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LARGE DEVIATIONS FOR PROCESSES WITH INDEPENDENT INCREMENTS

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This paper strengthens and generalizes some theorems proved earlier by Lynch and Sethuraman on large deviations (LD) for random processes with independent increments.

1. Introduction. Let $\xi(t)$, $t \geq 0$, be a stochastic process with stationary independent increments, $\mathbf{E}\xi(1) = 0$, $\sigma^2 = \mathbf{E}\xi(1)^2 > 0$. Denote by (λ_-, λ_+) the interval such that $\varphi(\lambda) = \mathbf{E}e^{\lambda\xi(1)} < \infty$ for $\lambda \in (\lambda_-, \lambda_+)$ and $\varphi(\lambda) = \infty$ for $\lambda \notin [\lambda_-, \lambda_+]$. Let $X_T = X_T(t) = \xi(tT)$, $0 \leq t \leq 1$, be the suitably scaled segment of $\xi(tT)$ and $\mathbf{P}_\lambda$ the distribution of its paths in the space $\mathbf{D}[0,1]$:

$$\mathbf{P}_\lambda(U) = \mathbf{P}(X_T/r \in U), \qquad \lambda = r^2/T, U \subseteq \mathbf{D}[0,1]. \tag{1.1}$$

A family of probability measures $(\mathbf{P}_\lambda)$ is said to obey the large deviation principle (LDP) with rate function I [see Lynch and Sethuraman (1987)], if for appropriate sets $U \subseteq \mathbf{D}[0,1]$,

$$(1/\lambda)\ln \mathbf{P}_\lambda(U) \sim -\inf_{x\in U} I(x) \tag{1.2}$$

as $\lambda \to \infty$ (for more precise definition see Section 2). The class of sets U in (1.2) is defined through the topology and is wider when the topology is stronger.

Lynch and Sethuraman (1987) proved that the family $(\mathbf{P}_\lambda)$ satisfies LDP if (a) $|\lambda_\pm| > 0$, and the sample paths of $\xi(t)$ are of bounded variation a.s.; (b) $r = r(T) = T$; (c) the class of sets U in (1.2) is defined through the weak topology in $\mathbf{D}[0,1]$.

We prove in this paper that $(\mathbf{P}_\lambda)$ satisfies LDP if: (a) $|\lambda_\pm| > 0$; (b) the dependence $r = r(T)$ is such that $r/T \leq \infty$, $r/(T)^{1/2} \to \infty$ as $T \to \infty$; (c) the class of sets U in (1.2) is defined through the Skorohod topology or the uniform-norm topology in $\mathbf{D}[0,1]$.

The first theorems on LD for stochastic processes were proved by Borovkov (1967), who considered the Wiener and Poisson processes and linear interpolations of random walks generated by i.i.d. random variables. Borovkov's initial results apply to LD in the range [of (1.1)]

$$\limsup_{T\to\infty} r/T < \infty, \qquad \lim_{T\to\infty} r/(T \ln T)^{1/2} = \infty.$$

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LD theorems for interpolation of random walks were proved by Mogulskii (1976) for the whole scope of deviations satisfying

$$\limsup_{T\to\infty} r/T < \infty, \qquad \lim_{T\to\infty} r/T^{1/2} = \infty,$$

and the class of sets U in (1.2) defined through the Skorohod topology.

For other results see Lynch and Sethuraman (1987).

2. Definitions and general results. We denote by $\mathbf{D}_w$ the space $\mathbf{D}[0,1]$ with the weak topology (a sequence f_n in $\mathbf{D}[0,1]$ converges in the weak sense to f if $f_n(t) \to f(t)$ for each point t, where f is continuous). $\mathbf{D}_c$ is the space $\mathbf{D}[0,1]$ with the uniform norm

$$\|f\| = \sup_{0\le t\le 1} |f(t)|.$$

We also consider the Skorohod metric

$$s(f,g) = \inf_{\lambda\in A}\left(\max\left(\sup_{0\le t\le 1}\left(|g(t) - f(\lambda(t))|, \sup_{0\le t\le 1}(|t-\lambda(t)|)\right)\right)\right),$$

where A is the class of continuous increasing mappings of $[0,1]$ into $[0,1]$. The metric space $(\mathbf{D}[0,1], s)$ is not complete [see Billingsley (1968)]. We denote by $\mathbf{D}_s$ the completion of $(\mathbf{D}[0,1], s)$. For $\varepsilon > 0$, $C_\varepsilon(f)$ and $S_\varepsilon(f)$ are, respectively, the ε-neighbourhoods of f in $\mathbf{D}_c$ and $\mathbf{D}_s$. Let $\mathbf{X}$ be a topological space.

Definition 2.1 [Lynch and Sethuraman (1987)]. A function $I(\cdot)$ is said to be a regular rate function (r.r.f.) in $\mathbf{X}$ if

(2.1) $$0 \le I(x) \le \infty,$$

(2.2) $I(x)$ is lower semicontinuous

and

(2.3) for each $t < \infty$, $\Gamma_t = \{x: I(x) \le t\}$ is compact on $\mathbf{X}$.

Let us repeat the definition of r.r.f. $I(x)$, $f \in \mathbf{D}[0,1]$ for measures $(\mathbf{P}_\lambda)$ [see Lynch and Sethuraman (1987); recall that var f is the variation of f]. For any element f in $\mathbf{D}[0,1]$ for which var $f < \infty$ and $f(0) = 0$, we define elements f_0, f_1, f_2 such that $f = f_0 + f_1 + f_2$, $f_0(0) = f_1(0) = f_2(0) = 0$, f_0 is an absolutely continuous function, $f_1 + f_2$ is the singular part of the function f, f_1 and f_2 are increasing and decreasing, respectively. Set

(2.4) $$\Lambda(t) = \sup_\lambda\{t\lambda - \ln \mathbf{E}e^{\lambda\xi}\}.$$

$$I(f) = \begin{cases} \int_0^1 \Lambda(f_0(t))\,dt + \lambda_+ f_2(1) + |\lambda_-|\,|f_2(1)|, & \text{if } f(0) = 0, \text{var } f < \infty, \\ \infty, & \text{otherwise}. \end{cases}$$

For any subset U of $\mathbf{D}[0,1]$,

$$I(U) = \inf_{f \in U} I(f).$$

If f is an ideal element of $\mathbf{D}_s$, that is, $f \in \mathbf{D}_s \setminus \mathbf{D}[0,1]$, set

$$I(f) = \lim_{\varepsilon \to \infty} I(S_\varepsilon(f)).$$

LEMMA 2.2. *Let $|\lambda_\pm| > 0$. Then $I(f)$ is a r.r.f. in $\mathbf{D}_w$, $\mathbf{D}_s$. If $|\lambda_\pm| = \infty$, then $I(f)$ is a r.r.f. in $\mathbf{D}_c$.*

We shall prove Lemma 2.2 in Section 3.

Let $0 < \sigma^2 < \infty$, where $\sigma^2 = \mathbf{E}\xi^2(1)$. Define

$$I_0(f) = \begin{cases} (1/2\sigma^2)\int_0^1 \left(f'(t)^2\right) dt, & \text{if } f(0) = 0,\ f \text{ is absolutely continuous}, \\ \infty, & \text{otherwise}. \end{cases}$$

The function $I_0(f)$ is a r.r.f. in $\mathbf{D}_s$ because it corresponds to the Gaussian process with stationary independent increments $\xi(t) = (1/\sigma^2)w(t)$, where $w(t)$ is the Wiener process.

Let $(\mathbf{P}_\lambda)$ be a family of probability measures on $(\mathbf{X}, \mathscr{F})$, where $\mathbf{X}$ is a topological space and $\mathscr{F}$ is the Borel σ-field of $\mathbf{X}$. For any U in $\mathscr{F}$, define

$$L^+(U) = \limsup (1/\lambda) \ln \mathbf{P}_\lambda(U),$$
$$L^-(U) = \liminf (1/\lambda) \ln \mathbf{P}_\lambda(U),$$

where, throughout the remainder of this paper, the limits are as $\lambda \to \infty$.

DEFINITION 2.3. The measures $(\mathbf{P}_\lambda)$ satisfy the upper large deviation principle (ULDP) with rate function $I(\cdot)$ if

$$I(\cdot) \quad \text{is a r.r.f. in } \mathbf{X} \tag{2.5}$$

and

$$\text{for each closed set } F,\ L^+(F) \le -I(F). \tag{2.6}$$

DEFINITION 2.4. The measures $(\mathbf{P}_\lambda)$ satisfy the lower large deviation principle (LLDP) with rate function $I(\cdot)$ in $\mathbf{X}$ if condition (2.5) holds as well as the condition

$$\text{for each open set } F,\ L^-(F) \ge -I(F). \tag{2.7}$$

The family $(\mathbf{P}_\lambda)$ is the family of path distributions on $\mathbf{D}_w$ (respectively, $\mathbf{D}_s$, $\mathbf{D}_c$) defined by (1.1).

THEOREM 2.5. *Let $|\lambda_\pm| > 0$, $r/T \to 1$ as $T \to \infty$. Then $(\mathbf{P}_\lambda)$ satisfies the ULDP with r.r.f. $I(\cdot)$ in $\mathbf{D}_s$. If, in addition $|\lambda_\pm| = \infty$, then $(\mathbf{P}_\lambda)$ satisfies the ULDP in $\mathbf{D}_c$.*

THEOREM 2.6. *Let* $|\lambda_{\pm}| > 0$, $r/T \to 0$, $r/(T)^{1/2} \to \infty$ *as* $T \to \infty$. *Then* $(\mathbf{P}_\lambda)$ *satisfies the ULDP with r.r.f.* $I_0(\cdot)$ *in* $\mathbf{D}_c$.

From standard results on infinitely divisible distributions [e.g., Skorohod (1986)] it follows that

$$\mathbf{E}e^{i\lambda\xi(t)} = e^{tK(\lambda)},$$

where

$$K(\lambda) = ia\lambda - (1/2)b^2\lambda^2 + \int(e^{i\lambda x} - 1 - i\lambda x I(|x| \le 1))\pi(dx).$$

Since $|\lambda_{\pm}| > 0$, it follows that

$$K(i\lambda) < \infty$$

for $\lambda \in (\lambda_-, \lambda_+)$. Chebyshev's inequality implies that

$$\limsup_{T\to\infty} 1/T \ln \pi(\pm T) \le -|\lambda_{\pm}|,$$

where, for $T > 0$,

$$\pi(T) = \Pi([T, \infty]), \qquad \pi(-T) = \pi((-\infty, T]).$$

We shall need the regularity condition

$$\lim_{T\to\infty} (1/T)\ln \pi(\pm T) = -|\lambda_{\pm}|. \tag{2.8}$$

THEOREM 2.7. *Let* $|\lambda_{\pm}| > 0$, $r/T \to 1$ *as* $T \to \infty$. *Then* $(\mathbf{P}_\lambda)$ *satisfies the LLDP with r.r.f.* $I(\cdot)$ *in* $\mathbf{D}_w$. *If, in addition,* (2.8) *holds, then* $(\mathbf{P}_\lambda)$ *satisfies the LLDP in* $\mathbf{D}_s$.

REMARK 2.8. Let $|\lambda_{\pm}| > 0$, $r/T \to 1$ as $T \to \infty$. Then for each open set G in $\mathbf{D}_s$,

$$L^-(G) \ge -I(G \cap \mathbf{C}[0,1]). \tag{2.9}$$

THEOREM 2.9. *Let* $|\lambda_{\pm}| > 0$, $r/T \to 0$, $r/(T)^{1/2} \to \infty$ *as* $T \to \infty$. *Then* $(\mathbf{P}_\lambda)$ *satisfies the LLDP with r.r.f.* $I_0(\cdot)$ *in* $\mathbf{D}_c$.

Using Theorems 2.5–2.9, we can obtain LD principles for the process $\xi(t)$ for the whole scope of large deviations. To state these results we need the following. Denote

$$I_\alpha(f) = \alpha^{-2}I(\alpha f), \qquad \alpha > 0.$$

The deviation function $\Lambda(\alpha)$ under conditions

$$\mathbf{E}\xi(1) = 0, \qquad 0 < \mathbf{E}\xi^2(1) = \sigma^2 < \infty, \qquad |\lambda_{\pm}| > 0,$$

satisfies the relations

$$\lim_{\alpha\to 0} \alpha^{-2}\Lambda(\alpha t) = \Lambda_0(t) \equiv t^2/(2\sigma)^2.$$

Hence

$$\lim_{\alpha\to\infty} I_\alpha(f) = I_0(f).$$

THEOREM 2.10. *Let* $|\lambda_\pm| > 0$, $r/(T)^{1/2} \to \infty$ *as* $T \to \infty$,

$$\Lambda_- \equiv \liminf_{T\to\infty} r/T \geq 0, \qquad \Lambda_+ \equiv \limsup_{T\to\infty} r/T < \infty.$$

Let the measurable set $G \subseteq \mathbf{D}[0,1]$ *satisfy the two conditions*

$$I_\alpha(G_w) = I_\alpha(\overline{G}_s) \quad \text{for all } a \in [\Lambda_-, \Lambda_+], \tag{2.10}$$

$$\lim_{\varepsilon\to\infty} I_\varepsilon(G) = I(G), \tag{2.11}$$

where G_w *is the interior of* G *in* $\mathbf{D}_w$ *and* $\overline{G}_s$ *is the closure of* G *in* $\mathbf{D}_s$. *Then*

$$\ln \mathbf{P}(X_T/T \in G) \sim -TI((r/T)G)$$

as $T \to \infty$, *where* $(r/T)G = \{f\colon (T/r)f \in G\}$.

REMARK 2.11. The statements of Theorem 2.10 remain valid under the condition

$$I_\alpha(G_c \cap \mathbf{C}[0,1]) = I_\alpha(\overline{G}_s), \qquad \alpha \in [\Lambda_-, \Lambda_+],$$

or (2.8) and the condition

$$I_\alpha(G_c) = I_\alpha(\overline{G}_s), \qquad \alpha \in [\Lambda_-, \Lambda_+],$$

instead of (2.10).

Lynch and Sethuraman (1987) study the processes $X_T^\alpha(t) \equiv X_T(\alpha(t))$, where $\alpha = \alpha(t)$ is a time deformation, and obtain LDP for the family

$$\mathbf{P}_\lambda^\alpha(G) = \mathbf{P}(X_T^\alpha/r \in G), \qquad \lambda = r^2/T.$$

But

$$\mathbf{P}_\lambda^\alpha(G) = \mathbf{P}_\lambda(G^\alpha),$$

where

$$G^\alpha = \{f = f(t) \equiv g(a(t)) : g \in G\}.$$

Therefore the LDP for $(\mathbf{P}_\lambda^\alpha)$ is a corollary of the LDP for $(\mathbf{P}_\lambda)$.

3. Proof of Lemma 2.2. The function $I(f)$ is a r.r.f. in $\mathbf{D}_w$ [see Lynch and Sethuraman (1987)]. Therefore $I(f)$ is lower semicontinuous in $\mathbf{D}_w$, $\mathbf{D}_s$, $\mathbf{D}_c$ (the topology in $\mathbf{D}_s$, $\mathbf{D}_c$ being stronger than the weak topology). If $|\lambda_\pm| = \infty$, the set Γ_T is compact in $\mathbf{D}_c$ [see Mogulskii (1976)].

To finish the proof of Lemma 2.2 we show that Γ_T is compact in $\mathbf{D}_s$ (if $|\lambda_\pm| > 0$). The set Γ_T is closed in $\mathbf{D}_s$, therefore it suffices to show that $\Gamma_T^0 \doteq \Gamma_T \cap \mathbf{D}[0,1]$ is relatively compact in $(\mathbf{D}[0,1], s)$.

It follows from the definition of the deviation function that, for all t,

$$\Lambda(t) \geq |t|\delta - \Delta, \tag{3.1}$$

where

$$0 < \delta < \min(|\lambda_+|, |\lambda_-|), \qquad \Delta \equiv \ln \mathbf{E} e^{\delta|\xi(1)|} < \infty.$$

Hence by (3.1) it follows for $f \in \mathbf{D}[0,1]$ that

$$\begin{aligned} I(f) &\geq -\Delta + \delta \operatorname{var} f_0 + \lambda_+ \operatorname{var} f_1 + |\lambda_-| \operatorname{var} f_2 \\ &\geq -\Delta + \delta \operatorname{var} f. \end{aligned} \tag{3.2}$$

Using (3.2) we obtain

$$\Gamma_T^0 \subseteq V((T+\Delta)/\delta) \equiv \{ f \in \mathbf{D}[0,1] \colon f(0) = 0, \operatorname{var} f \leq (T+\Delta)/\delta \}.$$

Therefore it is sufficient to show that $V(T)$ is relatively compact in $(\mathbf{D}[0,1], s)$ for all $T < \infty$. Denote by $B(\alpha, \beta, N)$ the finite set of step functions such that

$$f(t) = \sum_{j \geq 0} a_j I_{(j\beta, (j+1)\beta)}(t), \qquad a_j \in \{ \alpha i \colon -N \leq i \leq N \}.$$

Let $f \in V(T)$. Denote $f_\varepsilon(t) = \varepsilon[f_1(t)/\varepsilon] - \varepsilon[f_2(t)/\varepsilon]$, where $[a]$ is the integer part of a, $f(t) = f_1(t) - f_2(t)$ and $f_1(t)$, $f_2(t)$ are increasing. It is obvious that

$$s(f, f_\varepsilon) \leq |f - f_\varepsilon| \leq 2\varepsilon. \tag{3.3}$$

Denote by $0 < t_1 < t_2 < \cdots < t_k < 1$ all the discontinuity points of f_ε. Obviously, for $k \leq T/\varepsilon$:

$$\begin{aligned} k\varepsilon \leq \operatorname{var} f_\varepsilon &= \varepsilon \operatorname{var}[f_1/\varepsilon] + \varepsilon \operatorname{var}[f_2/\varepsilon] \leq \varepsilon(\operatorname{var} f_1 + \operatorname{var} f_2) \\ &= \varepsilon \operatorname{var} f \leq \varepsilon T. \end{aligned}$$

For all t_i there exists $\tilde{t}_i \in \{ j\varepsilon/k,\ j = 1, 2, \dots \}$ such that (a) $\tilde{t}_i \leq 1$, (b) $\max_{1 \leq i \leq k} |t_i + \tilde{t}_i| \leq \varepsilon$, (c) $\tilde{t}_i < \tilde{t}_j$ if $t_i < t_j$. Construct a piecewise linear function $\lambda(t)$ interpolating

$$(0,0), (t_1, \tilde{t}_1), \dots, (t_k, \tilde{t}_k), (1,1)$$

in the plane (x, t). It is obvious that $\lambda(t)$ satisfies

$$\sup_{0 \leq t \leq 1} |t - \lambda(t)| \leq \varepsilon. \tag{3.4}$$

All points of discontinuity of a function $\tilde{f}_\varepsilon(t) = f_\varepsilon(\lambda(t))$ lie in $\{ j\varepsilon/k,\ j = 1, 2, \dots; j\varepsilon/k \leq 1 \}$. Therefore

$$\tilde{f}_\varepsilon \in B(\varepsilon, \varepsilon/k, k).$$

Thanks to (3.4), the relation

$$s(\tilde{f}_\varepsilon, f_\varepsilon) \leq \varepsilon \tag{3.5}$$

follows. We obtain

$$s(\tilde{f}_\varepsilon, f) \leq 3\varepsilon$$

using (3.5) and (3.1). We have already seen that $B(\varepsilon, \varepsilon/k, k)$ is a 3ε-net in $V(t)$. □

4. Auxiliary lemmas.

LEMMA 4.1. *Let f, f_1, $g \in \mathbf{D}[0,1]$, $h \in \mathbf{C}[0,1]$, $\varepsilon = s(f, f_1)$, $\varepsilon_1 = |g - h|$. Then*

$$s(f+g, f_1+h) \le \varepsilon + \varepsilon_1 + \omega_h(\varepsilon),$$

where $\omega_h(t)$ is the continuity modulus of h.

PROOF. For every $\theta > 1$, there exists continuous increasing functions $\lambda(t)$, $\lambda(0) = 0$, $\lambda(1) = 1$, such that for all $t \in [0,1]$,

$$|\lambda(t) - t| \le \theta\varepsilon, \qquad |f(\lambda(t)) - f_1(t)| \le \theta\varepsilon.$$

Therefore,

$$\begin{aligned} s(f+g, f_1+h) &\le |f(\lambda(t)) + g(\lambda(t)) - f_1(t) - h(t)| \\ &\le |f(\lambda(t)) - f_1(t)| + |g(\lambda(t)) - h(\lambda(t))| + |h(\lambda(t)) - h(t)| \\ &\le \theta\varepsilon + \varepsilon_1 + \omega_h(\theta\varepsilon). \end{aligned}$$

We now complete the proof of Lemma 4.1 by letting $\theta \to 1$. □

Let $(\mathbf{E}, d)$ be a complete metric space, $(\mathbf{P}_\lambda)$ be a family of probability measures on $(\mathbf{E}, \mathscr{F})$, where $\mathscr{F}$ is the Borel σ-field of $(\mathbf{E}, d)$. Let $I(x)$ be a r.r.f. in $(\mathbf{E}, d)$.

LEMMA 4.2. *For any closed set F in $\mathbf{E}$,*

$$I(F) = \lim_{\varepsilon \to 0} I(D_\varepsilon(F)),$$

where

$$D_\varepsilon(F) = \{x \in E: d(x,y) < \varepsilon, y \in F\}.$$

PROOF. Set

$$M = \lim_{\varepsilon \to 0} I(D_\varepsilon(F)).$$

Since

$$I(F) \ge M,$$

it suffices to show that

$$I(F) \le M. \tag{4.1}$$

If $M = \infty$, then (4.1) is correct. If $M < \infty$, then there exists a sequence (x_n), $x_n \in \mathbf{E}$ such that

$$M = \lim_{n \to \infty} I(x_n), \qquad I(x_n) \le 2M, \qquad x_n \in \mathbf{D}_{1/n}(F).$$

Since Γ_{2M} is compact in $(\mathbf{E}, d)$ and $x_n \in \Gamma_{2M}$, there exists a subsequence (y_n), $y_n = x_{i_n}$, satisfying

$$y_0 \equiv \lim_{n\to\infty} y_n \in F.$$

By the semicontinuity of $I(\cdot)$ we have (4.1):

$$M = \lim_{n\to\infty} I(y_n) \geq I(y_0) \geq \inf_{y\in F} I(y) = I(F). \qquad \square$$

DEFINITION 4.3. The measures $(\mathbf{P}_\lambda)$ are weak large deviation tight (WLD tight) in $(\mathbf{E}, d)$, if for each $M < \infty$ and $\varepsilon > 0$ there exists a finite set $(x_1, \dots, x_m)$, $x_i \in \mathbf{E}$, such that

$$L^+\left(\left(\bigcup_{i=1}^m D_\varepsilon(x_i)\right)^c\right) \leq -M, \tag{4.2}$$

where $(U)^c$ is the complement of a set U.

It is obvious that the measures $(\mathbf{P}_\lambda)$ are WLD tight in $(\mathbf{E}, d)$, if for each $M < \infty$, there exists a compact set $K = K_M$ such that for each $\varepsilon > 0$,

$$L^+\left((D_\varepsilon(K))^c\right) \leq -M.$$

LEMMA 4.4. *Let $(\mathbf{P}_\lambda)$ be WLD tight in $(\mathbf{E}, D)$, let the function $I(x)$ be a r.r.f. in $(\mathbf{E}, D)$ and let E_0 be a dense set in $(\mathbf{E}, d)$ such that for each $x \in E_0$, $\varepsilon > 0$, $\delta > 0$,*

$$L^+(D_\varepsilon(x)) \leq -I(D_{\varepsilon+\delta}(x)). \tag{4.3}$$

Then $(\mathbf{P}_\lambda)$ satisfies the ULDP with r.r.f. $I(\cdot)$ in $(\mathbf{E}, d)$.

PROOF. If $I(x) = 0$, then (2.6) is correct. If $I(x) > 0$, denote

$$M = \min(2I(F), N),$$

where $N < \infty$. For this M, there exists a finite set $(x_1, \dots, x_m)$, $x_i \in E_0$ (thanks to WLD tightness) such that (4.2) is correct. Therefore,

$$\mathbf{P}_\lambda(F) \leq \sum_{i=1}^m \mathbf{P}_\lambda(F \cap D_\varepsilon(x_i)) + \mathbf{P}_\lambda\left(\left(\bigcup_{i=1}^m D_\varepsilon(x_i)\right)^c\right).$$

With the help of (4.2) and (4.3), we obtain that

$$\begin{aligned} L^+(F) &\leq -\min\big(M, \min\big(I(D_\varepsilon(x_i))\big)\colon F \cap D_\varepsilon(x_i) \neq \varnothing\big) \\ &\leq -\min\big(M, \min\big(I(D_{2\varepsilon}(F))\big)\big). \end{aligned}$$

We now complete the proof of Lemma 4.4 by letting M go to ∞ and ε go to 0 (thanks to Lemma 4.2). $\square$

LEMMA 4.5. *Let $I(\cdot)$ be a r.r.f. in $(\mathbf{E}, d)$ and $E_0 \subseteq \mathbf{E}$ a set such that for each open set G,*

$$I(G) = I(G \cap E_0). \tag{4.4}$$

If for each $x \in E_0$ *and* $\varepsilon > 0$,

$$L^-(D_\varepsilon) \geq -I(x),$$

then $(\mathbf{P}_\lambda)$ *satisfies the LLDP with r.r.f.* $I(\cdot)$ *in* $(\mathbf{E}, d)$.

PROOF. Obvious. □

5. Proof of the theorems.

PROOF OF THEOREM 2.5. For $k = 1, 2, \dots, \delta > 0$ we introduce the set

$$X(k,\delta) = \Big\{ f \in \mathbf{D}[0,1]: f(0) = 0, \sup_{0 \leq t \leq 1} |f((i+t)/k) - f(i/k)| < \delta, i = 0, 1, \dots, k-1 \Big\}.$$

It is obvious that

$$\mathbf{P}_\lambda\big((X(k,\delta))^c\big) \leq k\mathbf{P}\Big(\sup_{0 \leq t \leq T} |\xi(t/k)| \geq r\delta\Big).$$

It follows from a well-known inequality [Skorohod (1986), page 35] that

$$\mathbf{P}\Big(\sup_{0 \leq t \leq T} |\xi(t)| \geq 2x\Big) \leq (1-\alpha)^{-1}\mathbf{P}(|\xi(T)| \geq x), \tag{5.1}$$

where

$$\alpha = \sup_{0 \leq t \leq T} \mathbf{P}(|\xi(t)| \geq x) < 1.$$

Chebyshev's inequality implies the estimate

$$\mathbf{P}(|\xi(T)| \geq x) \leq e^{-T\Lambda(x/T)} + e^{-T\Lambda(-x/T)}. \tag{5.2}$$

The deviation function $\Lambda(\alpha)$ [see (2.4)] satisfies [Borovkov (1967)]

$$\Lambda(\alpha) = \int_a^\alpha \lambda(t)\,dt, \qquad a = \mathbf{E}\xi(1), \tag{5.3}$$

where $\lambda(t)$ is an increasing function, $\lambda(a) = 0$ and

$$\lim_{t \mapsto \pm\infty} \lambda(t) = \lambda_\pm. \tag{5.4}$$

It follows by (5.3) and (5.4) that

$$\lim_{t \to \pm\infty} \Lambda(t)/|t| = |\lambda_\pm|. \tag{5.5}$$

With the help of (5.1), (5.2) and (5.5), we obtain that for each $M < \infty$, $\delta > 0$, there exists an integer $k = k(M, \delta)$ such that

$$L^+\big((X(k,\delta))^c\big) \leq -M \tag{5.6}$$

and

$$\delta k(M,\delta) \to \infty \quad \text{as } \delta \to 0. \tag{5.7}$$

Let us consider the compact set in $\mathbf{C}[0,1]$:

$$K = \{f \in \mathbf{C}[0,1]\colon f(0) = 0, \omega_f(1/k) \le \delta, 0 \le \delta \le 1\},$$

where $\omega_f(t)$ is the modulus of continuity of f. K is, obviously, also compact in $\mathbf{D}_c$. Let $f_k = f_k(t)$ be the piecewise linear function interpolation points

$$(i/k, f(i/k)), \qquad i = 0, 1, \dots, k,$$

where $f \in X(k, \delta)$. It follows by (5.7) that for $k = k(M, \delta)$,

$$f_k \in K, \qquad |f - f_k| \le \delta$$

and

$$C_\delta(K) \supseteq X(k,\delta). \tag{5.8}$$

By (5.6) and (5.8), we obtain that

$$L^+\big((C_\delta(K))^c\big) \le L^+\big((X(k,\delta))^c\big) \le -M.$$

So, the measures $(\mathbf{P}_\lambda)$ are WLD tight.

Let Γ be the class of linear interpolations between nodes with a given finite set of time t, D be the class of step functions with finite sets of steps. Denote

$$\Gamma + D = \{f = g + h\colon g \in \Gamma, h \in D\}.$$

It is obvious that $\Gamma + D$ is a dense set in $\mathbf{D}_c$. We can obtain that for each $f \in \Gamma + D$, $\varepsilon > 0$, $\delta > 0$,

$$L^+(C_\varepsilon(f)) \le -I(C_{\varepsilon+\delta}(f)), \tag{5.9}$$

using the proof of Theorem 4.1 in Lynch and Sethuraman (1987). Therefore thanks to Lemma 4.4, the measures $(\mathbf{P}_\lambda)$ satisfy the ULDP with $I(\cdot)$ in $\mathbf{D}_c$.

Assume now the condition $|\lambda_\pm| > 0$. We can write

$$\xi(t) = \xi(t)^{(1)} + \xi(t)^{(2)}, \tag{5.10}$$

where $\xi^{(1)}$ and $\xi^{(2)}$ are two independent processes and

$$\begin{aligned} &\mathbf{E}\exp\{i\lambda\xi^{(1)}(t)\} \\ &\quad = \exp\left\{t\left(ia\lambda - b^2\lambda^2/2 + \int_{-N}^{N}(e^{i\lambda x} - 1 - i\lambda x I(|x| \le 1))\pi(dx)\right)\right\}, \end{aligned} \tag{5.11}$$

$$\mathbf{E}\exp\{i\lambda\xi^{(2)}(t)\} = \exp\left\{t\left(\int_{|x|>N}(e^{i\lambda x} - 1)\pi(dx)\right)\right\}.$$

Let $N = 1$. The process $\xi^{(1)}$ satisfies the condition $|\lambda_\pm^{(1)}| = \infty$. By the part of Theorem 2.5 already proved, the family $(\mathbf{P}_\lambda^{(1)})$ satisfies the ULDP in $\mathbf{D}_c$ (and in $\mathbf{D}_s$) with r.r.f. $I^{(1)}(\cdot)$.

Let us consider the process $\xi^{(2)}$, which is a generalized Poisson process. Denote T_i the subsequent jump times of $\xi^{(2)}$ and set $\tau_i = T_i - T_{i-1}$, $\xi_i = |\xi^{(2)}(T_i + 0) - \xi^{(2)}(T_i - 0)|$. Then (τ_i), (ξ_i) are two independent sequences of i.i.d. random variables. The total variation $\xi^{(2)}$ obviously satisfies the relations

$$\{\operatorname{var}(X_T^{(2)}/r) > N\} \subseteq \left\{\sum_{i=1}^{mT} \xi_i > N\right\} \cup \left\{\sum_{i=1}^{mT} \tau_i \geq T\right\}.$$

By Chebyshev's inequality,

$$\begin{aligned}\mathbf{P}(X_T^{(2)}/r \notin V(N)) \leq \exp\{-mT\Lambda_1(Nr/(mT))\}\\ + \exp\{-mT\Lambda_2(1/m)\},\end{aligned} \tag{5.12}$$

where Λ_1 and Λ_2 are the deviation functions for ξ_1 and τ_1, respectively. Using (5.3), we can show that

$$m\Lambda_2(1/m) \to \infty, \qquad \Lambda_1(N) \to \infty,$$

as $m, N \to \infty$. Therefore by (5.12), for each $M < \infty$ there exist $N < \infty$ such that $L^{(2)}((V(N))^c) \leq -M$. Hence, under the condition $|\lambda_\pm| > 0$, the family $(\mathbf{P}_\lambda^{(2)})$ is WLD tight in $\mathbf{D}_s$. Since the inequality (5.9) holds for $(\mathbf{P}_\lambda^{(2)})$, it follows easily by lemma 4.4 that $(\mathbf{P}_\lambda^{(2)})$ satisfies the ULDP with r.r.f. $I^{(2)}$ in $\mathbf{D}_s$.

Let $\varepsilon > 0$, $M < \infty$. As we proved, there exists a finite set $\{f_1, \dots, f_m\}$, $f_i \in \mathbf{C}[0,1]$, such that

$$L_1^+\left(\left(\cup C_{\varepsilon/3}(f_i)\right)^c\right) \leq -M.$$

Let $\nu > 0$ be such that $\nu \leq \varepsilon$ and

$$\max_{1 \leq i \leq m} \omega_f(\nu) \leq \varepsilon/3, \qquad f \in \{f_1, \dots, f_m\}.$$

For this ν there exists a finite set $(g_1, \dots, g_n)$, $g_i \in \mathbf{D}[0,1]$ such that [the family $(\mathbf{P}_\lambda^{(2)})$ is WLD tight in $\mathbf{D}_s$]

$$L_2^+\left(\left(\bigcup S_\nu(g_j)\right)^c\right) \leq -M.$$

It follows from Lemma 4.1 that

$$\left(\bigcup_{ij} S_\varepsilon(f_i + g_j)\right)^c \subseteq \left(\bigcup_j S_\nu(g_j)\right)^c \cup \left(\bigcup_i C_{\varepsilon/3}(f_i)\right)^c.$$

Hence we obtain that $(\mathbf{P}_\lambda)$ satisfies WLD tightness in $\mathbf{D}_s$:

$$L^+\left(\left(\bigcup_{ij} S_\varepsilon(f_i + g_j)\right)^c\right) \leq -M.$$

Since the inequality (4.3) holds for $(\mathbf{P}_\lambda)$, it follows easily by Lemma 4.4 that $(\mathbf{P}_\lambda)$ satisfies the ULDP in $\mathbf{D}_s$ with r.r.f. $I(\cdot)$. □

Proof of Theorem 2.6. This proof is similar to that of Theorem 2.5. □

Proof of Theorem 2.7. Let $|\lambda_\pm| = \infty$. We shall show that for each $f \in \Gamma$, $\varepsilon > 0$,

$$L^-(C_\varepsilon(f)) \geq -I(f). \tag{5.13}$$

If $I(f) = \infty$, then (5.13) is obvious. Let $I(f) < \infty$. Introduce the set

$$Y(k, \varepsilon) = \left\{ g \in \mathbf{D}[0,1]: \max_{1 \le i \le k} |g(i/k) - f(i/k)| < \varepsilon \right\}.$$

It is not difficult to prove [see Lynch and Sethuraman (1987)] that for each $f \in Y(k, \varepsilon)$,

$$L^-(Y(k,\varepsilon)) \ge -I(f). \tag{5.14}$$

We proved (see the proof of Theorem 2.5), that there exists a compact K in $\mathbf{D}_C$ such that for each $\delta > 0$,

$$L^-\left((C_\delta(K))^c\right) \le -(2I(f) + 1). \tag{5.15}$$

It is obvious that for each $\varepsilon > 0$, there exist $\delta > 0$ and $k < \infty$ such that

$$C_\varepsilon(f) \supseteq Y(k, \delta) \cap C_\delta(K).$$

Hence by (5.14) and (5.15) we obtain (5.13). Under the condition $|\lambda_{\pm}| = \infty$ the class Γ satisfies (4.4). Hence by Lemma 4.5 we obtain that $(\mathbf{P}_\lambda)$ satisfies the LLDP with r.r.f. I in $\mathbf{D}_C$.

Assuming the condition $|\lambda_{\pm}| > 0$, let us consider the sum [see (5.10) and (5.11)]

$$\xi(t) = \xi^{(1)}(t) + \xi^{(2)}(t),$$

where number $N \in [1, \infty)$ is arbitrary. The process $\xi^{(1)}(t)$ satisfies $|\lambda_{\pm}^{(1)}| = \infty$, hence $(\mathbf{P}_\lambda^{(1)})$ satisfies the LLDP in $\mathbf{D}_c$ (and in $\mathbf{D}_w$, obviously) with r.r.f. $I^{(1)}$. The process $\xi^{(2)}(t)$ satisfies $|\lambda_{\pm}^{(2)}| > 0$ and its sample paths belong to the space of functions of bounded variation. Thus, using Theorem 5.1 in Lynch and Sethuraman (1987) we obtain that $(\mathbf{P}_\lambda^{(2)})$ satisfies the LLDP in $\mathbf{D}_w$ with r.r.f. $I^{(2)}$.

Let G be an open set in $\mathbf{D}_w$, $f \in G$, $g \in \mathbf{C}[0,1]$, $g(0) = f(0) = 0$. There are $\varepsilon > 0$ and a neighbourhood G' of $f - g$ in $\mathbf{D}_w$ such that

$$G \supseteq G' \cap C_\varepsilon(g).$$

Therefore,

$$\mathbf{P}_\lambda(G) \ge \mathbf{P}_\lambda^{(1)}(C_\varepsilon(g))\mathbf{P}_\lambda^{(2)}(G').$$

We obtain

$$L^-(G) \ge -(I^{(1)}(g) + I^{(2)}(f - g)).$$

It is known [see the proof of Theorem 5.1 in Lynch and Sethuraman (1987)] that

$$I(f) = \inf_g \{ I^{(1)}(g) + I^{(2)}(f - g) \}.$$

Therefore for each $f \in G$,

$$L^-(G) \ge -I(f)$$

and $(\mathbf{P}_\lambda)$ satisfies the LLDP in $\mathbf{D}_w$ with $I(\cdot)$.

Under the condition (2.8) let us consider a function

$$f \in \Gamma + D, \qquad f = g + h,$$

where $f(0) = g(0) = h(0) = 0$, $g \in \Gamma$, $h \in D$. It is obvious (by Lemma 4.1), that there exists $\delta > 0$ such that

$$\{X_T/r \in S_\varepsilon(f)\} \supseteq \{X_T^{(1)}/r \in C_{\varepsilon/3}(g)\} \cap \{X_T^{(2)}/r \in S_\delta(h)\}.$$

Hence

$$L^-(S_\varepsilon(f)) \geq L_N^{(1)-}(C_{\varepsilon/3}(g)) + L_N^{(2)-}(S_\delta(h)), \tag{5.16}$$

where N is a level in (5.11). Thanks to (5.13) we obtain for each $g \in \Gamma$,

$$\lim_{N\to\infty} I_N^{(1)}(g) = I(g),$$

hence

$$\lim_{N\to\infty} L_N^{(1)-}(C_\varepsilon(g)) \geq -I(g). \tag{5.17}$$

To estimate $L_N^{(2)-}(S_\varepsilon(h))$, we assume for brevity that

$$h(0) = 0 \quad \text{as } 0 \leq t \leq t_1, \qquad h(t) = a \quad \text{as } t_1 \leq t \leq 1, \tag{5.18}$$

where $a > 0$, $0 < t_1 < 1$. Let $\eta(t)$ be the Poisson process with constant intensity 1. It is convenient to suppose that the processes $\xi^{(2)}(t)$ and $\eta(t)$ have the same jump times. Let us denote by $\xi_i^{(N)}$ the random jumps for $\xi^{(2)}(t)$. It is obvious that

$$\lim_{N\to\infty} \mathbf{P}(\xi_i^{(N)} = 0) = 1, \tag{5.19}$$

and (2.8) implies that for each $N < \infty$, $\varepsilon > 0$,

$$\liminf_{N\to\infty}(1/T)\ln \mathbf{P}(\xi_i^{(N)}/r \in (a-\varepsilon, a+\varepsilon) \geq -\lambda_+(a-\varepsilon)). \tag{5.20}$$

Let

$$A_\varepsilon = \{\eta((t_1+\varepsilon)T) - \eta((t_1-\varepsilon)T), \eta(T) \leq 2T\}.$$

It is obvious that

$$\mathbf{P}(X_T^{(2)}/r \in S_\varepsilon(h)) \geq \mathbf{P}(A_\varepsilon)\mathbf{P}(\xi_1^{(N)}/r \in (a-\varepsilon, a+\varepsilon)\mathbf{P}^{2T}(\xi_1^{(N)} = 0)).$$

Hence by (5.19) and (5.20), we obtain that

$$L_N^{(2)-}(S_\varepsilon(h)) \geq -\lambda_+(a-\varepsilon) \geq -\lambda_+ a = -I^{(2)}(h),$$

where h has the form (5.19). It is obvious that for each $h \in D$ the inequality

$$L_N^{(2)}(S_\varepsilon(h)) \geq -I^{(2)}(h) = I(h) \tag{5.21}$$

holds. Hence by (5.16), (5.17) and (5.21), it follows that

$$L^-(S_\varepsilon(F)) \geq -I(g) - I(h) = I(f).$$

The set $\Gamma + D$ satisfies (4.4) in $\mathbf{D}_s$. Hence by Lemma 4.5, we obtain that $(\mathbf{P}_\lambda)$ satisfies the LLDP in $\mathbf{D}_s$ [with r.r.f. $I(\cdot)$]. □

We have at the same time proved the proposition stated in Remark 2.8. □

PROOF OF THEOREM 2.9. This proof is similar to the proof of Theorem 2.7. □

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