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MARTINGALE FUNCTIONAL CENTRAL LIMIT THEOREMS FOR A GENERALIZED PÓLYA URN[1]

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In a generalized two-color Pólya urn scheme, allowing negative replacements, we use martingale techniques to obtain weak invariance principles for the urn process (W_n), where W_n is the number of white balls in the urn at stage n. The normalizing constants and the limiting Gaussian process are shown to depend on the ratio of the eigenvalues of the replacement matrix.

1. Introduction. We consider a two-color urn with W_0 white and B_0 black balls, $T_0 = W_0 + B_0$. Balls are drawn at random in succession, their color noticed and then replaced in the urn, together with new black and white balls. Replacements are controlled by a deterministic matrix $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as follows: If a white ball is drawn, it is returned to the urn with a white and b black balls. Otherwise, when a black ball is drawn, it is returned with c white and d black balls. Negative entries in R are allowed and correspond to removals.

Our purpose is to obtain weak invariance principles for the urn process (W_n) (number of white balls in the urn at time n), using martingale transforms and a standard version of the functional central limit theorem (FCLT) for discrete time martingales. We are motivated by a paper of Bagchi and Pal (1985), where asymptotic normality is obtained through the method of moments. We give here a positive answer to their question about asymptotic normality via martingale theory and we extend the convergence to a functional limit theorem. We also extend their results by proving strong and weak convergence for the particular case $bc = 0$, $\max(b, c) > 0$. Simple martingale arguments are also used in Gouet (1989) to obtain a strong law for W_n, when $bc > 0$.

Bagchi and Pal (1985) show the relevance of the urn model with negative replacements in a nice application to computer data structures known as 2-3 trees. However, in a recent paper, Aldous, Flannery and Palacios (1988) have modeled 2-3 trees using an urn scheme with nonnegative R. See also Mahmoud and Smythe (1991).

The traditional Pólya urn ($a = d$, $b = c = 0$) as well as many generalizations, have been extensively studied. A detailed discussion of these models and their applications can be found in Johnson and Kotz [(1977), Chapter 4].

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Asymptotic results under various replacement strategies are given by Athreya and Karlin (1968), Freedman (1965) and Holst (1979) but they do not include the case studied here. Weak invariance principles for Friedman's urn are due to Ito and Freedman. See Freedman (1965).

The paper is organized as follows: The tenable model is described in the introduction. Main results (FCLT) are presented next, in Propositions 2.1 and 2.2, and their proofs are given in Section 3. Finally, some technical results on the asymptotics of the normalizing constants of the FCLT are collected in the Appendix.

The tenable model. Let W_n (resp., B_n) be the number of white (resp., black) balls in the urn at stage n and $T_n = W_n + B_n$. Following Bagchi and Pal (1985) we state the assumptions that define the tenable urn process:

1. The urn is not initially empty, that is, $T_0 > 0$.
2. The total number of balls increases by the same amount $s \geq 1$ at every stage, that is, $T_n = ns + T_0$.
3. The urn process (W_n) is not deterministic, that is, $a \neq c$.
4. The process does not stop because of impossible removals. This requires that only balls of the same color as the ball drawn should be discarded from the urn, that is, $b \geq 0$ and $c \geq 0$. Further, if $a < 0$ then a divides W_0 and c. The analogous property also applies to d.

Convergence and notation. We assume that all random variables and processes are defined on a common probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Continuous time parameter processes are considered as random elements of $D = D[0, \infty)$, the space of right continuous real valued functions $x(t)$ on $[0, \infty)$ with left limits, endowed with the usual Skorohod topology on compact t sets. Weak convergence of a sequence of processes (X_n) to X in D, denoted by $X_n(t) \Rightarrow X(t)$, is understood as weak convergence of the induced probabilities on D.

The Wiener process is denoted by $W(t)$ and for every positive, continuous and strictly increasing function φ on $[0, \infty)$, such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, $W \circ \varphi(t)$ denotes the continuous Gaussian martingale with covariance function $K(s, t) = \varphi(\min(s, t))$.

Let (u_n) and (v_n) be two sequences of real numbers. We write $u_n \sim v_n$ when (u_n/v_n) has a nonzero finite limit, while $u_n \simeq v_n$ means that (u_n/v_n) has limit 1. The symbols $o(\cdot)$ and $O(\cdot)$ have their usual meanings.

2. Main results. The aim of this paper is to provide weak convergence results for the sequence of processes $W_{k_n(t)} - \mu T_{k_n(t)}$, $n \in \mathbb{N}$ and $t \geq 0$, where $\mu = c/(b + c) = \lim_{n \to \infty} W_n/T_n$ a.s. and $(k_n(t))$ is a sequence of deterministic time scales, that is, for every positive integer n, $k_n(\cdot)$ is a nondecreasing right continuous function taking nonnegative integer values.

The limiting process, the normalizing constants and some of the tools used in the proofs depend on two parameters of the urn process: $\rho = (a - c)/s$ (the ratio of the eigenvalues of R), and the product bc, which is related to the conditional variances of the martingale (M_n) defined in (12).

It should be noted that properties 3 and 4 of the tenable model imply that $\rho \neq 0$ and $\rho \leq 1$, respectively. However, unbounded negative values are possible. When $\rho > 1/2$ and $bc > 0$, results depend on a nondegenerate random variable Z, which is the a.s. limit of the martingale involved in the proof. In this case we apply a technique used by Heyde (1977) to obtain a FCLT even though we know nothing of the distribution of Z.

PROPOSITION 2.1. *Let (W_n) be a tenable urn process such that $bc > 0$. Then for $\rho > 1/2$,*

$$n^{-1/2}\Big(W_{[nt]} + (nt)^{\rho}\big(\beta_{[nt]} - Z\big)\Big) \Rightarrow t^{1-\rho} W \circ \varphi(t), \tag{1}$$

where $\varphi(t) = bc/(2\rho - 1)(\rho/(1-\rho))^2 t^{2\rho - 1}$, Z is a nondegenerate r.v. independent of W and β_k is defined in (14).

For $\rho = 1/2$,

$$\big(n^t \log(n)\big)^{-1/2}\big(W_{[n^t]} - \mu T_{[n^t]}\big) \Rightarrow W \circ \varphi(t), \tag{2}$$

where $\varphi(t) = bct$.

For $\rho < 1/2$,

$$n^{-1/2}\big(W_{[nt]} - \mu T_{[nt]}\big) \Rightarrow t^{\rho} W \circ \varphi(t), \tag{3}$$

where $\varphi(t) = bc/(1 - 2\rho)(\rho/(1-\rho))^2 t^{1-2\rho}$.

We consider in the following proposition a tenable scheme with replacement matrix $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $bc = 0$. Under this assumption, the results of Proposition 2.1 do not hold. If we exclude from our analysis the classical Pólya urn ($b = c = 0$), studied by Heyde (1977), we have two symmetric cases: $b > 0$, $c = 0$ and $b = 0$, $c > 0$. For the first one, it is easy to verify that a negative value of parameter a implies the extinction of white balls. Therefore, we are left with essentially one interesting case: $R = \begin{bmatrix} a & b \\ 0 & s \end{bmatrix}$, where $s > a > 0$ and $\rho = a/s$. However, since R is not irreducible, the asymptotic behavior of this process does not follow from results of Athreya and Karlin (1968). The strong convergence $W_n/T_n \to 0$, established in Gouet (1989), can be improved to $W_n/T_n^{\rho} \to Z$, where Z is a nondegenerate r.v., using the martingale convergence theorem. Information on the moments of Z can be obtained from the difference equations characterizing the moments of W_n.

PROPOSITION 2.2. *Let (W_n) be a tenable urn process such that $R = \begin{bmatrix} a & b \\ 0 & s \end{bmatrix}$, where $s > a > 0$ ($bc = 0$, $\max(b, c) > 0$). Then*

$$n^{-\rho/2}\big(W_{[nt^{1/\rho}]} - n^{\rho} t Z\big) \Rightarrow W \circ \varphi(t), \tag{4}$$

where $\varphi(t) = aZt$ and Z is a nondegenerate positive r.v., independent of W.

3. The proofs. The proofs of Propositions 2.1 and 2.2 are based on a simplified version of a well-known FCLT for martingale arrays, that we take from Durrett and Resnick [(1978), Theorem 2.5]: Let $(\mathscr{F}_n)$ be a sequence of increasing sub σ-fields of $\mathscr{F}$, $\{M_n = \sum_{k=1}^n \xi_k, \mathscr{F}_n\}$ a square integrable martingale and (b_n) an increasing divergent sequence of norming constants. If for all $t > 0$,

$$A_n(t) = b_n^{-2} \sum_{k=1}^{k_n(t)} \mathbf{E}(\xi_k^2 | \mathscr{F}_{k-1}) \to_P \varphi(t) \tag{5}$$

with $\mathbf{P}(\varphi \text{ continuous}) = 1$; and for all $\varepsilon > 0$, $t > 0$,

$$B_n(\varepsilon) = b_n^{-2} \sum_{k=1}^{k_n(t)} \mathbf{E}\left(\xi_k^2 \mathbb{1}_{\{|\xi_k| > \varepsilon b_n\}} | \mathscr{F}_{k-1}\right) \to_P 0, \tag{6}$$

where $\to_P$ denotes convergence in $\mathbf{P}$ probability and $\mathbb{1}_{\{\cdot\}}$ is the indicator function of the set $\{\cdot\}$, then, as $n \to \infty$,

$$b_n^{-1} M_{k_n(t)} = b_n^{-1} \sum_{k=1}^{k_n(t)} \xi_k \Rightarrow W \circ \varphi(t). \tag{7}$$

It can be easily seen that the tenable urn process (W_n) is a Markov chain with $\mathbf{P}(W_{n+1} = W_n + a | W_0 \cdots W_n) = W_n/T_n$, $n = 0, 1 \dots$. Furthermore, W_n/T_n converges a.s. to $\mu = c/(b + c)$ for any matrix R such that $\max(b, c) > 0$.

Define the indicator variable $I_n = 1$ if the ball drawn at stage n is white and zero otherwise. Then $\mathbf{E}(I_n | I_0 \cdots I_{n-1}) = \mathbf{E}(I_n | W_0 \cdots W_n) = W_n/T_n$. This shows that $N_n = \sum_{k=0}^n \eta_k = \sum_{k=0}^n (I_k - W_k/T_k)$ is a martingale with respect to the filtration $\mathscr{F}_n = \sigma(I_0 \cdots I_n)$ $n = 1, 2 \dots$. Further, since the increments of (N_n) are uniformly bounded, condition (6) is verified and we obtain a first FCLT for (W_n).

Proposition 3.1. *Let (W_n) be a tenable urn process such that $bc > 0$ and let $\varphi(t) = bc(\rho/(1-\rho))^2 t$. Then as $n \to \infty$,*

$$n^{-1/2}\left(W_{[nt]} - \mu T_{[nt]} - \rho s \sum_{k=0}^{[nt]} (W_k/T_k - \mu)\right) \Rightarrow W \circ \varphi(t). \tag{8}$$

Proof. Let $\psi(t) = \mu(1-\mu)t$. Then, (5) is readily checked since, as $n \to \infty$, for all $t > 0$,

$$n^{-1} \sum_{k=0}^{[nt]} \mathbf{E}\left((I_k - W_k/T_k)^2 | \mathscr{F}_{k-1}\right) = n^{-1} \sum_{k=0}^{[nt]} (W_k/T_k)(1 - W_k/T_k) \to \psi(t) \text{ a.s.}$$

Hence,

$$n^{-1/2} N_{[nt]} = n^{-1/2} \sum_{k=0}^{[nt]} (I_k - W_k/T_k) \Rightarrow W \circ \psi(t). \tag{9}$$

Next, note that

$$W_{n+1} - W_n = aI_n + c(1 - I_n) = \rho s I_n + c \quad \text{for } n = 0, 1, \dots \tag{10}$$

and hence that

$$W_n = W_0 + \rho s \sum_{k=0}^{n-1} I_k + nc \quad \text{for } n = 1, 2 \dots . \tag{11}$$

The result (8) now follows from (9), (11) and the identity $(\rho s)^2 \mu(1-\mu) = bc(\rho/(1-\rho))^2$. $\square$

We cannot obtain a FCLT for (W_n) with deterministic centering from (8) since $n^{-1/2}\Sigma_{k=0}^{n}(W_k/T_k - \mu)$ does not converge to zero in probability. To prove Proposition 2.1 we will consider a martingale transform of (N_n) such that the norming for the FCLT is deterministic.

Let (x_n) be a predictable sequence of random variables, that is, adapted to $(\mathscr{F}_{n-1})$. The transform of the $(\mathscr{F}_n)$ martingale $N_n = \Sigma_{k=0}^{n}\eta_k$ by (x_n) is given by

$$M_n = \sum_{k=0}^{n} x_k \eta_k = \sum_{k=0}^{n} \xi_k. \tag{12}$$

Full details about martingale transforms can be found in Burkholder (1966) and Neveu [(1972), Chapter VII-3].

In the context of the martingale (N_n) related to the urn process, we wish to find deterministic sequences (x_n), (α_n) and (β_n) such that (M_n) is of the form

$$M_{n-1} = \alpha_n W_n + \beta_n - \alpha_0 W_0 \tag{13}$$

for $n = 1, 2, \dots$.

LEMMA 3.2. *A solution to the problem stated above is given by*

$$x_n = \rho s \alpha_{n+1}, \qquad \alpha_n = \frac{\Gamma(T_n/s)}{\Gamma(\rho + T_n/s)} \quad \textit{and} \quad \beta_n = -c \sum_{k=1}^{n} \alpha_k. \tag{14}$$

PROOF. Let Δ be the forward difference operator. Then,

$$\begin{aligned} \Delta M_{n-1} &= M_n - M_{n-1} = x_n(I_n - W_n/T_n) \\ &= x_n((\Delta W_n - c)/\rho s - W_n/T_n), \end{aligned} \tag{15}$$

where the last equality follows from (10). On the other hand, if (13) is to hold,

$$\Delta M_{n-1} = \Delta(\alpha_n W_n + \beta_n) = \alpha_{n+1} \Delta W_n + W_n \Delta \alpha_n + \Delta \beta_n. \tag{16}$$

Finally, we identify the coefficients of W_n, ΔW_n and the constant terms of (15) and (16) to get

$$\alpha_{n+1} = (\rho s)^{-1} x_n, \qquad \Delta \alpha_n = -x_n/T_n, \quad \text{and} \quad \Delta \beta_n = -c(\rho s)^{-1} x_n. \tag{17}$$

The desired formulas follow at once from (17). $\square$

In the next lemma we establish the weak convergence of (M_n).

LEMMA 3.3. *Let* (W_n) *be a tenable urn process such that* $bc > 0$. *Let* $(\alpha_n), (\beta_n), (x_n)$ *be the sequences defined in* (14) *of Lemma* 3.2 *and* (M_n) *the martingale of* (12) *and* (13). *Then, for* $\rho > 1/2$,

$$n^{\rho-1/2}\big(\alpha_{k_n(t)}W_{k_n(t)} + \beta_{k_n(t)} - Z\big) \Rightarrow W \circ \varphi(t), \tag{18}$$

where $k_n(t) = [nt^{1/(1-2\rho)}]$, $\varphi(t) = bc/(2\rho - 1)(\rho/(1-\rho))^2 t$ *and* Z *is a nondegenerate r.v., independent of* W.

For $\rho = 1/2$,

$$(\log(n))^{-1/2}\big(\alpha_{k_n(t)}W_{k_n(t)} + \beta_{k_n(t)}\big) \Rightarrow W \circ \varphi(t), \tag{19}$$

where $k_n(t) = [n^t]$ *and* $\varphi(t) = bct$.

For $\rho < 1/2$,

$$n^{-1/2}\big(\alpha_{k_n(t)}W_{k_n(t)} + \beta_{k_n(t)}\big) \Rightarrow W \circ \varphi(t), \tag{20}$$

where $k_n(t) = [nt]$ *and* $\varphi(t) = bc/(1-2\rho)(\rho/(1-\rho))^2 t^{1-2\rho}$.

PROOF. From Corollary 4.2 we have $x_n \simeq \rho s n^{-\rho}$ as $n \to \infty$. Then, since W_n/T_n converges to μ a.s., the asymptotic behaviour of $A_n(t)$, defined in (5), can be easily established as follows:

$$\begin{aligned}
\mathbf{E}\big(\xi_n^2 | \mathscr{F}_{n-1}\big) &= x_n^2 \mathbf{E}\left(\left(I_n - \frac{W_n}{T_n}\right)^2 | \mathscr{F}_{n-1}\right) \\
&= x_n^2 \left(\frac{W_n}{T_n}\right)\left(1 - \frac{W_n}{T_n}\right) \\
&\simeq (\rho s)^2 \mu(1-\mu) n^{-2\rho} \\
&= bc\left(\frac{\rho}{1-\rho}\right)^2 n^{-2\rho} \quad \text{a.s. as } n \to \infty.
\end{aligned}$$

Then, as $n \to \infty$, for each $t > 0$ and $\rho < 1/2$,

$$A_n(t) \simeq bc\left(\frac{\rho}{1-\rho}\right)^2 b_n^{-2} \sum_{k=1}^{k_n(t)} k^{-2\rho} \simeq \frac{bc}{1-2\rho}\left(\frac{\rho}{1-\rho}\right)^2 b_n^{-2} k_n(t)^{1-2\rho} \tag{21}$$

and for $\rho = 1/2$,

$$A_n(t) \simeq bc\left(\frac{\rho}{1-\rho}\right)^2 b_n^{-2} \sum_{k=1}^{k_n(t)} k^{-1} \simeq bcb_n^{-2} \log(k_n(t)). \tag{22}$$

Now let $\rho > 1/2$. As a first step toward (18), we note that $\sum_{k=1}^n \mathbf{E}(\xi_k^2 | \mathscr{F}_{k-1}) \sim \sum_{k=1}^n k^{-2\rho}$ as $n \to \infty$, which is convergent. Then, by the martingale convergence theorem [see Neveu (1972), VII-2-3], there exists a nondegenerate r.v. Z

such that

$$M_n = \sum_{k=0}^{n} \xi_k \to Z \quad \text{a.s. as } n \to \infty. \tag{23}$$

The FCLT will be derived from an idea of Heyde (1977) that gives a FCLT for the tail of an a.s. convergent martingale. Since we do not present our result using his time scale $k(n,t)$, [Heyde (1977), page 762] we apply Barbour's transform directly, following as much as possible the notation of Heyde but omitting many details.

Let T be the subspace of functions $x(t)$ in $D[0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{|x(t)|}{t} = 0, \qquad \int_1^\infty \frac{|x(t)|}{t^2}\,dt < \infty \quad \text{and} \quad \int_0^1 \frac{|x(t)|}{t}\,dt < \infty,$$

equipped with the topology defined as follows: $x_n \to x$ in T if there exists a sequence (λ_n) of strictly increasing continuous mappings on $[0,\infty)$, converging uniformly to the identity, such that

$$\begin{gathered} \frac{x_n(\lambda_n(t)) - x(t)}{t+1} \to 0 \quad \text{uniformly on } t, \\ \int_1^\infty \frac{|x_n(\lambda_n(t)) - x(t)|}{t^2}\,dt \to 0 \quad \text{and} \quad \int_0^1 \frac{|x_n(\lambda_n(t)) - x(t)|}{t}\,dt \to 0. \end{gathered} \tag{24}$$

Let $g\colon T \to T_1^*$ be the mapping given by

$$g(x)(t) = \int_{1/t}^{\infty} s^{-1} x(ds), \qquad 0 < t < \infty, \tag{25}$$

which is defined by its integration by parts formula

$$g(x)(t) = -tx(t^{-1}) + \int_{1/t}^{\infty} s^{-2} x(s)\,ds$$

and is shown to be continuous. See Heyde [(1977), page 761], for a description of the space T_1^* and the topologies on T and T_1^*.

Let us consider the transform of martingale $N_n = \sum_{k=0}^n \eta_k$ by the sequence $y_n = n^{2\rho-1} x_n$, with x_n defined in (14), and let

$$Y_n(t) = n^{1/2-\rho} \sum_{k=1}^{k_n(t)} y_k \eta_k$$

with $k_n(t) = [nt^{1/(2\rho-1)}]$. To obtain (18) we first show that for every $n > 0$, $Y_n(\cdot)$ belongs to T, then we check conditions (5) and (6) of the martingale FCLT and show that convergence holds in T with topology (24). Finally we apply Barbour's transform g, given by (25).

Following Heyde's arguments closely (pages 765 and 766), we show that $Y_n(t) = o(t^{1/2+\varepsilon})$ a.s. for every $\varepsilon > 0$ as $t \to \infty$, which clearly implies that $Y_n(\cdot)$

belongs to T. For this purpose we note that as $n \to \infty$ and for each positive ε,

$$\mathbf{E}\Big(\big(n^{(1/2+\varepsilon)(1-2\rho)}y_n\eta_n\big)^2|\mathscr{F}_{n-1}\Big) \sim n^{2\varepsilon(1-2\rho)-1} \quad \text{a.s.}$$

Therefore, the martingale $\sum_{k=0}^{n} k^{(1/2+\varepsilon)(1-2\rho)}y_k\eta_k$ converges a.s. and from Kronecker's lemma we get $\sum_{k=0}^{n} y_k\eta_k = o(n^{(1/2+\varepsilon)(2\rho-1)})$ a.s. as $n \to \infty$, which implies that $Y_n(t) = o((t^{1/(2\rho-1)})^{(1/2+\varepsilon)(2\rho-1)}) = o(t^{1/2+\varepsilon})$ a.s. as $t \to \infty$.

The verification of (5) and (6) is straightforward. The first one follows from

$$\sum_{k=0}^{[nt^{1/(2\rho-1)}]} \mathbf{E}\Big((y_k\eta_k)^2|\mathscr{F}_{k-1}\Big) \simeq [nt^{1/(2\rho-1)}]^{2\rho-1}\frac{(\rho s)^2}{2\rho-1}\mu(1-\mu)$$
$$\simeq \frac{bc}{2\rho-1}\left(\frac{\rho}{1-\rho}\right)^2 n^{2\rho-1}t.$$

For condition (6) we let $b_n^2 = n^{2\rho-1}$ and obtain

$$\mathbf{E}\Big((y_k\eta_k)^{-2}(y_k\eta_k)^4\mathbb{1}_{\{|y_k\eta_k|>\varepsilon b_n\}}|\mathscr{F}_{k-1}\Big)$$
$$\leq \varepsilon^{-2}b_n^{-2}\mathbf{E}\big\{(y_k\eta_k)^4|\mathscr{F}_{k-1}\big\} \sim b_n^{-2}k^{4(2\rho-1)}x_k^4$$
$$\sim b_n^{-2}k^{4\rho-4} \quad \text{a.s. as } k\to\infty.$$

Then, for some constant C, $B_n(\varepsilon) \leq Cb_n^{-4}n^{4\rho-3} \sim n^{-1}$ a.s. as $n \to \infty$ and we conclude that $Y_n(t) \Rightarrow W \circ \varphi(t)$ with $\varphi(t) = bc/(2\rho-1)(\rho/(1-\rho))^2 t$.

Next we apply the continuous mapping theorem with g defined in (25), which is continuous when T is equipped with topology (24), stronger than Skorohod's. Therefore, the convergence of $Y_n(t)$ has to be strengthened accordingly and this is done, as in Lemma 3.4 of Whitt (1972), with the following complementary tightness condition.

Let $1/2 < \alpha < 1$ be a fixed constant. Then for each positive ε and η, there exists t_0 such that

$$\mathbf{P}\left\{\sup_{t\geq t_0}\frac{|Y_n(t)|}{t^\alpha} > \varepsilon\right\} \leq \eta. \tag{26}$$

The integrals in (24) suggest that a condition like $\mathbf{P}\{\sup_{t\geq t_0}\int_t^\infty|Y_n(s)|/s^2\,ds > \varepsilon\} \leq \eta$ should also be required for tightness. However, it is easy to verify that (26) implies this integral condition. On the other hand, the integral on $[0,1]$ is not a matter of concern since it reduces to a finite sum.

The proof of (26) can be carried out as in Müller [(1968), page 177], using the Hájek–Rényi inequality [see Chow and Teicher (1988)] and the fact that $\sum_{i=1}^{n}\mathbf{E}(y_i\eta_i)^2 \sim n^{2\rho-1}$ as $n \to \infty$. Note that $\alpha > 1/2$ is necessary in order to apply Müller's argument.

Finally, we determine

$$g(Y_n)(t) = \int_{1/t}^{\infty}\frac{Y_n(ds)}{s} = \sum_{s\geq 1/t}\frac{\Delta Y_n(s)}{s},$$

where $\Delta Y_n(s) = Y_n(s) - Y_n(s^-)$ is the jump of Y_n at s.

It is clear that $\Delta Y_n(s) = n^{1/2-\rho} y_k \eta_k$ if $ns^{1/(2\rho-1)} = k$ and 0 otherwise. Then

$$g(Y_n)(t) = n^{1/2-\rho} \sum_{k > k_n(t)} (n/k)^{2\rho-1} y_k \eta_k = n^{\rho-1/2} \sum_{k > k_n(t)} x_k \eta_k,$$

where $k_n(t) = [nt^{1/(1-2\rho)}]$.

Since g is continuous and $g(W) = W$ in distribution [see Heyde (1977), pages 761, 767] we get

$$\text{(27)} \quad n^{\rho-1/2} \sum_{k > k_n(t)} x_k \eta_k \Rightarrow g(W \circ \varphi)(t) = W \circ \varphi(t) \quad \text{(in distribution)}$$

with $\varphi(t) = bc/(2\rho - 1)(\rho/(1-\rho))^2 t$. Combining (27) with (23) and identity (13), we get the convergence of (18).

To prove (19) ($\rho = 1/2$), we note that (5) of the FCLT follows from (22) with $b_n^2 = \log(n)$, $k_n(t) = [n^t]$ and $\varphi(t) = bct$. Convergence of $B_n(\varepsilon)$ to zero is immediate since the increments of (M_n) are bounded when ρ is positive.

To establish (20) ($\rho < 1/2$), we see that (5) is deduced from (21), with $b_n^2 = n^{1-2\rho}$, $k_n(t) = [nt]$ and $\varphi(t) = (\rho s)^2/(1-2\rho)\mu(1-\mu)t^{1-2\rho} = bc/(1-2\rho)(\rho/(1-\rho))^2 t^{1-2\rho}$ while (6) follows at once for positive ρ. For negative values of ρ we have $|\xi_k| \sim |x_k| \sim k^{-\rho}$ a.s. as $k \to \infty$ and

$$\mathbf{E}\left(\xi_k^2 \mathbb{1}_{\{|\xi_k| > \varepsilon b_n\}} | \mathscr{F}_{k-1}\right) = \mathbf{E}\left(\xi_k^{-2} \xi_k^4 \mathbb{1}_{\{|\xi_k| > \varepsilon b_n\}} | \mathscr{F}_{k-1}\right) \le \varepsilon^{-2} b_n^{-2} x_k^4.$$

Then, as $n \to \infty$, for each positive ε, $b_n^{-4} \sum_{k=1}^n k^{-4\rho} \sim n^{-(2-4\rho)} n^{1-4\rho} \sim n^{-1}$ and $B_n(\varepsilon) \le Cn^{-1}$ a.s., for some constant C. $\square$

Proof of Proposition 2.1. We combine the convergence results of Lemma 3.3 with the change of coefficients of Lemma 4.3 to obtain (1), (2) and (3) of Proposition 2.1.

Let us consider first the simpler case $\rho = 1/2$. From (19) and (38) we have $(\log(n))^{-1/2}(n^{-\rho t} W_{[n^t]} - \mu n^{-\rho t} T_{[n^t]}) \Rightarrow W \circ \varphi(t)$, with $\varphi(t) = bct$, and (2) follows clearly.

From (20) and (39) we have $t^{-\rho} n^{-1/2}(W_{[nt]} - \mu T_{[nt]}) \Rightarrow W \circ \varphi(t)$. Result (3) is obtained using the continuous mapping theorem with h_ρ: $D[0,\infty) \to D[0,\infty)$, given by $h_\rho(x)(t) = t^\rho x(t)$, which is clearly continuous for $0 < \rho < 1/2$. However, for negative values of ρ we have a more elaborate argument, based on the classical two stage proof of weak convergence of processes: convergence of finite dimensional distributions plus tightness.

It can be easily checked, using the Cramér–Wold device, that the finite dimensional distributions of $n^{-1/2}(W_{[nt]} - \mu T_{[nt]})$ converge to those of $t^\rho W \circ \varphi(t)$.

Tightness follows from (8) in Proposition 3.1. Indeed, (8) implies the tightness of $n^{-1/2}(W_{[nt]} - \mu T_{[nt]})$ if we prove that $n^{-1/2} \sum_{k=0}^{[nt]} (W_k/T_k - \mu)$ is tight. This is done in Lemma 4.5 of the Appendix and the proof of (3) is now complete.

It remains to show that (1) holds. From (18) and (37) we get

$$t^{\rho/(2\rho-1)}n^{-1/2}\Big(W_{k_n(t)} + n^{\rho}t^{\rho/(1-2\rho)}\big(\beta_{k_n(t)} - Z\big)\Big) \Rightarrow W \circ \varphi(t),$$

where $\varphi(t) = bc/(2\rho - 1)(\rho/(1-\rho))^2 t$ and $k_n(t) = [nt^{1/(1-2\rho)}]$.

Next, we apply the mapping $x(t) \to tx(t^{-1})$, which is continuous on the subspace T of D [see Heyde (1977), page 761], to obtain

$$t^{1-\rho/(2\rho-1)}n^{-1/2}\Big(W_{k_n(t)} + n^{\rho}t^{\rho/(1-2\rho)}\big(\beta_{k_n(t)} - Z\big)\Big) \Rightarrow tW \circ \varphi(t^{-1}),$$

with $k_n(t) = [nt^{1/(2\rho-1)}]$.

Finally we note that $tW(t^{-1})$ has the same distribution as $W(t)$ [see Hida (1980), Proposition 2.1] and we apply the continuous mapping $x(t) \to t^{(1-\rho)/(2\rho-1)}x(t)$ to get

$$n^{-1/2}\Big(W_{k_n(t)} + n^{\rho}t^{\rho/(1-2\rho)}\big(\beta_{k_n(t)} - Z\big)\Big) \Rightarrow t^{(1-\rho)/(2\rho-1)}W \circ \varphi(t),$$

with $k_n(t) = [nt^{1/(2\rho-1)}]$. We make the change of variable $t \to t^{2\rho-1}$ and the proof of (1) is now complete. □

The next result is central in the proof of Proposition 2.2.

LEMMA 3.4. *Let* (W_n) *be a tenable urn process with* $R = \begin{bmatrix} a & b \\ 0 & s \end{bmatrix}$, *where* $s > a > 0$ $[bc = 0, \max(b,c) > 0]$. *Let* (α_n) *and* (x_n) *be the sequences of* (14). *Then, there exists a nondegenerate* r.v. Z, *independent of the Wiener process* W, *such that*

$$n^{\rho/2}\big(\alpha_{k_n(t)}W_{k_n(t)} - Z\big) \Rightarrow W \circ \varphi(t) \tag{28}$$

with $k_n(t) = [nt^{-1/\rho}]$ *and* $\varphi(t) = aZt$.

PROOF. The argument closely parallels the proof of convergence (18). We note first that $(\alpha_n W_n)$ is a positive martingale, converging a.s. to a nondegenerate r.v. Z as $n \to \infty$. It can be verified that convergence also holds in L_2. See Neveu [(1972), Proposition VII-2-3].

In order to use Barbour's technique we introduce the process $Y_n(t)$, which is shown to belong to T,

$$Y_n(t) = n^{-\rho/2}\sum_{k=1}^{k_n(t)} y_k\eta_k$$

with $k_n(t) = [nt^{1/\rho}]$, $y_k = k^{\rho}x_k$ and $\eta_k = I_k - W_k/T_k$. See (14), where x_k is defined.

It is easy to check that, as $k \to \infty$,

$$\mathbf{E}\big((y_k\eta_k)^2 | \mathscr{F}_{k-1}\big) \simeq y_k^2 W_k/T_k \simeq a\rho k^{\rho-1}Z \quad \text{a.s.}$$

and also that $A_n(t) \simeq aZt$ and $B_n(\varepsilon) \sim n^{-\rho}$ a.s. as $n \to \infty$, with $b_n = n^{\rho/2}$. Therefore,

$$Y_n(t) \Rightarrow W \circ \varphi(t),$$

where $\varphi(t) = aZt$ and the Wiener process W is independent of Z. See Durrett and Resnick (1978). As in Lemma 3.3, convergence of $Y_n(t)$ is shown to hold in T with the topology of (24). It is easy to verify that the reasoning of Lemma 3.3 can be carried out here with ρ instead of $2\rho - 1$.

Next, applying Barbour's transform to Y_n we obtain

$$g(Y_n)(t) = n^{\rho/2} \sum_{k > k_n(t)} x_k \eta_k \Rightarrow g(W \circ \varphi)(t) = W \circ \varphi(t) \tag{29}$$

(in distribution),

where $k_n(t) = [nt^{-1/\rho}]$ and $\varphi(t) = aZt$.

The conclusion (28) follows from (29), the strong convergence of the martingale $(\alpha_n W_n)$ to Z and identity (13). □

Proof of Proposition 2.2. From (28) of Lemma 3.4 and (40) of Lemma 4.4, we obtain

$$n^{-\rho/2} t\left(W_{k_n(t)} - t^{-1} n^{\rho} Z\right) \Rightarrow W \circ \varphi(t)$$

with $k_n(t) = [nt^{-1/\rho}]$ and $\varphi(t) = aZt$. Then, we apply the mapping $x(t) \to tx(t^{-1})$, which is continuous in T_1^*, to get

$$n^{-\rho/2}\left(W_{k_n(t)} - n^{\rho} t Z\right) \Rightarrow tW \circ \varphi(t^{-1}) = W \circ \varphi(t) \quad \text{(in distribution)}$$

with $k_n(t) = [nt^{1/\rho}]$ and $\varphi(t) = aZt$. □

4. Appendix. To characterize the asymptotic behaviour of α_n we apply Stirling's expansion of Γ [see de Bruijn (1970), page 70, Equation 4.5.2]:

$$\Gamma(t) e^t t^{1/2-t} (2\pi)^{-1/2} = 1 + \tfrac{1}{12} t^{-1} + O(t^{-2}) \quad \text{as } t \to \infty. \tag{30}$$

Lemma 4.1. *For each real number ρ, as $t \to \infty$,*

$$\frac{\Gamma(t)}{\Gamma(t+\rho)} - t^{-\rho} = O(t^{-\rho-1}). \tag{31}$$

Proof. Let $A(t) = \Gamma(t) e^t t^{1/2-t} (2\pi)^{-1/2}$. *Then as $t \to \infty$,*

$$\frac{A(t)}{A(t+\rho)} = \frac{\Gamma(t)}{\Gamma(t+\rho)} B(t) = \frac{1 + Kt^{-1} + O(t^{-2})}{1 + K(t+\rho)^{-1} + O(t^{-2})} = 1 + O(t^{-2}) \tag{32}$$

with

$$B(t) = e^{-\rho} t^{1/2-t} (t+\rho)^{t+\rho-(1/2)} = t^{\rho}\left(1 + O(t^{-1})\right). \tag{33}$$

Equation (31) follows now from (32) and (33). □

COROLLARY 4.2. *For each $\rho < 1$, as $n \to \infty$,*

$$\alpha_n - n^{-\rho} = O(n^{-\rho-1}) \tag{34}$$

and

$$\beta_n + \mu s n^{1-\rho} = O(1 + n^{-\rho}). \tag{35}$$

PROOF. Equation (34) follows directly from (31) with $t = T_n$. To obtain (35), we note from (34) that $\alpha_n \simeq n^{-\rho}$ as $n \to \infty$, so that $\beta_n = -c\sum_{k=1}^{n}\alpha_k \simeq -(c/(1-\rho))n^{1-\rho} = -\mu s n^{1-\rho}$. □

In the following two lemmas we consider the substitutions of the α and β coefficients in the FCLT of Lemmas 3.3 and 3.4. Following Corollary 4.2, $\alpha_{k_n(t)}$ and $\beta_{k_n(t)}$ will be replaced, respectively, by functions $\alpha(n,t)$ and $\beta(n,t)$ given essentially by $k_n(t)^{-\rho}$ and $-\mu s k_n(t)^{1-\rho}$. We show that the remainders converge to zero a.s., uniformly on compact t sets, so that the FCLT also holds for the new processes. That is, for any positive τ, $D_n^{\alpha} + D_n^{\beta} = o(1)$ a.s. as $n \to \infty$, where

$$D_n^{\alpha} = b_n^{-1} \sup_{t\in[0,\tau]} \left\{W_{k_n(t)}|\alpha_{k_n(t)} - \alpha(n,t)|\right\}$$

and

$$D_n^{\beta} = b_n^{-1} \sup_{t\in[0,\tau]} \left\{|\beta_{k_n(t)} - \beta(n,t)|\right\}.$$

We note first that $W_n/T_n \le 1$ a.s. and $T_n = ns + T_0$ imply that

$$D_n^{\alpha} \le K b_n^{-1} \sup_{t\in[0,\tau]} \left\{k_n(t)|\alpha_{k_n(t)} - \alpha(n,t)|\right\} \quad \text{a.s.,} \tag{36}$$

where $K = T_0 + s$.

Let us define, for $k \in \mathbb{N}$,

$$S_1(k) = k\max\left\{|\alpha_k - k^{-\rho}|, |\alpha_k - (k+1)^{-\rho}|\right\},$$

$$S_2(k) = \max\left\{|\beta_k + \mu s k^{1-\rho}|, |\beta_k + \mu s k(k+1)^{-\rho}|\right\},$$

$$S_3(k) = k^{\rho} S_1(k),$$

$$S_4(k) = k^{\rho} S_2(k).$$

Let $S_i = \sup_{k\in\mathbb{N}} S_i(k)$, $i = 1, \dots, 4$. Then, it follows from (34) and (35) that $S_1 < \infty$ and $S_2 < \infty$ when $\rho > 0$; $S_3 < \infty$ for any ρ and $S_4 < \infty$ for $\rho < 0$. This simple consequence of Corollary 4.2 will be useful in the proofs of Lemmas 4.3 and 4.4.

LEMMA 4.3. *For any of the following three sets of parameters, corresponding to different values of ρ, $D_n^\alpha + D_n^\beta = o(1)$ as $n \to \infty$, a.s.*

$$(37)\qquad \rho > \frac{1}{2} \quad \begin{cases} b_n = n^{1/2-\rho},\ k_n(t) = [nt^\delta],\ \delta = \dfrac{1}{1-2\rho} < 0, \\ \alpha(n,t) = (nt^\delta)^{-\rho}, \end{cases}$$

$$(38)\qquad \rho = \frac{1}{2} \quad \begin{cases} b_n = (\log(n))^{1/2},\ k_n(t) = [n^t], \\ \alpha(n,t) = n^{-\rho t} \text{ and } \beta(n,t) = -\mu s[n^t]n^{-\rho t}, \end{cases}$$

$$(39)\qquad \rho < \frac{1}{2} \begin{cases} b_n = n^{1/2-\rho},\ k_n(t) = [nt], \\ \alpha(n,t) = (nt)^{-\rho} \text{ and } \beta(n,t) = -\mu s[nt](nt)^{-\rho}. \end{cases}$$

PROOF. Let $\rho > 1/2$ and J_k be the interval $]((k+1)/n)^{1-2\rho}, (k/n)^{1-2\rho}]$. Then, for all positive τ, as $n \to \infty$,

$$\begin{aligned} D_n^\alpha &\le K b_n^{-1} \sup_{t\in[0,\tau]} \left\{k_n(t)^{-\rho}\right\} \sup_{t\in[0,\tau]} \left\{k_n(t)^{1+\rho} |\alpha_{k_n(t)} - \alpha(n,t)|\right\} \\ &\le K b_n^{-1}[n\tau^\delta]^{-\rho} \sup_{k \ge [n\tau^\delta]} \left\{k^{1+\rho} \sup_{t\in J_k} \left\{|\alpha_k - n^{-\rho}t^{-\delta\rho}|\right\}\right\} \\ &\le K b_n^{-1}[n\tau^\delta]^{-\rho} \sup_{k\ge[n\tau^\delta]} \{S_3(k)\} \le K b_n^{-1}[n\tau^\delta]^{-\rho} S_3 = O(n^{-1/2}). \end{aligned}$$

Next, let $\rho = 1/2$ and the interval $J_k = [\log(k)/\log(n), \log(k+1)/\log(n)[$. Then, as $n \to \infty$,

$$\begin{aligned} D_n^\alpha &\le K b_n^{-1} \sup_{k \le n^\tau} \left\{k \sup_{t\in J_k} \left\{|\alpha_k - n^{-\rho t}|\right\}\right\} \\ &\le K b_n^{-1} \sup_{k\le n^\tau} \{S_1(k)\} \le K b_n^{-1} S_1 = O\big((\log(n))^{-1/2}\big). \end{aligned}$$

Convergence for D_n^β follows from

$$D_n^\beta \le b_n^{-1} \sup_{k\le n^\tau} \{S_2(k)\} \le b_n^{-1} S_2 = O\big((\log(n))^{-1/2}\big).$$

Finally, let $\rho < 1/2$ and $J_k = [k/n, (k+1)/n[$. We consider first the case $0 < \rho < 1/2$. Then, as $n \to \infty$,

$$\begin{aligned} D_n^\alpha &\le K b_n^{-1} \sup_{k \le n\tau} \left\{k \sup_{t\in J_k} \left\{|\alpha_k - n^{-\rho}t^{-\rho}|\right\}\right\} \\ &\le K b_n^{-1} \sup_{k\le n\tau} \{S_1(k)\} \le K b_n^{-1} S_1 = O(n^{\rho-(1/2)}), \\ D_n^\beta &\le b_n^{-1} \sup_{k \le n\tau} \{S_2(k)\} \le b_n^{-1} S_2 = O(n^{\rho-(1/2)}). \end{aligned}$$

To deal with $\rho < 0$ we use the same approach of the case $\rho > 1/2$. That is, we factor out $\sup_{t\in[0,\tau]}\{k_n(t)\}^{-\rho}$ and we have, as $n\to\infty$,

$$D_n^\alpha \le Kb_n^{-1}\sup_{t\in[0,\tau]}\{[nt]^{-\rho}\}\sup_{t\in[0,\tau]}\left\{[nt]^{1+\rho}|\alpha_{[nt]}-\alpha(n,t)|\right\}$$
$$\le Kb_n^{-1}[n\tau]^{-\rho}\sup_{k\le n\tau}\{S_3(k)\}\le Kb_n^{-1}[n\tau]^{-\rho}S_3 = O(n^{-1/2}),$$
$$D_n^\beta \le b_n^{-1}[n\tau]^{-\rho}\sup_{k\le n\tau}\{S_4(k)\}\le b_n^{-1}[n\tau]^{-\rho}S_4 = O(n^{-1/2}). \qquad \square$$

LEMMA 4.4. *For the urn model of Proposition* 2.2 *and the set of parameters* (40), $D_n^\alpha = o(1)$ *as* $n\to\infty$, *a.s.*

$$\rho > 0 \qquad \begin{cases} b_n = n^{-\rho/2},\ k_n(t) = [nt^{-1/\rho}], \\ \alpha(n,t) = n^{-\rho}t. \end{cases} \tag{40}$$

PROOF. Note first that ρ is always positive for the matrix R of Proposition 2.2.

We proceed as in Lemma 4.3, defining the interval $J_k =]((k+1)/n)^{-\rho}, (k/n)^{-\rho}]$. Then, for all positive τ, as $n\to\infty$,

$$D_n^\alpha \le Kb_n^{-1}\sup_{t\in[0,\tau]}\{k_n(t)^{-\rho}\}\sup_{t\in[0,\tau]}\left\{k_n(t)^{1+\rho}|\alpha_{k_n(t)}-\alpha(n,t)|\right\}$$
$$\le Kb_n^{-1}[n\tau^{-1/\rho}]^{-\rho}\sup_{k\ge[n\tau^{-1/\rho}]}\left\{k^{1+\rho}\sup_{t\in J_k}\{|\alpha_k - n^{-\rho}t|\}\right\}$$
$$\le Kb_n^{-1}[n\tau^{-1/\rho}]^{-\rho}\sup_{k\ge[n\tau^{-1/\rho}]}\{S_3(k)\}\le Kn^{\rho/2}[n\tau^{-1/\rho}]^{-\rho}S_3$$
$$= O(n^{-\rho/2}). \qquad \square$$

LEMMA 4.5. *For* $\rho < 1/2$ *the sequence of processes*

$$X_n(t) = n^{-1/2}\sum_{k=0}^{[nt]}(W_k/T_k - \mu), \qquad t>0,\ n\in\mathbb{N},$$

with paths in $D[0,\infty)$ *is tight.*

PROOF. Let $\zeta_k = W_k/T_k - \mu$ and $S_n = \sum_{k=1}^n \zeta_k$. Then, (X_n) is tight if for each positive ε there exists $\delta > 1$ and $n_0\in\mathbb{N}$ such that $\mathbf{P}(\max_{1\le k\le n}|S_{m+k} - S_m| > \delta n^{1/2}) \le \varepsilon/\delta^2$ for all $n\ge n_0$ and $m\in\mathbb{N}$. See Billingsley [(1968), pages 59, 138]. Tightness follows if we show that

$$\mathbf{P}\left(\sum_{m+1}^{m+n}|\zeta_k| > \delta n^{1/2}\right)\le \varepsilon/\delta^2. \tag{41}$$

To obtain (41) we compute the third moment of $\sum_{m+1}^{m+n}|\zeta_k|$ and apply Tchebychev's inequality. Let $A = \mathbf{E}(\sum_{m+1}^{m+n}|\zeta_k|)^3 = \sum_{m+1}^{m+n}\mathbf{E}(|\zeta_p\zeta_q\zeta_r|)$. Then, from the Cauchy–Schwarz inequality we obtain

$$\mathbf{E}(|\zeta_p\zeta_q\zeta_r|) \le \mathbf{E}(\zeta_p^2)^{1/2}\mathbf{E}(\zeta_q^2\zeta_r^2)^{1/2}$$
$$\le \mathbf{E}(\zeta_p^2)^{1/2}\mathbf{E}(\zeta_q^4)^{1/4}\mathbf{E}(\zeta_r^4)^{1/4},$$
$$A \le \sum_{m+1}^{m+n} \mathbf{E}(\zeta_p^2)^{1/2}\left(\sum_{m+1}^{m+n} \mathbf{E}(\zeta_q^4)^{1/4}\right)^2 = B.$$

Bagchi and Pal [(1985), page 399)] show that for every integer $r \ge 1$ and $\rho < 1/2$, $\mathbf{E}(\zeta_n^{2r}) \sim n^{-r}$ as $n \to \infty$. Then, there exist positive constants C_1 and C_2 such that

$$B \le C_1\left(\sum_{m+1}^{m+n} k^{-1/2}\right)^3 \le C_2(\sqrt{m+n} - \sqrt{m})^3 \le C_2 n^{3/2}.$$

From Chebychev's inequality we get for $\delta \ge C_2/\varepsilon$,

$$\mathbf{P}\left(\sum_{m+1}^{m+n} |\zeta_k| > \delta n^{1/2}\right) \le A/(\delta^3 n^{3/2}) \le C_2/\delta^3 \le \varepsilon/\delta^2. \qquad \square$$

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REFERENCES

ALDOUS, D., FLANNERY, B. and PALACIOS, J. L. (1988). Two applications of urn processes. The fringe analysis of search trees and the simulation of quasi-stationary distributions of Markov chains. *Probab. Engrg. Inform. Sci.* **2** 193–307.

ATHREYA, K. B. and KARLIN, S. (1968). Embedding of urn schemes into continuous time Markov branching processes and related limit theorems. *Ann. Math. Statist.* **39** 1801–1817.

BAGCHI, A. and PAL, A. K. (1985). Asymptotic normality in the generalized Pólya–Eggenberger urn model, with an application to computer data structures. *SIAM J. Algebraic Discrete Methods* **6** 394–405.

BURKHOLDER, D. (1966). Martingale transform. *Ann. Math. Statist.* **37** 1494–1504.

CHOW, S. C. and TEICHER, H. (1988). *Probability Theory: Independence, Interchangeability, Martingales.* Springer, New York.

DE BRUIJN, N. D. (1970). *Asymptotic Methods in Analysis*, 3rd ed. North-Holland, Amsterdam.

DURRETT, R. and RESNICK, S. (1978). Functional limit theorems for dependent variables. *Ann. Probab.* **6** 829–846.

FREEDMAN, D. A. (1965). Bernard Friedman's urn. *Ann. Math. Statist.* **36** 956–970.

GOUET, R. (1989). A martingale approach to strong convergence in a generalized Pólya–Eggenberger urn model. *Statist. Probab. Lett.* **8** 225–228.

HIDA, T. (1980). *Brownian Motion.* Springer, New York.

HEYDE, C. C. (1977). On the central limit and iterated logarithm supplements to the martingale convergence theorem. *J. Appl. Probab.* **14** 758–775.

HOLST, L. (1979). A unified approach to limit theorems for urn models. *J. Appl. Probab.* **16** 154–162.

JOHNSON, N. L. and KOTZ, S., (1977). *Urn Models and Their Applications*. Wiley, New York.

MAHMOUD, H. and SMYTHE, R. T. (1991). On the distribution of leaves in rooted subtrees of recursive trees. *Ann. Appl. Probab.* **1** 406–418.

MÜLLER, D. W. (1968). Verteilungs-Invarianzprinzipien für das starke Gesetz der grossen Zahl. *Z. Wahrsch. Verw. Gebiete* **10** 173–192.

NEVEU, J. (1972). *Martingales à Temps Discret*. Masson, Paris.

WHITT, W. (1972). Stochastic abelian and tauberian theorems. *Z. Wahrsch. Verw. Gebiete* **22** 251–267.

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