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A NEW APPROACH TO THE MARTIN BOUNDARY VIA DIFFUSIONS CONDITIONED TO HIT A COMPACT SET

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Let L generate a transient diffusion $X(t)$ on R^d and let D be an exterior domain. Let h be the smallest positive solution of $Lh = 0$ in D and $h = 1$ on ∂D. Define $X^h(t)$ to be the process $X(t)$ conditioned to hit ∂D. By Doob's h-transform theory, $X^h(t)$ is also a Markov diffusion and its generator L^h is defined by $L^h f = (1/h)L(hf)$. Letting τ_D be the hitting time of ∂D, define the harmonic measure for $X^h(t)$ on ∂D starting from $x \in D$ by $\mu_x^h(dy) = P_x^h(X^h(\tau_D) \in dy)$. Let $\{x_n\}_{n=1}^\infty \subset D$ be a sequence satisfying $\lim_{n\to\infty} |x_n| = \infty$ for which $\mu_{x_n}^h$ converges weakly. Call two such sequences $\{x_n\}_{n=1}^\infty$ and $\{x'_n\}_{n=1}^\infty$ equivalent if $\lim_{n\to\infty} \mu_{x_n}^h = \lim_{n\to\infty} \mu_{x'_n}^h$. We call the set of equivalence classes thus generated the *harmonic measure boundary at infinity* for L^h. This boundary is independent of the particular exterior domain D. We prove that the harmonic measure boundary at infinity for L^h coincides with the Martin boundary for $\tilde{L}$ on R^d, the formal adjoint of the operator L on R^d. In the case that L generates a reversible diffusion, the Martin boundaries of L and $\tilde{L}$ coincide and hence the harmonic measure boundary of L^h coincides with the Martin boundary for L on R^d. A similar probabilistic description of the Martin boundary for L on R^d can be given in the nonreversible case. These results are then used to give explicit representations of the Martin boundaries of L and $\tilde{L}$ for several classes of diffusion processes.

1. Statement and discussion of results. A complete probabilistic description of the bounded harmonic functions for a diffusion generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x}$$

can be given in terms of the diffusion process $X(t)$ generated by L. There exist nonconstant bounded harmonic functions if and only if the invariant σ-field for $X(t)$ is nontrivial. An equivalent but more graphic description can be given in terms of the directions through which $X(t)$ exits R^d. For any Borel set $B \subset R^d$, call the bounded harmonic function $u_B(x) \equiv P_x(X(t)$ is eventually in $B)$ an "exit probability function." Then the subspace of harmonic functions that is obtained by taking pointwise limits of linear combinations of "exit probability functions" will contain every bounded harmonic function [9]. If L does not admit any nonconstant bounded harmonic functions, then clearly this subspace contains only the constants. Since there are simple examples for which L admits nonconstant positive harmonic functions but not nonconstant

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bounded harmonic functions, it follows that the above subspace does not necessarily contain all the positive harmonic functions (see the end of this section for more elaboration).

The positive harmonic functions may be represented in terms of the Martin boundary for L. In this paper we will give a probabilistic characterization of the Martin boundary for L and for its formal adjoint $\tilde{L}$ as operators on R^d in terms of diffusions conditioned to hit a compact set. We also use this characterization to calculate explicitly the Martin boundaries for L and $\tilde{L}$ for several classes of operators. (The formal adjoint is defined by

$$\tilde{L}f = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}f) - \sum_{i=1}^{d}\frac{\partial}{\partial x}(b_i f).)$$

We begin by recalling the Martin boundary construction. For simplicity, we shall assume that the coefficients of the operator L defined above satisfy the following conditions: $a_{ij} \in C^{2+\alpha}(R^d)$, $b_i \in C^{1+\alpha}(R^d)$ for some $\alpha > 0$ and $a(x) = \{a_{ij}(x)\}_{i,j=1}^{d}$ is positive definite for each $x \in R^d$. Let $C_L = \{u \in C^2(R^d)\colon Lu = 0$ and $u > 0$ in $R^d\}$ denote the cone of positive L-harmonic functions on all of R^d. If L generates a recurrent diffusion, then it is not hard to show via an application of Itô's formula that C_L is one-dimensional; it will contain only the positive constants. Thus from here on we shall assume that L generates a transient diffusion or, equivalently, that L possesses a Green's function.

The Green's function $G(x,y)$ for L is L-harmonic in x for $x \neq y$ and $\tilde{L}$-harmonic in y for $y \neq x$. The Green's function $\tilde{G}(x,y)$ for $\tilde{L}$ is given by $\tilde{G}(x,y) = G(y,x)$. Define the Martin kernel for L by $k(x,y) \equiv G(x,y)/G(x_0,y)$, where x_0 is a fixed reference point. Then $k(x,y)$ is L-harmonic in x for $x \neq y$ and by Schauder estimates ([12], Chapter 6) and Harnack's inequality ([12], Chapter 9) it follows that $k(\cdot,y)$ is relatively compact as $|y| \to \infty$ in the topology of uniform convergence on compacts of all derivatives up to order 2. A second application of Harnack's inequality shows that any limit function will also be strictly positive. Thus, for a sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}|y_n| = \infty$ and $\lim_{n\to\infty}k(x,y_n)$ exists in the above topology, we obtain a positive L-harmonic function $k(x,\xi)$, where ξ denotes the sequence $\{y_n\}_{n=1}^{\infty}$. Two sequences $\{y_n\}_{n=1}^{\infty}$ and $\{y'_n\}_{n=1}^{\infty}$ are called *equivalent* if $\lim_{n\to\infty}k(x,y_n) = \lim_{n\to\infty}k(x,y'_n)$ for all $x \in R^d$. If equality holds for all x in some open set U, then in fact equality holds for all $x \in R^d$. Furthermore, the equivalence classes are independent of the fixed reference point x_0. The set of all equivalence classes is called the Martin boundary and is designated by Δ. Define a metric on $R^d \cup \Delta$ by

$$\rho(z_1,z_2) = \int_U \frac{|k(x,z_1) - k(x,z_2)|}{1 + |k(x,z_1) - k(x,z_2)|}\,dx,$$

where $U \subset R^d$ is an arbitrary bounded open set. The set $R^d \cup \Delta$ with the topology induced by this metric is called the Martin compactification.

A function $u \in C_L$ is called minimal if the only functions $v \in C_L$ which satisfy $v \le u$ are in fact multiples of u. Define the minimal Martin boundary by $\Delta_1 = \{\xi \in \Delta\colon k(x, \xi) \text{ is minimal}\}$. The Martin representation theorem states that all minimal elements of C_L are of the form $k(x, \xi)$ with $\xi \in \Delta_1$ and that any function $u \in C_L$ has a unique representation of the form $u(x) = \int_{\Delta_1} k(x, \xi)\mu_u(d\xi)$, where $\mu_u(d\xi)$ is a finite measure on Δ_1 [21, 17]. We shall call Δ trivial or nontrivial according to whether it consists of one point or more than one point. In particular, if Δ is trivial, then it follows from the Martin representation that C_L will be one-dimensional; it will contain only the positive constants as was the case for recurrent generators. The same type of construction may also be implemented for $\tilde{L}$ using the Green's function $\tilde{G}(x, y) = G(y, x)$ and the kernel $\tilde{k}(x, y) = \tilde{G}(x, y)/\tilde{G}(x_0, y)$.

An explicit calculation of the Martin boundary is usually very difficult and has only been carried out in a few special cases. Anderson and Schoen [2] proved that the Laplace–Beltrami operator on a manifold with curvature bounded between two negative constants has the sphere S^{d-1} as its Martin boundary. ($\Delta = S^{d-1}$ means that sequences $\{y_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|y_n| = \infty$ are Martin sequences if and only if $\lim_{n\to\infty} y_n/|y_n|$ exists on S^{d-1}.) Ancona [1] proved that for certain more general elliptic operators on the same type of manifolds, the Martin boundary is either homeomorphic to the sphere S^{d-1} or one point. Cranston, Orey and Rösler [7] considered two-dimensional Ornstein–Uhlenbeck operators, $L = \frac{1}{2}\Delta + Bx \cdot \nabla$, for a constant matrix B, and found that the Martin boundary is either homeomorphic to the sphere S^{d-1} or one point. Murato [23] considered Schrödinger operators of the form $L = \Delta + V(|x|)$ and proved that the Martin boundary is either the sphere S^{d-1} or one point. Any elliptic operator with constant coefficients may be reduced to the form $\Delta + c$, where c is a constant, by suitable transformations which preserve the Martin boundary. Thus, from Murato's result, it follows that the Martin boundary in the constant coefficient case is either the sphere S^{d-1} or one point.

We now turn to our probabilistic characterization of the Martin boundary via diffusions conditioned to hit a compact set. Let $D \subset R^d$ be a smooth exterior domain (i.e., D^c is compact). For $\psi \in C(\partial D)$, denote by u_ψ the smallest positive solution of the exterior Dirichlet problem

$$
\begin{aligned}
Lu &= 0 \quad \text{in } D, \\
u &= \psi \quad \text{on } \partial D.
\end{aligned}
\tag{1.1}
$$

Such a solution u_ψ exists and is in fact the pointwise limit as $n \to \infty$ of the solutions $u_{\psi,n}$ to the Dirichlet problem $Lu_{\psi,n} = 0$ in $D \cap \{|x| < n\}$, $u_{\psi,n} = \psi$ on ∂D and $u_{\psi,n} = 0$ on $|x| = n$. From this it follows that u_ψ has the stochastic representation $u_\psi(x) = E_x(\psi(X(\tau_D)); \tau_D < \infty)$, where $\tau_D = \inf(t \ge 0\colon X(t) \in \partial D\}$. In the special case $\psi = 1$, we will use the notation $h(x) \equiv u_1(x) = P_x(\tau_D < \infty)$.

Let $X^h(t)$ be the process $X(t)$ conditioned to hit ∂D. That is, $X^h(\cdot) = \{X(\cdot)|\tau_D < \infty\}$. Then following Doob's h-transform theory [8], $X^h(t)$, $0 \le t \le \tau_D$, is also a Markov diffusion process and its generator is given by $L^h = L +$

$a\nabla h/h \cdot \nabla$ [i.e., $L^h f = (1/h)L(hf)$]. For $x \in D$, let $\mu_x^h(dy) = P_x^h(X^h(\tau_D) \in dy)$ denote the harmonic (probability) measure on ∂D for the process $X^h(t)$ (or the operator L^h) starting from $x \in D$. Note that $\{\mu_x^h;\, x \in D\}$ is a tight family since ∂D is compact. Let $\{x_n\}_{n=1}^{\infty} \subset D$ be a sequence satisfying $\lim_{n\to\infty} |x_n| = \infty$ for which $\mu_{x_n}^h(dy)$ converges weakly. Call two such sequences, $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$, equivalent if $\lim_{n\to\infty} \mu_{x_n}^h(dy) = \lim_{n\to\infty} \mu_{x'_n}^h(dy)$. We call the set of equivalence classes thus generated the *harmonic measure boundary at infinity* for L^h. It follows from Theorem 1 below that this definition is independent of D.

THEOREM 1. *The Martin boundary of* $\tilde{L}$ *coincides with the harmonic measure boundary at infinity for* L^h.

Recall that L generates a reversible diffusion if there exists a σ-finite measure $\phi(x)\,dx$ on R^d such that $\phi(x)p(t,x,y) = \phi(y)p(t,y,x)$, where $p(t,x,y)$ denotes the transition probability density for L. Equivalently, L is self-adjoint on $L^2(R^d, \phi(x)\,dx)$ with an appropriate domain of definition. Now, since $G(x,y) = \int_0^\infty p(t,x,y)\,dt$ and $\tilde{G}(x,y) = \int_0^\infty p(t,y,x)\,dt$, it follows that in the reversible case, $G(x,y) = [\phi(y)/\phi(x)]\tilde{G}(x,y)$. Consequently, the Martin kernels $k(x;y)$ and $\tilde{k}(x,y)$ differ only by a multiplicative function of x and thus define identical Martin boundaries. We thus have the following corollary.

COROLLARY 1. *If* L *is self-adjoint with respect to some reference measure or, equivalently, if* L *generates a reversible diffusion, then the Martin boundary of* L *coincides with the harmonic measure boundary at infinity for* L^h.

Consider rotationally symmetric operators of the form

$$L = \frac{1}{2}a(r)\frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r} + \frac{1}{2m^2(r)}\Delta_{S^{d-1}},$$

where $\Delta_{S^{d-1}}$ is the Laplace–Beltrami operator on S^{d-1}. As an application of Theorem 1, we will prove the following theorem.

THEOREM 2. *Let*

$$L = \frac{1}{2}a(r)\frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r} + \frac{1}{2m^2(r)}\Delta_{S^{d-1}}$$

generate a transient diffusion process. Then the Martin boundary of L *is either one point or the sphere* S^{d-1} *according to whether the integral*

$$\int_1^\infty dr\frac{1}{a(r)m^2(r)}\exp\left(\int_1^r 2\frac{b}{a}(s)\,ds\right)\int_r^\infty dt\,\exp\left(-\int_1^t 2\frac{b}{a}(s)\,ds\right)$$

is infinite or finite.

After submitting this paper for publication, we discovered that actually, after making the appropriate transformations, this theorem is essentially equivalent to Murato's result mentioned above [23]. Murato used entirely different nonprobabilistic methods.

We now turn to a companion theorem to Theorem 1, which characterizes the Martin boundary for the operator L in the nonreversible case. Analogous to (1.1), let $\tilde{u}_\psi$ denote the smallest positive solution to the exterior Dirichlet problem

$$\begin{aligned} \tilde{L}\tilde{u} &= 0 \quad \text{in } D, \\ \tilde{u} &= \psi \quad \text{on } \partial D. \end{aligned} \tag{1.2}$$

Such a solution exists and is the pointwise limit as $n \to \infty$ of $\tilde{u}_{\psi,n}$, where $\tilde{u}_{\psi,n}$ solves $\tilde{L}\tilde{u}_{\psi,n} = 0$ in $D \cap \{|x| < n\}$, $\tilde{u}_{\psi,n} = \psi$ on ∂D and $\tilde{u}_{\psi,n} = 0$ on $|x| = n$. In the special case $\psi \equiv 1$, we will write $\tilde{h}(x) \equiv \tilde{u}_1(x)$. We want to give a probabilistic representation for $\tilde{u}_\psi$ as we did for u_ψ. To this end, let $c = \sum_{i,j=1}^d (a_{ij})_{x_i x_j} - \nabla \cdot b$ denote the zeroth-order part of the operator $\tilde{L}$, let $\hat{L} = \tilde{L} - c$ and let $\hat{X}(t)$ denote the diffusion process generated by $\hat{L}$. Now apply the Feynman–Kac formula to $G(x,y)$ as a function of $y \in D$ for fixed $x \in (\overline{D})^c$. Since $G(x,y)$ is $\tilde{L}$-harmonic as a function of $y \in D$, we obtain

$$\begin{aligned} G(x,y) &= \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c(\hat{X}(s))\,ds\right)G(x,\hat{X}(\tau_D));\ \tau_D < t \wedge \tau_n\right) \\ &\quad + \hat{E}_x\left(\exp\left(\int_0^{t\wedge\tau_n} c(\hat{X}(s))\,ds\right)G(x,\hat{X}(t\wedge\tau_n));\ \tau_D \ge t \wedge \tau_n\right). \end{aligned} \tag{1.3}$$

Both terms on the right-hand side of (1.3) are positive and thus, letting $t \to \infty$ and then $n \to \infty$ and using monotone convergence and the uniform positivity of $G(x,y)$ for $y \in \partial D$, we conclude that

$$\hat{E}_x\left(\exp\left(\int_0^{\tau_D} c(\hat{X}(s))\,ds\right);\ \tau_D < \infty\right) < \infty. \tag{1.4}$$

In light of (1.4) we can define

$$\tilde{v}_\psi(x) = \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c(\hat{X}(s))\right)ds\right)\psi(\hat{X}(\tau_D));\ \tau_D < \infty\Big), \tag{1.5}$$

for $\psi \in C(\partial D)$. By the Feynman–Kac formula, $\tilde{v}_\psi$ satisfies (1.2). Furthermore, if w is any positive solution to (1.2), then applying (1.3) to w instead of G shows that $\tilde{v}_\psi \le w$. Thus in fact $\tilde{v}_\psi(x) = \tilde{u}_\psi(x)$ and the right-hand side of (1.5) gives a probabilistic representation of $\tilde{u}_\psi$.

In particular,

$$\tilde{h}(x) = \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c(\hat{X}(s))\,ds\right);\ \tau_D < \infty\right). \tag{1.6}$$

[Note that $\hat{L}$ may be recurrent in which case $\{\tau_D < \infty\}$ is superfluous in (1.5) and (1.6).]

Analogous to the way we defined $\mu_x^h(dy)$, define $\tilde{\mu}_x^{\tilde{h}}(dy)$, the harmonic (probability) measure on ∂D starting at $x \in D$ for $\tilde{L}^{\tilde{h}}$, by

$$(1.7)\quad \tilde{u}_x^{\tilde{h}}(dy) = \frac{1}{\tilde{h}(x)} \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c(\hat{X}(s))\,ds\right); \tau_D < \infty,\ \hat{X}(\tau_D) \in dy\right).$$

Analogous to the way we defined the harmonic measure boundary at infinity for L^h, define the harmonic measure boundary at infinity for $\tilde{L}^{\tilde{h}}$ as the collection of equivalence classes determined by those sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ for which $\tilde{\mu}_{x_n}^{\tilde{h}}(dy)$ converges weakly, calling two such sequences equivalent if they give rise to the same limiting measure.

THEOREM 3. *The Martin boundary of L coincides with the harmonic measure boundary at infinity for* $\tilde{L}^{\tilde{h}}$.

This characterization is not as appealing probabilistically as Theorem 1 because the harmonic measures $\tilde{\mu}_x^{\tilde{h}}(dy)$ are not exit measures of diffusions but rather exit measures of diffusions with potentials. In particular, this means that the calculation of $\tilde{\mu}_x^{\tilde{h}}(dy)$ involves a functional integral depending on the whole path of the diffusion and not just on its position at the exit time. Nonetheless, we are able to use Theorem 3 to prove part (a) of Theorem 4 below.

If one is interested not in the complete characterization of the Martin boundary, but just in whether or not there exist nonconstant positive harmonic functions for L, that is, whether or not the Martin boundary is trivial for L, then we suspect that its suffices to consider the more probabilistically appealing Theorem 1. More precisely, we make the following conjecture.

CONJECTURE. *The Martin boundary for L is trivial if and only if the Martin boundary for* $\tilde{L}$ *is trivial, or, equivalently by Theorem* 1, *the Martin boundary for L is trivial if and only if the harmonic measure boundary at infinity for* L^h *is trivial.*

REMARK. The harmonic measure boundary at infinity for L^h plays an important role in determining the asymptotic behavior of the solution of the exterior Dirichlet problem for an elliptic operator perturbed by a small drift; see [25] and [14]. In particular, the conjecture above has significant implications with respect to the main result in [25]. See Remark 2 following Theorem 1 in that paper.

As an application of Theorems 1 and 3, consider the class of operators

$$L = \frac{1}{2}\Delta + r^{-\delta}\frac{\partial}{\partial r} + r^{-k}\frac{1}{r}\frac{\partial}{\partial \theta}$$

on R^2 with $-1 < \delta < 1$ and $-\infty < k < \infty$. A simple calculation reveals that

$$\tilde{L} = \frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} - r^{-k}\frac{1}{r}\frac{\partial}{\partial\theta} - (1-\delta)r^{-\delta-1}.$$

Define

$$(1.8)\qquad U(x) = U(r,\theta) = \begin{cases} \left(\dfrac{r^{\delta-k}}{\delta-k} - \theta\right) \bmod 2\pi, & \text{if } k \neq \delta, \\ (\log r - \theta) \bmod 2\pi, & \text{if } k = \delta. \end{cases}$$

Then, for $0 \le c < 2\pi$, the trajectories $U = c$ form distinct spirals running out to ∞ counterclockwise which are, in fact, the trajectories of the deterministic dynamical system

$$(1.9)\qquad \{\bar{r}'(t) = \bar{r}^{-\delta}(t),\ \bar{\theta}'(t) = \bar{r}^{-k-1}(t)\}$$

obtained by ignoring the diffusion part of the generator L. [Actually, if $k > \delta$, the spirals are degenerate in the sense that they only wrap around the origin finitely many times as $t \to \infty$. That is, $\bar{\theta}(t)$ converges as $t \to \infty$.] Also define

$$(1.10)\qquad V(r,\theta) = U(r,-\theta).$$

Then, for $0 \le c < 2\pi$, the trajectories $V = c$ form distinct spirals running out to ∞ clockwise which are the trajectories of the dynamical system

$$(1.11)\qquad \{\bar{r}'(t) = \bar{r}^{-\delta}(t),\ \bar{\theta}'(t) = -\bar{r}^{-k-1}(t)\}.$$

(Again, if $k > \delta$, the spirals are degenerate.)

Now, consider sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ and for which $\lim_{n\to\infty} U(x_n)$ exists. Call two such sequences, $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$, U-equivalent if $\lim_{n\to\infty} U(x_n) = \lim_{n\to\infty} U(x'_n)$. Define V-equivalence analogously. It follows that the collection of U-equivalence classes can be identified with the spirals $U = c$, $0 \le c < 2\pi$ corresponding to the solutions of (1.9) and that the collection of V-equivalence classes can be identified with the spirals $V = c$, $0 \le c < 2\pi$ corresponding to the solutions of (1.11).

THEOREM 4. *Let*

$$L = \frac{1}{2}\Delta + r^{-\delta}\frac{\partial}{\partial r} + r^{-k}\frac{1}{r}\frac{\partial}{\partial\theta}$$

with formal adjoint

$$\tilde{L} = \frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} - r^{-k}\frac{1}{r}\frac{\partial}{\partial\theta} - (1-\delta)r^{-\delta-1}.$$

(a) (i) *If* $k > \delta - \frac{1}{2}(1-\delta)$, *then the Martin boundary of L coincides with the U-equivalence classes of counterclockwise spirals corresponding to the solutions of* (1.9).

(ii) *If* $k \le \delta - \frac{1}{2}(1-\delta)$, *then the Martin boundary of L consists of a single point.*

(b) (i) *If* $k > \delta - \frac{1}{2}(1 - \delta)$, *then the Martin boundary of* $\tilde{L}$ *coincides with the V-equivalence classes of clockwise spirals corresponding to the solutions of* (1.11).

(ii) *If* $k \le \delta - \frac{1}{2}(1 - \delta)$, *then the Martin boundary of* $\tilde{L}$ *consists of a single point.*

REMARK 1. Note that, with the exception of Ancona's work, all the results on the Martin boundary quoted above concern reversible diffusions or, equivalently, self-adjoint operators. This result appears to be the first one in the literature where the Martin boundaries for a non-self-adjoint elliptic operator L and its formal adjoint $\tilde{L}$ are calculated explicitly and are found to differ.

REMARK 2. Part (a) of Theorem 4 significantly extends a result in [24]. There it was shown that if $k > \delta - \frac{1}{2}(1 - \delta)$, then the Martin boundary is at least as large as the U-equivalence classes and that if $k \le \delta - (1 - \delta)$ (note the gap), then the Martin boundary is one point.

In the first paragraph of this article, we noted that the subspace of harmonic functions that is obtained via pointwise limits of "exit probability functions" contains all of the bounded harmonic functions, but does not necessarily contain all of the positive harmonic functions. To see why, recall that the Doob–Hunt theory applied to diffusion processes [8, 13, 17, 27] states that as $t \to \zeta^-$ (ζ is the lifetime of the process and is equal to ∞ if the process does not explode), $X(t)$ converges in the Martin topology to a point on the minimal Martin boundary Δ_1, and that in fact for any Borel set $B \subset \Delta_1$,

$$P_x\Big(\lim_{t \to \zeta^-} X(t) \in B\Big) = \int_B k(x, \xi)\mu_1(d\xi),$$

where μ_1 is the unique (probability) measure appearing in the representation for the harmonic function $u \equiv 1$. Thus, in exiting R^d, $X(t)$ can "distinguish" between any two points in $\operatorname{supp}\mu_1$ but cannot distinguish points in $\Delta - \Delta_1$. Consequently, a positive harmonic function u will belong to the subspace of harmonic functions generated by the "exit probability functions" if and only if the measure μ_u in its Martin representation satisfies $\operatorname{supp}\mu_u \subseteq \operatorname{supp}\mu_1$.

To illustrate this phenomenon, consider the two generators $L_1 = \frac{1}{2}\Delta$ and $L_2 = \frac{1}{2}\Delta + b \cdot \nabla$, where $b \neq 0$ is a constant vector, and let $X_1(t)$ and $X_2(t)$ denote the diffusions they generate. Then $X_1(t) = B(t)$, where $B(t)$ is a standard Brownian motion and $X_2(t) = B(t) + bt$. It is well known that there are no positive harmonic functions for $L_1 = \frac{1}{2}\Delta$, that is, that the Martin boundary for L_1 is one point. Thus $\lim_{t \to \infty} X_1(t)$ converges to the unique point on the Martin boundary. On the other hand, for each v on the sphere $|b + v| = |b|$, the function $u_v(x) = e^{x \cdot v}$ is a minimal nonconstant positive harmonic function for L_2. If $v = 0$, we obtain $u_0 \equiv 1$; otherwise u_v is unbounded. It is known that for L_2, $\Delta = \Delta_1 = S^{d-1}$. (This is the constant coefficients case which, as mentioned above, is covered by [23].) It turns out that the point $b/|b| \in S^{d-1}$ corresponds to the minimal positive harmonic function $u_0 \equiv 1$. And, by the ergodic theorem, $\lim_{t \to \infty} X_2(t)/|X_2(t)| = b/|b|$ a.s. Thus both $X_1(t)$ and $X_2(t)$ "see" only one point on the Martin boundary. In the former case, that point is the unique point on the Martin boundary; in the latter case, that point is only one point out of a whole sphere of Martin boundary points.

It might be interesting to attempt to connect the theory developed in this paper with the description of the Martin boundary in terms of the Ray–Knight compactification and the entrance boundary of the time-reversed process [26, 18, 27].

Theorems 1 and 3 will be proved in Section 2 and Theorem 2 will be proved in Section 3. In Section 4 we prove a result concerning one-dimensional conditioned diffusions which is needed for the proof of Theorem 4 but which also seems to be of some independent interest. The proof of Theorem 4 is given in Section 5.

For the proof of Theorem 1, we will need the following technical result.

THEOREM 5. *Let $D \subset R^d$ be a smooth exterior domain and let*

$$\mathscr{A} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

generate a diffusion for which $P_x(\tau_D < \infty) = 1$ for $x \in D$. Assume that $a_{ij} \in C^{1,\alpha}(R^d)$, $b_i \in C^{\alpha}(R^d)$ and that $a(x)$ is nonsingular for each $x \in R^d$. Also assume that ∂D is a $C^{1,\alpha}$-boundary. Then the harmonic measures on ∂D, $\{\mu_x(dy)\}_{x \in D}$, and the corresponding collection of harmonic measures at infinity for $\mathscr{A}$ relative to D are all mutually absolutely continuous with respect to Lebesgue measure on ∂D and possess strictly positive densities which are in $C^{\alpha}(\partial D)$.

REMARK. Actually for the proof of Theorem 1, we need only the existence of a bounded density.

PROOF. The existence of such a density in the case that D is bounded follows from [22], page 79. The extension to the present situation is via a fairly straightforward application of the strong Markov property; we leave this to the reader. □

2. Proofs of Theorems 1 and 3.

PROOF OF THEOREM 1. For $x \in (\overline{D})^c$, $\tilde{G}(x,y) = G(y,x)$ is L-harmonic as a function of $y \in D$. Define for $x \in (\overline{D})^c$, $\psi_x(z) \equiv \tilde{G}(x,z)|_{z \in \partial D}$. By Itô's formula,

$$\tilde{G}(x,y) = E_y\big(\psi_x(X(\tau_D)); \tau_D \le t\big) + E_y\big(\tilde{G}(x, X(t)); \tau_D > t\big).$$

Letting $t \to \infty$, we obtain $\tilde{G}(x,y) \ge E_y(\psi_x(X(\tau_D)); \tau_D < \infty)$. However, the right-hand side of the above inequality is L-harmonic as a function of $y \in D$ and equals ψ_x on ∂D. Since by the definition of the Green's function, $\tilde{G}(x,y)$ is the smallest positive L-harmonic function satisfying this, we conclude that

$$\tilde{G}(x,y) = E_y(\psi_x(X(\tau_D)); \tau_D < \infty), \quad \text{for } x \in (\overline{D})^c \text{ and } y \in D. \tag{2.1}$$

Pick a reference point $x_0 \in (\overline{D})^c$. Let $\tilde{k}(x,y)$ denote the Martin kernel for $\tilde{L}$. Using (2.1) and the definition of the conditioned process $X^h(t)$, we have

$$\begin{aligned}\tilde{k}(x,y) &= \frac{\tilde{G}(x,y)}{\tilde{G}(x_0,y)} = \frac{\tilde{G}(x,y)/h(y)}{\tilde{G}(x_0,y)/h(y)} \\ &= \frac{E_y^h\psi_x(X^h(\tau_D))}{E_y^h\psi_{x_0}(X^h(\tau_D))} = \frac{\int_{\partial D}\psi_x(z)\mu_y^h(dz)}{\int_{\partial D}\psi_{x_0}(z)\mu_y^h(dz)},\end{aligned} \tag{2.2}$$

for $x \in (\overline{D})^c$ and $y \in D$.

Now we recall that if $\lim_{n\to\infty} k(x,y_n)$ exists for $x \in (\overline{D})^c$ and some sequence $\{y_n\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty}|y_n| = \infty$, then in fact $\lim_{n\to\infty} k(x,y_n)$ exists for all $x \in R^d$. For, if not, then by the weak compactness, there would exist two subsequences $\xi_1 \equiv \{y'_n\}_{n=1}^\infty$ and $\xi_2 \equiv \{y''_n\}_{n=1}^\infty$ such that $k(x,\xi_1) = k(x,\xi_2)$ for $x \in (\overline{D})^c$ but $k(x,\xi_1) \not\equiv k(x,\xi_2)$. But then the difference of these two functions would be an L-harmonic function vanishing on $(\overline{D})^c$ but not identically 0. This is impossible. From this fact and (2.2), it follows immediately that if $\{y_n\}_{n=1}^\infty$ is a harmonic measure boundary point (calculated for the domain D), then it is also a Martin boundary point.

We will show now that in fact the two boundaries coincide. This will then also prove that the harmonic measure boundary at infinity is independent of D. We must show that if $\{z_n\}_{n=1}^\infty$ is a Martin boundary point, then it is also a harmonic measure boundary point. Assume to the contrary. Then there exist two subsequences, $\{y_n\}_{n=1}^\infty$ and $\{y'_n\}_{n=1}^\infty$, of $\{z_n\}_{n=1}^\infty$, corresponding to two distinct points on the harmonic measure boundary at infinity. That is, $\nu_1(dz) \equiv \lim_{n\to\infty}\mu_{y_n}^h(dz)$ and $\nu_2(dz) \equiv \lim_{n\to\infty}\mu_{y'_n}^h(dz)$ are distinct. Since $\{y_n\}_{n=1}^\infty$ and $\{y'_n\}_{n=1}^\infty$ are subsequences of $\{z_n\}_{n=1}^\infty$, they correspond to one and the same Martin boundary point. Plugging $\{y_n\}_{n=1}^\infty$ and $\{y'_n\}_{n=1}^\infty$ into (2.2) and recalling that $\psi_x(z) = \tilde{G}(x,z)$, we obtain

$$\frac{\int_{\partial D}\tilde{G}(x,z)\nu_1(dz)}{\int_{\partial D}\tilde{G}(x_0,z)\nu_1(dz)} = \frac{\int_{\partial D}\tilde{G}(x,z)\nu_2(dz)}{\int_{\partial D}\tilde{G}(x_0,z)\nu_2(dz)}, \tag{2.3}$$

for $x \in (\overline{D})^c$. Define the Green potentials

$$u_1(x) = \int_{\partial D}\tilde{G}(x,z)\nu_1(dz) \quad \text{and} \quad u_2(x) = \int_{\partial D}\tilde{G}(x,z)\nu_2(dz),$$

for $x \in R^d$ and let $\gamma = u_1(x_0)/u_2(x_0)$. We can rewrite (2.3) as

$$u_1(x) = \gamma u_2(x), \quad \text{for } x \in (\overline{D})^c. \tag{2.4}$$

By Theorem 5, $\nu_1(dz)$ and $\nu_2(dz)$ possess bounded densities. From this, from the fact that the singularity in $\tilde{G}(x,z)$ is of order $1/|x-z|^{d-2}$ if $d \geq 3$ and of order $\log|x-z|$ if $d = 2$ and from the fact that ∂D is a $(d-1)$-dimensional hypersurface, it follows that $u_1(x)$ and $u_2(x)$ are continuous on ∂D and thus on all R^d. Then, from (2.4), it follows that $u_1(x) = \gamma u_2(x)$ for $x \in \partial D$. Define $\psi(x) = u_1(x) = \gamma u_2(x)$ for $x \in \partial D$. Note that $u_1(x)$ and $u_2(x)$ are $\tilde{L}$-harmonic in D. Let $\tilde{v}_\psi$ denote the smallest positive $\tilde{L}$-harmonic function on D that equals ψ on ∂D. From (1.5),

$$\tilde{v}_\psi = \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c\big(\hat{X}(s)\big)\,ds\right)\psi\big(\hat{X}(\tau_D)\big);\ \tau_D < \infty\right), \quad \text{for } x \in \overline{D}.$$

We will show that $u_1(x) = \gamma u_2(x) = \tilde{v}_\psi$ for $x \in \overline{D}$. From this and (2.4), it will then follow that $u_1(x) = \gamma u_2(x)$ for all $x \in R^d$. But then by the uniqueness part of the Riesz decomposition theorem for superharmonic functions [4], it will follow that $\nu_1 = \gamma \nu_2$. Since ν_1 and ν_2 are probability measures, in fact $\gamma = 1$ and $\nu_1 = \nu_2$. This is a contradiction.

It remains to show that that $u_1 = \gamma u_2 = \tilde{v}_\psi$ for $x \in \overline{D}$. Let $\tilde{G}_n(x, y)$ denote the Green's function for the operator $\tilde{L}$ in $B_n = \{|x| < n\}$ with the Dirichlet boundary condition on ∂B_n. Define $u_{i,n}(x) = \int_{\partial D} \tilde{G}_n(x, z)\nu_i(dz)$, $i = 1, 2$, if $D^c \subset B_n$ and let $\psi_{i,n}(x) = u_{i,n}(x)|_{\partial D}$. Then $u_{i,n}(x)$ is $\tilde{L}$-harmonic in $B_n - \overline{D}^c$ and $u_{i,n} = 0$ on ∂B_n. Since $\tilde{G}_n(x, y) \uparrow \tilde{G}(x, y)$ as $n \to \infty$, for all $x \in R^d - \{y\}$, it follows that $u_{i,n}(x) \uparrow u_i(x)$ as $n \to \infty$, for all $x \in R^d$ and $i = 1, 2$. In particular, $\psi_{1,n}(x) \uparrow \psi(x)$ and $\psi_{2,n}(x) \uparrow \psi(x)/\gamma$ as $n \to \infty$, for $x \in \partial D$. By the Feynman–Kac formula,

$$u_{i,n}(x) = \hat{E}_x\left(\exp\left(\int_0^{\tau_D} c\big(\hat{X}(s)\big)\,ds\right)\psi_{i,n}\big(\hat{X}(\tau_D)\big);\ \tau_D < \tau_{B_n}\right),$$

for $x \in B_n - D^c$. Letting $n \to \infty$ gives $u_1(x) = \gamma u_2(x) = \tilde{v}_\psi(x)$, for $x \in \overline{D}$ and $i = 1, 2$. □

PROOF OF THEOREM 3. The proof of Theorem 3 is essentially the same as that of Theorem 1—one just interchanges the roles of L and $\tilde{L}$. There is one point, though, that requires comment. In the proof showing that the Green potentials are continuous on ∂D, use was made of the fact that $\nu_1(dz) = \lim_{n \to \infty} \mu^h_{y_n}(dz)$ and $\nu_2(dz) = \lim_{n \to \infty} \mu^h_{y'_n}(dz)$ possess bounded densities. This followed from Theorem 5. In the proof of Theorem 3, we thus need to know that the weak limits of $\tilde{u}^h_{y_n}(dz)$ possess bounded densities. One can prove this similarly to the proof of Theorem 5. □

3. Proof of Theorem 2. The harmonic measure boundary at infinity which we defined in Section 1 for the generator L^h can be defined in the exact same manner for any diffusion generator $\mathscr{A}$ for which the corresponding diffusion process satisfies $P_x(\tau_D < \infty) = 1$. For the proof of Theorem 2, we need the following theorem.

THEOREM 6. *Let*

$$\mathscr{A} = \frac{1}{2}a(r)\frac{\partial^2}{\partial r^2} + b(r)\frac{\partial}{\partial r} + \frac{1}{2m^2(r)}\Delta_{S^{d-1}}$$

generate a diffusion in $D = \{|x| > 1\}$ *and assume that* $P_x(\tau_D < \infty) = 1$ *for* $x \in D$. *Then the harmonic measure boundary at infinity for* $\mathscr{A}$ *is a single point or the sphere* S^{d-1} *according to whether*

$$\int_1^\infty dr \exp\left(-\int_1^r 2\frac{b}{a}(s)\,ds\right)\int_r^\infty dt \frac{1}{a(t)m^2(t)} \exp\left(\int_1^t 2\frac{b}{a}(s)\,ds\right)$$

is infinite or finite.

For the proof of Theorem 6, we need to introduce the concept of "explosion inward from infinity." For a process satisfying $P_x(\tau_D < \infty) = 1$ for all $x \in D$, consider the following two possibilities:

(i) $\lim_{|x|\to\infty} P_x(\tau_D < t) = 0$ for all $t > 0$;
(ii) condition (i) fails.

Following Azencott [3], condition (i) is necessary and sufficient for the semigroup corresponding to the process to leave invariant $C_0(R^d)$, the class of continuous functions vanishing at ∞. Note that case (ii) means that there exists a fixed time t, a number $\varepsilon > 0$ and arbitrarily large x's such that the process starting from x will hit the compact set D^c before time t with probability at least ε. In analogy with explosion, we will say that in case (ii) the diffusion *explodes inward from infinity*. In the one-dimensional case, in the language of Feller [10], explosion inward from infinity is equivalent to ∞ being an entrance boundary for the process.

A test exists for explosion inward from infinity (see Azencott [3], Proposition 4.3).

Test for explosion inward from infinity. A one-dimensional diffusion generated by

$$\mathscr{L} = \frac{1}{2}a(r)\frac{d^2}{dr^2} + b(r)\frac{d}{dr} \tag{3.1}$$

on $D = [1, \infty)$ and satisfying $P_r(\tau_D < \infty) = 1$ for $r \in D$, explodes inward from infinity if and only if

$$\int_1^\infty dr \exp\left(-\int_1^r 2\frac{b}{a}(y)\,dy\right)\int_r^\infty dy \frac{1}{a(y)} \exp\left(\int_1^y 2\frac{b}{a}(z)\,dz\right) < \infty.$$

To prove Theorem 6, we will utilize the following skew product representation for a diffusion $X(t) = (r(t), \phi(t))$ generated by a radially symmetric generator $\mathscr{A}$ as in the statement of the theorem:

$$r(t) = r(t; r_0) = r_0 + \int_0^t \sigma(r(s; r_0))\,d\omega(s) + \int_0^t b(r(s; r_0))\,ds,$$

$$\phi(t) = \phi(t; r_0, \phi_0) = \theta(\rho(t; r_0); \phi_0),$$

where $\omega(\cdot)$ is a one-dimensional Brownian motion, $\theta(\cdot; \phi_0)$ is a Brownian motion on S^{d-1} starting from ϕ_0 and independent of $\omega(\cdot)$, $\sigma = a^{1/2}$ and

$$\rho(t; r_0) = \int_0^t \frac{ds}{m^2(r(s; r_0))}.$$

Note that $r(t; r_0)$ is generated by $\mathscr{L}$ as in (3.1). Let $\tau_y = \tau_y(r_0) = \inf\{t \geq 0: r(t; r_0) = y\}$ and let $(\Omega, \mathscr{F}, P)$ denote the probability space on which $\omega(\cdot)$ and $\theta(\cdot\,; \phi_0)$ are defined. The key to the proof of Theorem 6 is the following lemma. [For ease of notation, we will write τ_y for $\tau_y(r_0)$ in the sequel.]

LEMMA 1. *Let $D = [1, \infty)$ and assume that the process generated by $\mathscr{L}$ as in (3.1) satisfies $P_x(\tau_D < \infty) = 1$ for $x \in D$. Then $P(\lim_{r_0 \to \infty} \rho(\tau_1; r_0) < \infty) = 1$ or 0 according to whether*

$$\int_1^\infty dr \exp\left(-\int_1^r 2\frac{b}{a}(s)\,ds\right)\int_r^\infty dt \frac{1}{a(t)m^2(t)} \exp\left(\int_1^t 2\frac{b}{a}(s)\,ds\right)$$

is finite or infinite.

PROOF. Since $\rho(t; r_0)$ is increasing in t, we have $\rho(\tau_1; r_0) = \inf\{\rho(t; r_0): r(t; r_0) = 1\} = \inf\{t: r(\sigma(t; r_0); r_0) = 1\}$, where $\sigma(t; r_0)$ is the inverse of $\rho(t; r_0)$. Thus $\rho(\tau_1; r_0)$ is the first hitting time of 1 by the process $r(\sigma(t; r_0); r_0)$. The generator of this process is

$$\frac{1}{2} m^2 a \frac{d^2}{dr^2} + m^2 b \frac{d}{dr}.$$

Applying the test for explosion inward from infinity, we see that $r(\sigma(t; r_0); r_0)$ explodes inward from infinity if and only if the integral appearing in the statement of the lemma is finite.

On the other hand, from the above representation, which identifies $\rho(\tau_1; r_0)$ with the first hitting time of 1 by $r(\sigma(t; r_0); r_0)$, it follows that this process does not explode inward from infinity if $\lim_{r_0 \to \infty} \rho(\tau_1; r_0) = \infty$ a.s. and does explode inward from infinity otherwise, that is, if $P(\lim_{r_0 \to \infty} \rho(\tau_1; r_0) < \infty) > 0$. The lemma will be proved if we show that $P(\lim_{r_0 \to \infty} \rho(\tau_1; r_0) < \infty) = 0$ or 1. We can represent $\rho(\tau_1; r_0)$ by

$$\rho(\tau_1; r_0) = \sum_{j=0}^{[r_0]-1} \int_0^{\tau_{[r_0]-j}} \frac{dt}{m^2(r(t; r_0))}. \tag{3.2}$$

By the strong Markov property, the above summands are independent. Thus $\lim_{r_0 \to \infty} \rho(\tau_1; r_0)$ is actually an infinite sum of independent random variables. Since $\lim_{r_0 \to \infty} \rho(\tau_1; r_0) < \infty$ is a tail event, it follows from the Kolomogorov zero–one law that $P(\lim_{r_0 \to \infty} \rho(\tau_1; r_0) < \infty) = 0$ or 1 [5]. □

PROOF OF THEOREM 6. Let $x_0 = (r_0, \phi_0)$. Then

$$\mu_{x_0}(dy) = \mu_{(r_0, \phi_0)}(dy) \equiv P(\phi(\tau_1; r_0, \phi_0) \in dy) = P(\theta(\rho(\tau_1; r_0); \phi_0) \in dy).$$

Since $\theta(\cdot\,;\phi_0)$ is ergodic on S^{d-1} and since $\theta(\cdot\,;\phi_0)$ and $\rho(\tau_1;r_0)$ are independent, it is clear that if $\lim_{r_0\to\infty}\rho(\tau_1;r_0)=\infty$ a.s., then $\lim_{r_0\to\infty}\mu_{(r_0,\phi_0)}(dy)$ will be normalized Lebesgue measure on S^{d-1}, for all $\phi_0\in S^{d-1}$. On the other hand, if $\lim_{r_0\to\infty}\rho(\tau_1;r_0)<\infty$ a.s., then

$$\lim_{r_0\to\infty}\mu_{(r_0,\phi_0)}(dy)=\int_0^\infty P(\theta(t;\phi_0)\in dy)\gamma(dt),$$

where $\gamma(dt)$ is the distribution of $\lim_{r_0\to\infty}\rho(\tau_1;r_0)$. Since $\theta(t;\phi_0)$ is "centered" at ϕ_0, that is, its density attains its unique maximum at ϕ_0, it follows that in this case $\lim_{r_0\to\infty}\mu_{(r_0,\phi_0)}(dy)$ is distinct for each $\phi_0\in S^{d-1}$. Thus the theorem follows from Lemma 1. □

Proof of Theorem 2. If we take $D=\{|x|>1\}$, then $h(x)=P_x(\tau_D<\infty)$ is a function of $r=|x|$ and thus L^h is also radially symmetric. It then follows immediately from Theorem 6 applied to L^h and Corollary 1 applied to the reversible generator L, that the Martin boundary of L must be either one point or the sphere S^{d-1}. It remains to show that the integral condition in the statement of the theorem determines which of these two possibilities occurs.

Write the diffusion process $X(t)$ generated by L in polar coordinates: $X(t)=(r(t),\phi(t))$. In a paper by March [20], it was shown that $\lim_{t\to\zeta^-}\phi(t)$ exists a.s. P_x and is supported on all of S^{d-1} if the integral appearing in the statement of the theorem is finite, and that $P_x(\phi(t)\in B$ eventually$)=0$ for all $B\subset S^{d-1}$ such that B^c has positive Lebesgue measure, if the integral is infinite. (March's technique was similar to that used to prove Theorem 6. Whereas Theorem 6 relies on Azencott's integral test for explosion inward from infinity, March's proof relies on Feller's integral test for explosion [10].) Now, if the integral appearing in the statement of the theorem is finite so that $\lim_{t\to\zeta^-}\phi(t)$ exists a.s. P_x and is supported on all of S^{d-1}, then for each $B\subset S^{d-1}$ such that B and B^c have positive Lebesgue measure, the function $u(x)=P_x(\lim_{t\to\zeta^-}\phi(t)\in B)$ is a nonconstant bounded harmonic function. Thus the Martin boundary is nontrivial and from the dichotomy noted at the beginning of the proof, it follows that the Martin boundary is S^{d-1}.

On the other hand, if the integral in the statement of the theorem is infinite so that $P_x(\phi(t)\in B$ eventually$)=0$ for all B such that B^c has positive Lebesgue measure, then clearly

$$P_x\Big(\lim_{t\to\zeta^-}\phi(t)\text{ exists}\Big)=0. \tag{3.3}$$

But, as noted in the exposition at the end of Section 1, the Doob–Hunt theory states that $X(t)$ converges as $t\to\zeta^-$ to a point on the minimal Martin boundary. Thus, if the Martin boundary were S^{d-1}, (3.3) could not hold. Hence we conclude by the dichotomy noted at the beginning of the proof that the Martin boundary is one point. This proves Theorem 2. □

4. An auxiliary result on one-dimensional conditioned diffusions. Consider a one-dimensional diffusion on $D = [1, \infty)$ generated by

$$L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

and which, for convenience, is reflected at $x = 1$.

Assume that the diffusion is transient and let $\tau_0 = \inf\{t \geq 0: X(t) = 1\}$. Then $h(x) = P_x(\tau_D < \infty)$ satisfies $Lh = 0$ in D, $h(1) = 0$ and $\lim_{x\to\infty} h(x) = 0$. In fact,

$$h(x) = \frac{\int_x^\infty dy \exp\left(-\int_1^y \frac{2b}{a}(z)\,dz\right)}{\int_1^\infty dy \exp\left(-\int_1^y \frac{2b}{a}(z)\,dz\right)}.$$

Note that the integrals appearing in the definition of h are finite by the transience assumption ([11], Chapter 9). The h-transformed generator is now given by

$$L^h = \frac{1}{2}a\frac{d^2}{dx^2} + D(b)\frac{d}{dx},$$

where the new drift, which we denote by $D(b)$, satisfies

$$D(b)(x) = b(x) + a(x)\frac{h'(x)}{h(x)} = b(x) - a(x)\frac{\exp\left(-\int_1^x \frac{2b}{a}(z)\,dz\right)}{\int_x^\infty dy \exp\left(-\int_1^y \frac{2b}{a}(z)\,dz\right)}.$$

We have the following proposition.

PROPOSITION 1. *Let*

$$L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} \quad \text{on } [1,\infty),$$

where

$$\int_1^\infty dy \exp\left(-\int_1^y \frac{2b}{a}(z)\,dz\right) < \infty.$$

Assume that

(i) $$\frac{a(x)}{x} = o(b(x));$$

(ii) $$\liminf_{x\to\infty}\left(\frac{b}{a}\right)^{-1}(x)\exp\left(-\int_1^x \frac{2b}{a}(y)\,dy\right) = 0;$$

(iii) $$\limsup_{x\to\infty}\frac{x\left|\left(\frac{b}{a}\right)'(x)\right|}{\left|\left(\frac{b}{a}\right)(x)\right|} < \infty.$$

Let $D(b)$ be as defined above. Then

$$D(b)(x) = -b(x) + O\left(\frac{a(x)}{x}\right) \quad \textit{as } x \to \infty. \tag{4.1}$$

REMARK. Note that this result indicates that the drift of the conditioned process is roughly equal to the negative of the drift of the original process.

PROOF. Let $\phi(x; f) = \int_x^\infty \exp(-\int_1^y 2f(z)\,dz)$. Then

$$\phi'(x; f) = -\exp\left(-\int_1^x 2f(z)\,dz\right)$$

and

$$D(b)(x) = b + a\frac{\phi'(x; b/a)}{\phi(x; b/a)}.$$

Using the extended mean value theorem and a bit of standard analysis, one can show that

$$\lim_{x\to\infty} \frac{-\phi'(x; b/a)}{2(b/a)(x)\phi(x; b/a)} = 1. \tag{4.2}$$

Note that (4.2) is equivalent to $\lim_{x\to\infty} D(b)(x)/b(x) = -1$ which, by condition (i), is weaker than the proposition. Then, using (4.2) and again the extended mean value theorem and a bit of analysis, one shows that

$$\limsup_{n\to\infty}\left|2x\frac{b}{a}(x) + x\frac{\phi'(x; b/a)}{\phi(x; b/a)}\right| < \infty.$$

This proves the proposition. □

5. Proof of Theorem 4. We will first prove part (b). Then, relying on the proof of part (b), we can give the proof of part (a) relatively quickly.

PROOF OF PART (b). To prove that the Martin boundary of $\tilde{L}$ is given by the V-equivalence classes of clockwise spirals, we will show that the harmonic measure boundary at infinity for the conditioned process L^h is given by the V-equivalence classes. Part (b) will then follow from Theorem 1. We choose $D = \{|x| > 1\}$. We can write the generator of the process $X(t)$ in the form

$$\begin{aligned} L &= \frac{1}{2}\Delta + r^{-\delta}\frac{\partial}{\partial r} + r^{-k-1}\frac{\partial}{\partial \theta} \\ &= \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2r}\frac{\partial}{\partial r} + r^{-\delta}\frac{\partial}{\partial r} + \frac{1}{2r^2}\frac{\partial^2}{\partial \theta^2} + r^{-k-1}\frac{\partial}{\partial \theta}. \end{aligned}$$

The function $h(x) = P_x(\tau_D < \infty)$ depends on r alone and is given by

$$h(r) = \frac{\int_r^\infty ds \exp\left(-\int_1^s 2\left(z^{-\delta} + \frac{1}{2z}\right) dz\right)}{\int_1^\infty ds \exp\left(-\int_1^s 2\left(z^{-\delta} + \frac{1}{2z}\right) dz\right)} = \frac{\int_r^\infty ds \frac{1}{s} \exp\left(-\frac{2}{1-\delta} s^{1-\delta}\right)}{\int_1^\infty ds \frac{1}{s} \exp\left(-\frac{2}{1-\delta} s^{1-\delta}\right)}.$$

Let $b(r) = r^{-\delta} + 1/r$ and let $D(b) = b + h'/h$. Then

$$L^h = \frac{1}{2}\frac{\partial^2}{\partial r^2} + D(b)(r)\frac{\partial}{\partial r} + \frac{1}{2r^2}\frac{\partial^2}{\partial \theta^2} + r^{-k-1}\frac{\partial}{\partial \theta}.$$

By Proposition 1,

$$D(b)(r) = -b(r) + O\left(\frac{1}{r}\right) = -r^{-\delta} + O\left(\frac{1}{r}\right). \tag{5.1}$$

We now make an abuse of notation. We will denote the polar coordinate representation of $X^h(t)$, the process generated by L^h, by $(r(t), \theta(t))$ instead of by $(r^h(t), \theta^h(t))$. Since $X(t)$, the process generated by L, is not mentioned again in this paper, this should cause no confusion. We will realize $(r(t), \theta(t))$ by the following stochastic differential equations:

(5.2)

$$\text{(a)}\quad r(t) = r(t; r_0) = r_0 + \omega_1(t) + \int_0^t D(b)(r(s; r_0))\, ds,$$

$$\text{(b)}\quad \theta(t) = \theta(t; r_0, \theta_0) = \theta_0 + \int_0^t \frac{d\omega_2(s)}{r(s; r_0)} + \int_0^t r^{-k-1}(s; r_0)\, ds,$$

where $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are independent Brownian motions on a space $(\Omega, \mathscr{F}, P)$. [Of course, $\theta(t)$ as defined in (5.2) lives on R. It is convenient to consider $\theta(t)$ on R for the analysis which follows. The "true" $\theta(t)$ is obtained by taking $\theta(t) \bmod 2\pi$.]

Let $\tau_1 = \tau_1(r_0) = \inf\{t \geq 0: r(t; r_0) = 1\}$. For ease of notation we will write τ_1 for $\tau_1(r_0)$ in the sequel. Then we have

(5.3)

$$\mu^h_{r_0, \theta_0}(dy) = P(\theta(\tau_1; r_0, \theta_0) \in dy)$$
$$= P\left(\theta_0 + \int_0^{\tau_1} \frac{d\omega_2(s)}{r(s; r_0)} + \int_0^{\tau_1} r^{-k-1}(s; r_0)\, ds \in dy\right).$$

Define

$$\psi(r) = \begin{cases} \dfrac{r^{\delta-k}}{\delta - k}, & \text{if } \delta \neq k, \\ \log r, & \text{if } \delta = k. \end{cases}$$

Then from (1.10), $V(r,\theta) = (\psi(r) + \theta) \bmod 2\pi$. Consequently,

$$\lim_{r\to\infty} V(r,\theta(r)) = c \in [0,2\pi)$$

exists if and only if $\theta(r) = c - \psi(r) + \varepsilon(r)$, where $\lim_{r\to\infty} \varepsilon(r) \bmod 2\pi = 0$. To prove (b)(i), we must show that if $k > \delta - \frac{1}{2}(1-\delta)$ and if $\theta_0(r_0) = c - \psi(r_0) + \varepsilon(r_0)$, where $\varepsilon(r_0)$ is as above, then $\lim_{r_0\to\infty} \mu^h_{r_0,\theta_0(r_0)}$ exists and is distinct for each $c \in [0, 2\pi)$. To prove (b)(ii), we will show that if $k \le \delta - \frac{1}{2}(1-\delta)$, then for each $\theta_0 \in S^{d-1}$, $\lim_{r_0\to\infty} \mu^h_{r_0,\theta_0}$ = normalized Lebesgue measure.

By changing variables and using the fact that

$$\int_x^\infty y^\gamma \exp(-cy^2)\,dy = O(x^{\gamma-1}\exp(-cx^2)) \quad \text{as } x \to \infty, \tag{5.4}$$

one can check that the integral appearing in Lemma 1 in Section 3 is finite if one substitutes r, 1 and $-r^{-\delta}$ respectively for the $m(r)$, $a(r)$ and $b(r)$ appearing there. Using the fact that $D(b)(r) = -r^{-\delta} + O(1/r)$, one can show that the integral is also finite if one substitutes r, 1 and $D(b)(r)$. We thus conclude from Lemma 1 that

$$\lim_{r_0\to\infty} \int_0^{\tau_1} \frac{ds}{r^2(s;r_0)} < \infty \quad \text{a.s.}$$

It then follows that

$$\lim_{r_0\to\infty} \int_0^{\tau_1} \frac{d\omega_2(s)}{r(s;r_0)} \quad \text{exists a.s. as a finite random variable.} \tag{5.5}$$

The key step in the proof consists of proving the following points: If $k > \delta - \frac{1}{2}(1-\delta)$, then

$$\begin{aligned} &\text{(a)} \quad \lim_{r_0\to\infty} \left(E\int_0^{\tau_1} \frac{ds}{r^{k+1}(s;r_0)} - \psi(r_0) \right) \text{ exists and is finite;} \\ &\text{(b)} \quad \lim_{r_0\to\infty} \operatorname{Var}\left(\int_0^{\tau_1} \frac{ds}{r^{k+1}(s;r_0)} \right) < \infty. \end{aligned} \tag{5.6}$$

If $k \le \delta - \frac{1}{2}(1-\delta)$, then

$$\lim_{r_0\to\infty} \operatorname{Var}\left(\int_0^{\tau_1} \frac{ds}{r^{k+1}(s;r_0)} \right) = \infty. \tag{5.7}$$

We will now prove the theorem using (5.6) and (5.7) and then go back to prove these statements.

PROOF OF PART (i). For $0 \le c < 2\pi$, let $\theta_0(r_0) = c - \psi(r_0) + \varepsilon(r_0)$, where $\lim_{r_0 \to \infty} \varepsilon(r_0) = 0$. We must show that $\mu^h_{r_0, \theta_0(r_0)}$ possesses a weak limit as $r_0 \to \infty$ and that this limit is different for each $0 \le c < 2\pi$. From (5.3), we have

$$\begin{aligned} \theta(\tau_1; r_0, \theta_0(r_0)) &= c + \varepsilon(r_0) + \int_0^{\tau_1} \frac{d\omega_2(s)}{r(s; r_0)} \\ &\quad + \left(\int_0^{\tau_1} \frac{ds}{r^{k+1}(s; r_0)} - E\int_0^{\tau_1} \frac{ds}{r^{k+1}(s; r_0)} \right) \\ &\quad + \left(E\int_0^{\tau_1} \frac{ds}{r^{k+1}(s; r_0)} - \psi(r_0) \right) \\ &\equiv c + \varepsilon(r_0) + \mathrm{I}_{r_0} + \mathrm{II}_{r_0} + \mathrm{III}_{r_0}. \end{aligned} \tag{5.8}$$

By (5.5), $\mathrm{I}_\infty \equiv \lim_{r_0 \to \infty} \mathrm{I}_{r_0}$ exists as a finite random variable. Using the technique in (3.2), II_{r_0} may be thought of as a sum of independent mean zero random variables. But then (5.6)(b) is a sufficient condition for the a.s convergence of II_{r_0} as $r_0 \to \infty$. Thus $\mathrm{II}_\infty \equiv \lim_{r_0 \to \infty} \mathrm{II}_{r_0}$ exists as a finite random variable. Finally, by (5.6)(a), $\mathrm{III}_\infty = \lim_{r_0 \to \infty} \mathrm{III}_{r_0}$ is finite. From these facts we conclude that $\lim_{r_0 \to \infty} \theta(\tau_1; r_0, \theta_0(r_0))| \bmod 2\pi$ exists a.s.

It remains to show that the limiting distributions obtained above with $c = c_i \in [0, 2\pi)$, $i = 1, 2$, are distinct if $c_1 \ne c_2$. We argue as follows. Consider the above analysis on $D_n = \{|x| > n\}$ instead of on $D = \{|x| > 1\}$. Then in (5.8) we must replace τ_1 by τ_n. We denote the new right-hand side of (5.8) by $c + \varepsilon(r_0) + \mathrm{I}^{(n)}_{r_0} + \mathrm{II}^{(n)}_{r_0} + \mathrm{III}^{(n)}_{r_0}$. Now by Theorem 1, the harmonic measure boundary at infinity is independent of the particular exterior domain D used in its calculation. Thus the measures above on ∂D_n corresponding to c_1 and c_2 will either be distinct for all n or the same for all n. Hence it suffices to show that the measures obtained above on ∂D_n corresponding to c_1 and c_2 are distinct for sufficiently large n. The analysis above which proved the convergence of I_{r_0} and II_{r_0} as $r_0 \to \infty$ also proves that

$$\lim_{n \to \infty} \lim_{r_0 \to \infty} \mathrm{I}^{(n)}_{r_0} \stackrel{d}{=} \lim_{n \to \infty} \lim_{r_0 \to \infty} \mathrm{II}^{(n)}_{r_0} \stackrel{d}{=} \delta_0,$$

the δ-mass at 0. This, along with the fact that $\mathrm{III}^{(n)}_{r_0}$ is a constant, allows us to conclude that the measures are distinct for c_1 and c_2 as long as n is sufficiently large.

PROOF OF PART (ii). We will show that $\lim_{r_0 \to \infty} \theta(\tau_1; r_0, \theta_0) \bmod 2\pi =$ normalized Lebesgue measure uniformly over $\theta_0 \in S^{d-1}$. Introducing the notation

$$\rho(t; r_0) = \int_0^t \frac{ds}{r^{k+1}(s; r_0)}, \tag{5.9}$$

we can write

$$\theta(\tau_1; r_0, \theta_0) = \theta_0 + \int_0^{\tau_1} \frac{d\omega_2(s)}{r(s; r_0)} + \rho(\tau_1; r_0).$$

In fact, it is enough to show that

(5.10) $$\lim_{r_0 \to \infty} \rho(\tau_j; r_0) \bmod 2\pi \overset{d}{=} \text{normalized Lebesgue measure}, \quad \text{for } j \geq 1.$$

To see this, use the strong Markov property to write

$$\theta(\tau_1; r_0, \theta_0) = \theta_0 + \left(\int_0^{\tau_j} \frac{d\omega_2(s)}{r(s; r_0)} + \rho(\tau_j; r_0) \right) + \left(\int_{\tau_j}^{\tau_1} \frac{d\omega_2(s)}{r(s; r_0)} + \int_{\tau_j}^{\tau_1} \frac{ds}{r^{k+1}(s; r_0)} \right),$$

where $1 < j < r_0$. The two expressions in parentheses are independent of one another. Now, if X_n and Y_n are independent random variables for each n and $X_n \bmod 2\pi$ converges as $n \to \infty$ to the uniform distribution, then $(X_n + Y_n) \bmod 2\pi$ also converges to the uniform distribution as $n \to \infty$. Using this fact, (5.10) and the fact [noted at the end of the proof of part (i)] that

$$\lim_{j \to \infty} \lim_{r_0 \to \infty} \int_0^{\tau_j} \frac{d\omega_2(s)}{r(s; r_0)} \overset{d}{=} \delta_0,$$

it is easy to conclude that $\lim_{r_0 \to \infty} \theta(\tau_1; r_0, \theta_0) \bmod 2\pi =$ normalized Lebesgue measure.

It remains to prove (5.10). We will prove (5.10) for $j = 1$; the same proof holds for any j. Now although, in light of (5.7), it seems intuitively clear that (5.10) should hold, a direct proof eludes us and appears difficult. We give the following argument which is along the lines of an argument given by Cox and Rösler [6] when they were faced with a similar situation.

Let $\sigma(t; r_0)$ be the inverse of $\rho(t; r_0)$. Then, as in the proof of Lemma 1, $\rho(\tau_1; r_0)$ may be identified with the first hitting time of $r = 1$ for the process $r(\sigma(t; r_0))$. This process is generated by

$$L_0 \equiv \frac{1}{2} r^{k+1} \frac{d^2}{dr^2} + r^{k+1} D(b)(r) \frac{d}{dr}.$$

We make the following parallel construction. Let $(\hat{r}(t; r_0), \hat{\theta}(t; r_0, \theta_0))$ denote the process on $\Sigma_{r_0} = \{|x| \leq r_0\}$ starting from (r_0, θ_0) which is generated by L and is normally reflected at $\partial \Sigma_{r_0}$, and let $\hat{\tau}_1$ denote the corresponding hitting time of $\partial D = \{|x| = 1\}$. Define

$$\hat{\rho}(t, r_0) = \int_0^t \frac{ds}{\hat{r}^{k+1}(s; r_0)},$$

analogously to (5.9). Analogously to the nonreflected case, if we let $\hat{\sigma}(t; r_0)$ denote the inverse of $\hat{\rho}(t; r_0)$, then $\hat{\rho}(\hat{\tau}_1; r_0)$ may be identified with the first hitting time of $r = 1$ for the process $\hat{r}(\hat{\sigma}(t; r_0); r_0)$ which is generated by L_0 as above, and which is reflected at $r = r_0$. We make use of a result for hitting time distributions of reflected one-dimensional processes which start at the point of reflection. This result was first proved by Kac and Krein [15] (see also the recent paper [16]).

KAC AND KREIN [15]. The distribution of $\hat{\rho}(\tau_1; r_0)$ is a convolution of exponential distributions.

Now (5.7) states that $\lim_{r_0 \to \infty} \mathrm{Var}(\rho(\tau_1; r_0)) = \infty$. In fact, we also have

$$\lim_{r_0 \to \infty} \mathrm{Var}(\hat{\rho}(\tau_1; r_0)) = \infty. \tag{5.11}$$

The proof of this is by the same method we will employ for the proof of (5.7). [One replaces $T_1(r)$ which appears in (5.18) by $T_1^{r_0}(r)$ which is obtained by replacing the upper limit ∞ with r_0 in the inside integral and one replaces $T_2(r)$, appearing in (5.21), by $T_2^{r_0}(r)$ which is obtained by replacing the upper limit ∞ with r_0 in the inside integral and by replacing $T_1(z)$ by $T_1^{r_0}(z)$. One now shows that $\lim_{r \to \infty} r(T_2^r(r) - (T_1^r(r))^2)' > 0$, analogous to (5.23). We ought to note that indeed we do not use (5.7) at all in our proof, but rather its analog (5.11). However, since the proof of (5.7) is similar to that of (5.11), and since it can be presented simultaneously with the proof of (5.6)(b), in the interest of brevity we will prove (5.7) rather than (5.11).] One can check explicitly that for distributions which are convolutions of exponential distributions, if the variance goes to ∞, then the distribution modulo 2π converges weakly to the uniform distribution. Thus

$$\lim_{r_0 \to \infty} \hat{\rho}(\hat{\tau}_1; r_0) \bmod 2\pi = \text{normalized Lebesgue measure}. \tag{5.12}$$

In other words, we have proven the analog of (5.10) for the reflected process. Since (5.2)(b) holds with $\hat{r}(s; r_0)$ and $\hat{\theta}(s; r_0, \theta_0)$ in place of $r(s; r_0)$ and $\theta(s; r_0, \theta_0)$, and since (5.5) holds with $\hat{r}$ in place of r (by using the analog of Lemma 1 for reflected processes), it follows from (5.12) and the argument following (5.10) that

$$\begin{gathered}\lim_{r_0 \to \infty} \hat{\theta}(\hat{\tau}_1; r_0, \theta_0) \bmod 2\pi \overset{d}{=} \text{normalized Lebesgue measure} \\ \text{uniformly for } \theta_0 \in S^{d-1}.\end{gathered} \tag{5.13}$$

(The uniformity follows from symmetry considerations.)

We claim that for each $\theta_0 \in S^1$, there exists some distribution $\mu_{r_0, \theta_0}(d\theta)$ on the circle such that

$$\theta(\tau_1; r_0, \theta_0) \overset{d}{=} \int_{S^1} \hat{\theta}(\hat{\tau}_1; r_0, \theta) \mu_{r_0, \theta_0}(d\theta). \tag{5.14}$$

The desired conclusion that $\lim_{r_0 \to \infty} \theta(\tau_1; r_0, \theta_0) \bmod 2\pi$ = normalized Lebesgue measure now follows from (5.13) and (5.14). It remains to show

(5.14). By recurrence and the strong Markov property, it follows that for any $0 < \varepsilon < r_0 - 1$, and each $\theta_0 \in S^1$, there exist distributions $\mu_{r_0, \theta_0, \varepsilon}(d\theta)$ and $\hat{\mu}_{r_0, \theta_0, \varepsilon}(d\theta)$ on the circle such that

$$(5.15)\quad \begin{aligned} &P(\theta(\tau_1; r_0, \theta_0) \in d\theta) \\ &\quad = \int_{S^1} P(\theta(\tau_1; r_0 - \varepsilon, s) \in d\theta | \tau_1 < \tau_{r_0}) \mu_{r_0, \theta_0; \varepsilon}(ds) \end{aligned}$$

and

$$(5.16)\quad \begin{aligned} &P(\hat{\theta}(\hat{\tau}_1; r_0, \theta_0) \in d\theta) \\ &\quad = \int_{S^1} P(\hat{\theta}(\hat{\tau}_1; r_0 - \varepsilon, s) \in d\theta | \hat{\tau}_1 < \hat{\tau}_{r_0}) \hat{\mu}_{r_0, \theta_0; \varepsilon}(ds). \end{aligned}$$

But

$$(5.17)\quad \begin{aligned} &P(\hat{\theta}(\hat{\tau}_1; r_0 - \varepsilon, s) \in d\theta | \hat{\tau}_1 < \hat{\tau}_{r_0}) \\ &\quad = P(\theta(\tau_1; r_0 - \varepsilon, s) \in d\theta | \tau_1 < \tau_{r_0}). \end{aligned}$$

Furthermore, since by Theorem 5 the distributions $P(\theta(\tau_{r_0 - \varepsilon}; r_0, \theta) \in ds)$ and $P(\hat{\theta}(\hat{\tau}_{r_0 - \varepsilon}; r_0, \theta) \in ds)$ are mutually absolutely continuous with respect to Lebesgue measure for each $\theta \in S^{d-1}$, it follows from the definitions of $\mu_{r_0, \theta_0, \varepsilon}$ and $\hat{\mu}_{r_0, \theta_0, \varepsilon}$ that each of these is also mutually absolutely continuous with respect to Lebesgue measure. Thus $\mu_{r_0, \theta_0, \varepsilon}$ and $\hat{\mu}_{r_0, \theta_0, \varepsilon}$ are mutually absolutely continuous. This fact, together with (5.15), (5.16) and (5.17), gives (5.14).

To complete the proof of part (b), it remains to prove (5.6) and (5.7). We will use the notation

$$T_1(r) = E\int_0^{\tau_1} \frac{ds}{r^{k+1}(s; r)} \quad \text{and} \quad T_2(r) = E\left(\int_0^{\tau_1} \frac{ds}{r^{k+1}(s; r)}\right)^2.$$

We first prove (5.6)(a). By Itô's formula,

$$T_1^N(r) \equiv E\int_0^{\tau_1 \wedge \tau_N} \frac{ds}{r^{k+1}(s; r)}$$

solves

$$\frac{1}{2}(T_1^N(r))'' + D(b)(r)(T_1^N(r))' = -\frac{1}{r^{k+1}},$$

with boundary conditions $T_1^N(1) = T_1^N(N) = 0$. Now $T_1(r) = \lim_{N \to \infty} T_1^N(r)$. Calculating $T_1^N(r)$ explicitly and taking a limit, we arrive at

$$(5.18)\quad T_1(r) = \int_1^r ds \exp\left(-\int_1^s 2D(b)(z)\, dz\right) \int_s^\infty dz \frac{2}{z^{k+1}} \exp\left(\int_1^z 2D(b)(y)\, dy\right).$$

To prove (5.6)(a), it suffices to show that $\lim_{r \to \infty} r^{1+\varepsilon}(T'(r) - \psi'(r)) = 0$, for sufficiently small $\varepsilon > 0$. We have

$$r^{1+\varepsilon}(T'(r) - \psi'(r))$$

$$(5.19)\qquad = \frac{\int_r^\infty ds \frac{2}{s^{k+1}} \exp\left(\int_1^s 2D(b)(y)\,dy\right) - r^{\delta-k-1}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}{r^{-1-\varepsilon}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}.$$

From (5.1), it is clear that the right-hand side of (5.19) is the indeterminate form $0/0$. Applying l'Hôpital's rule, we obtain, after some algebra,

$$\lim_{r\to\infty} r^{1+\varepsilon}|T'(r) - \psi'(r)|$$

$$= \lim_{r\to\infty}\left|\frac{-2r^{\delta-k+\varepsilon} - 2D(b)(r)r^{2\delta-k+\varepsilon} - (\delta - k - 1)r^{2\delta-k-1+\varepsilon}}{2D(b)(r)r^{\delta} - (1+\varepsilon)r^{-1+\delta}}\right|.$$

But $-1 + \delta < 0$ and by (5.1), $\lim_{r\to\infty} D(b)(r)r^\delta = -1$. Furthermore, since by assumption $k > \delta - \frac{1}{2}(1 - \delta)$, it follows that $2\delta - k - 1 + \varepsilon < 0$ for small ε. Thus, to complete the proof, we must show that for small ε,

$$(5.20)\qquad \lim_{r\to\infty} |-2r^{\delta-k+\varepsilon} - 2D(b)(r)r^{2\delta-k-\varepsilon}| = 0.$$

We write

$$-2r^{\delta-k+\varepsilon} - 2D(b)(r)r^{2\delta-k-\varepsilon} = 2r^{\delta-k+\varepsilon}\left[-1 - r^\delta D(b)(r)\right].$$

From (5.1) again, $-1 - r^\delta D(b)(r) = O(r^{\delta-1})$. Now (5.20) will follow if $(\delta - k + \varepsilon) + (\delta - 1) < 0$, that is, if $2\delta - k - 1 + \varepsilon < 0$. But, as noted above, this last inequality holds for small ε. This completes the proof of (5.6)(a).

We now consider (5.6)(b) and (5.7). Let

$$T_2^N(r) \equiv E\left(\int_0^{\tau_1\wedge\tau_N} \frac{ds}{r^{k+1}(s;r)}\right)^2.$$

Then $T_2^N(r)$ solves

$$\frac{1}{2}\left(T_2^N(r)\right)'' + D(b)(r)\left(T_2^N(r)\right)' + \frac{2T_1^N(r)}{r^{k+1}} = 0,$$

with $T_2^N(1) = T_2^N(N) = 0$, where $T_1^N(r)$ is as in the proof of (5.6)(a) above (see Mandl [19], page 108, Lemma 1, for the case $k + 1 = 0$). Solving explicitly for $T_2^N(r)$ and using the fact that $T_2(r) = \lim_{N\to\infty} T_2^N(r)$, we arrive at

$$(5.21)\qquad \begin{aligned} T_2(r) = &\int_1^r ds \exp\left(-\int_1^s 2D(b)(y)\,dy\right) \\ &\times \int_s^\infty dz \frac{4T_1(z)}{z^{k+1}} \exp\left(\int_1^z 2D(b)(y)\,dy\right). \end{aligned}$$

Now

$$\mathrm{Var}\left(\int_0^{\tau_1}\frac{ds}{r^{k+1}(s;r_0)}\right) = T_2(r_0) - T_1^2(r_0).$$

To prove (5.6)(b) and (5.7), we will show that

$$\lim_{r\to\infty} r^{1+\varepsilon}\left(T_2(r) - (T_1(r))^2\right)' = 0, \quad \text{if } k > \delta - \tfrac{1}{2}(1-\delta), \tag{5.22}$$

for sufficiently small $\varepsilon > 0$ and

$$\lim_{r\to\infty} r\left(T_2(r) - (T_1(r))^2\right)' > 0 \quad \text{if } k \le \delta - \tfrac{1}{2}(1-\delta). \tag{5.23}$$

Let $\varepsilon \ge 0$. We have, after some algebra,

$$\begin{aligned}
&\lim_{r\to\infty} r^{1+\varepsilon}\left(T_2(r) - (T_1(r))^2\right)' \\
&\quad = \lim_{r\to\infty} r^{1+\varepsilon}\left(T_2'(r) - 2T_1(r)T_1'(r)\right) \\
&\quad = \lim_{r\to\infty}\left[\frac{\int_r^\infty dz\,\frac{4T_1(z)}{z^{k+1}}\exp\left(\int_1^z 2D(b)(y)\,dy\right)}{r^{-1-\varepsilon}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}\right. \\
&\qquad \left. - \frac{2\left(\int_1^r dz\exp\left(-\int_1^z 2D(b)(y)\,dy\right)\int_z^\infty ds\,\frac{2}{s^{k+1}}\exp\left(\int_1^s 2D(b)(y)\,dy\right)\right)\times\int_r^\infty dz\,\frac{2}{z^{k+1}}\exp\left(\int_1^z 2D(b)(y)\,dy\right)}{r^{-1-\varepsilon}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}\right].
\end{aligned} \tag{5.24}$$

We leave it to the reader to make the appropriate estimates using (5.1) and (5.4), similar to ones we have already made, to show that the rightmost expression in (5.24) is the indeterminate form $0/0$. Applying l'Hôpital's rule to this, we obtain, after cancellations,

$$\begin{aligned}
&\lim_{r\to\infty} r^{1+\varepsilon}\left(T_2(r) - (T_1(r))^2\right)' \\
&\quad = \lim_{r\to\infty}\frac{-2\left(\int_r^\infty dz\,\frac{2}{z^{k+1}}\exp\left(\int_1^z 2D(b)(y)\,dy\right)\right)^2}{\left(2r^{-1-\varepsilon}D(b)(r) - (1+\varepsilon)r^{-2-\varepsilon}\right)\exp\left(\int_1^r 4D(b)(y)\,dy\right)}.
\end{aligned} \tag{5.25}$$

It suffices to consider the asymptotics of the square root of the right-hand side of (5.25). In fact, since by (5.1) the term $r^{-1-\varepsilon}D(b)(r)$ is on the order of $r^{-1-\varepsilon-\delta}$ which dominates $r^{-2-\varepsilon}$ (recall that $\delta < 1$), it follows that we need only consider the asymptotics of

$$\frac{\int_r^\infty dz \frac{1}{z^{k+1}} \exp\left(\int_1^z 2D(b)(y)\,dy\right)}{r^{-1/2-\varepsilon/2}(-D(b)(r))^{1/2}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}. \tag{5.26}$$

Note that Proposition 1 in Section 4 can be formulated as follows:

$$\frac{h'(x)}{h(x)} = \frac{-2b(x)}{a(x)} + O\left(\frac{1}{x}\right), \tag{5.27}$$

where $h(x)$ is as given in the paragraph preceding the statement of the proposition. Also note that

$$\frac{1}{z^{k+1}}\exp\left(\int_1^z 2D(b)(y)\,dy\right) = \exp\left(\int_1^z 2\left(D(b)(y) - \frac{k+1}{2y}\right)dy\right).$$

Substituting $(-D(b)(r) + (k+1)/2r)$ and 1 respectively for the b and the a of Proposition 1, one can check that the conditions of that proposition are met for these expressions. Thus, by (5.27) with b and a replaced by $(-D(b)(r) + (k+1)/2r)$ and 1 respectively, and by (5.1), we have

$$\begin{aligned}\frac{\frac{1}{r^{k+1}}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}{\int_r^\infty dz\frac{1}{z^{k+1}}\exp\left(\int_1^z 2D(b)(y)\,dy\right)} &= 2\left(-D(b)(r) + \frac{k-1}{2r}\right) + O\left(\frac{1}{r}\right)\\ &= 2r^{-\delta} + O\left(\frac{1}{r}\right).\end{aligned}$$

Applying this and (5.1) to the expression (5.26), we have

$$\begin{aligned}&\frac{\int_r^\infty dz \frac{1}{z^{k+1}} \exp\left(\int_1^z 2D(b)(y)\,dy\right)}{r^{-1/2-\varepsilon/2}(-D(b)(r))^{1/2}\exp\left(\int_1^r 2D(b)(y)\,dy\right)}\\ &= \left(r^{-1/2-\varepsilon/2}(-D(b)(r))^{1/2}\right)^{-1}\left(r^{k+1}\left(2r^{-\delta} + O\left(\frac{1}{r}\right)\right)\right)^{-1}\\ &= \left(r^{-1/2-\varepsilon/2}\left(r^{-\delta} + O\left(\frac{1}{r}\right)\right)^{1/2}\right)^{-1}\left(r^{k+1}\left(2r^{-\delta} + O\left(\frac{1}{r}\right)\right)\right)^{-1}\\ &= \frac{1}{2}r^{3\delta/2-1/2-k+\varepsilon/2} + \text{lower-order terms}.\end{aligned} \tag{5.28}$$

Now, if $k > \delta - \frac{1}{2}(1-\delta)$, then $3\delta/2 - 1/2 - k + \varepsilon/2 < 0$ for sufficiently small $\varepsilon > 0$ and, consequently, the left-hand side of (5.28) goes to 0 as $r \to \infty$. This gives (5.22) and proves (5.6)(b). On the other hand, if $k \le \delta - \frac{1}{2}(1-\delta)$, then with $\varepsilon = 0$, we have $3\delta/2 - 1/2 - k + \varepsilon/2 \ge 0$ and, consequently, the left-hand side of (5.28) with $\varepsilon = 0$ is bounded away from 0 as $r \to \infty$. This gives (5.23) and proves (5.7). This completes the proof of part (b).

Proof of part (a). Recall the notation introduced prior to the statement of Theorem 3: c is the zeroth-order part of $\tilde{L}$ and $\hat{L} = \tilde{L} - c$. Since

$$\tilde{L} = \frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} - r^{-k}\frac{1}{r}\frac{\partial}{\partial\theta} - (1-\delta)r^{-1-\delta},$$

we have $c(r) = -(1-\delta)r^{-1-\delta}$ and

$$\hat{L} = \frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} - r^{-k}\frac{1}{r}\frac{\partial}{\partial\theta},$$

which generates a recurrent diffusion, $\hat{X}(t) = (\hat{r}(t), \hat{\theta}(t))$. By Theorem 3, the Martin boundary of L coincides with the harmonic measure boundary at infinity of $\tilde{L}^{\tilde{h}}$. From (1.7) and the fact that $\hat{P}(\tau_D < \infty) = 1$, it follows that this boundary is determined by the sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ for which

$$\lim_{n\to\infty}\frac{1}{\tilde{h}(x_n)}\hat{E}_{x_n}\exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)\psi\big(\hat{X}(\tau_D)\big)$$

exists for all $\psi \in C(\partial D)$, where $\tilde{h}(x)$ is as in (1.6) with $c(r) = -(1-\delta)r^{-1-\delta}$. It follows from (5.32) below and the strong Markov property that $\lim_{|x|\to\infty}\tilde{h}(x)$ exists. Thus, in fact, the harmonic measure boundary at infinity of $\tilde{L}^{\tilde{h}}$ is determined by the sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$, for which

$$\lim_{n\to\infty}\hat{E}_{x_n}\exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)\psi\big(\hat{X}(\tau_D)\big) \quad \text{exists for all } \psi \in C(\partial D). \tag{5.29}$$

Now, consider for a moment the boundary determined by the sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ for which

$$\lim_{n\to\infty}\hat{E}_{x_n}\psi\big(\hat{X}(\tau_D)\big) \text{ exists for all } \psi \in C(\partial D). \tag{5.30}$$

This boundary is exactly the counterclockwise spirals generated by U as in (1.8) if $k > \delta - \frac{1}{2}(1-\delta)$ and is a single point if $k \le \delta - \frac{1}{2}(1-\delta)$. Indeed, the proof of this is almost exactly like the proof above of part (b), only slightly easier. To see this, recall that in part (b) we proved that if $k > \delta - \frac{1}{2}(1-\delta)$, then the clockwise spirals generated by V as in (1.10) form the harmonic measure boundary at infinity for L^h, that is, the boundary determined by the sequences $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty}|x_n| = \infty$ for which

$$\lim_{n\to\infty}E^h\psi\big(X^h(\tau_D)\big) \text{ exists for all } \psi \in C(\partial D). \tag{5.31}$$

We also proved that if $k \le \delta - \frac{1}{2}(1-\delta)$, then this boundary is a single point. But compare (5.30) to (5.31). The process $\hat{X}(t)$ is generated by

$$\hat{L} = \frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} - r^{-k}\frac{1}{r}\frac{\partial}{\partial \theta},$$

and the process $X^h(t)$ is generated by

$$L^h = \frac{1}{2}\Delta + D(b)\frac{\partial}{\partial r} + r^{-k}\frac{1}{r}\frac{\partial}{\partial \theta}.$$

Recall that $D(b)(r) = -r^{-\delta} + O(1/r)$. Now, if $D(b)(r)$ were exactly equal to $-r^{-\delta}$, the proof of part (b) would still have gone through—in fact a bit more easily. Thus the harmonic measure boundary at infinity for

$$\frac{1}{2}\Delta - r^{-\delta}\frac{\partial}{\partial r} + r^{-k}\frac{1}{r}\frac{\partial}{\partial \theta}$$

is the same as that of L^h. Clearly then, the same proof shows that the harmonic measure boundary for $\hat{L}$ is either the collection of counterclockwise spirals or one point according to whether $k > \delta - \frac{1}{2}(1-\delta)$ or $k \le \delta - \frac{1}{2}(1-\delta)$.

To complete the proof of part (a), it remains to show that the boundaries determined by (5.29) and (5.30) coincide. To prove this, let $\tau_n = \inf\{t \ge 0\colon \hat{r}(t) = n\}$. Then, letting $x = (r, \theta)$, we will show that

$$\lim_{n\to\infty}\lim_{r\to\infty}\sup_{\theta\in S^1}\hat{P}_{r,\theta}\left(\int_0^{\tau_n}\hat{r}^{-1-\delta}(s)\,ds > \varepsilon\right) = 0. \tag{5.32}$$

Note that the probability appearing in (5.32) is actually independent of θ; the supremum over θ is thus trivial.

We prove (5.32) as follows. The process $\hat{r}(t)$ is Markov and can be represented as follows. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\omega(t)$ a Brownian motion on Ω. Define

$$\hat{r}(t; r_0) = r_0 + \omega(t) + \int_0^t\left(\frac{1}{2\hat{r}(s;r_0)} - \hat{r}^{-\delta}(s;r_0)\right)ds.$$

Apply Lemma 1 in Section 3 to $\rho(\tau_1; r_0) \equiv \int_0^{\tau_1}\hat{r}^{-1-\delta}(s; r_0)\,ds$. In the notation of that lemma, $a(r) = 1$, $b(r) = 1/2r - r^{-\delta}$ and $m^2(r) = r^{1+\delta}$. Changing variables and using (5.4) and the fact that $\delta > -1$ show that the integral in Lemma 1 is finite. Thus Lemma 1 gives

$$P\left(\lim_{r_0\to\infty}\int_0^{\tau_1}\hat{r}^{-1-\delta}(s;r_0)\,ds < \infty\right) = 1. \tag{5.33}$$

Now (5.32) follows immediately from (5.33).

We write (5.29) in the form

$$\begin{aligned}
&\lim_{n\to\infty} \hat{E}_{x_n} \exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)\psi\left(\hat{X}(\tau_D)\right) \\
(5.34)\qquad &= \lim_{n\to\infty} \hat{E}_{x_n}\hat{E}_{x_n}\left(\exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)\psi\left(\hat{X}(\tau_D)\right)|\hat{\theta}(\tau_D)\right) \\
&= \lim_{n\to\infty}\int_0^{2\pi}\psi(\theta)\hat{E}_{x_n}\left(\exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)|\hat{\theta}(\tau_D)=\theta\right)\hat{\mu}_{x_n}(d\theta).
\end{aligned}$$

Now, by Theorem 5, the measures $\hat{\mu}_x(d\theta)$, $x \in D$, and the limiting measures $\lim_{n\to\infty}\hat{\mu}_{x_n}$ are all mutually absolutely continuous with respect to Lebesgue measure on $\partial D = S^1$. Thus, from (5.32), it follows that for almost every $\theta \in S^1$,

$$\lim_{|x|\to\infty} \hat{E}_x\left(\exp\left(-\int_0^{\tau_D}(1-\delta)\hat{r}^{-1-\delta}(s)\,ds\right)|\hat{\theta}(\tau_D)=\theta\right)$$

exists (and is in fact independent of θ). This fact and the representation for (5.29) given on the right-hand side of (5.34) show that the boundaries of (5.29) and (5.30) coincide. □

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REFERENCES

[1] Ancona, A. (1987). Negatively curved manifolds, elliptic operators, and the Martin boundary. *Ann. Math.* **125** 495–536.

[2] Anderson, M. T. and Schoen, R. (1985). Positive harmonic functions on complete manifolds of negative curvature. *Ann. Math.* **121** 429–461.

[3] Azencott, R. (1974). Behavior of diffusion semigroups at infinity. *Bull. Soc. Math. France* **102** 193–240.

[4] Brelot, M. (1971). *On Topologies and Boundaries in Potential Theory. Lecture Notes in Math.* **175**. Springer, Berlin.

[5] Chung, K. I. (1974). *A Course in Probability Theory*. Academic, Orlando.

[6] Cox, J. T. and Rösler, U. (1983). A duality relation for entrance and exit laws for Markov processes. *Stochastic Process. Appl.* **16** 141–156.

[7] Cranston, M., Orey, S. and Rösler, U. (1982). The Martin boundary of two dimensional Ornstein–Uhlenbeck processes. *London Math. Soc. Lecture Note Ser.* **79** 63–78.

[8] Doob, J. L. (1957). Conditioned Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France* **85** 431–458.

[9] Dynkin, E. B. (1965). *Markov Processes* **2**. Springer, Berlin.

[10] Feller, W. (1952). The parabolic differential equations and the associated semigroups of transformations. *Ann. Math.* **55** 483–519.

[11] Friedman, A. (1975). *Stochastic Differential Equations and Applications* **1**. Academic, New York.

[12] Gilbarg, D. and Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order* 2nd ed. Springer, Berlin.

[13] Hunt, G. A. (1960). Markoff chains and Martin boundaries. *Illinois J. Math.* **4** 313–340.

[14] Ioffe, D. (1993). Recurrent perturbations of certain transient radially symmetric diffusions. *Ann. Probab.* To appear.

[15] KAC, I. S. and KREIN, M. G. (1968). *O spektralnych funkcii struny*. Appendix to the Russian translation of Atkinson, *Discrete and Continuous Boundary Problems*. Mir, Moscow.

[16] KELLER, G., KERSTING, G. and RÖSLER, U. (1988). On the asymptotic behaviour of first passage times for discussions [diffusions]. *Probab. Theory Related Fields* **77** 379–395.

[17] KUNITA, H. and WATANABE, T. (1965). Markov processes and Martin boundaries. I. *Illinois J. Math.* **9** 485–526.

[18] KUNITA, H. and WATANABE, T. (1966). On certain reversed processes and their applications to potential theory and boundary theory. *J. Math. Mech.* **15** 393–434.

[19] MANDL, P. (1968). *Analytical Treatment of One-Dimensional Markov Processes*. Springer, Berlin.

[20] MARCH, P. (1986). Brownian motion and harmonic functions on rotationally symmetric manifolds. *Ann. Probab.* **14** 793–801.

[21] MARTIN, R. S. (1941). Minimal positive harmonic functions. *Trans. Amer. Math. Soc.* **49** 137–172.

[22] MIRANDA, C. (1970). *Partial Differential Equations of Elliptic Type*, 2nd ed. Springer, Berlin.

[23] MURATO, M. (1986). Structure of positive solutions to $(-\Delta + V) = 0$ in R^d. *Duke Math. J.* **53** 869–943.

[24] PINSKY, R. G. (1987). Recurrence, transience and bounded harmonic functions for diffusions in the plane. *Ann. Probab.* **15** 954–984.

[25] PINSKY, R. G. (1990). The asymptotic behavior of the solution of the exterior Dirichlet problem for Brownian motion perturbed by a small parameter drift. *Ann. Probab.* **18** 1602–1618.

[26] RAY, D. B. (1959). Resolvents, transition functions and strongly Markovian processes. *Ann. Math.* **70** 43–72.

[27] WILLIAMS, D. (1979). *Diffusions, Markov Processes, and Martingales*. Wiley, New York.

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