - r)\psi(s)^2\,ds\right).$$

Using the Cauchy–Schwarz inequality gives

$$E_1\left[\zeta_{\sigma^\varepsilon}(\psi)^r\right] \leq E_1\left[\zeta_{\sigma^\varepsilon}(2r\psi)\right]^{1/2} \cdot E_1\left[\exp\left(\varepsilon\int_0^{\sigma^\varepsilon}(2r^2 - r)\psi(s)^2\,ds\right)\right]^{1/2}.$$

The first factor on the right is just 1, by the same reasoning as (4.6). Using the assumed bound on ψ for the second factor, (4.3) implies

$$E_1\left[\zeta_{\sigma^\varepsilon}(\psi)^r\right] \leq E_1\left[e^{\varepsilon(2r^2+r)K^2\sigma^\varepsilon}\right]^{1/2} \to 1,$$

completing the proof. □

We want next to apply these results to justify our first simplification in Section 2: dropping the $\mathscr{O}(\varepsilon)$ drift terms from (2.18). Let $\beta^{\triangleright}(s) = (\beta_1^{\triangleright}(s), \beta_2^{\triangleright}(s))$ be a two-dimensional Brownian motion with respect to some probability measure $P^{\triangleright}$. [Initially we assume no connection with the P and β_i of (2.18).] With respect to these, let $\rho^\varepsilon(s)$, $\theta^\varepsilon(s)$ be the solutions of (3.1). According to Corollary 3.5,

$$E^{\triangleright}\left[e^{\varepsilon c \sigma_{0,R}^\varepsilon}\right] \to 1$$

for any $c > 0$. Thus we can apply Theorem 4.3 to $E^{\triangleright}$ with $\sigma_{0,R}^\varepsilon$ playing the role of σ^ε. We want to use a Girsanov change of measure to transform (3.1) into (2.18). Construct P and β_i from $P^{\triangleright}$ and $\beta_i^{\triangleright}$ via the following formulas:

$$\frac{dP}{dP^{\triangleright}} = \zeta_{\sigma_{0,R}^\varepsilon} = \zeta_{\sigma_{0,R}^\varepsilon}^{(2)}\zeta_{\sigma_{0,R}^\varepsilon}^{(1)}, \tag{4.7}$$

where

$$\zeta_{\sigma_{0,R}^\varepsilon}^{(1)} = \exp\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}\frac{\alpha_1}{\sigma_1^2}(\rho^\varepsilon, \theta^\varepsilon)\,d\beta_1^{\triangleright} - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}\left(\frac{\alpha_1}{\sigma_1^2}\right)^2 ds\right),$$

$$\zeta_{\sigma_{0,R}^\varepsilon}^{(2)} = \exp\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}\frac{\alpha_1}{\sigma_1\sigma_2}(\rho^\varepsilon, \theta^\varepsilon)\,d\beta_2^{\triangleright} - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}\left(\frac{\alpha_2}{\sigma_1\sigma_2}\right)^2 ds\right);$$

and

$$\beta_1(s) = \beta_1^{\triangleright}(s) - \varepsilon^{1/2}\int_0^{s\wedge\sigma_{0,R}^\varepsilon}\frac{\alpha_1}{\sigma_1^2}(\rho^\varepsilon, \theta^\varepsilon)\,ds,$$

$$\beta_2(s) = \beta_2^{\triangleright}(s) - \varepsilon^{1/2}\int_0^{s\wedge\sigma_{0,R}^\varepsilon}\frac{\alpha_2}{\sigma_1\sigma_2}(\rho^\varepsilon, \theta^\varepsilon)\,ds.$$

The results of this section imply that

$$P \overset{\approx 1}{\leftarrow} P^{\triangleright}. \tag{4.8}$$

Since we have only formulated Theorem 4.3 for a single Brownian motion we need to carry this out in two stages. First construct the intermediate probability P^* with $dP^*/dP^{\triangleright} = \zeta^{(1)}_{\sigma^\varepsilon_{0,R}}$. From Theorem 4.3 and Lemma 4.2, we have

$$P^* \overset{\approx 1}{\leftarrow} P^{\triangleright} \quad \text{and} \quad E^*[e^{\varepsilon c \sigma^\varepsilon_{0,R}}] \to 1 \quad \text{all } c > 0.$$

Since $\beta_2^{\triangleright}$ remains a Brownian motion with respect to P^*, we can now apply Theorem 4.3 to P^* with $\zeta^{(2)}_{\sigma^\varepsilon_{0,R}}$ to get

$$P \overset{\approx 1}{\leftarrow} P^*.$$

Lemma 4.1 now implies (4.8) and

$$E[e^{\varepsilon c \sigma^\varepsilon_{0,R}}] \to 1 \tag{4.9}$$

follows from Lemma 4.2. With respect to the P and β_i thus constructed, $(\rho^\varepsilon, \theta^\varepsilon)$ satisfies (2.18).

Actually the preceding reasoning is backwards. We really want to start with the P, β_i that we assumed for (2.18) and construct $P^{\triangleright}$, $\beta_i^{\triangleright}$ from them. However to do this we need to know (4.9) initially. Note that (4.9) depends only on the distribution of $\sigma^\varepsilon_{0,R}$ as determined by (2.18), which does *not* depend on what Brownian motion is used in place of $\beta^{\triangleright}$. Thus we can start with an arbitrary $P^{\triangleright}$ and $\beta_i^{\triangleright}$ and argue as above to establish (4.9). With (4.9) now in hand we can discard our initial choice of $P^{\triangleright}$, $\beta_i^{\triangleright}$ and proceed to construct $P^{\triangleright}$ and $\beta_i^{\triangleright}$ from the P and β_i of (2.18) by means of a Girsanov transformation (the inverse of the one used above) which turns (2.18) into (3.1). The above formulas remain valid and serve to recover the original P, β_i from the newly constructed $P^{\triangleright}$, $\beta_i^{\triangleright}$. In the sequel, $P^{\triangleright}$, $\beta_i^{\triangleright}$ refer to those constructed in this way.

5. Conditional exit asymptotics. We are now ready to address the issues necessary for a successful application of the conditional approach to the exit problem described in Section 2. Specifically, we want to establish (2.6) and describe the asymptotic behavior of the conditional exit kernel $Q^\varepsilon(\cdot, \cdot)$ of (2.3).

Let P denote the underlying probability measure with respect to which our original process $(\rho^\varepsilon, \theta^\varepsilon)$ is described by (2.18). Define $\overline{P}$ to be the corresponding conditional probability measure on $\{\sigma_0^\varepsilon < \sigma_R^\varepsilon\}$:

$$\overline{P}[A] = P[A | \sigma_0^\varepsilon < \sigma_R^\varepsilon].$$

With $x_0 = (\rho_0, \theta_0) = (\rho^\varepsilon(0), \theta^\varepsilon(0))$, we have from (2.23):

$$Q^\varepsilon(x_0, A) = \overline{P}[(0, \theta^\varepsilon(\sigma_0^\varepsilon)) \in A], \qquad A \subseteq \partial G. \tag{5.1}$$

Equations (2.18) describe $(\rho^\varepsilon, \theta^\varepsilon)$ under P. Our goal is to describe the distribution of σ_0^ε and the dynamics of $\theta^\varepsilon(s)$, $s < \sigma_0^\varepsilon$, when considered subject to $\overline{P}$.

The following diagram is helpful in describing the organization of our argument.

(5.2)
$$\begin{tikzcd}
P \arrow[d, "\tau_D^\varepsilon < \tau_C^\varepsilon"'] & P^{\triangleright} \arrow[l, "\simeq 1"'] \arrow[d, "\sigma_0^\varepsilon < \sigma_R^\varepsilon"] \arrow[r, equal] & P^{\triangleright} \arrow[d, "\sigma_0^\varepsilon < \infty"] & \\
\bar{P} & \bar{P}^{\triangleright} \arrow[l, "\simeq 1"'] & P^{\circ} \arrow[l, "\simeq 1"'] & P^{\bullet} \arrow[l, "\simeq 1"']
\end{tikzcd}$$

The arrows represent absolute continuity of probability measures. The notation $\simeq 1$ over an arrow stands for asymptotic replacement as defined as in Section 4. $P^{\triangleright}$ is the probability measure defined at the end of Section 4, with respect to which $(\rho^\varepsilon, \theta^\varepsilon)$ is described by (3.1) instead of (2.18). Thus the arrow from $P^{\triangleright}$ to P is what we established in (4.8).

The downward arrows correspond to conditioning on the indicated events. Thus, as already defined in Section 4,

$$\bar{P}^{\triangleright}[A] = P^{\triangleright}[Q|\sigma_0^\varepsilon < \sigma_R^\varepsilon]$$

and

$$P^{\circ}[A] = P^{\triangleright}[A|\sigma_0^\varepsilon < \infty].$$

We have a good understanding of the behavior of ρ^ε under P° from Propositions 3.1 and 3.3. The arrow from P° to $\bar{P}^{\triangleright}$ is Proposition 3.2. The measure $P^{\bullet}$ will be defined at the end of this section. It gives the cleanest representation of the conditioned process in the original x-variables; see (5.11).

The arrow from $\bar{P}^{\triangleright}$ to $\bar{P}$ is Theorem 5.4 below. Its significance is that we can substitute $\bar{P}^{\triangleright}$ for $\bar{P}$ in (5.1), for purposes of determining the asymptotic behavior of Q^ε. Consequently, P° and $P^{\bullet}$ are also acceptable replacements. Thus either (5.9), (5.10) or (5.11), whichever is more convenient, can be used to describe the asymptotic behavior of Q^ε. This conclusion, stated for P° in Theorem 5.6 below, has been the goal of all our work.

The first theorem of this section, Theorem 5.3, addresses (2.6). To understand its source, begin with the observation that

$$P[\sigma_0^\varepsilon < \sigma_R^\varepsilon] = E^{\triangleright}\left[\zeta_{\delta_{0,R}^\varepsilon}; \sigma_0^\varepsilon < \sigma_R^\varepsilon\right],$$

where $\zeta_{\sigma_{0,R}^\varepsilon}$ is the density $dP/dP^{\triangleright}$ as given in (4.7). For any $A \in \mathscr{F}$ we have

$$\bar{P}[A] = E^{\triangleright}\left[1_A \zeta_{\sigma_{0,R}^\varepsilon}; \sigma_0^\varepsilon < \sigma_R^\varepsilon\right] / E^{\triangleright}\left[\zeta_{\sigma_{0,R}^\varepsilon}; \sigma_0^\varepsilon < \sigma_R^\varepsilon\right] = \bar{E}^{\triangleright}\left[1_A \bar{\zeta}\right],$$

where

$$\bar{\zeta} = \frac{\zeta_{\sigma_{0,R}^\varepsilon}}{\bar{E}^{\triangleright}\left[\zeta_{\sigma_{0,R}^\varepsilon}\right]}.$$

In other words,

$$\bar{\zeta} = \frac{d\bar{P}}{d\bar{P}^{\triangleright}}. \tag{5.3}$$

The fact behind both theorems of this section is the following.

PROPOSITION 5.1. *Let g be the function defined on ∂G by*

$$g(x_0) = \exp\left(-2\int_0^\infty \rho^*(s)\cdot \frac{\alpha_1}{\sigma_1^2}(\rho^*,\theta^*)\,ds\right),$$

where

$$\dot\rho^*(s) = -\rho^*(s),$$
$$\dot\theta^*(s) = \frac{\langle \nabla\theta, \mathbf{b}\rangle}{\alpha_1^2}(\rho^*,\theta^*),$$

with $x_0 = (\rho^*(0), \theta^*(0)) = (\rho_0, \theta_0)$. Then, for any $r \ge 1$,

$$E^\circ\left[\left|\zeta_{\sigma_0^\varepsilon, R} - g(x_0)\right|^r\right] \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Moreover this convergence is uniform with respect to $x^\varepsilon(0) = x_0 \in \partial G$.

Notice that $\rho^*(s) = \rho_0 e^{-s}$. Since α_1/σ_1^2 is bounded and continuous, it follows that g is well-defined, continuous and positive. Before proving the proposition, we make some observations and note some consequences. First, Proposition 3.2 (in which there is no θ_0-dependence) and an argument analogous to that for Lemma 4.1 yield the following corollary.

COROLLARY 5.2. *For each $r \ge 1$,*

$$\overline{E}^{\triangleright}\left[\left|\zeta_{\sigma_0^\varepsilon, R} - g(x_0)\right|^r\right] \to 0 \quad \text{as } \epsilon \downarrow 0,$$

uniformly over $x_0 \in \partial G$. In particular,

$$\overline{E}^{\triangleright}\left[\zeta_{\sigma_0^\varepsilon, R}\right] \to g(x_0)$$

uniformly over ∂G.

Next we have the result which justifies (2.6).

THEOREM 5.3. *The function g of the preceding lemma is positive and continuous over ∂G and*

$$\frac{P_{x_0}[\tau_D^\varepsilon < \tau_C^\varepsilon]}{\gamma^\varepsilon} \to g(x_0)$$

uniformly over $x_0 \in \partial G$, where $\gamma^\varepsilon = h_\varepsilon(\rho_0)$ and h_ε is as in (3.4).

PROOF. Having already noted the properties of g, we only need consider the convergence

$$\begin{aligned} P_{x_0}[\tau_D^\varepsilon < \tau_C^\varepsilon] &= P[\sigma_0^\varepsilon < \sigma_R^\varepsilon] \\ &= E^{\triangleright}\left[\zeta_{\sigma_0^\varepsilon, R}; \sigma_0^\varepsilon < \sigma_R^\varepsilon\right] = \overline{E}^{\triangleright}\left[\zeta_{\sigma_0^\varepsilon, R}\right]\cdot P^{\triangleright}\left[\sigma_0^\varepsilon < \sigma_R^\varepsilon\right] \\ &= \overline{E}^{\triangleright}\left[\zeta_{\sigma_0^\varepsilon, R}\right]\cdot P^\circ[\sigma_0^\varepsilon < \sigma_R^\varepsilon]P^{\triangleright}\left[\sigma_0^\varepsilon < \infty\right]. \end{aligned} \tag{5.4}$$

Now $P^{\triangleright}[\sigma_0^\varepsilon < \infty] = h_\varepsilon(\rho_0) = \gamma^\varepsilon$, from (3.2). We also have from (3.16) that

$$P^{\circ}[\sigma_0^\varepsilon < \sigma_R^\varepsilon] \to 1,$$

and this is independent of $x_0 \in \partial G$, since the first equation of (3.8) does not involve θ^ε. The first factor in (5.4) converges uniformly to $g(x_0)$ by Corollary 5.2. This proves the theorem. □

The function g has a natural expression in the original space and time variables x and t. To see this, return to the original time scale with

$$ds = \sigma_1^2(\rho^*, \theta^*)\, dt.$$

In this setting (2.11) implies

$$\begin{aligned} \dot{\rho}^*(t) &= -\rho^* \sigma_1^2(\rho^*, \theta^*) \\ &= -\langle \nabla\rho, \mathbf{b}\rangle(\rho^*, \theta^*), \\ \dot{\theta}^*(t) &= \langle \nabla\theta, \mathbf{b}\rangle(\rho^*, \theta^*) \end{aligned} \tag{5.5}$$

and

$$g(x_0) = \exp\left(-2\int_0^\infty \rho^*(t)\alpha_1(\rho^*, \theta^*)\, dt\right).$$

Notice that (5.5) are the equations for the extremals associated with the boundary quasipotential $W = \rho^2$ of (2.13). In the original x-coordinates the extremals solve

$$\begin{aligned} \dot{x}^* &= H_p(x^*, -\nabla W(x^*)) \\ &= \mathbf{b}(x^*) - a\nabla W(x^*) \\ &= \mathbf{b}(x^*) - 2\rho(x^*)a\nabla\rho(x^*). \end{aligned} \tag{5.6}$$

When (2.11) and (2.14) are used to convert this to (ρ, θ)-coordinates, we obtain (5.5):

$$\begin{aligned} \dot{\rho}^* &= \langle \nabla\rho, \mathbf{b}\rangle - 2\rho\langle \nabla\rho, a\nabla\rho\rangle = -\langle \nabla\rho, \mathbf{b}\rangle, \\ \dot{\theta}^* &= \langle \nabla\theta, \mathbf{b}\rangle - 2\rho\langle \nabla\theta, a\nabla\rho\rangle = \langle \nabla\theta, \mathbf{b}\rangle. \end{aligned}$$

The integrand in the exponent for g can be expressed in terms of W, evaluated along the extremal x^*, using $(\rho^*, \theta^*) = (\rho(x^*), \theta(x^*))$ and $W = \rho^2$:

$$\begin{aligned} 2\rho^*(t)\alpha_1(\rho^*, \theta^*) &= \frac{1}{2}\sum a_{ij}\frac{2\rho\,\partial^2\rho}{\partial x_i\,\partial x_j}(x^*) \\ &= \frac{1}{2}\sum a_{ij}\frac{\partial^2 W}{\partial x_i\,\partial x_j}(x^*) - \sigma_1^2(x^*), \end{aligned}$$

where by (2.12) and (5.6),

$$\sigma_1^2 = \frac{\langle \nabla W, a\nabla W\rangle}{4W} = -\frac{\langle \nabla W, \mathbf{b} - a\nabla W\rangle}{2W} = -\frac{1}{2}\frac{\langle \nabla W, \dot{x}^*\rangle}{W}.$$

Thus we can write

$$g(x_0) = \exp\left(-\frac{1}{2}\int_0^\infty \left[\frac{\langle \nabla W, \dot{x}^* \rangle}{W} + \sum a_{ij}\frac{\partial^2 W}{\partial x_i \partial x_j}(x^*)\right] dt\right),$$

where x^* is the extremal (5.6) starting at x_0.

We return now to the proof of the proposition itself.

PROOF OF PROPOSITION 5.1. It suffices to show that $\zeta_{\sigma_{0,R}^\varepsilon} \to g$ in probability and that for any $r > 0$,

$$\limsup_{\varepsilon \downarrow 0} E^\circ\left[\left(\zeta_{\sigma_{0,R}^\varepsilon}\right)^r\right] < \infty. \tag{5.7}$$

[This implies uniformly integrability of $|\zeta_{\sigma_{0,R}^\varepsilon} - g(x_0)|$ to any power.] Using the factorization (4.7) it suffices to prove

$$\zeta^{(1)}_{\sigma_{0,R}^\varepsilon} \to g \quad \text{and} \quad \zeta^{(2)}_{\sigma_{0,R}^\varepsilon} \to 1$$

in P°-probability and, for each of $i = 1, 2$,

$$\limsup E^\circ\left[\left(\zeta_{i,\sigma_{0,R}^\varepsilon}\right)^r\right] < \infty.$$

According to Proposition 3.1, $\beta_2^\triangleright = \beta_2^\circ$ remains a Brownian motion under P°. Thus Theorem 4.3 implies what we need for $\zeta^{(2)}_{\sigma_{0,R}^\varepsilon}$. We concentrate therefore on $\zeta^{(1)}_{\sigma_{0,R}^\varepsilon}$.

Define ϕ to be the function

$$\phi(\rho, \theta) = \frac{\alpha_1}{\sigma_1^2}(\rho, \theta).$$

Our hypotheses and extended definitions (2.22) imply that this is bounded and continuous. Let K be a bound:

$$|\phi| \le K.$$

Let $\phi^\varepsilon(s)$ be the process obtained by evaluating ϕ along the path of $(\rho^\varepsilon, \theta^\varepsilon)$:

$$\phi^\varepsilon(s) = \phi(\rho^\varepsilon(s), \theta^\varepsilon(s)).$$

We can rewrite $\zeta^{(1)}_{\sigma_{0,R}^\varepsilon}$ using Proposition 3.1:

$$\begin{aligned} \zeta^{(1)}_{\sigma_{0,R}^\varepsilon} &= \exp\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon} \phi^\varepsilon \, d\beta_1^\triangleright - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon} \phi^\varepsilon(s)^2 \, ds\right) \\ &= \exp\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon} \phi^\varepsilon \, d\beta_1^\circ - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon} (\phi^\varepsilon)^2 \, ds + \varepsilon\int_0^{\sigma_{0,R}^\varepsilon} \phi^\varepsilon \frac{h'_\varepsilon}{h_\varepsilon}(\rho^\varepsilon)\, ds\right). \end{aligned} \tag{5.8}$$

Since $h'_\varepsilon \le 0$ and $-K \le \phi^\varepsilon$, we have

$$\varepsilon\phi^\varepsilon \frac{h'_\varepsilon}{h_\varepsilon} \le -K\varepsilon\frac{h'_\varepsilon}{h_\varepsilon}.$$

Using (3.13),

$$\varepsilon\int_0^{\sigma_{0,R}^\varepsilon}\phi^\varepsilon\frac{h'_\varepsilon}{h_\varepsilon}\,ds \le -2K\int_0^{\sigma_{0,R}^\varepsilon} b_\varepsilon^\circ(\rho_\varepsilon(s))\,ds$$

$$= -2K\left[\rho^\varepsilon(\sigma_{0,R}^\varepsilon) - \rho^\varepsilon(0) - \varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon} d\beta_1^\circ\right].$$

Therefore

$$\begin{aligned}\log\left(\zeta_{\sigma_{0,R}^\varepsilon}^{(1)}\right) &\le \varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)\,d\beta_1^\circ \\ &\quad - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon)^2\,ds - 2K(\rho^\varepsilon(\sigma_{0,R}^\varepsilon) - \rho_0) \\ &= \varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)\,d\beta_1^\circ - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)^2\,ds \\ &\quad + \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(4\phi^\varepsilon K + 4K^2)\,ds - 2K(\rho^\varepsilon(\sigma_{0,R}^\varepsilon) - \rho_0) \\ &\le \varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)\,d\beta_1^\circ \\ &\quad - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)^2\,ds + \left(\varepsilon 4K^2\sigma_{0,R}^\varepsilon + 2K\rho_0\right).\end{aligned}$$

This gives the inequality

$$\zeta_{\sigma_{0,R}^\varepsilon}^{(1)} \le \xi_{\sigma_{0,R}^\varepsilon}\exp(\varepsilon 4K^2\sigma_{0,R}^\varepsilon)\exp(2K\rho_0),$$

where

$$\xi_{\sigma_{0,R}^\varepsilon} = \exp\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)\,d\beta_1^\circ - \frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon + 2K)^2\,ds\right).$$

For any $r > 0$,

$$E^\circ\left[\left(\zeta_{\sigma_{0,R}^\varepsilon}^{(1)}\right)^r\right] \le e^{2rK\rho_0}\cdot E^\circ\left[\left(\xi_{\sigma_{0,R}^\varepsilon}\right)^{2r}\right]^{1/2} E^\circ\left[\exp(\varepsilon 8rK^2\sigma_{0,R}^\varepsilon)\right]^{1/2}.$$

The third factor of this $\to 1$ as $\varepsilon \downarrow 0$, by Corollary 3.7. We can apply Theorem 4.3 to see that the second factor also $\to 1$. This establishes (5.7).

We now look at the terms of $\zeta_{\sigma_{0,R}^\varepsilon}^{(1)}$ from (5.8) individually to see that $\zeta_{\sigma_{0,R}^\varepsilon}^{(1)} \to g$ in P°-probability. First,

$$\frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon)^2\,ds \to 0.$$

This follows from Corollary 3.7 and

$$\left|\frac{\varepsilon}{2}\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon)^2\,ds\right| \le \varepsilon\frac{K^2}{2}\sigma_{0,R}^\varepsilon.$$

For the first term of (5.8),

$$E^\circ\left[\left(\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}\phi^\varepsilon\,d\beta_1^\circ\right)^2\right] = E^\circ\left[\varepsilon\int_0^{\sigma_{0,R}^\varepsilon}(\phi^\varepsilon)^2\,ds\right] \to 0$$

by Corollary 3.7 again, and so

$$\varepsilon^{1/2}\int_0^{\sigma_{0,R}^\varepsilon}\phi^\varepsilon\,d\beta_1^\circ \to 0$$

in probability as well. Finally, it follows from Proposition 3.9 that

$$\varepsilon\int_0^{\sigma_{0,R}^\varepsilon}\phi^\varepsilon\frac{h'_\varepsilon}{h_\varepsilon}(\rho^\varepsilon)\,ds \to -2\int_0^\infty\phi(\rho^*,\theta^*)\rho^*\,ds$$

in P°-probability. Putting these together then, we have

$$\zeta_{\sigma_{0,R}^\varepsilon}^{(1)} \to \exp\left[0+0-2\int_0^\infty\rho^*\phi(\rho^*,\theta^*)\,ds\right] = g$$

in P^0-probability as claimed.

The uniformity of the convergence in the lemma follows from the observation that the above arguments all hold if $x^\varepsilon(0)$ is not fixed but is allowed to depend on ϵ: $x^\varepsilon(0) = x_0^\varepsilon \to x_0$. □

THEOREM 5.4. *$\overline{P}$ is asymptotically replaceable by $\overline{P}^\triangleright$, that is,*

$$\overline{P} \overset{\simeq 1}{\rightleftarrows} \overline{P}^\triangleright.$$

PROOF. This is immediate from Corollary 5.2 and (5.3):

$$\overline{E}^\triangleright\left[|\zeta - 1|^r\right] = \frac{\overline{E}^\triangleright\left[|\zeta_{\sigma_{0,R}^\varepsilon} - \overline{E}^\triangleright\left[\zeta_{\sigma_{0,R}^\varepsilon}\right]|^r\right]}{\overline{E}^\triangleright\left[\zeta_{\sigma_{0,R}^\varepsilon}\right]^r} \to 0. \qquad \square$$

Proposition 3.2 and Lemma 4.1 result in the following.

COROLLARY 5.5. *$\overline{P} \overset{\simeq 1}{\rightleftarrows} P^\circ$, uniformly over $(\rho_0,\theta_0)\in\partial G$.*

Consider now the significance of this for the approach to the exit problem discussed in Section 2, (2.9) in particular. The corollary says that, uniformly over $x_0 = (\rho_0,\theta_0)\in\partial C$,

$$Q^\varepsilon(x_0, A) - P_{\rho_0,\theta_0}^\circ\left[(0,\theta^\varepsilon(\sigma_0^\varepsilon))\in A\right] \to 0 \quad \text{as } \varepsilon\downarrow 0.$$

[Recall that $\rho^\varepsilon(0) = \rho_0$ is fixed for $x_0\in\partial C$.] Thus Q^ε is given asymptotically by the $P_{\rho_0,\theta_0}^\circ$-distribution of $\theta^\varepsilon(\sigma_0^\varepsilon)$.

THEOREM 5.6. *Let $(\beta_1^\circ(s), \beta_2^\circ(s))$ be a pair of independent Brownian motions with respect to a probability measure P°. For a given $x_0 = (\rho_0,\theta_0)\in\partial G$ (using the coordinate system of Section 2), let $(\rho^\varepsilon(s),\theta^\varepsilon(s))$ solve*

$$(5.9)\qquad \begin{aligned} d\rho^\varepsilon(s) &= -\left[\rho^\varepsilon + \varepsilon^{1/2}d^*(\varepsilon^{-1/2}\rho^\varepsilon)\right]ds + \epsilon^{1/2}\,d\beta_1^\circ(s); \qquad \rho^\varepsilon(0) = \rho_0,\\ d\theta^\varepsilon(s) &= \frac{\langle\nabla\theta,\mathbf{b}\rangle}{\sigma_1^2}\,ds + \varepsilon^{1/2}\frac{\sigma_2}{\sigma_1}\,d\beta_2^\circ(s); \qquad \theta^\varepsilon(0) = \theta_0,\end{aligned}$$

and σ_0^ε be the first hitting time of 0 *by $\rho^\varepsilon(s)$. The conditional exit kernel Q^ε is given asymptotically by*

$$Q^\varepsilon(x_0, A) = P^\circ_{(\rho_0, \theta_0)}[\theta^\varepsilon(\sigma_0^\varepsilon) \in A] + o(1),$$

where $o(1) \to 0$ *uniformly over* $x_0 = (\rho_0, \theta_0) \in \partial G$ *as* $\varepsilon \downarrow 0$. *Moreover the distribution of σ_0^ε does not depend on θ_0 and is given asymptotically by Proposition* 3.3 *above.*

We can get another representation for Q^ε by converting back to the original t-time scale. This results in

$$Q^\varepsilon(x_0, A) = P^\circ[\theta^\varepsilon(\tau_D^\varepsilon) \in A] + o(1),$$

with

$$\begin{aligned} d\rho^\varepsilon(t) &= \left[-\langle \nabla\rho, \mathbf{b}\rangle + \varepsilon^{1/2} d^*(\varepsilon^{-1/2}\rho^\varepsilon)\sigma_1^2\right] dt + \varepsilon^{1/2}\sigma_1\, dw_1^\circ(t), \\ d\theta^\varepsilon(t) &= \langle \nabla\theta, \mathbf{b}\rangle\, dt + \varepsilon^{1/2}\sigma_2\, dw_2^\circ(t). \end{aligned} \tag{5.10}$$

The $\sigma_i = \sigma_i(\rho^\varepsilon, \theta^\varepsilon)$ are still as in (2.15) with the extensions of (2.22) and the t-Brownian motions $w_i^\circ(t)$ are related to the $\beta_i{}^\circ(s)$ just as in (2.17). Because $(\rho^\varepsilon, \theta^\varepsilon)$ are not legitimate as the coordinates of a point x^ε if $\rho^\varepsilon > R$, we must take (2.21) as the definition of τ_D^ε here. The asymptotic distribution of τ_D^ε is not as clean now because the behavior of θ^ε enters into (2.21).

Expressing this in the original x coordinate system introduces $\mathscr{O}(\varepsilon)$ drift terms again. These can be dropped with another application of Theorem 4.3, resulting in another probability measure $P^\bullet$, with $P^\circ \overset{\simeq 1}{\leftarrow} P^\bullet$, and a two-dimensional $P^\bullet$-Brownian motion $w^\bullet(t)$ such that

$$Q^\varepsilon(x_0, A) = P^\bullet_{x_0}[x^\varepsilon(\tau_D^\varepsilon) \in A; \tau_D^\varepsilon < \tau_C^\varepsilon] + o(1),$$

uniformly over $x_0 \in \partial G$, where

$$\begin{aligned} dx^\varepsilon(t) &= \left[\mathbf{b}(x^\varepsilon) - a\nabla W(x^\varepsilon) - a\epsilon^{1/2} d^*(\varepsilon^{-1/2}\rho(x^\varepsilon))\nabla\rho\right] dt \\ &\quad + \varepsilon^{1/2}\sigma(x^\varepsilon)\, dw^\bullet(t) \end{aligned} \tag{5.11}$$

for $t \le \tau_D^\varepsilon \wedge \tau_C^\varepsilon$. Here σ is the original 2×2 matrix function from (1.2). Including the restriction $\tau_D^\varepsilon < \tau_C^\varepsilon$ keeps the process in the range for which $(\rho^\varepsilon, \theta^\varepsilon)$ can be interpreted as coordinates of $x^\varepsilon \in \overline{D} \setminus C$. Now τ_D^ε and τ_C^ε can be interpreted geometrically as in (1.3) and (2.1). The difference between $\{\tau_D^\varepsilon < \tau_C^\varepsilon\} = \{\sigma_0^\varepsilon > \sigma_R^\varepsilon\}$ and $\{\tau_D^\varepsilon < \infty\} = \{\sigma_0^\varepsilon < \infty\}$ is negligible by (3.16).

It is interesting to compare (5.11) with the original (1.2). We see that the drift $\mathbf{b}$ has been replaced by the vector field for the extremals (5.6) plus the boundary layer correction involving d^*. Any of the formulations (5.9), (5.10) or (5.11) give the correct asymptotic behavior of Q^ε. We will use the P° representation (5.9) for the examples in Section 6. For those examples, the θ^ε equation will completely uncouple from the ρ^ε equation. This reduces the study of Q^ε to properties of a one-dimensional diffusion θ^ε on the appropriate time scale.

6. A class of examples. We turn our attention now to a special class of examples for which the approach of the preceding sections can be worked out more explicitly. These are essentially the examples from [8], slightly generalized. The difference is that in [8] the examples were not actually of the form (1.2), so that the $\mathscr{O}(\varepsilon)$ contributions to the drift in (2.16) did not occur. Here the examples are actually of the form (1.2). We will apply the results of the preceding sections to reduce these examples to the point that the discussion of Sections 4 and 5 of [8] applies to them; see (6.6) below.

We take D to be the unit disk

$$D = \{x \in \mathbb{R}^2 \colon |x| < 1\}.$$

Let the diffusion coefficient of (1.2) be the identity

$$\sigma(\cdot) \equiv I.$$

Assume that there exists a boundary strip

$$D_0 = \{x \colon 1 - \delta < |x| \le 1\} \quad \text{for some } 0 < \delta < 1,$$

in which drift $\mathbf{b}$ can be expressed as follows:

$$\mathbf{b}(x) = b_1(r)\frac{\partial x}{\partial r} + r^{-2}b_0(\theta)\frac{\partial x}{\partial \theta}. \tag{6.1}$$

Here (r, θ) are the usual polar coordinates, so that

$$\frac{\partial x}{\partial r} = \frac{x}{|x|}, \qquad \frac{\partial x}{\partial \theta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x. \tag{6.2}$$

In other words,

$$\mathbf{b} \cdot \frac{\partial}{\partial x} = b_1(r)\frac{\partial}{\partial r} + r^{-2}b_0(\theta)\frac{\partial}{\partial \theta}.$$

The functions b_1 and b_0 are assumed to be smooth, with $b_0(\theta)$ being 2π-periodic. In addition to the stability of origin for (1.1) in D, we assume

$$b_1(r) < 0 \quad \text{for } r < 1$$

and that there exists a constant $B > 0$ so that

$$b_1(r) \sim (r-1)B \quad \text{as } r \uparrow 1. \tag{6.3}$$

In particular, (6.3) implies that $b_1 = 0$ on $\partial D = \{r = 1\}$, so that this *is* a characteristic boundary problem, that is, (1.4) holds.

The fact that the radial component b_1 is independent of θ allows us to verify the W-regularity hypotheses of Section 2 by explicit calculation. Observe that

$$W(r) = -2\int_r^1 b_1(u)\,du$$

solves $H(x, -\nabla W(x)) = \frac{1}{2}(W'(r))^2 - W'(r)b_1(r) = 0$, with $W(1) = 0$ and $W(r) > 0$ for $1 - \delta < r < 1$. It follows that

$$W(r) \sim B(1-r)^2 \quad \text{as } r \uparrow 1,$$

and so

$$\rho(r) = W(r)^{1/2} \sim \sqrt{B}(1-r) \quad \text{as } r \uparrow 1.$$

It is easy to check that $b_1 \in C^2$ implies ρ is C^2 in $1 - \delta < r \le 1$ with

$$\rho'(1) = -\sqrt{B} \quad \text{and} \quad \rho''(1) = \frac{-b_1''(1)}{3\sqrt{B}}.$$

In particular,

$$\sigma_1^2 = \langle \nabla\rho, a\nabla\rho \rangle = (\rho'(r))^2$$

is uniformly positive in D_0, as desired. Note that on $\partial D = \{\rho = 0\}$ we have

$$\sigma_1^2 = B. \tag{6.4}$$

We also note that the lack of θ-dependence in b_1 implies that the function g of (2.6) is constant.

We pick $0 < \rho_0 < R$ so that $\{0 \le \rho \le R\} \subseteq D_0$ and take $C \subseteq G$ to be the disks

$$C = D \setminus \{0 \le \rho \le R\} \quad \text{and} \quad G = D \setminus \{0 \le \rho \le \rho_0\}.$$

Equation (2.14) is satisfied since ρ is independent of θ. Thus the natural coordinate system of Section 2 consists of $\rho = \rho(r)$ in place of the radial coordinate r, together with the usual angular coordinate θ. We have

$$\sigma_2 = \langle \nabla\theta, a\nabla\theta \rangle^{1/2} = \frac{1}{r} \quad \text{and} \quad \langle \nabla\theta, \mathbf{b} \rangle = r^{-2} b_0(\theta).$$

Thus the θ^ε equation from (5.9) becomes, on the s-time scale,

$$d\theta^\varepsilon(s) = b_0(\theta^\varepsilon)\frac{1}{(r\sigma_1)^2}\, ds + \varepsilon^{1/2}\frac{1}{r\sigma_1}\, d\beta_2^\circ(s).$$

This explains our inclusion of r^2 in the angular part of (6.1); it allows us to decouple θ^ε from ρ^ε using a random time change. Let

$$u = \int_0^s \frac{1}{(r\sigma_1)^2}\, ds,$$

$$\beta_1^*(u) = \int_0^s \frac{1}{r\sigma_1}\, d\beta_2^\circ(s).$$

Of course $r\sigma_1$ is evaluated along the path of $x^\varepsilon = (\rho^\varepsilon, \theta^\varepsilon)$ in these expressions. On this time scale we have

$$d\theta^\varepsilon(u) = b_0(\theta^\varepsilon)\, du + \varepsilon^{1/2}\, d\beta^*(u), \tag{6.5}$$

and the exit time is given by

$$\begin{aligned}\tau^* &= \int_0^{\sigma_0^\varepsilon} \frac{1}{(r\sigma_1)^2}\, ds \\ &= \inf\{u > 0 : \rho^\varepsilon(u) = 0\}.\end{aligned}$$

The corresponding $\rho^\varepsilon(u)$ equation is completely uncoupled from (6.5), so that the random variable τ^* is independent of $\theta^\varepsilon(\cdot)$. The analysis of the preceding sections tells us that the conditional kernel Q^ε of (2.3) and (2.9) is given by

$$Q^\varepsilon((\rho_0, \theta_0), A) = P^\circ_{\rho_0, \theta_0}[(0, \theta^\varepsilon(\tau^*)) \in A] + o(1),$$

where $o(1) \to 0$ uniformly over θ_0 and $A \subseteq \partial D$ as $\varepsilon \downarrow 0$. The asymptotic distribution of τ^* follows from Propositions 3.3 and 3.8.

PROPOSITION 6.1. *In the setting described above, as* $\varepsilon \downarrow 0$,

$$\tau^* - B^{-1}\log(\varepsilon^{-1/2}) - c \Rightarrow B^{-1}\mathscr{E},$$

where c is the constant

$$c = B^{-1}\log(\rho_0) + \int_0^\infty \left[\frac{1}{r^2\sigma_1^2}(\rho^*(s)) - B^{-1}\right] ds,$$

$\rho^*(s) = \rho_0 e^{-s}$ *and* $\mathscr{E}$ *is a random variable with the density* $q(s)$ *of Proposition* 3.3.

PROOF. First observe that

$$\begin{aligned}\tau^* - B^{-1}\sigma_0^\varepsilon &= \int_0^{\sigma_0^\varepsilon}\left[\frac{1}{r^2\sigma_1^2}(\rho^\varepsilon(s)) - B^{-1}\right] ds \\ &= \int_0^{\sigma_0^\varepsilon} f(\rho^\varepsilon(s))\, ds,\end{aligned}$$

where

$$f(\rho) = \frac{1}{r^2\sigma_1^2}(\rho) - B^{-1}.$$

Since this is a C^1 function with $f(0) = 0$ by (6.4), Proposition 3.8 tells us that

$$\tau^* - B^{-1}\sigma_0^\varepsilon \to \int_0^\infty f(\rho^*(s))\, ds$$

in probability. The present proposition follows from this and Proposition 3.3. □

We come to the conclusion that, for our class of examples, the exit measure μ^ε is described asymptotically by

$$\mu^\varepsilon(A) \sim P^\circ_{\nu^\varepsilon}\left[\left(0, \theta^\varepsilon(B^{-1}\log(\varepsilon^{-1/2}) + c + B^{-1}\mathscr{E})\right) \in A\right], \tag{6.6}$$

with ν^ε the averaged first exit distribution from G, as in Section 2, $\theta^\varepsilon(\cdot)$ given by (6.5) and $\mathscr{E}$ an independent random variable as above. This is (except for changing the values of some constants) the situation studied in Sections 4 and 5 of [8]. Thus all the phenomena described there also occur in the class of examples of (1.2) described in this section.

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