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STRONG LIMIT THEOREMS FOR LARGE AND SMALL INCREMENTS OF l^p-VALUED GAUSSIAN PROCESSES

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Based on the well-known Borell inequality and on a general theorem for large and small increments of Banach space valued stochastic processes of Csáki, Csörgő and Shao, we establish some almost sure path behaviour of increments in general, and moduli of continuity in particular, for l^p-valued, $1 \le p < \infty$, Gaussian processes with stationary increments. Applications to l^p-valued fractional Wiener and Ornstein–Uhlenbeck processes are also discussed. Our results refine and extend those of Csáki, Csörgő and Shao.

1. Introduction. A function $f(x)$ on (a, b) is called quasiincreasing on (a, b) if there exists a positive c such that

$$f(x) \le cf(y) \quad \text{for all } a < x < y < b.$$

It is clear that if $f(x) = x^\alpha L(x)$, $\alpha > 0$, $x \in (0, 1)$, where $L(\cdot)$ is slowly varying at zero, then $f(x)/x^{\alpha/2}$ is quasiincreasing on $(0, 1)$. Similarly, if $f(x) = x^\alpha L(x)$, $\alpha > 0$, $x \in (1, \infty)$, where $L(\cdot)$ is slowly varying at infinity, then $f(x)/x^{\alpha/2}$ is quasiincreasing on $(1, \infty)$.

For further use we quote a general estimate for the increments of Banach space valued processes, due to Csáki, Csörgő and Shao (1992) (cf. Lemma 2.1 and Theorem 3.1 of the just mentioned paper).

THEOREM A. *Let $\mathscr{B}$ be a separable Banach space with norm $\|\ \|$ and let $\{\Gamma(t),\ t \ge 0\}$ be a stochastic process with values in $\mathscr{B}$. Let P be the probability measure generated by $\Gamma(\cdot)$. Let a_T be a positive continuous function and b_T be a nonnegative continuous function. Put $a^* = \sup_{T>0} a_T$. Assume that $\Gamma(\cdot)$ is P-almost surely continuous with respect to $\|\ \|$ and that there exist nonnegative monotone nondecreasing continuous functions $\sigma_1(h)$ and $\sigma_2(h)$ and $x^* > 0$ such that for every $t \ge 0$, $x \ge x^*$, $0 < h \le a^*$,*

$$P\{\|\Gamma(t+h) - \Gamma(t)\| \ge x\sigma_1(h) + \sigma_2(h)\} \le K\exp(-\gamma x^\beta), \tag{1.1}$$

with some $K, \gamma, \beta > 0$. Suppose also that $\sigma_1(h)/h^\alpha$ and $\sigma_2(h)/h^\alpha$ are quasi-

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increasing on $(0, a^*)$ *for some* $\alpha > 0$ *and that*

$$\frac{b_T}{a_T} + \sigma_1(a_T) + \frac{1}{\sigma_1(a_T)} \to \infty \quad \text{as } T \to \infty.$$

Then, we have

$$\limsup_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \beta_T \|\Gamma(t+s) - \Gamma(t)\| \le 1 \quad a.s., \tag{1.2}$$

where

$$\beta_T^{-1} = \sigma_1(a_T)\left(\frac{1}{\gamma}\left(\log\frac{b_T}{a_T} + \log\log\left(\sigma_1(a_T) + \frac{1}{\sigma_1(a_T)}\right)\right)\right)^{1/\beta} + \sigma_2(a_T).$$

The usefulness of Theorem A alone was demonstrated in Csáki, Csörgő and Shao (1991, 1992) for studying some path properties of l^p-valued, $1 \le p \le 2$, Gaussian processes. Here we combine Theorem A with the well-known Borell inequality, as given in Adler (1990), via the dual-space idea as used in Marcus and Rosen (1992), and thus we succeed in refining the earlier results of Csáki, Csörgő and Shao (1992, 1991), as well as in extending them to l^p-valued, $1 \le p < \infty$, Gaussian processes, having stationary increments.

Our main, general results are summarized in Section 2, and their proofs are given in Section 3. In Section 4 we demonstrate the use of our approach in proving laws of the iterated logarithm (LIL) for l^p-valued, $1 \le p < \infty$, Gaussian processes. As further applications of our theorems, we study sample path properties of l^p-valued, $1 \le p < \infty$, fractional Wiener processes in Section 5, and those of l^p-valued, $1 \le p < \infty$, fractional Ornstein–Uhlenbeck processes in Section 6.

2. Increments for l^p-valued Gaussian processes. Let $\{Y(t), t \ge 0\} = \{X_k(t), t \ge 0\}_{k=1}^\infty$ be a sequence of independent Gaussian processes with $EX_k(t) = 0$ and stationary increments $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, where throughout this paper $\sigma_k(h)$ is assumed to be a non-decreasing continuous function for each $k \ge 1$. Put

$$\sigma(p,h) = \left(\sum_{k=1}^\infty \sigma_k^p(h)\right)^{1/p}, \qquad p \ge 1, \tag{2.1}$$

$$\sigma^*(h) = \max_{k\ge 1} \sigma_k(h), \tag{2.2}$$

$$\tilde\sigma(p,h) = \begin{cases} \sigma\left(\dfrac{2p}{2-p}, h\right), & \text{if } 1 \le p < 2, \\ \sigma^*(h), & \text{if } p \ge 2, \end{cases} \tag{2.3}$$

$$\delta_p^p = E|N(0,1)|^p = \frac{2^{p/2}}{\sqrt{\pi}} \int_0^\infty x^{(p-1)/2} e^{-x}\, dx, \qquad p \ge 1. \tag{2.4}$$

Since $E\|Y(t+h)-Y(t)\|_{l^p}^p = \delta_p^p \sum_{k=1}^{\infty} \sigma_k^p(h)$, $Y(t+h)-Y(t) \in l^p$, $1 \le p < \infty$, almost surely for fixed t and h if and only if

$$\sigma(p,h) < \infty, \tag{2.5}$$

and $Y(t) \in l^p$ almost surely for every t if and only if we have (2.5) and also

$$\sum_{k=1}^{\infty} E|X_k(0)|^p < \infty. \tag{2.6}$$

Extending the results of Csáki and Csörgő (1992), Csáki, Csörgő and Shao (1991, 1992) investigated the moduli of continuity for l^p-valued Gaussian processes with $1 \le p \le 2$. The aim of this section is to apply the well-known Borell inequality and our Theorem A to studying the increments for l^p-valued Gaussian processes for every $p \ge 1$. Our main results are as follows.

THEOREM 2.1. *Let a_T, $T > 0$, be a positive continuous function. Put $a^* = \sup_{T>0} a_T$. Assume $\tilde{\sigma}(p,h)/h^{\alpha}$ and $\sigma(p,h)/h^{\alpha}$ are quasiincreasing on $(0, a^*)$ for some $\alpha > 0$ and also that*

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty, \tag{2.7}$$

$$\sigma(p,a_T) = o\left(\tilde{\sigma}(p,a_T)\left(\log\frac{T}{a_T}\right)^{1/2}\right) \quad \text{as } T\to\infty, \tag{2.8}$$

$$\limsup_{T\to\infty} \max_{(T/a_T)^{\varepsilon}\le j\le T/a_T} \max_{k\ge 1}\{\sigma_k^{-2}(a_T) \times E[(X_k(a_T)-X_k(0))(X_k(ja_T)-X_k((j+1)a_T))]\} \le 0 \tag{2.9}$$

for each $\varepsilon > 0$. Then we have

$$\lim_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,a_T)(2\log(T/a_T))^{1/2}} = 1 \quad a.s. \tag{2.10}$$

THEOREM 2.2. *Let a_T, $T > 0$, be a positive continuous function satisfying (2.7). Assume $\sigma(p,h)/h^{\alpha}$ and $\tilde{\sigma}(p,h)/h^{\alpha}$ are quasiincreasing on $(0, a^*)$ for some $\alpha > 0$ and*

$$\tilde{\sigma}(p,a_T)\left(\log\frac{T}{a_T}\right)^{1/2} = o(\sigma(p,a_T)) \quad \text{as } T\to\infty. \tag{2.11}$$

Then we have

$$\lim_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p\sigma(p,a_T)} = 1 \quad a.s.$$

THEOREM 2.3. *Assume that* $\tilde{\sigma}(p,h)/h^{\alpha}$ *is quasiincreasing on* $(0,1)$ *for some* $\alpha > 0$. *Moreover, suppose that*

$$\sigma(p,h) = o\left(\tilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2}\right) \quad \text{as } h \to 0, \tag{2.13}$$

$$\limsup_{h\to 0} \max_{h^{-\varepsilon}\le j\le h^{-1}} \max_{k\ge 1} \frac{E[(X_k(h) - X_k(0))(X_k(jh) - X_k((j-1)h)]}{\sigma_k^2(h)} \le 0, \tag{2.14}$$

for each $\varepsilon > 0$. *Then we have*

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} = 1 \quad a.s. \tag{2.15}$$

THEOREM 2.4. *Assume that* $\sigma(p,h)/h^{\alpha}$ *is quasiincreasing on* $(0,1)$ *for some* $\alpha > 0$ *and that*

$$\tilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2} = o(\sigma(p,h)) \quad \text{as } h \to 0. \tag{2.16}$$

Then we have

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p,h)} = 1 \quad a.s. \tag{2.17}$$

REMARK 2.1. If

$$E\{(X_k(b) - X_k(a))(X_k(d) - X_k(c))\} \le 0 \tag{2.18}$$

for every $0 \le a \le b \le c \le d < \infty$ and for every $k \ge 1$, then, obviously, (2.9) and (2.14) are satisfied. In particular, if $\sigma_k^2(h)$ is concave on $(0,\infty)$ for each $k \ge 1$, then (2.18) is true and hence (2.9) and (2.14) are satisfied. On the other hand, as we will see in Sections 5 and 6, condition (2.9) or (2.14) is really much weaker than (2.18).

REMARK 2.2. We call attention to the normalizing constants in Theorems 2.1 and 2.3 being completely different from those of Theorems 2.2 and 2.4. The conclusions of the latter two theorems may appear to be somewhat surprising at first sight. We should, however, note that, under the conditions of Theorems 2.2 and 2.4, respectively, we have

$$\|Y(a_T) - Y(0)\|_{l^p} \sim \delta_p \sigma(p, a_T), \qquad T \to \infty$$

and

$$\|Y(h) - Y(0)\|_{l^p} \sim \delta_p \sigma(p,h), \qquad h \to 0$$

and, consequently, their conclusions are like laws of large numbers. On the other hand, the conclusions of Theorems 2.1 and 2.3, respectively, may be compared to large and small increments of a standard Wiener process [cf. Chapter 1 of Csörgő and Révész (1981)].

3. Proofs of Theorems 2.1–2.4. Throughout this section we assume $p \geq 1$, $\{Y(t), t \geq 0\} = \{X_k(t), t \geq 0\}_{k=1}^{\infty}$ to be a sequence of independent Gaussian processes with mean zero and stationary increments,

$$\sigma_k^2(h) = E\big(X_k(t+h) - X_k(t)\big)^2,$$

and

$$\sigma(p,h), \tilde{\sigma}(p,h) \quad \text{and} \quad \delta_p$$

are defined as in (2.1), (2.3) and (2.4), respectively.

Lemma 3.1. *Let $p \geq 1$, $\{\xi_n, n \geq 1\}$ be independent normal random variables with $E\xi_n = 0$ and $\sum_{i=1}^{\infty}(E\xi_i^2)^{p/2} < \infty$. Then*

$$P\left\{\left|\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p} - E\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p}\right| \geq x\right\}$$

$$\leq \begin{cases} 2\exp\left(-\dfrac{x^2}{2\left(\sum_{i=1}^{\infty}\left(E\xi_i^2\right)^{p/(2-p)}\right)^{(2-p)/p}}\right), & \text{if } 1 \leq p < 2, \\ 2\exp\left(-\dfrac{x^2}{2\max_{i\geq 1} E\xi_i^2}\right), & \text{if } p \geq 2, \end{cases} \tag{3.1}$$

for every $x > 0$.

Proof. The idea of the proof is from that of Lemma 2.2 of Marcus and Rosen (1992). It is well known that

$$\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p} = \sup_{\|a\|_{l^q}\leq 1}\sum_{i=1}^{\infty}\xi_i a_i, \tag{3.2}$$

where $q = p/(p-1)$, $a = (a_1, a_2, \dots) \in l^q$. Using (3.2) and the Borell inequality [cf. Adler (1990)], we have

$$\begin{aligned} &P\left\{\left|\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p} - E\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p}\right| \geq x\right\} \\ &\quad = P\left\{\left|\sup_{\|a\|_{l^q}\leq 1}\sum_{i=1}^{\infty}a_i\xi_i - E\sup_{\|a\|_{l^q}\leq 1}\sum_{i=1}^{\infty}a_i\xi_i\right| \geq x\right\} \\ &\quad \leq 2\exp\left(-\frac{x^2}{2\sup_{\|a\|_{l^q}\leq 1}E\left(\sum_{i=1}^{\infty}a_i\xi_i\right)^2}\right) \\ &\quad = 2\exp\left(-\frac{x^2}{2\sup_{\|a\|_{l^q}\leq 1}\sum_{i=1}^{\infty}a_i^2E\xi_i^2}\right). \end{aligned} \tag{3.3}$$

Noting that

$$\sup_{\|a\|_{l^q}\le 1}\sum_{i=1}^{\infty} a_i^2 E\xi_i^2 \le \begin{cases} \left(\sum_{i=1}^{\infty}\left(E\xi_i^2\right)^{q/(q-2)}\right)^{(q-2)/q} \sup_{\|a\|_{l^q}\le 1}\left(\sum_{i=1}^{\infty}|a_i|^q\right)^{2/q}, & \text{if } 1 \le p < 2, \\ \max_{i\ge 1} E\xi_i^2 \sup_{\|a\|_{l^q}\le 1}\sum_{i=1}^{\infty} a_i^2, & \text{if } p \ge 2, \end{cases}$$

$$= \begin{cases} \left(\sum_{i=1}^{\infty}\left(E\xi_i^2\right)^{p/(2-p)}\right)^{(2-p)/p}, & \text{if } 1 \le p < 2, \\ \max_{i\ge 1} E\xi_i^2, & \text{if } p \ge 2, \end{cases}$$

we arrive at (3.1) by (3.3), as desired. □

Since

$$E\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty}E|\xi_i|^p\right)^{1/p} = \delta_p\left(\sum_{i=1}^{\infty}\left(E\xi_i^2\right)^{p/2}\right)^{1/p}$$

by the Hölder inequality, it follows immediately from Lemma 3.1 that we also have the following:

LEMMA 3.2. *With $p \ge 1$, we have*

$$P\left\{\|Y(t+h) - Y(t)\|_{l^p} \ge \delta_p \sigma(p,h) + x\tilde{\sigma}(p,h)\right\} \le 2\exp\left(-\frac{x^2}{2}\right) \tag{3.4}$$

for each $t, x, h \ge 0$.

Before proving our Theorems 2.1–2.4 we present some more general results. The first one is the almost sure continuity of l^p-valued Gaussian processes, while the second and third ones are, respectively, concerned with upper and lower bounds for the increments of such Gaussian processes.

THEOREM 3.1. *Assume* (2.6),

$$\int_1^{\infty}\frac{\sigma(p,e^{-z})}{z}\,dz < \infty \tag{3.5}$$

and

$$\int_1^{\infty}\tilde{\sigma}\left(p,e^{-z^2}\right)dz < \infty \tag{3.6}$$

are satisfied. Then, $Y(\cdot) \in l^p$, $p \ge 1$, has a.s. continuous sample paths.

PROOF. The proof is along the lines of that of Theorem 2.6 of Csáki, Csörgő and Shao (1991), using (3.4) instead of their Lemma 2.4. We omit the details. □

THEOREM 3.2. *Let a_T, $T > 0$, be a positive continuous function and b_T, $T > 0$, be a nonnegative continuous function. Put $a^* = \sup_{T>0} a_T$. Assume that $\sigma(p,h)/h^\alpha$ and $\tilde{\sigma}(p,h)/h^\alpha$ are quasiincreasing on $(0, a^*)$ for some $\alpha > 0$ and that*

$$\frac{1 + b_T}{a_T} + a_T \to \infty \quad \text{as } T \to \infty. \tag{3.7}$$

Then we have

$$\limsup_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \beta(p,T)\|Y(t+s) - Y(t)\|_{l^p} \le 1 \quad a.s., \tag{3.8}$$

where $\beta(p,T)^{-1} = \delta_p\sigma(p, a_T) + \tilde{\sigma}(p, a_T)(2(\log(b_T/a_T) + \log\log(a_T + 1/a_T)))^{1/2}$.

PROOF. Recalling that

$$\tilde{\sigma}(p,h) = \begin{cases} \delta_{2p/(2-p)}^{-1}\left(E\left(\sum_{k=1}^{\infty} |X_k(t+h) - X_k(t)|^{2p/(2-p)}\right)\right)^{(2-p)/2p}, & \text{if } 1 \le p < 2, \\ \max_{k\ge1}\left(E\left(X_k(t+h) - X_k(t)\right)^2\right)^{1/2}, & \text{if } p \ge 2, \end{cases}$$

and using the Minkowski inequality, we obtain

$$\tilde{\sigma}(p, 2h) \le 2\tilde{\sigma}(p,h) \quad \text{for each } h > 0 \text{ and } p \ge 1. \tag{3.9}$$

From (3.9) it follows easily that

$$\tilde{\sigma}(p,h) + \frac{1}{\tilde{\sigma}(p,h)} \le 4\left(h + \frac{1}{h}\right)\left(\tilde{\sigma}(p,1) + \frac{1}{\tilde{\sigma}(p,1)}\right) \quad \text{for each } h > 0. \tag{3.10}$$

By (3.4), (3.10), Theorem 3.1 and Theorem A, we conclude that (3.8) holds. □

REMARK 3.1. Let $\sigma_*(p,h)$ and $\tilde{\sigma}_*(p,h)$ be nondecreasing functions such that $\sigma(p,h) \le \sigma_*(p,h)$ and $\tilde{\sigma}(p,h) \le \tilde{\sigma}_*(p,h)$ for each $h > 0$. Assume that $\sigma_*(p,h)/h^\alpha$ and $\tilde{\sigma}_*(p,h)/h^\alpha$ are quasiincreasing on $(0, a^*)$ for some $\alpha > 0$. Clearly, (3.4) remains true if $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ are replaced by $\sigma_*(p,h)$ and $\tilde{\sigma}_*(p,h)$, respectively. Hence, (3.8) remains valid with $\sigma_*(p,a_T)$ and $\tilde{\sigma}_*(p,a_T)$ instead of $\sigma(p,a_T)$ and $\tilde{\sigma}(p,a_T)$, respectively.

THEOREM 3.3. *Let a_T and b_T be positive continuous functions. Assume that* (2.5), *as well as*

$$\frac{\log(b_T/a_T)}{\log\log(a_T + (1/a_t))} \to \infty \quad \text{as } T \to \infty \tag{3.11}$$

and

$$\begin{aligned}\limsup_{T\to\infty} \max_{(b_T/a_T)^\varepsilon \le j \le b_T/a_T} \max_{k\ge 1} \{&\sigma_k^{-2}(a_T)\\ &\times E[(X_k(a_T) - X_k(0))(X_k(ja_T) - X_k((j-1)a_T))]\} \le 0\end{aligned} \tag{3.12}$$

for each $\varepsilon > 0$, are satisfied. Then we have

$$\liminf_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde\sigma(p, a_T)(2\log(b_T/a_T))^{1/2}} \ge 1 \quad a.s. \tag{3.13}$$

PROOF. Let $1 < \theta < 65/64$. Define

$$A_k = \left\{T\colon 2^k \le \frac{b_T}{a_T} \le 2^{k+1}\right\}, \qquad k \ge 0,$$

$$A_{k,j} = \{T\colon \theta^{j-1} \le \tilde\sigma(p, a_t) \le \theta^j, T \in A_k\}, \qquad -\infty < j < \infty,$$

$$b(T_{k,j}) = \inf\{b_T\colon T \in A_{k,j}\}, \qquad a_{k,j} = a(T^*_{k,j}) = \inf\{a_T\colon T \in A_{k,j}\}.$$

By (3.10) and (3.11), one finds that

$$A_{k,j} = \varnothing \quad \text{for every } |j| \ge e^k, \tag{3.14}$$

provided that k is sufficiently large. It is also easy to see that

$$2^k \le \frac{b(T_{k,j})}{a(T_{k,j})} \le \frac{b(T_{k,j})}{a(T^*_{k,j})} \le \frac{b(T^*_{k,j})}{a(T^*_{k,j})} \le 2^{k+1}. \tag{3.15}$$

Therefore

$$\begin{aligned}&\liminf_{T\to\infty} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde\sigma(p, a_T)(2\log(b_T/a_T))^{1/2}}\\
&\quad\ge \liminf_{k\to\infty} \inf_j \inf_{T\in A_{k,j}} \sup_{0\le t\le b_T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde\sigma(p, a_T)(2\log(b_T/a_T))^{1/2}}\\
&\quad\ge \liminf_{k\to\infty} \min_{|j|\le e^k} \sup_{0\le t\le b(T_{k,j})} \sup_{0\le s\le a_{k,j}} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\theta^j(2\log 2^{k+1})^{1/2}}\\
&\quad\ge \liminf_{k\to\infty} \min_{|j|\le e^k} \max_{0\le i\le 2^{k(2-\theta)}} \frac{\|Y(i2^{k(\theta-1)}a_{k,j} + a_{k,j}) - Y(i2^{k(\theta-1)}a_{k,j})\|_{l^p}}{\theta\tilde\sigma(p, a_{k,j})(2\log 2^k)^{1/2}}.\end{aligned} \tag{3.16}$$

We proceed with the proof by considering the two cases of $1 \le p < 2$ and $2 \le p < \infty$ separately.

CASE I. $1 \le p < 2$. In this case, by (3.2) with $q = p/(p-1)$, we have

$$\begin{aligned}
&\left\| Y\left(i2^{k(\theta-1)}a_{k,j} + a_{k,j}\right) - Y\left(i2^{k(\theta-1)}a_{k,j}\right)\right\|_{l^p} \\
(3.17)\qquad &\ge \left[\sum_{\nu=1}^{\infty} \sigma_\nu(a_{k,j})^{2p/(2-p)}\right]^{-(p-1)/p} \cdot \sum_{\nu=1}^{\infty} \sigma_\nu(a_{k,j})^{2(p-1)/(2-p)} \\
&\quad \times \left(X_\nu\left(i2^{k(\theta-1)}a_{k,j} + a_{k,j}\right) - X_\nu\left(i2^{k(\theta-1)}a_{k,j}\right)\right).
\end{aligned}$$

Consider

$$\begin{aligned}
&\xi(k,j;i) \\
&= \frac{\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2(p-1)/(2-p)}\left(X_\nu\left(i2^{k(\theta-1)}a_{k,j}+a_{k,j}\right) - X_\nu\left(i2^{k(\theta-1)}a_{k,j}\right)\right)}{\tilde{\sigma}(p, a_{k,j})\left(\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2p/(2-p)}\right)^{(p-1)/p}} \\
&= \frac{\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2(p-1)/(2-p)}\left(X_\nu\left(i2^{k(\theta-1)}a_{k,j}+a_{k,j}\right) - X_\nu\left(i2^{k(\theta-1)}a_{k,j}\right)\right)}{\left(\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2p/(2-p)}\right)^{1/2}},
\end{aligned}$$

$k = 1, 2, \dots$, $|j| \le e^k$, $0 \le i \le 2^{k(2-\theta)}$. For j, k fixed and $0 \le i < m \le 2^{k(2-\theta)}$, we have

$$\begin{aligned}
&E\{\xi(k,j;i)\xi(k,j;m)\} \\
(3.18)\qquad &= \left(\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2p/(2-p)}\right)^{-1}\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{4(p-1)/(2-p)} \cdot E\{(X_\nu(a_{k,j}) - X_\nu(0)) \\
&\qquad \times \left(X_\nu((m-i)2^{k(\theta-1)}a_{k,j} + a_{k,j}) - X_\nu((m-i)2^{k(\theta-1)}a_{k,j})\right)\},
\end{aligned}$$

by the fact that $\{X_\nu(t),\ t \ge 0\}_{\nu=1}^{\infty}$ is a sequence of independent Gaussian processes with stationary increments. Noting that

$$\sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{4(p-1)/(2-p)}\sigma_\nu^2(a_{k,j}) = \sum_{\nu=1}^{\infty}\sigma_\nu(a_{k,j})^{2p/(2-p)},$$

and using the assumption (3.12), we deduce from (3.18) that

$$(3.19)\qquad E\xi(k,j;i)\xi(k,j;m) \le \theta - 1 \quad \text{for every } |j| \le e^k,\ 0 \le i < m \le 2^{k(2-\theta)},$$

provided that k is sufficiently large. Also, clearly

$$(3.20)\qquad E\xi^2(k,j;i) = 1.$$

Let $\{\eta_i,\ 0 \le i \le 2^{k(2-\theta)}\}$ and τ be independent normal random variables with mean zero and with $E\eta_i^2 = 2 - \theta$, $0 \le i \le 2^{k(2-\theta)}$ and $E\tau^2 = \theta - 1$. Define $\tau_i = \tau + \eta_i$, $0 \le i \le 2^{k(2-\theta)}$. Note that

$$E\xi^2(k,j;i) = E\tau_i^2 = 1, \qquad 0 \le i \le 2^{k(2-\theta)},$$

$$E\{\xi(k,j;i)\xi(k,j;m)\} \le E\{\tau_i\tau_m\}, \qquad 0 \le i \ne m \le 2^{k(2-\theta)},$$

for k sufficiently large. Therefore, by the Slepian inequality [cf. Adler (1990)],

$$\begin{aligned}
&P\left\{\max_{0\le i\le 2^{k(2-\theta)}} \xi(k,j;i) \le \left((2-\theta)^2 - 2(\theta-1)^{1/2}\right)\left(2\log 2^k\right)^{1/2}\right\}\\
&\quad\le P\left\{\max_{0\le i\le 2^{k(2-\theta)}} \tau_i \le \left((2-\theta)^2 - 2(\theta-1)^{1/2}\right)\left(2\log 2^k\right)^{1/2}\right\}\\
&\quad\le P\left\{\max_{0\le i\le 2^{k(2-\theta)}} \eta_i \le (2-\theta)^2\left(2\log 2^k\right)^{1/2}\right\}\\
&\qquad+ P\left\{\tau \ge 2(\theta-1)^{1/2}\left(2\log 2^k\right)^{1/2}\right\}\\
&\quad\le \left(\Phi\left((2-\theta)^{3/2}\left(2\log 2^k\right)^{1/2}\right)\right)^{2^{k(2-\theta)}} + \exp\left(-4\log 2^k\right) \qquad (3.21)\\
&\quad\le 2^{-4k} + \left(1 - \frac{\exp\left(-(2-\theta)^3\log 2^k\right)}{3\left(1 + (2-\theta)^{3/2}\left(2\log 2^k\right)^{1/2}\right)}\right)^{2^{k(2-\theta)}}\\
&\quad\le 2^{-4k} + \exp\left(-\frac{2^{k(2-\theta)}\cdot 2^{-k(2-\theta)^3}}{k}\right)\\
&\quad\le 2^{-4k} + \exp\left(-\frac{2^{k(2-\theta)(\theta-1)}}{k}\right)
\end{aligned}$$

for every k big enough.

Putting the above inequalities together and applying the Borel–Cantelli lemma, we conclude

$$\begin{aligned}
&\liminf_{k\to\infty} \min_{|j|\le e^k} \max_{0\le i\le 2^{k(2-\theta)}} \frac{\|Y\left(i2^{k(\theta-1)}a_{k,j} + a_{k,j}\right) - Y\left(i2^{k(\theta-1)}a_{k,j}\right)\|_{l^p}}{\theta\tilde{\sigma}(p, a_{k,j})\left(2\log 2^k\right)^{1/2}} \qquad (3.22)\\
&\qquad\ge \frac{(2-\theta)^2 - 2(\theta-1)^{1/2}}{\theta} \quad \text{a.s.},
\end{aligned}$$

which yields (3.13), by (3.16) and the arbitrariness of $1 < \theta < 65/64$.

CASE II. $p \ge 2$. Take $N_{k,j}$ such that $\sigma_{N_{k,j}}(a_{k,j}) = \sigma^*(a_{k,j})$. Clearly

$$\begin{aligned}
&\frac{\|Y\left(i2^{k(\theta-1)}a_{k,j} + a_{k,j}\right) - Y\left(i2^{k(\theta-1)}a_{k,j}\right)\|_{l^p}}{\tilde{\sigma}(p, a_{k,j})}\\
&\qquad\ge \frac{X_{N_{k,j}}\left(i2^{k(\theta-1)}a_{k,j} + a_{k,j}\right) - X_{N_{k,j}}\left(i2^{k(\theta-1)}a_{k,j}\right)}{\sigma_{N_{k,j}}(a_{k,j})}.
\end{aligned}$$

Along the lines of the proof of Case I, we conclude that (3.13) remains true in this case as well. □

REMARK 3.2. From the proof of Theorem 3.3, one can conclude also that (3.13) remains true if (3.11) is replaced by

$$\log\log\left(a_T + \frac{1}{a_T}\right) = O\left(\log\frac{b_T}{a_T}\right) \quad \text{and} \quad \frac{b_T}{a_T} \to \infty \quad \text{as } T \to \infty.$$

REMARK 3.3. If the conditions (3.11) and (3.12) are replaced by

$$\frac{\log(b_T/a_T)}{\log\log\log(a_T + 1/a_T)} \to \infty \quad \text{as } T \to \infty$$

and

$$E\{(X_k(a) - X_k(0))(X_k(c) - X_k(b))\} \le 0$$

for each $k \ge 1$, $0 \le a \le b \le c$, respectively, then (3.13) holds true.

PROOF OF THEOREM 2.1. This is an immediate consequence of Theorems 3.2 and 3.3. □

PROOF OF THEOREM 2.2. By Theorem 3.2, we have

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \sup_{0\le s\le a_T} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p, a_T)} \le 1 \quad \text{a.s.}$$

So it suffices to show that

$$\liminf_{T\to\infty} \sup_{0\le s\le a_T} \frac{\|Y(s) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_T)} \ge 1 \quad \text{a.s.} \tag{3.23}$$

Let $1 < \theta < 65/64$. Define

$$B_k = \left\{T : 2^k \le \frac{T}{a_T} \le 2^{k+1}\right\}, \qquad k \ge 0,$$

$$B_{k,j} = \{T : \theta^j \le \sigma(p, a_T) \le \theta^{j+1}, T \in B_k\}, \qquad -\infty < j < \infty,$$

$$a_{k,j} = a(T_{k,j}) = \inf\{a_T : T \in B_{k,j}\}.$$

Similarly to (3.10), we have

$$\sigma(p, a_T) \leq 2(1 + a_T)\sigma(p, 1)$$

and hence

$$B_{k,j} = \varnothing \quad \text{if } |j| > e^k$$

by (2.7), provided k is sufficiently large. Therefore

$$\liminf_{T\to\infty} \sup_{0\leq s\leq a_T} \frac{\|Y(s) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_T)}$$

$$\geq \liminf_{k\to\infty} \min_{|j|\leq e^k} \inf_{T\in B_{k,j}} \sup_{0\leq s\leq a_T} \frac{\|Y(s) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_T)} \tag{3.24}$$

$$\geq \liminf_{k\to\infty} \min_{|j|\leq e^k} \frac{\|Y(a_{k,j}) - Y(0)\|_{l^p}}{\theta\delta_p \sigma(p, a_{k,j})}.$$

Applying Hölder's inequality, that is,

$$E|X|^p \leq (E|X|)^{p/(2p-1)} \left(E|X|^{2p}\right)^{(p-1)/(2p-1)}$$

for every $p \geq 1$ and any random variable X, we find that

$$E\|Y(a_T) - Y(0)\|_{l^p} \geq \frac{\left(E\|Y(a_T) - Y(0)\|_{l^p}^p\right)^{(2p-1)/p}}{\left(E\|Y(a_T) - Y(0)\|_{l^p}^{2p}\right)^{(p-1)/p}}$$

$$\geq \frac{\left(\delta_p^p \sum_{k=1}^\infty \sigma_k^p(a_T)\right)^{(2p-1)/p}}{\left(\left(\delta_p^p \sum_{k=1}^\infty \sigma_k^p(a_T)\right)^2 + \delta_{2p}^{2p} \sum_{k=1}^\infty \sigma_k^{2p}(a_T)\right)^{(p-1)/p}}$$

$$\geq \frac{\left(\delta_p^p \sum_{k=1}^\infty \sigma_k^p(a_T)\right)^{(2p-1)/p}}{\left(\left(\delta_p^p \sum_{k=1}^\infty \sigma_k^p(a_T)\right)^2 + \delta_{2p}^{2p} \sigma^{*p}(a_T) \sum_{k=1}^\infty \sigma_k^p(a_T)\right)^{(p-1)/p}} \tag{3.25}$$

$$= \frac{\delta_p \sigma(p, a_T)}{\left(1 + \delta_{2p}^{2p} \delta_p^{-p} \left(\sigma^*(a_T)/\sigma(p, a_t)\right)^p\right)^{(p-1)/p}}.$$

Therefore, by (2.11),

$$\liminf_{T\to\infty} \frac{E\|Y(a_T) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_T)} \geq 1. \tag{3.26}$$

From (3.1) and (3.26) it follows that for every k sufficiently large, $|j| \le e^k$,

$$
\begin{aligned}
&P\left\{\frac{\|Y(a_{k,j}) - Y(0)\|_{l^p}}{\delta_p \sigma(p, a_{k,j})} \le 2 - \theta\right\} \\
&\quad \le P\left\{\|Y(a_{k,j}) - Y(0)\|_{l^p} - E\|Y(a_{k,j}) - Y(0)\|_{l^p} \le -\frac{\theta - 1}{2}\delta_p \sigma(p, a_{k,j})\right\} \\
&\quad \le 2\exp\left(-\frac{(\theta - 1)^2 \delta_p^2 \sigma^2(p, a_{k,j})}{\delta \tilde{\sigma}^2(p, a_{k,j})}\right) \\
&\quad \le 2\exp\left(-4\log\frac{T_{k,j}}{a(T_{k,j})}\right) \\
&\quad \le 2 \cdot 2^{-4k},
\end{aligned}
\tag{3.27}
$$

which, together with the Borel–Cantelli lemma, implies

$$
\liminf_{k \to \infty} \min_{|j| \le e^k} \frac{\|Y(a_{k,j}) - Y(0)\|_{l^p}}{\theta \delta_p \sigma(p, a_{k,j})} \ge \frac{2 - \theta}{\theta} \quad \text{a.s.} \tag{3.28}
$$

This proves (3.23) by (3.24), (3.28) and the arbitrariness of $\theta > 1$. □

Proof of Theorem 2.3. From Theorem 3.3, we have

$$
\liminf_{h \to 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \frac{\|Y(t + s) - Y(t)\|_{l^p}}{\tilde{\sigma}(p, h)(2\log(1/h))^{1/2}} \ge 1 \quad \text{a.s.}
$$

It suffices to prove that

$$
\limsup_{h \to 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \frac{\|Y(t + s) - Y(t)\|_{l^p}}{\tilde{\sigma}(p, h)(2\log(1/h))^{1/2}} \le 1 \quad \text{a.s.} \tag{3.29}
$$

For any fixed $\varepsilon > 0$, put $\sigma_*(p, h) = \varepsilon \sup_{0 \le s \le h} \tilde{\sigma}(p, h)(\log(1/s))^{1/2}$, $0 < h \le 1$. Noting that $\tilde{\sigma}(p, h)/h^\alpha$ is quasiincreasing, one can see that there exists a constant c_0, independent of ε, such that

$$
\begin{aligned}
\varepsilon \sigma(p, h)\left(\log\frac{1}{h}\right)^{1/2} &\le \sigma_*(p, h) = \varepsilon \sup_{0 \le s \le h} \frac{\tilde{\sigma}(p, s)}{s^\alpha} s^\alpha \left(\log\frac{1}{s}\right)^{1/2} \\
&\le \varepsilon c_0 \tilde{\sigma}(p, h)\left(\log\frac{1}{h}\right)^{1/2}
\end{aligned}
\tag{3.30}
$$

for $0 < h < 1$. Moreover, $\sigma_*(p,h)$ is nondecreasing, $\sigma_*(p,h)/h^{\alpha/2}$ is quasi-increasing and $\sigma(p,h) \le \sigma_*(p,h)$ by (2.13), provided that h is sufficiently small. Hence, using Remark 3.1, we obtain

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2} + \sigma_*(p,h)} \le 1 \quad \text{a.s.}$$

Thus, by (3.30),

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} \le 1 + \varepsilon c_0.$$

This proves (3.29) by the arbitrariness of ε. □

PROOF OF THEOREM 2.4. Similarly to the proof of (3.29), by Theorem 3.2 we have

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s)-Y(t)\|_{l^p}}{\delta_p \sigma(p,h)} \le 1 \quad \text{a.s.} \tag{3.31}$$

On the other hand, along the lines of the proof of (3.23), we can also obtain

$$\liminf_{h\to 0} \sup_{0\le s\le h} \frac{\|Y(s)-Y(0)\|_{l^p}}{\delta_p \sigma(p,h)} \ge 1 \quad \text{a.s.} \tag{3.32}$$

A combination of (3.31) with (3.32) yields (2.17), as desired. □

4. LIL for l^p-valued Gaussian processes. For results on the law of the iterated logarithm, in general, we refer to the insightful review of Bingham (1986) and, for real valued Gaussian processes, to Nisio (1967), Marcus (1970) and Lai (1973). In this section we are interested in proving LIL results for l^p-valued Gaussian processes. We deal first with stationary Gaussian processes.

Let $\{Y(t), t \ge 0\} = \{X_k(t), t \ge 0\}_{k=1}^{\infty}$ be a sequence of independent stationary Gaussian processes with mean zero, $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$, $\sigma_k^2 = EX_k^2(t)$. Let $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ be defined as in (2.1) and (2.3). Put

$$\tilde{\sigma}(p) = \begin{cases} \left(\displaystyle\sum_{k=1}^{\infty} \sigma_k^{2p/(2-p)}\right)^{(2-p)/2p}, & \text{if } 1 \le p < 2, \\ \displaystyle\max_{k\ge 1} \sigma_k, & \text{if } p \ge 2. \end{cases} \tag{4.1}$$

THEOREM 4.1. *Let $p \ge 1$, $\{Y(t), t \ge 0\} = \{X_k(t), t \ge 0\}_{t=1}^{\infty}$ be independent stationary Gaussian processes defined as above. Assume that the following* *conditions are satisfied*:

$$\sum_{k=1}^{\infty} \sigma_k^p < \infty, \tag{4.2}$$

$$\int_1^{\infty} \frac{\sigma(p, e^{-z})}{z}\, dz < \infty, \tag{4.3}$$

$$\int_1^{\infty} \tilde{\sigma}\left(p, e^{-z^2}\right) dz < \infty, \tag{4.4}$$

$$\limsup_{a \to \infty} E\{X_k(0) X_k(a)\} \le 0 \quad \textit{for each } k \ge 1. \tag{4.5}$$

Then we have

$$\lim_{T \to \infty} \sup_{0 \le s \le T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p)(2 \log T)^{1/2}} = 1 \quad a.s., \tag{4.6}$$

$$\limsup_{T \to \infty} \frac{\|Y(t)\|_{l^p}}{\tilde{\sigma}(p)(2 \log T)^{1/2}} = 1 \quad a.s. \tag{4.7}$$

Our next theorems show that the situation may be very different if $X_k(\cdot)$ is nonstationary.

THEOREM 4.2. *Let $p \ge 1$, $\{Y(t),\ t \ge 0\} = \{X_k(t),\ t \ge 0\}_{k=1}^{\infty}$ be independent Gaussian processes with $X_k(0) = 0$, $EX_k(t) = 0$, and with stationary increments $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$. Assume that $\sigma(p,h)/h^{\alpha}$ and $\tilde{\sigma}(p,h)/h^{\alpha}$ are quasiincreasing on $(0, \infty)$ for some $\alpha > 0$. Moreover, suppose that*

$$\sigma(p, T) = o\left(\tilde{\sigma}(p, T)(\log\log T)^{1/2}\right) \quad \textit{as } T \to \infty, \tag{4.8}$$

$$\lim_{a \to \infty} \max_{j \ge \log a} \max_{k \ge 1} \frac{E\{X_k(a)(X_k(ja) - X_k(a))\}}{\sigma_k(a)\sigma_k(ja)} \le 0. \tag{4.9}$$

Then we have

$$\limsup_{T \to \infty} \sup_{0 \le s \le T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p, T)(2 \log\log T)^{1/2}} = 1 \quad a.s., \tag{4.10}$$

$$\limsup_{T \to \infty} \frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p, T)(2 \log\log T)^{1/2}} = 1 \quad a.s. \tag{4.11}$$

THEOREM 4.3. *Assume the conditions of Theorem* 4.2, *except that* (4.8) *and* (4.9) *are to be replaced by*

$$\tilde{\sigma}(p, T)(\log\log T)^{1/2} = o(\sigma(p, T)) \quad \textit{as } T \to \infty. \tag{4.12}$$

Then we have

$$\lim_{T\to\infty} \sup_{0\le s\le T} \frac{\|Y(s)\|_{l^p}}{\delta_p \sigma(p,T)} = 1 \quad a.s., \tag{4.13}$$

$$\lim_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\delta_p \sigma(p,T)} = 1 \quad a.s. \tag{4.14}$$

PROOF OF THEOREM 4.1. Let $0 < \varepsilon < 1/8$. Clearly, (4.3) and (4.4) imply

$$\int_1^\infty \frac{\tilde{\sigma}(p, e^{-z})}{z}\, dz < \infty, \qquad \sigma(p,h) + \tilde{\sigma}(p,h) \to 0 \quad \text{as } h \to 0.$$

So, we can take $0 < \delta < 1$ such that

$$\tilde{\sigma}(p,\delta) + 16\int_1^\infty \frac{\tilde{\sigma}(p,\delta e^{-z})}{z}\, dz < \varepsilon^2. \tag{4.15}$$

Notice that

$$\begin{aligned}
&\limsup_{T\to\infty} \sup_{0\le s\le T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} \\
&\quad= \limsup_{k\to\infty} \sup_{0\le s\le e^k} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p)(2k)^{1/2}} \\
&\quad\le \limsup_{k\to\infty} \max_{0\le i\le e^i/\delta} \frac{\|Y(i\delta)\|_{l^p}}{\tilde{\sigma}(p)(2k)^{1/2}} \\
&\qquad+ \limsup_{k\to\infty} \max_{0\le i\le e^k/\delta} \sup_{0\le s\le\delta} \frac{\|Y(i\delta+s) - Y(i\delta)\|_{l^p}}{\tilde{\sigma}(p)(2k)^{1/2}}.
\end{aligned} \tag{4.16}$$

Using (3.1), we get that for every k sufficiently large,

$$\begin{aligned}
&P\left\{ \frac{\|Y(i\delta)\|_{l^p}}{\tilde{\sigma}(p)(2k)^{1/2}} \ge 1 + 2\varepsilon \right\} \\
&\quad\le P\left\{ \|Y(i\delta)\|_{l^p} - E\|Y(i\delta)\|_{l^p} \ge (1+\varepsilon)\tilde{\sigma}(p)(2k)^{1/2} \right\} \\
&\quad\le 2\exp\left(-(1+\varepsilon)^2 k\right).
\end{aligned}$$

Hence

$$\limsup_{k\to\infty} \max_{0\le i\le e^k/\delta} \frac{\|Y(i\delta)\|_{l^p}}{\tilde{\sigma}(p)(2k)^{1/2}} \le 1 + 2\varepsilon \quad \text{a.s.} \tag{4.17}$$

by the Borel–Cantelli lemma.

In terms of (3.4) and Lemma 2.1 of Csáki, Csörgő and Shao (1992), we have

$$P\left\{\max_{0\le i\le e^k/\delta}\sup_{0\le s\le\delta}\|Y(s+i\delta)-Y(i\delta)\|_{l^p}\ge x\left(\tilde\sigma(p,\delta)+16\int_1^\infty\frac{\tilde\sigma(p,\delta e^{-z})}{z}\,dz\right.\right.$$

$$\left.\left.+48\int_1^\infty\tilde\sigma\left(p,\delta e^{-z^2}\right)dz+\delta_p\sigma(p,\delta)+\delta_p\int_1^\infty\frac{\sigma(p,\delta e^{-z})}{z}\,dz\right)\right\}$$

$$\le 2^{19}\cdot e^k\cdot\delta^{-1}\cdot\exp\left(-\frac{x^2}{2}\right)$$

for every $x>0$. Therefore, by (4.15), (4.3) and (4.4) we have

$$P\left\{\max_{0\le i\le e^k/\delta}\sup_{0\le s\le\delta}\|Y(s+i\delta)-Y(i\delta)\|_{l^p}\ge\varepsilon\tilde\sigma(p)k^{1/2}\right\}$$

$$\le P\left\{\max_{1\le i\le e^k/\delta}\sup_{0\le s\le\delta}\|Y(s+i\delta)-Y(i\delta)\|_{l^p}\right.$$

$$\ge\frac{\varepsilon^{-1}}{2}\tilde\sigma(p)k^{1/2}\left(\tilde\sigma(p,\delta)+16\int_1^\infty\frac{\tilde\sigma(p,\delta e^{-z})}{z}\,dz\right.$$

$$\left.\left.+48\int_1^\infty\tilde\sigma\left(p,\delta e^{-z^2}\right)dz+\delta_p\sigma(p,\delta)+\delta_p\int_1^\infty\frac{\sigma(p,\delta e^{-z})}{z}\,dz\right)\right\}$$

$$\le 2^{19}\cdot e^k\cdot\delta^{-1}\exp\left(-\frac{k}{8\varepsilon^2}\right)$$

$$\le e^{-k}$$

for every k big enough. This proves

$$\limsup_{k\to\infty}\max_{0\le i\le e^k/\delta}\sup_{0\le s\le\delta}\frac{\|Y(i\delta+s)-Y(i\delta)\|_{l^p}}{\tilde\sigma(p)(2k)^{1/2}}\le\varepsilon\quad\text{a.s.}\tag{4.18}$$

by the Borel–Cantelli lemma again. We conclude from (4.16), (4.17), (4.18) and the arbitrariness of ε that

$$\limsup_{T\to\infty}\sup_{0\le s\le T}\frac{\|Y(s)\|_{l^p}}{\tilde\sigma(p)(2\log T)^{1/2}}\le 1\quad\text{a.s.}\tag{4.19}$$

We prove next that

$$\liminf_{T\to\infty}\sup_{0\le s\le T}\frac{\|Y(s)\|_{l^p}}{\tilde\sigma(p)(2\log T)^{1/2}}\ge 1\quad\text{a.s.}\tag{4.20}$$

We divide the proof into two cases.

CASE I. $1 \le p < 2$. For any fixed $0 < \varepsilon < 1/8$, take N such that

$$(4.21) \qquad \left(\sum_{k=1}^{N} \sigma_k^{2p/(2-p)} \right)^{(2-p)/2p} \ge (1 - \varepsilon)\tilde{\sigma}(p).$$

Note that

$$\begin{aligned}
&\liminf_{T\to\infty} \sup_{0\le s\le T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} \\
&\quad \ge \liminf_{T\to\infty} \sup_{0\le s\le T} \frac{\left(\sum_{k=1}^{N}|X_k(s)|^p\right)^{1/p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} \\
(4.22) \qquad &\quad \ge \liminf_{T\to\infty} \sup_{0\le s\le T} \frac{|\sum_{k=1}^{N}\sigma_k^{2(p-1)/(2-p)}X_k(s)|}{\tilde{\sigma}(p)(2\log T)^{1/2}\left(\sum_{k=1}^{N}\sigma^{2p/(2-p)}\right)^{(p-1)/p}} \\
&\quad \ge (1-\varepsilon)\liminf_{T\to\infty} \sup_{0\le s\le T} \frac{|\sum_{k=1}^{N}\sigma^{2(p-1)/(2-p)}X_k(s)|}{\left(\sum_{k=1}^{N}\sigma^{2p/(2-p)}\right)^{1/2}(2\log T)^{1/2}},
\end{aligned}$$

by (3.2) and (4.21) [cf. (3.17)].

Clearly, $\{Y_N(s),\ s \ge 0\} = \{\sum_{k=1}^{N}\sigma_k^{2(p-1)/(2-p)} \cdot X_k(s)/(\sum_{k=1}^{N}\sigma_k^{2p/(2-p)})^{1/2},\ s \ge 0\}$ is a stationary Gaussian process with $EY_N(s) = 0$, $EY_N^2(s) = 1$. Furthermore, it follows from (4.5) that $\limsup_{a\to\infty} E\{Y_N(0)Y_N(a)\} \le 0$. Consequently, using a theorem of Nisio (1967) [cf. Lai (1973)], we obtain

$$(4.23) \qquad \lim_{T\to\infty} \sup_{0\le s\le T} \frac{|Y_N(s)|}{(2\log T)^{1/2}} = 1 \quad \text{a.s.},$$

which, together with (4.22), implies (4.20), by the arbitrariness of ε, as desired.

CASE II. $p \ge 2$. In this case we can take N such that

$$\sigma_N = \max_{k\ge1} \sigma_k = \tilde{\sigma}(p).$$

Obviously, $\|Y(s)\|_{l^p} \ge |X_N(s)|$. The rest of the proof is exactly the same as that of Case I.

Putting the above inequalities together, we arrive at (4.6).

To show (4.7), it suffices to verify that

$$(4.24) \qquad \limsup_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} \ge 1 \quad \text{a.s.}$$

For each $0 < \varepsilon < 1$, we have

$$
\begin{aligned}
&P\left(\bigcup_{T\le t\le 2T}\left\{\frac{\|Y(t)\|_{l^p}}{\tilde{\sigma}(p)(2\log t)^{1/2}} \ge 1-\varepsilon\right\}\right)\\
&\quad\le P\left(\bigcup_{T\le t\le 2T}\left\{\frac{\|Y(t)\|_{l^p}}{\tilde{\sigma}(p)} \ge 1(1-\varepsilon)(2\log(2T))^{1/2}\right\}\right)\\
&\quad= 1-P\left(\sup_{0\le t\le T}\frac{\|Y(t)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} < (1-\varepsilon)\left(\frac{\log(2T)}{\log T}\right)^{1/2}\right)\\
&\quad\to 1 \quad \text{as } T\to\infty,
\end{aligned}
$$

by (4.6). This proves (4.24). This also completes the proof of Theorem 4.1. □

PROOF OF THEOREM 4.2. Using Theorem 3.2 with $b_T = 0$ and $a_T = T$, we obtain immediately

$$
\limsup_{T\to\infty}\sup_{0\le s\le T}\frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p,T)(2\log\log T)^{1/2}} \le 1 \quad \text{a.s.} \tag{4.25}
$$

So, it suffices to show that

$$
\limsup_{T\to\infty}\frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p,t)(2\log\log T)^{1/2}} \ge 1 \quad \text{a.s.} \tag{4.26}
$$

Again, we consider two cases.

CASE I. $1 \le p < 2$. For $0 < \varepsilon < 1/8$, take $T_k = e^{k^{1+\varepsilon}}$. Similarly to (3.17), write

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p,T)(2\log\log T)^{1/2}}\\
&\quad\ge \limsup_{k\to\infty}\frac{\|Y(T_k)\|_{l^p}}{\tilde{\sigma}(p,T_k)(2\log\log T_k)^{1/2}}\\
&\quad\ge \limsup_{k\to\infty}\frac{\sum_{i=1}^{\infty}\sigma_i(T_k)^{2(p-1)/(2-p)}X_i(T_k)}{\left(\sum_{i=1}^{\infty}\sigma_i(T_k)^{2p/(2-p)}\right)^{1/2}(2(1+\varepsilon)\log k)^{1/2}}.
\end{aligned}
\tag{4.27}
$$

Put

$$
\xi_k = \frac{\sum_{i=1}^{\infty}\sigma_i(T_k)^{2(p-1)/(2-p)}X_i(T_k)}{\left(\sum_{i=1}^{\infty}\sigma_i(T_k)^{2p/(2-p)}\right)^{1/2}}, \qquad k = 1,2,\dots .
$$

Then, $\{\xi_k,\ k \geq 1\}$ is a Gaussian sequence with $E\xi_k = 0$, $E\xi_k^2 = 1$ and

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{m-k\geq n} E\{\xi_k\xi_n\}\\
&= \lim_{n\to\infty} \sup_{m-k\geq n} \frac{\begin{Bmatrix}\sum_{i=1}^{\infty}\sigma_i^{2p/(2-p)}(T_k) + \sum_{i=1}^{\infty}\sigma_i(T_k)^{2(p-1)/(2-p)} \\ \times \sigma_i(T_m)^{2(p-1)/(2-p)} E[X_i(T_k)(X_i(T_m) - X_i(T_k))]\end{Bmatrix}}{\left(\sum_{i=1}^{\infty}\sigma_i(T_k)^{2p/(2-p)}\right)^{1/2}\left(\sum_{i=1}^{\infty}\sigma_i(T_m)^{2p/(2-p)}\right)^{1/2}}\\
&\leq 0
\end{aligned}$$

by (4.9), the fact that $T_k \log T_k = o(T_{k+1})$ and by the assumption of $\tilde{\sigma}(p,h)/h^{\alpha}$ being quasiincreasing. Applying Nisio's methods (1967) [cf. Theorem 2 of Lai (1973)], we have

$$\limsup_{k\to\infty} \frac{\xi_k}{(2\log k)^{1/2}} = 1 \quad \text{a.s.}, \tag{4.28}$$

which, together with (4.27), yields (4.26) by the arbitrariness of ε.

Case II. $p \geq 2$. Let $T_k = e^{k^{1+\varepsilon}}$, $\varepsilon > 0$. Choose N_k such that $\sigma_{N_k}(T_k) = \tilde{\sigma}(p, T_k)$. Then

$$\limsup_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p,T)(2\log\log T)^{1/2}} \geq \limsup_{k\to\infty} \frac{X_{N_k}(T_k)}{\sigma_{N_k}(T_k)(2(1+\varepsilon)\log k)^{1/2}}.$$

Along the lines of the proof of Case I, one can arrive also at (4.26). This completes the proof of Theorem 4.2. □

Proof of Theorem 4.3. According to Theorem 3.2 with $b_T = 0$ and $a_T = T$, we have

$$\limsup_{T\to\infty} \sup_{0\leq s\leq T} \frac{\|Y(s)\|_{l^p}}{\delta_p\sigma(p,T)} \leq 1 \quad \text{a.s.} \tag{4.29}$$

Hence, we need to show only that

$$\liminf_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\delta_p\sigma(p,T)} \geq 1 \quad \text{a.s.} \tag{4.30}$$

Set

$$T_k = e^{k/\log k}, \quad k = 1, 2, \ldots .$$

It is easy to see that

$$T_{k+1} - T_k = o(T_k), \qquad \log\frac{T_{k+1}}{T_{k+1} - T_k} = o(\log\log(T_{k+1} - T_k)). \tag{4.31}$$

We have also

$$
\begin{aligned}
&\liminf_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\delta_p \sigma(p,T)} \\
(4.32)\qquad &\geq \liminf_{k\to\infty} \frac{\|Y(T_{k+1})\|_{l^p}}{\delta_p \sigma(p,T_{k+1})} - \limsup_{k\to\infty} \sup_{T_k \leq T \leq T_{k+1}} \frac{\|Y(T) - Y(T_{k+1})\|_{l^p}}{\delta_p \sigma(p,T_{k+1})} \\
&\geq \liminf_{k\to\infty} \frac{\|Y(T_{k+1})\|_{l^p}}{\delta_p \sigma(p,T_{k+1})} \\
&\quad - 2\limsup_{k\to\infty} \sup_{0\leq t\leq T_{k+1}} \sup_{0\leq s\leq T_{k+1}-T_k} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p,T_{k+1})}.
\end{aligned}
$$

Similarly to the proof of (3.23), using (4.12) we can obtain that

$$
\liminf_{k\to\infty} \frac{\|Y(T_{k+1})\|_{l^p}}{\delta_p \sigma(p,T_{k+1})} \geq 1 \quad \text{a.s.} \tag{4.33}
$$

From (4.12) and Theorem 3.2 it follows that

$$
\limsup_{k\to\infty} \sup_{0\leq t\leq T_{k+1}} \sup_{0\leq s\leq T_{k+1}-T_k} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p,T_{k+1}-T_k)} \leq 1 \quad \text{a.s.} \tag{4.34}
$$

Since $\sigma(p,T)/T^\alpha$ is quasiincreasing on $(0,\infty)$, we find that $\sigma(p,T_{k+1}-T_k) = o(\sigma(p,T_{k+1}))$, as $k\to\infty$, by (4.31). Therefore

$$
\limsup_{k\to\infty} \sup_{0\leq t\leq T_{k+1}} \sup_{0\leq s\leq T_{k+1}-T_k} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p,T_{k+1})} = 0 \quad \text{a.s.} \tag{4.35}
$$

This proves (4.30) by (4.32), (4.33) and (4.35), as desired. □

REMARK 4.1. Corresponding to Theorem 4.2, we have the following conclusion. Let $p\geq 1$, $\{Y(t),\ t\geq 0\} = \{X_k(t),\ t\geq 0\}_{k=1}^\infty$ be independent Gaussian processes with $X_k(0)=0$, $EX_k(t)=0$, $\sigma_k^2(h) = E(X_k(t+h)-X_k(t))^2$. Assume that $\tilde\sigma(p,h)/h^\alpha$ is quasiincreasing on $(0,1)$ for some $\alpha>0$. Suppose also that

$$
\sigma(p,h) = o\left(\tilde\sigma(p,h)\left(\log\log\frac{1}{h}\right)^{1/2}\right) \quad \text{as } h\to 0, \tag{4.36}
$$

$$
\limsup_{a\to 0} \max_{1/a\geq j\geq \log(1/a)} \max_{k\geq 1} \frac{EX_k(a)(X_k(ja)-X_k(a))}{\sigma_k(a)\sigma_k(ja)} \leq 0. \tag{4.37}
$$

Then we have

$$
\limsup_{h\to 0} \frac{\|Y(h)\|_{l^p}}{\tilde\sigma(p,h)(2\log\log(1/h))^{1/2}} = 1 \quad \text{a.s.} \tag{4.38}
$$

REMARK 4.2. Similarly to Theorem 4.3, we present the following result. Let $\{Y(t),\ t \ge 0\} = \{X_k(t),\ t \ge 0\}_{k=1}^{\infty}$ be independent Gaussian processes with $X_k(0) = 0$, $E\,X_k(t) = 0$, $\sigma_k^2(h) = E(X_k(t+h) - X_k(t))^2$. Assume that $\sigma(p,h)/h^{\alpha}$ is quasiincreasing on $(0,1)$ for some $\alpha > 0$ and that

$$\tilde{\sigma}(p,h)\left(\log\log\frac{1}{h}\right)^{1/2} = o(\sigma(p,h)) \quad \text{as } h \to 0. \tag{4.39}$$

Then we have

$$\lim_{h\to 0}\frac{\|Y(h)\|_{l^p}}{\delta_p\sigma(p,h)} = 1 \quad \text{a.s.} \tag{4.40}$$

5. Fractional Wiener processes. Let $\{\xi(t),\ t \ge 0\}$ be a centered Gaussian process with stationary increments. $\xi(t)$ is called a fractional Wiener process (or Gaussian self-similar process) of order γ if $E\xi^2(t) = t^{2\gamma}$, where $0 < \gamma < 1$. When $\gamma = 1/2$, $\xi(t)$ is the well-known Wiener process. For the increments, as well as the Lévy moduli of continuity for Wiener process, we refer to Csörgő and Révész (1979, 1981) and to Révész (1982). Similar quantities for general fractional Wiener processes were studied by Ortega (1984), Grill (1991) and many others (cf. e.g., the references cited therein). As an application of our previous theorems, this section is devoted to studying sample path properties of l^p-valued fractional Wiener processes.

Let $p \ge 1$, $\{c_n,\ n \ge 1\}$ be nonnegative numbers. Put

$$c(p) = \left(\sum_{k=1}^{\infty} c_k^p\right)^{1/p}, \tag{5.1}$$

$$\tilde{c}(p) = \begin{cases} c\left(\dfrac{2p}{2-p}\right), & \text{if } 1 \le p < 2, \\ \max\limits_{k\ge 1} c_k, & \text{if } p \ge 2. \end{cases} \tag{5.2}$$

Let $\{Y(t),\ t \ge 0\} = \{c_k\xi_k(t),\ t \ge 0\}_{k=1}^{\infty}$, where $\xi_k(t)$ are independent fractional Wiener processes of order γ, $0 < \gamma < 1$. Set $\sigma_k^2(h) = c_k^2 E\xi_k^2(h) = c_k^2 h^{2\gamma}$. Define $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ as in (2.1) and (2.3), respectively. Clearly, we have

$$\sigma(p,h) = h^{\gamma}c(p), \qquad \tilde{\sigma}(p,h) = h^{\gamma}\tilde{c}(p). \tag{5.3}$$

Assume

$$0 < \sum_{k=1}^{\infty} c_k^p < \infty. \tag{5.4}$$

Then, by Theorem 3.1, $Y(\cdot) \in l^p$ has a.s. continuous sample paths. Noting that for each $k \geq 1$, $a > 0$, $j > 2$,

$$\begin{aligned}\frac{Ec_k\xi_k(a)(c_k\xi_k(ja) - c_k\xi_k((j-1)a))}{\sigma_k^2(a)} &= \frac{E\xi_1(a)(\xi_1(ja) - \xi_1((j-1)a))}{E\xi_1^2(a)} \\ &= \frac{1}{2}\left((j+1)^{2\gamma} + (j-1)^{2\gamma} - 2j^{2\gamma}\right)\end{aligned}$$

and

$$\lim_{j\to\infty}\left((j+1)^{2\gamma} + (j-1)^{2\gamma} - 2j^{2\gamma}\right) = 0$$

we see that conditions (2.9) and (2.14) are satisfied. We can also verify that (4.9) and (4.37) are satisfied. Hence, from Theorem 2.1, 2.3, 4.2 and Remark 4.1, we obtain Theorem 5.1 immediately.

THEOREM 5.1. *Let $p \geq 1$, $\{\xi_k(t), t \geq 0\}$ be independent fractional Wiener processes of order γ, $0 < \gamma < 1$. Let $\{Y(t), t \geq 0\} = \{c_k\xi_k(t), t \geq 0\}_{k=1}^{\infty}$. Assume that* (5.4) *is satisfied. Then we have*

$$\limsup_{h\to 0}\frac{\|Y(h)\|_{l^p}}{h^{\gamma}\tilde{c}(p)(2\log\log(1/h))^{1/2}} = 1 \quad a.s., \tag{5.5}$$

$$\lim_{h\to 0}\sup_{0\leq t\leq 1}\sup_{0\leq s\leq h}\frac{\|Y(t+s) - Y(t)\|_{l^p}}{h^{\gamma}\tilde{c}(p)(2\log(1/h))^{1/2}} = 1 \quad a.s., \tag{5.6}$$

$$\limsup_{T\to\infty}\frac{\|Y(T)\|_{l^p}}{T^{\gamma}\tilde{c}(p)(2\log\log T)^{1/2}} = 1 \quad a.s. \tag{5.7}$$

and

$$\lim_{T\to\infty}\sup_{0\leq t\leq T}\sup_{0\leq s\leq a_T}\frac{\|Y(t+s) - Y(t)\|_{l^p}}{a_T^{\gamma}\tilde{c}(p)(2\log(T/a_T))^{1/2}} = 1 \quad a.s., \tag{5.8}$$

for any positive continuous function a_T with $\lim_{T\to\infty}\log(T/a_T)/\log\log T = \infty$.

A particular case of (5.7) with $c_k = 1$ for $1 \leq k \leq d$, $c_k = 0$ for $k > d$ and $\gamma = 1/2$ may be of independent interest.

COROLLARY 5.1. *Let $\mathbf{W}(t) = (W_1(t), \dots, W_d(t))$, $t \geq 0$ be a standard d-dimensional Wiener process. Then we have*

$$\limsup_{T\to\infty}\frac{\left(\sum_{i=1}^d|W_i(T)|^p\right)^{1/p}}{(2T\log\log T)^{1/2}} = d^{(2-p)/2p} \quad a.s. \text{ if } 1 \leq p < 2, \tag{5.9}$$

$$\limsup_{T\to\infty}\frac{\left(\sum_{i=1}^d|W_i(T)|^p\right)^{1/p}}{(2T\log\log T)^{1/2}} = 1 \quad a.s. \text{ if } p \geq 2. \tag{5.10}$$

6. Fractional Ornstein–Uhlenbeck processes. Let $\{Y(t),\ t \geq 0\} = \{X_k(t),\ t \geq 0\}_{k=1}^{\infty}$ be a sequence of independent Ornstein–Uhlenbeck processes with coefficients γ_k and λ_k, that is, the $X_k(\cdot)$ are centered stationary Gaussian processes with

$$(6.1) \qquad E\{X_k(s)X_k(t)\} = \frac{\gamma_t}{\lambda_k}\exp(-\lambda_k|t-s|), \qquad k = 1, 2, \dots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

The process $Y(\cdot)$ was introduced by Dawson (1972) as the stationary solution of the infinite array of stochastic differential equations

$$dX_k(t) = -\lambda_k X_k(t)\,dt + (2\gamma_k)^{1/2}\,dW_k(t), \qquad k = 1, 2, \dots,$$

where $\{W_k(t),\ t \geq 0\}_{k=1}^{\infty}$ are independent standard Wiener processes. The properties of $Y(\cdot)$ have been extensively studied in the literature. Continuity properties of $Y(\cdot)$ were investigated by Dawson (1972), Iscoe and McDonald (1986, 1989), Schmuland (1987, 1988), Marcus (1988), Fernique (1989, 1990, 1991) and Iscoe, Marcus, McDonald, Talagrand and Zinn (1990). Moduli of continuity of $Y(\cdot)$ as an l^p-valued process for $1 \leq p \leq 2$ were discussed by Schmuland (1990) and by Csáki, Csörgő and Shao (1991, 1992).

It is easy to see that

$$\{X_k(t), t \geq 0\}_{k=1}^{\infty} \quad \text{and} \quad \left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, t \geq 0\right\}_{k=1}^{\infty}$$

have the same distribution, where $\{W_k(t)\}_{k=1}^{\infty}$ are independent standard Wiener processes. Hence, without loss of generality, we can write

$$(6.2) \qquad X_k(t) = \left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, \qquad t \geq 0, k = 1, 2, \dots .$$

and keep the path property of $Y(\cdot)$ without change. This relationship and the notion of fractional Wiener processes lead in a natural way to introducing fractional Ornstein–Uhlenbeck processes and to studying their path behaviour, as we do in this section.

Let $\{\xi(t),\ t \geq 0\}$ be a fractional Wiener process of order γ, where $0 < \gamma < 1$. A stationary Gaussian process $\{X(t),\ t \geq 0\}$ is called a fractional Ornstein–Uhlenbeck process of order γ with coefficients a and b if

$$\{X(t), t \geq 0\} \quad \text{and} \quad \left\{\left(\frac{a}{b}\right)^{1/2}\frac{\xi(e^{2bt})}{e^{2\gamma bt}}, t \geq 0\right\}$$

have the same distribution, that is, $EX(t) = 0$, and

$$(6.3) \quad E\{X(t)X(s)\} = \frac{a}{2b}\left(e^{2\gamma b(t-s)} + e^{2\gamma b(s-t)} - |e^{b(t-s)} - e^{b(s-t)}|^{2\gamma}\right),$$

for all t, $s \geq 0$, where $a \geq 0$, $b > 0$.

Clearly, $\{X(t),\ t \geq 0\}$ is the usual Ornstein–Uhlenbeck process if $\gamma = 1/2$.

In what follows, we will always let $\{Y(t),\ t \geq 0\} = \{X_k(t),\ t \geq 0\}_{k=1}^{\infty}$ be a sequence of independent fractional Ornstein–Uhlenbeck processes of order γ with coefficients γ_k and λ_k, where $0 < \gamma < 1$, $\gamma_k \geq 0$, $\lambda_k > 0$. Put

$$\begin{aligned}\sigma_k^2(h) &= E\big(X_k(t+h) - X_k(t)\big)^2 \\ &= \frac{\gamma_k}{\lambda_k}\Big(2 + \big(e^{\lambda_k h} - e^{-\lambda_k h}\big)^{2\gamma} - e^{2\gamma\lambda_k h} - e^{-2\gamma\lambda_k h}\Big),\end{aligned} \tag{6.4}$$

for $h \geq 0$, $k = 1, 2, \dots$. Let $p \geq 1$. Define $\sigma(p,h)$, $\tilde{\sigma}(p,h)$ and δ_p as in (2.1), (2.3) and (2.4), respectively. As consequences of our previous theorems, we have the following results.

THEOREM 6.1. *Assume*

$$0 < \sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k}\right)^{p/2} < \infty, \tag{6.5}$$

$$\int_0^{\infty} \frac{\sigma(p, e^{-z})}{z}\, dz < \infty, \tag{6.6}$$

$$\int_0^{\infty} \tilde{\sigma}\big(p, e^{-z^2}\big)\, dz < \infty. \tag{6.7}$$

Then

$$\lim_{T\to\infty} \sup_{0 \leq s \leq T} \frac{\|Y(s)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} = 1 \quad a.s., \tag{6.8}$$

$$\limsup_{T\to\infty} \frac{\|Y(T)\|_{l^p}}{\tilde{\sigma}(p)(2\log T)^{1/2}} = \quad a.s., \tag{6.9}$$

where $\tilde{\sigma}(p) = \big(\sum_{k=1}^{\infty}(\gamma_k/\lambda_k)^{p/(2-p)}\big)^{(2-p)/2p}$ *if* $1 \leq p < 2$ *and* $\tilde{\sigma}(p) = \max_{k \geq 1}(\gamma_k/\lambda_k)^{1/2}$ *if* $p \geq 2$.

THEOREM 6.2. *Assume that* $\tilde{\sigma}(p,h)/h^{\alpha}$ *is quasiincreasing on* $(0,1)$ *for some* $\alpha > 0$. *If*

$$\sigma(p,h) = o\left(\tilde{\sigma}(p,h)\left(\log \frac{1}{h}\right)^{1/2}\right) \quad \text{as } h \to 0, \tag{6.10}$$

then

$$\lim_{h\to 0} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log(1/h))^{1/2}} = 1 \quad a.s. \tag{6.11}$$

If (6.10) *is replaced by*

$$\sigma(p,h) = o\left(\tilde{\sigma}(p,h)\left(\log\log \frac{1}{h}\right)^{1/2}\right) \quad \text{as } h \to 0, \tag{6.12}$$

then

$$\limsup_{h\to 0} \frac{\|Y(h) - Y(0)\|_{l^p}}{\tilde{\sigma}(p,h)(2\log\log(1/h))^{1/2}} = 1 \quad a.s. \tag{6.13}$$

THEOREM 6.3. *Assume that $\sigma(p,h)/h^\alpha$ is quasiincreasing on $(0,1)$ for some $\alpha > 0$. If*

$$\tilde{\sigma}(p,h)\left(\log\log\frac{1}{h}\right)^{1/2} = o(\sigma(p,h)) \quad \text{as } h \to 0, \tag{6.14}$$

then

$$\lim_{h\to 0} \frac{\|Y(h) - Y(0)\|_{l^p}}{\delta_p \sigma(p,h)} = 1 \quad a.s. \tag{6.15}$$

If (6.14) *is replaced by*

$$\tilde{\sigma}(p,h)\left(\log\frac{1}{h}\right)^{1/2} = o(\sigma(p,h)) \quad \text{as } h \to 0, \tag{6.16}$$

then

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \sigma(p,h)} = 1 \quad a.s. \tag{6.17}$$

The corollaries below give specific meaning to our theorems above in terms of the coefficients γ_k, λ_k and of the order γ of the fractional Ornstein–Uhlenbeck processes involved.

COROLLARY 6.1. *Assume*

$$0 < \sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k}\right)^{p/2} \lambda_k^{\gamma p} < \infty. \tag{6.18}$$

Then we have

$$\lim_{h\to 0} \sup_{0\le t\le 1} \sup_{0\le s\le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\Gamma(p,\gamma) h^\gamma (2\log(1/h))^{1/2}} = 1 \quad a.s., \tag{6.19}$$

$$\limsup_{h\to 0} \frac{\|Y(h) - Y(0)\|_{l^p}}{\Gamma(p,\gamma) h^\gamma (2\log\log(1/h))^{1/2}} = 1 \quad a.s., \tag{6.20}$$

where

$$\Gamma(p,\gamma) = \begin{cases} \left(\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k}\right)^{p/(2-p)} (2\lambda_k)^{2\gamma p/(2-p)}\right)^{(2-p)/2p}, & \text{if } 1 \le p < 2, \\ \max_{k\ge 1} \left(\frac{\gamma_k}{\lambda_k}\right)^{1/2} \cdot (2\lambda_k)^{\gamma}, & \text{if } p \ge 2. \end{cases}$$

COROLLARY 6.2. *Assume* $\gamma_k = k^a$, $\lambda_k = k^b$, *where* $b > 0$. *If* $b - a > (2/p) + 2\gamma b$, *then*

$$\lim_{h \to 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\Gamma(p,\gamma) h^\gamma (2 \log(1/h))^{1/2}} = 1 \quad a.s., \tag{6.21}$$

$$\limsup_{h \to 0} \frac{\|Y(h) - Y(0)\|_{l^p}}{\Gamma(p,\gamma) h^\gamma (2 \log\log(1/h))^{1/2}} = 1 \quad a.s. \tag{6.22}$$

If $2/p < b - a < (2/p) + 2\gamma b$, *then*

$$\lim_{h \to 0} \sup_{0 \le t \le 1} \sup_{0 \le s \le h} \frac{\|Y(t+s) - Y(t)\|_{l^p}}{\delta_p \Lambda(p,\gamma) h^{((b-a)/2b) - 1/pb}} = 1 \quad a.s., \tag{6.23}$$

$$\lim_{h \to 0} \frac{\|Y(h) - Y(0)\|_{l^p}}{\delta_p \Lambda(p,\gamma) h^{((b-a)/2b) - 1/pb}} = 1, \quad a.s., \tag{6.24}$$

where

$$\Gamma(p,\gamma) = \begin{cases} 2^\gamma \left(\sum_{k=1}^{\infty} k^{(a + (1-2\gamma)b)p/(2-p)} \right)^{(2-p)/2p}, & \text{if } 1 \le p < 2, \\ \cdot 2^\gamma, & \text{if } p \ge 2, \end{cases}$$

$$\Lambda(p,\gamma)^p = \frac{1}{b} \int_0^\infty y^{-(((b-a)p-2)/2b) - 1} \left(2 + (e^y - e^{-y})^{2\gamma} - e^{-2\gamma y} - e^{2\gamma y}\right)^{p/2} dy.$$

PROOF OF THEOREM 6.1. Let

$$f(x) := f(\gamma, x) = (e^x - e^{-x})^{2\gamma} - e^{2\gamma x} - e^{-2\gamma x}, \qquad x \ge 0, 0 < \gamma < 1. \tag{6.25}$$

It is easy to see that

$$f(x) + 2 \sim (2x)^{2\gamma} \quad \text{as } x \to 0 \tag{6.26}$$

and

$$f(x) \sim -2\gamma e^{-2(1-\gamma)x} - e^{-2\gamma x} \quad \text{as } x \to \infty. \tag{6.27}$$

By (6.3), we have

$$E\{X_k(0) X_k(a)\} = -\frac{\gamma_k}{2\lambda_k} f(\lambda_k a).$$

Therefore, (4.5) is satisfied by (6.27). The conclusion now follows from Theorem 4.1 immediately. □

PROOF OF THEOREM 6.2. By Theorem 2.3 and Remark 4.1, it suffices to verify that

$$\limsup_{h \to 0} \max_{\log(1/h) \le j \le 1/h} \max_{k \ge 1} \times \frac{E\{(X_k(h) - X_k(0)(X_k(jh) - X_k((j-1)h))\}}{\sigma_k^2(h)} \le 0 \tag{6.28}$$

and

$$\limsup_{h\to 0} \max_{\log(1/h)\le j\le 1/h} \max_{k\ge 1} \times \frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k(h))\}}{\sigma_k(h)\sigma_k(jh)} \le 0. \tag{6.29}$$

We have

$$\sigma_k^2(h) = \frac{\gamma_k}{\lambda_k}(2+f(\lambda_k h)), \tag{6.30}$$

$$\begin{aligned} f'(x) &= (2\gamma)\left((e^x-e^{-x})^{2\gamma-1}(e^x+e^{-x})+e^{-2\gamma x}-e^{2\gamma x}\right)\\ &= 2\gamma e^{2\gamma x}\left((1-e^{-2x})^{2\gamma-1}(1+e^{-2x})-1+e^{-4\gamma x}\right)\\ &\ge 2\gamma e^{2\gamma x}((1-e^{-2x})(1+e^{-2x})-1+e^{-4\gamma x})\\ &= 2\gamma e^{2\gamma x}(e^{-4\gamma x}-e^{-4x})>0 \quad \text{for all } x>0, \end{aligned} \tag{6.31}$$

$$\begin{aligned} f''(x) = 2\gamma\big((2\gamma-1)&(e^x-e^{-x})^{2\gamma-2}(e^x+e^{-x})^2\\ &+(e^x-e^{-x})^{2\gamma}-2\gamma e^{2\gamma x}-2\gamma e^{-2\gamma x}\big). \end{aligned} \tag{6.32}$$

It is easy to see that

$$f''(x)<0 \quad \text{for all } x>0 \text{ if } 0<\gamma\le \tfrac{1}{2}. \tag{6.33}$$

That is, $\sigma_k^2(h)$ is concave on $(0,\infty)$ if $0<\gamma\le 1/2$. Hence, according to Remark 2.1, (6.28) and (6.29) are satisfied in this case.

We consider below the case of $1/2<\gamma<1$. We have

$$\begin{aligned} &\frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k((j-1)h)\}}{\sigma_k^2(h)}\\ &\qquad = \frac{f(j\lambda_k h)+f((j-2)\lambda_k h)-2f((j-1)\lambda_k h)}{2(2+f(\lambda_k h))}\\ &\qquad = \frac{f''(\xi)(\lambda_k h)^2}{2(2+f(\lambda_k h))}, \end{aligned} \tag{6.34}$$

for every $h>0$, $j\ge 6$, $k\ge 1$ and for some $(j-2)\lambda_k h\le \xi\le j\lambda_k h$, by (6.3) and Taylor's formula. Clearly, $f(x)$ is an increasing continuous function with $2+f(x)>0$ for each $x>0$. Hence it follows from (6.26) that there exists a constant $C=C(\gamma)>0$ such that

$$2+f(x)\ge \min(x^{2\gamma},C) \quad \text{for all } x>0. \tag{6.35}$$

We deduce from (6.32) that

$$
\begin{aligned}
f''(x) &\le 2\gamma\big((2\gamma-1)(e^x-e^{-x})^{2\gamma-2}(e^x+e^{-x})^2+e^{2\gamma x}-2\gamma e^{2\gamma x}\big)\\
&= (2\gamma)(2\gamma-1)e^{2\gamma x}\left((1-e^{-2x})^{2\gamma}\left(1+\frac{2e^{-2x}}{1-e^{-2x}}\right)^2-1\right)\\
&\le 2\gamma(2\gamma-1)e^{2\gamma x}\left((1-e^{-2x})^{2\gamma}\left(1+\frac{8e^{-2x}}{(1-e^{-2x})^2}\right)-1\right)\\
&\le \frac{16\gamma(2\gamma-1)e^{-2(1-\gamma)x}}{(1-e^{-2x})^{2-2\gamma}}\\
&\le \frac{64e^{-2(1-\gamma)x}}{(\min(x,1))^{2-2\gamma}}\\
&\le \frac{C_\gamma e^{-(1-\gamma)x}}{x^{2-2\gamma}},
\end{aligned}
\tag{6.36}
$$

for all $x>0$, where C_γ is a positive constant, depending only on γ. A combination of (6.34), (6.35) and (6.36) yields that for $j\ge 6$, $h>0$, $k\ge 1$, and for $(j-2)\lambda_k h\le \xi\le j\lambda_k h$,

$$
\begin{aligned}
\frac{f''(\xi)(\lambda_k h)^2}{2+f(\lambda_k h)} &\le \frac{4C_\gamma e^{-(1-\gamma)(j/2)\lambda_k h}\cdot(\lambda_k h)^2}{(j\lambda_k h)^{2-2\gamma}\cdot\min\big((\lambda_k h)^{2\gamma},C\big)}\\
&= \frac{4C_\gamma e^{-(1-\gamma)(j/2)\lambda_k h}\cdot(\lambda_k h)^{2\gamma}}{j^{2-2\gamma}\min\big((\lambda_k h)^{2\gamma},C\big)}\\
&\le \frac{4C_\gamma}{j^{2-2\gamma}}\left(1+\sup_{x>0}\frac{x^{2\gamma}\cdot e^{-(1-\gamma)x}}{C}\right).
\end{aligned}
\tag{6.37}
$$

This proves (6.28) by (6.34) and (6.37).

In the rest we check for (6.29) in the case of $1/2<\gamma<1$. For $h>0$, $j\ge 6$, $k=1,2,\dots,$ by (6.3), we can write

$$
\begin{aligned}
&\frac{E\{(X_k(h)-X_k(0))(X_k(jh)-X_k(h))\}}{\sigma_k(h)\sigma_k(jh)}\\
&\quad= \frac{f(\lambda_k jh)-f((j-1)\lambda_k h)-f(\lambda_k h)-2}{2(2+f(\lambda_k h))^{1/2}(2+f(j\lambda_k h))^{1/2}}\\
&\quad\le \frac{f(\lambda_k jh)-f((j-1)\lambda_k h)}{2(2+f(\lambda_k h))^{1/2}(2+f(j\lambda_k h))^{1/2}}\\
&\quad= \frac{f'(\xi)\lambda_k h}{2(2+f(\lambda_k h))^{1/2}(2+f(j\lambda_k h))^{1/2}},
\end{aligned}
\tag{6.38}
$$

for some $(j-1)\lambda_k h\le\xi\le j\lambda_k h$.

In terms of (6.31), we have

$$
\begin{aligned}
f'(x) &\le \begin{cases} 8(e^x - e^{-x})^{2\gamma-1}, & \text{if } 0 < x \le 1, \\ 2(e^{(2\gamma-1)x}(e^x + e^{-x}) + e^{-2\gamma x} - e^{2\gamma x}, & \text{if } x \ge 1, \end{cases} \\
&\le \begin{cases} 48 \cdot x^{2\gamma-1}, & \text{if } 0 < x \le 1, \\ 4 \cdot e^{-2(1-\gamma)x}, & \text{if } x \ge 1, \end{cases} \\
&\le 150 x^{2\gamma-1} e^{-2(1-\gamma)x} \quad \text{for all } x > 0.
\end{aligned} \tag{6.39}
$$

Similarly to (6.37), one can arrive at (6.29) by (6.38), (6.35) and (6.39). This completes the proof of Theorem 6.2. □

PROOF OF THEOREM 6.3. This is an immediate consequence of Theorem 2.4 and Remark 4.2. □

PROOF OF COROLLARY 6.1. By Theorem 6.2, it suffices to show that, as $h \to 0$,

$$
\sigma(p, h) \sim h^\gamma \left(\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2\lambda_k)^{\gamma p} \right)^{1/p} \tag{6.40}
$$

and

$$
\tilde{\sigma}(p, h) \sim \Gamma(p, \gamma) h^\gamma. \tag{6.41}
$$

For any $0 < \varepsilon < 1/2$, by (6.26), there exists $\eta > 0$ such that

$$
(1 - \varepsilon)(2x)^{2\gamma} \le 2 + f(x) \le (1 + \varepsilon)(2x)^{2\gamma} \quad \text{for } 0 < x < \eta. \tag{6.42}
$$

Write

$$
\begin{aligned}
\sigma(p, h)^p &= \sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\
&= \sum_{\lambda_k h \le \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\
&\quad + \sum_{\lambda_k h > \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\
&:= \Sigma_1 + \Sigma_2.
\end{aligned} \tag{6.43}
$$

From (6.42), we get

$$
\begin{aligned}
&(1 - \varepsilon) h^{p\gamma} \sum_{\lambda_k h \le \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2\lambda_k)^{\gamma p} \\
&\quad \le \Sigma_1 \le (1 + \varepsilon) h^{p\gamma} \sum_{\lambda_k h \le \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2\lambda_k)^{\gamma p}.
\end{aligned} \tag{6.44}
$$

As to Σ_2, we have

$$\Sigma_2 \le 2^{p/2} \sum_{\lambda_k h > \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} \le \frac{2^{p/2} h^{\gamma p}}{\eta^{\gamma p}} \sum_{\lambda_k > \eta/h} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} \lambda_k^{\gamma p}. \tag{6.45}$$

Since

$$\sum_{\lambda_k h \le \eta} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2\lambda_k)^{\gamma p} \to \sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2\lambda_k)^{\gamma p} \quad \text{as } h \to 0$$

and

$$\sum_{\lambda_k > \eta/h} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} \lambda_k^{\gamma p} = o(1) \quad \text{as } h \to 0,$$

we conclude from (6.43)–(6.45) that (6.40) holds true. Similarly, we have (6.41). It follows from (6.40) and (6.41) that (6.10) and (6.12) are satisfied and that $\tilde{\sigma}(p,h)/h^{\gamma}$ is quasiincreasing on $(0,1)$. This proves (6.19) and (6.20), by Theorem 6.2. □

Proof of Corollary 6.2. When $b - a > (2/p) + 2\gamma b$, it is easy to see that (6.18) is satisfied. Therefore, (6.21) and (6.22) hold true by Corollary 6.1. We next deal with the case

$$\frac{2}{p} < b - a < \frac{2}{p} + 2\gamma b. \tag{6.46}$$

Noting that

$$\Lambda(p,\gamma)^p = \frac{1}{b} \int_0^{\infty} y^{-(((b-a)p-2)/2b)-1} \cdot (2 + f(y))^{p/2}\, dy,$$

when $f(y)$ is defined by (6.25), we see that

$$0 < \Lambda(p,\gamma) < \infty, \tag{6.47}$$

by (6.26) and (6.46).

We first show that

$$\sigma(p,h) \sim \Lambda(p,\gamma) h^{((b-a)/2b)-1/pb} \quad \text{as } h \to 0. \tag{6.48}$$

By (6.26) again, there exists a positive ε_0 such that

$$f(x) + 2 \le 8x^{2\gamma} \quad \text{for all } 0 < x \le \varepsilon_0. \tag{6.49}$$

For any $0 < \varepsilon < \min(\varepsilon_0, 1/2)$, write

$$\begin{aligned} \sigma(p,h)^p &= \sum_{\lambda_k h \le \varepsilon} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\ &\quad + \sum_{\varepsilon < \lambda_k h < 1/\varepsilon} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\ &\quad + \sum_{\lambda_k h \ge 1/\varepsilon} \left(\frac{\gamma_k}{\lambda_k} \right)^{p/2} (2 + f(\lambda_k h))^{p/2} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{6.50}$$

Using (6.49), we have

$$I_1 \le 3^p h^{\gamma p} \sum_{k^b h \le \varepsilon} k^{((a+(2\gamma-1)b)/2)p} \le C\varepsilon^{(2+2b\gamma p-(b-a)p)/2b} \cdot h^{((b-a)p-2)/2b}, \tag{6.51}$$

where, and in the sequel, C denotes a positive constant, depending only on a, b, p, and γ, whose value may be different from time to time. As to I_3, we can obtain

$$I_3 \le 2^p \sum_{k^b h \ge 1/\varepsilon} k^{-(b-a)p/2} \le C\varepsilon^{((b-a)p-2)/2b} \cdot h^{((b-a)p-2)/2b}. \tag{6.52}$$

Since $f(x)$ is an increasing continuous function, we have

$$\begin{aligned} I_2 &= \sum_{\varepsilon < k^b h < 1/\varepsilon} k^{-(b-a)p/2}\bigl(2+f(k^b h)\bigr)^{p/2} \\ &\sim \int_{(\varepsilon/h)^{1/b}}^{(1/\varepsilon h)^{1/b}} y^{-(b-a)p/2}\bigl(2+f(y^b h)\bigr)^{p/2}\,dy \qquad (6.53) \\ &= \frac{h^{((b-a)p-2)/2b}}{b}\int_{\varepsilon^{1/b}}^{\varepsilon^{-1/b}} y^{-(((b-a)p-2)/2b)-1}\bigl(2+f(y)\bigr)^{p/2}\,dy, \end{aligned}$$

as $h \to 0$. Now (6.48) follows from (6.50)–(6.53) and the arbitrariness of ε. Correspondingly, one can derive

$$\tilde\sigma(p,h) = \begin{cases} O\left(h^{\min(\gamma,((b-a)/2b)-(2-p)/2pb)} \cdot \log\dfrac{1}{h}\right), & \text{if } 1 \le p < 2, \\ O\bigl(h^{\min(\gamma,(b-a)/2b)}\bigr), & \text{if } p \ge 2. \end{cases} \tag{6.54}$$

Consequently, (6.14) and (6.16) are satisfied by (6.48), (6.54) and (6.46), and our conclusions, (6.23) and (6.24), follow from Theorem 6.3 and (6.48). The proof of Corollary 6.2 is now complete. □

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