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SYMMETRIC STABLE PROCESSES STAY IN THICK SETS[1]

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Let $X(t)$ be the symmetric α-stable process in $\mathbb{R}^d (0 < \alpha < 2, d \ge 2)$. Then let $W(f)$ be the thorn $\{x \in \mathbb{R}^d : 0 < x_1 < 1, (x_2^2 + \cdots + x_d^2)^{1/2} < f(x_1)\}$ where $f : (0,1) \to (0,1)$ is continuous, increasing with $f(0^+) = 0$. Recently Burdzy and Kulczycki gave an exact integral condition on f for the existence of a random time s such that $X(t)$ remains in the thorn $X(s) + \overline{W(f)}$ for all $t \in [s, s+1)$. We extend their theorem to general open sets W with $0 \in \partial W$. In general, α-processes may stay in sets which are quite lacunary and are not locally connected at 0.

1. Introduction. Let $X(t)$ be the symmetric α-stable process in $\mathbb{R}^d (0 < \alpha < 2, d \ge 2)$, $f : (0,1) \to (0,\infty)$ be a nondecreasing left-continuous function satisfying $f(0^+) = 0$ and $W(f)$ be the thorn $\{x \in \mathbb{R}^d : 0 < x_1 < 1, (x_2^2 + \cdots + x_d^2)^{1/2} < f(x_1)\}$. In [4], Burdzy and Kulczycki give an exact integral condition on f for the existence of a random time s such that $X(t)$ remains in the thorn $X(s) + \overline{W(f)}$ for all $t \in [s, s+1)$.

In this note we extend their theorem on thorns to general open sets having 0 on the boundary. These sets need not be locally connected at 0 and can be quite lacunary; this is possible due to the jumping property of the symmetric α-stable process.

This line of investigation is motivated by the existence of cone points for Brownian paths. For literature and some unsolved cases, see [3].

Let W be an open set in $\mathbb{R}^d$ that contains 0 on its boundary, (Ω, P) be the probability space on which $X(t)$ is defined, $t_0 > 0$ and

$$A(W) = \{\omega \in \Omega : \exists s = s(\omega) \ge 0 \text{ such that } X(t,\omega) \in X(s,\omega) + \overline{W} \text{ for all } t \in [s, s+t_0)\}.$$

We say $\omega \in \Omega$ has a W-point if $\omega \in A(W)$ for some $t_0 > 0$.

Let

$$I(f) = \int_0^1 \frac{f(r)^{\alpha+d-1}}{r^{\alpha+d}}\, dr.$$

The theorem of Burdzy and Kulczycki [4] says that if $I(f) = \infty$, then a symmetric α-stable process has $W(f)$-thorn points a.s., and if $I(f) < \infty$, then an α-process has no $W(f)$-thorn points a.s.

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THEOREM A. *For any* $t_0 > 0$,

(i) $P(A(W(f))) = 1$ *if* $I(f) = \infty$ *and*
(ii) $P(A(W(f))) = 0$ *if* $I(f) < \infty$.

It is clear that $I(f) < \infty$ if and only if $\sum_{k=1}^{\infty} \frac{f(2^{-k})^{\alpha+d-1}}{(2^{-k})^{\alpha+d-1}} < \infty$.

For an arbitrary open set W with $0 \in \partial W$, we give in Theorem 1 a thickness condition on W under which $P(A(W)) = 1$ and in Theorem 2 a thinness condition on W under which $P(A(W)) = 0$. These are natural extensions of Theorem A, and the proofs follow the same structure. The proof in [4] uses very precise harmonic measure estimates obtained by comparing sections of thorns with cylinders; here we must rely on very general estimates and make more use of the jumps. Unlike thorns, general sets do not point in a specific direction, and the uncertainty of the starting time $s(\omega)$ gives rise to a problem which cannot be solved by shifting the set W along an axis; these complications are handled by putting bands around W.

The conditions in Theorems 1 and 2 do not match and are complicated (see Section 3); however, in the case of thorns and also the examples below, they are sharp.

EXAMPLE 1 (Lacunary rings). Let $W = \bigcup_{j=1}^{\infty} \{2^{-j} < |x| < 2^{-j}(1+\delta_j)\}$ with $0 \le \delta_j < \frac{1}{2}$ satisfying

$$\delta_j 2^{-j} < \delta_i 2^{-i} \qquad \text{whenever } \delta_i, \delta_j > 0 \quad \text{and} \quad j > i.$$

Then:

(i) $P(A(W)) = 1$ if $\sum \delta_j^{\alpha+1} = \infty$ and
(ii) $P(A(W)) = 0$ if $\sum \delta_j^{\alpha+1} < \infty$.

In this example, we allow δ_j to be 0 infinitely often.

EXAMPLE 2 (Blocks of varying shape). Let $m(j)$ be integers in $[1, d]$ and δ_j be numbers in $[0, \frac{1}{2})$ satisfying

$$\delta_j 2^{-j} < \delta_i 2^{-i} \qquad \text{whenever } \delta_i, \delta_j > 0 \quad \text{and} \quad j > i. \tag{1.1}$$

Let Q_j be a rectangular cube contained in $\{\frac{5}{8}2^{-j} < |x| < \frac{7}{8}2^{-j}\}$ obtained by translation and rotation of $(0, \delta_j 2^{-j-5}/\sqrt{d})^{m(j)} \cdot (0, 2^{-j-5}/\sqrt{d})^{d-m(j)}$ ($Q_j = \phi$ when $\delta_j = 0$); and let $W = \bigcup_1^{\infty} Q_j$. Then:

(i) $P(A(W)) = 1$ if $\sum \delta_j^{\alpha+m(j)} = \infty$ and
(ii) $P(A(W)) = 0$ if $\sum \delta_j^{\alpha+m(j)} < \infty$.

In this example, we allow δ_j to be 0 infinitely often.

EXAMPLE 3 (Scattered cubes). Let $\{r_k\}_0^\infty$ and $\{\varepsilon_k\}_0^\infty$ be decreasing sequences of positive numbers so that $r_0 = \varepsilon_0 = 1$, $\varepsilon_k < \frac{1}{10}$, $(\varepsilon_k r_k)^{-1}$ is a power of 2, $N_k \equiv \varepsilon_{k-1} r_{k-1}/r_k$ is an odd integer and $\varepsilon_k^{d+\alpha} < N_k^{-\alpha}$, for any $k \geq 1$.

All cubes here have edges parallel to the coordinate axes. Let $Q_0 = (-\frac{1}{2}, \frac{1}{2})^d$, $\mathcal{C}_0 = \{Q_0\}$ and $\mathcal{C}'_0 = \phi$. After $Q_j, \mathcal{C}_j$ and $\mathcal{C}'_j$ have been defined for $0 \leq j \leq k-1$ with $\ell(Q_j) = \varepsilon_j r_j$, we subdivide Q_k into a collection $\mathcal{S}_k$ of N_k^d subcubes of side length r_k each. $\mathcal{C}_k$ consists of those cubes having side length $\varepsilon_k r_k$ and concentric to those in $\mathcal{S}_k$; let Q_k be the cube in $\mathcal{C}_k$ that contains the origin 0 and $\mathcal{C}'_k = \mathcal{C}_k \setminus \{Q_k\}$. For future discussion, we also choose and fix one cube from $\mathcal{C}'_k$ that is closest to Q_k; call it Q'_k. Let

$$W = \bigcup_{k=1}^{\infty} \bigcup_{Q \in \mathcal{C}'_k} Q.$$

Then

(i) $P(A(W)) = 1$ if $\sum \varepsilon_k^{\alpha+d} = \infty$ and
(ii) $P(A(W)) = 0$ if $\sum \varepsilon_k^{\alpha+d} < \infty$.

Section 2 contains properties of symmetric α-stable processes needed later, Section 3 contains the main theorems; proofs of Theorems 1, 2 and examples are given in Sections 4, 5 and 6, respectively.

2. Preliminaries. A symmetric α-stable process X on $\mathbb{R}^d$ is a Lévy process (homogeneous independent increments) whose transition density $p(t,x)$ is uniquely determined by its Fourier transform, $\int_{\mathbb{R}^d} e^{ix\cdot\xi} p(t,x)\,dx = e^{-t|\xi|^\alpha}$. Here α must be in $(0,2]$. When $\alpha = 2$, it is the Brownian motion except for a linear time change. From now on, symmetric α-stable processes are restricted to the case $0 < \alpha < 2$. Denote by (Ω, P) the probability space on which $X(t)$ is defined. Sample paths are discontinuous, and are right continuous with left limits a.s. [1, 2].

In the following, $B(x,r)$ is the ball centered at x of radius r, and $|S|$ is the Lebesgue measure (volume) of the set S. We use c (or c') to denote positive constants depending at most on d and α, $c(\cdot)$ to denote positive constants depending on d, α and the variables in the parentheses and C_j, $j = 1, 2, \ldots,$ to denote specific constants depending on d and α only. We write $a \lesssim b$ when $a/b \leq c$ for some constant c, and $a \cong b$ when $a \lesssim b$ and $b \lesssim a$.

As usual E^x is the expectation with respect to the process starting from $x \in \mathbb{R}^d$. For any open set D in $\mathbb{R}^d$, X^D is the symmetric α-stable process killed upon leaving D and $\tau_D = \inf\{t > 0 : X(t) \notin D\}$ is the first exit time.

For any $x \in D$, the α-harmonic measure $\mu^x(\cdot, D)$ is a measure on D^c defined by

$$\mu^x(A, D) = P^x\big(X(\tau_D) \in A\big), \qquad A \subseteq D^c;$$

it is monotone in D; that is,

$$\mu^x(A, D) \le \mu^x(A, \tilde{D}) \qquad \text{if } D \subseteq \tilde{D}.$$

In the case of a ball $B = B(0, r)$, it was shown by M. Riesz that

$$d\mu^x(y, B) = k_B(x, y)\, dy, \tag{2.1}$$

where

$$k_B(x, y) = \begin{cases} C_1\left(\dfrac{r^2 - |x|^2}{|y|^2 - r^2}\right)^{\alpha/2} |x - y|^{-d}, & |y| > r, \\ 0, & |y| \le r. \end{cases}$$

Note, from (2.1) and the monotonicity that

$$\mu^x(S, D) = 0 \qquad \text{if } S \text{ is a sphere in } D^c.$$

Denote by G the Green function of X; that is,

$$G(x, y) = \int_0^\infty p(t, x - y)\, dt = C_2|x - y|^{-d+\alpha}$$

and denote by $G_D(x, y)$ the Green function of X^D, that is,

$$G_D(x, y) = C_2\left[|x - y|^{-d+\alpha} - \int_{D^c} |y - z|^{-d+\alpha}\, d\mu^x(z, D)\right] \qquad \forall x, y \in D, x \ne y.$$

$G_D(x, x) = \infty$ if $x \in D$ and $G_D(x, y) = 0$ in $(D \times D)^c$ and the Green function has the scaling property

$$G_D(x, y) = a^{-\alpha+d} G_{aD}(ax, ay), \qquad a > 0;$$

and for any measurable $f \ge 0$ on D,

$$E^x\left[\int_0^{\tau_D} f(X(s))\, ds\right] = \int_D G_D(x, y) f(y)\, dy \qquad \forall x \in D.$$

In particular,

$$E^x(\tau_D) = \int_D G_D(x, y)\, dy \qquad \forall x \in D.$$

It is well known that

$$E^x\left(\tau_{B(x,r)}\right) = C_3 r^\alpha \tag{2.2}$$

and

$$E^x(\tau_D) \lesssim |D|^{\alpha/d}. \tag{2.3}$$

For any bounded measurable $\phi \ge 0$ on D^c,

$$E^x[\phi(X(\tau_D)) : X(\tau_D) \ne X(\tau_{D}-)] = C_4 \int_{D^c}\int_D \frac{G_D(x, y)}{|y - z|^{d+\alpha}}\, dy\phi(z)\, dz, \tag{2.4}$$

where $X(\tau_{D}-) = \lim_{t\uparrow\tau_D} X(t)$ exists a.s. [5]. Note from (2.4) and $X(\tau_{D}-) \in \overline{D}$ that for $x \in D$ and $A \subseteq \overline{D}^c$,

$$\mu^x(A, D) = C_4 \int_A \int_D \frac{G_D(x, y)}{|y - z|^{d+\alpha}}\, dy\, dz \tag{2.5}$$

and

$$\mu^x(A, D) \lesssim E^x(\tau_D) \operatorname{dist}(A, D)^{-\alpha-d}|A|. \tag{2.6}$$

When $\max\{\operatorname{diam} D, \operatorname{diam} A\} \le a \operatorname{dist}(A, D)$, we obtain from (2.5) the following estimate:

$$\mu^x(A, D) \cong c(a) E^x(\tau_D) \operatorname{dist}(A, D)^{-\alpha-d}|A|. \tag{2.7}$$

We shall use (2.7) repeatedly for X^D having certain prescribed jumps.

3. Theorems. Let W be an open set with $0 \in \partial W$.

THEOREM 1. *Suppose that*

$$\int_W E^x(\tau_W)|x|^{-\alpha-d}\, dx = \infty, \tag{3.1}$$

then $P(A(W)) = 1$.

In the case of a thorn $W(f)$, $E^x(\tau_{W(f)}) \cong f(x_1)^\alpha$ for any x satisfying $(x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2} < f(x_1)/2$; hence

$$\int_{W(f)} E^x\big(\tau_{W(f)}\big)|x|^{-\alpha-d} \cong \int_0^1 \frac{f(r)^{\alpha+d-1}}{r^{\alpha+d}}\, dr.$$

Therefore for thorns, Theorem 1 is equivalent to Theorem A(i).

For general open sets W, it is unclear whether

$$\int_W E^x(\tau_W)|x|^{-\alpha-d}\, dx < \infty \tag{3.2}$$

implies $P(A(W)) = 0$.

Before stating the thinness conditions under which $P(A(W)) = 0$, we need a few definitions. For any positive integers j and n, let

$$\begin{aligned}
W(j) &= W \cap \{|x| < 2^{-j}\},\\
W^*(j) &= W \cap \{2^{-j-1} \le |x| < 2^{-j}\},\\
p(j) &= \max\{i \le j-2 : W^*(i) \ne \phi\},\\
W_n &= \{x : \operatorname{dist}(x, W) < 2^{-n}\} = W + B(0, 2^{-n}),\\
W_n(j) &= W_n \cap \{|x| < 2^{-j}\},\\
W_n^*(j) &= W_n \cap \{2^{-j-2} \le |x| < 2^{-j}\}
\end{aligned}$$

and

$$p_n(j) = \max\{i \le j-2 : W_n^*(i) \ne \phi\}.$$

For $x \in W(j)$, define

$$\lambda^x(W, j) = \mu^x\big(W^*(p(j)), W(j-1)\big)2^{-p(j)(d+\alpha)}|W^*(p(j))|^{-1} \tag{3.3}$$

and

$$\Lambda(W, j) = \sup\{\lambda^x(W, j) : x \in W^*(j)\} \tag{3.4}$$

for $x \in W_n(j)$; the expressions $\lambda^x(W_n, j)$ and $\Lambda(W_n, j)$ are defined analogously.

REMARK 1. The quantity $\lambda^x(W, j)$ is a substitute for $E^x(\tau_W)$ and is comparable to $E^x(\tau_W)$ when $W(j-1)$ and $W^*(p(j))$ are separated by a large ring. In fact,

$$\lambda^x(W, j) \cong E^x\big(\tau_{W(j-1)}\big) \qquad \text{if } p(j) < j-2 \tag{3.5}$$

and

$$\lambda^x(W, j) \gtrsim E^x\big(\tau_{W(j-1)}\big) \qquad \text{if } p(j) = j-2;$$

the equivalence relation in the case $p(j) < j-2$ follows from (2.7) and the fact that $|y-z| \cong 2^{-p(j)}$ for $y \in W(j-1)$ and $z \in W^*(p(j))$. When $p(j) = j-2$ and $W(j-1)$ and $W^*(j-2)$ are separated by a ring $\{a < |x| < b\}$ of width $b-a$ at least $\beta 2^{-j}$, we have

$$\lambda^x(W, j) \cong c(\beta)E^x\big(\tau_{W(j-1)}\big). \tag{3.6}$$

THEOREM 2. *Let W be an open set with $0 \in \partial W$. Suppose that there is an infinite collection $\mathcal{A}$ of (n, i) with integers $n > i > 0$, satisfying $W^*(i) \ne \phi$*

$$\mu^0\big(W_n^*(i), W_n(i+1)\big) \cong \mu^0\big(W_n^*(i), B(0, 2^{-n})\big) \tag{3.7}$$

and for each $\varepsilon > 0$, there exists K so that

$$\sum_{j=K}^{i} \Lambda(W_n, j)(2^{-j})^{-d-\alpha}|W_n^*(j)| < \varepsilon \qquad \forall\,(n, i) \in \mathcal{A}. \tag{3.8}$$

Then $P(A(W)) = 0$.

Condition (3.8) measures the thinness of W in the manner of (3.2). Condition (3.7) is introduced for technical reasons; it says that the probability of the process landing in $W_n^*(i)$ upon leaving $W_n(i+1)$ is equivalent to that of the process jumping directly from the ball $B(0, 2^{-n})$ to $W_n^*(i)$. It would be desirable to remove (3.7) or to replace it by a geometric condition.

The reason for expanding W to W_n is to surround the path $X(t)$, $t > s(\omega)$, when the initial position $X(s(\omega))$ can only be located to within a ball of radius 2^{-n}. For sets with certain geometric characteristics, for example, thorns or those in Examples 1–3, the enlargement plays a minor role. However, when the set is scattered, W_n can be substantially larger that W. An assumption such as (3.2) does not guarantee the boundedness of $\sum_{j=i}^{n} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)|$; and the series $\sum_{j=n+1}^{\infty} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)|$ is always infinite. For this reason, the portion of W in $\{2^{-n} \le |x| \le 2^{-j}\}$ needs to be considered separately, using (3.7).

Conditions (3.7) and (3.8) are used for all open sets; therefore they are complicated and the geometrical implications are less apparent. We now examine these conditions on sets having special characteristics.

(A) When volumes $|W_n^*(j)|$ change very regularly,

$$c^{-1} < |W_n^*(j)|/|W_n^*(j+1)| < c \qquad \forall n, j \ge 1.$$

Note from (3.3) and (3.4) that (3.8) is equivalent to

$$\sum_{j=K}^{i} \sup_{x \in W_n^*(j)} \mu^x\bigl(W_n^*(j-2), W_n(j-1)\bigr) < \varepsilon \qquad \forall (n, i).$$

(B) For open sets W whose complement $\mathbb{R}^d \setminus W$ contains a sequence of uniformly fat rings going to 0, for example,

$$\mathbb{R}^d \setminus W \supseteq \bigcup_{j=1}^{\infty} \{\tfrac{3}{4} 2^{-j} < |x| < 2^{-j}\},$$

it follows from (3.5) and (3.6) that (3.8) is equivalent to

$$\sum_{j=K}^{i} \Bigl(\sup_{x \in W_n^*(j)} E^x\bigl(\tau_{W_n(j-1)}\bigr)\Bigr)(2^{-j})^{-\alpha-d}|W_n^*(j)| < \varepsilon \qquad \forall (n, i).$$

(C) For thorns $W(f)$, $I(f) < \infty$ implies (3.7) and (3.8). Consider only pairs (n, i) satisfying

$$f(2^{-i})/2 \le 2^{-n} < f(2^{-i}). \tag{3.9}$$

Lemma 4.5 in [4] yields (3.7). Kulczycki has shown that for all thorns, with no assumption on $I(f)$,

$$\mu^x\bigl(W^*(j-2), W(j-1)\bigr) \lesssim E^x\bigl(\tau_{W(j-1)}\bigr)(2^{-j})^{-\alpha-d}|W^*(j-2)| \qquad \forall x \in W^*(j).$$

Since $E^x(\tau_{W(j-1)}) \lesssim f(2^{-j+1})^\alpha$ and $|W^*(j-2)| \lesssim f(2^{-j+2})^{d-1} 2^{-j}$, we obtain, from $I(f) < \infty$,

$$\sum_{j=1}^{\infty} \Lambda(W, j)(2^{-j})^{-\alpha-d}|W^*(j)| < \infty.$$

Since (3.9) implies $W_n \cap \{|x| \geq 2^{-i}\} \subseteq \{x : (x_2^2 + \cdots + x_n^2)^{1/2} < 3f(x_1)\}$, condition (3.8) holds for all such pairs (n, i).

4. Proof of Theorem 1. We follow the proof of Theorem A(i) in [4] and give details at two crucial points for general open sets. The key to the proof is (4.1); roughly it says that when W is thick at 0, in order to travel from $W \cap \{|x| < \varepsilon\}(\varepsilon > 0$ small) to $W \cap \{|x| > \frac{1}{8}\}$ without leaving W, at least half of the paths must pass W "section by section" without making extremely long jumps. The reasoning which leads to (4.1) for general sets uses harmonic measure estimates for paths with prescribed jumps (2.7). For $\omega \in \Omega$, the starting time $s(\omega)$ for the path $X(t, \omega)$ to stay in $X(s(\omega)) + \overline{W}$ for a given period of time is chosen as a limit of a sequence; and the continuity (4.6) of X at $s(\omega)$ is essential. Details on the continuity are given for the sake of completeness, since W need not be locally connected at 0.

We assume as we may that $W \subseteq \{|x| < \frac{1}{4}\}$, and let $\{a_n\}$ be a sequence of integers with $a_1 = 2$ and $a_{n+1} > 5 + a_n$. Let

$$W[n] = W \cap \{|x| < 2^{-a_n}\}$$

and

$$W^*[n] = W \cap \{2^{-a_{n+1}} \leq |x| < 2^{-a_n}\}.$$

Note that $W = W[1]$, $W[n] = W(a_n)$ and $W^*[n] \neq W^*(a_n)$. Let also $a_0 = 0$, $W^*[0] = \{\frac{1}{2} < |x| < 1\}$.

Define

$$F_1 = \{X_{\tau_{W[1]}} \in W^*[0]\}$$

and

$$F_{n+1} = \{X_{\tau_{W[n+1]}} \in W^*[n]\} \cap \theta^{-1}_{\tau_{W[n+1]}} F_n, \qquad n \geq 1,$$

where θ is the shift operator. Note on the set $\{X_{\tau_{W[n+1]}} \in W^*[n]\}$, we have $\theta^{-1}_{\tau_{W[n+1]}}(\{X_{\tau_{W[n]}} \in W^*[n-1]\}) = \{X_{\tau_{W[n]}} \in W^*[n-1]\}$. So

$$F_{n+1} = \bigcap_{m=1}^{n+1} \{X_{\tau_{W[m]}} \in W^*[m-1]\}.$$

LEMMA 1. *Under assumption* (3.1), *the sequence* $\{a_n\}$ *can be chosen so that*

(4.1) $$P^x(F_n) \geq \tfrac{1}{2} P^x(F_1) \qquad \forall n \in \mathbb{N}_+ \quad \textit{and} \quad x \in W[n].$$

PROOF. Let $H_n = F_1 \setminus F_n$. Inequality (4.1) follows from the following:

(4.2) $$P^x(H_n) \leq \frac{n}{n+1} P^x(F_n) \qquad \forall n \in N_+ \quad \text{and} \quad x \in W[n].$$

Recall that $a_0 = 0$ and $a_1 = 2$, and that (4.2) holds trivially for $n = 1$. Suppose that a_n's have been selected and (4.2) has been verified for $n = 1, 2, \ldots, m$; we shall choose a_{m+1} and verify (4.2) for $m + 1$. Consider any $a_{m+1} > 5 + a_m$ and $x \in W[m + 1]$. Then

$$\begin{aligned} P^x(F_{m+1}) &= \sum_{k=a_m}^{-1+a_{m+1}} E^x\bigl(X_{\tau_{W[m+1]}} \in W^*(k);\, P^{X_{\tau_{W[m+1]}}}(F_m)\bigr) \\ &\geq \sum_{k=3+a_m}^{-2+a_{m+1}} E^x\bigl(X_{\tau_{W[m+1]}} \in W^*(k);\, \tfrac{1}{2} P^{X_{\tau_{W[m+1]}}}(F_1)\bigr). \end{aligned}$$

Note from (2.4) that

$$P^x(F_{m+1}) \gtrsim \sum_{k=3+a_m}^{-2+a_{m+1}} \int_{W[m+1]} \int_{W^*(k)} \frac{G_{W[m+1]}(x, y)}{|y - z|^{d+\alpha}} P^z(F_1)\, dz\, dy.$$

Since $\operatorname{dist}(z, W^*[0]) \cong 1$ and $|y - z| \cong |z|$ for $z \in W^*(k)$ and $y \in W[m + 1]$, and $P^z(F_1) = \mu^z(W^*[0], W) \cong E^z(\tau_W)$ by (2.7), we have

$$P^x(F_{m+1}) \gtrsim E^x\bigl(\tau_{W[m+1]}\bigr) \sum_{k=3+a_m}^{-2+a_{m+1}} \int_{W^*(k)} \frac{E^z(\tau_W)}{|z|^{d+\alpha}}\, dz. \tag{4.3}$$

On the other hand, it follows from (2.6) and the induction hypothesis that for any $x \in W[m + 1]$,

$$\begin{aligned} P^x(H_{m+1}) &= P^x\bigl(F_1, X_{\tau_{W[m+1]}} \in W^*[m], \bigl(\theta^{-1}_{\tau_{W[m+1]}} F_m\bigr)^c\bigr) \\ &\quad + P^x\bigl(F_1, X_{\tau_{W[m+1]}} \notin W^*[m]\bigr) \\ &\leq \frac{m}{m+1} P^x(F_{m+1}) + c(2^{-a_m})^{-d-\alpha} E^x\bigl(\tau_{W[m+1]}\bigr). \end{aligned} \tag{4.4}$$

The argument is adopted from (3.4) and (3.5) in [4], where only the boundedness of the thorn is used in the proof. From (4.3), (4.4) and the assumption (3.1), it follows that if a_{m+1} is large enough then

$$P^x(H_{m+1}) \leq \frac{m+1}{m+2} P^x(F_{m+1}) \qquad \forall\, x \in W[m + 1].$$

This completes the proof of Lemma 1. □

Fix $\{a_n\}_0^\infty$ as in Lemma 1, and choose a point y_n in each $W^*[n]$. As in [4], define for $1 \leq k \leq n$,

$$S_k^n = \inf\{t \geq 0 : X(t) \notin X(0) - y_n + W[n - k + 1]\}.$$

Then $S_1^n \leq S_2^n \leq \cdots \leq S_n^n$. Let R_n be the first S_k^n such that $X(S_k^n) \notin X(0) - y_n + W^*[n - k]$ if it exists; otherwise let $R_n = \inf\{t \geq 0 : X(t) \notin X(0) - y_n + W\}$.

Following the argument of Lemma 3.3 in [4] and using the Markov property, (2.6) and Lemma 1 above (in place of Lemma 3.2 in [4]), we obtain

$$E(R_n) \cong E^{y_n}(\tau_W) \lesssim c(W, t_0)\ P(R_n \geq t_0). \tag{4.5}$$

Define for $n \geq 1$, a sequence of stopping times as follows: $T(0, n) = 0$,

$$T(j+1, n) = \begin{cases} T(j, n) + (R_n \wedge t_0) \circ \theta_{T(j,n)}, & \text{if } T(j, n) < t_0, \\ T(j, n), & \text{if } T(j, n) \geq t_0; \end{cases}$$

define also

$$F(j, n) = \{\omega \in \Omega : T(j+1, n) - T(j, n) = t_0\}$$

and

$$H_n = \bigcup_{j=0}^{\infty} F(j, n).$$

LEMMA 2. *There exists a positive constant $c(W, t_0)$ so that*

$$P(H_n) \geq c(W, t_0) \qquad \forall n \geq 1.$$

PROOF. Unlike the situation in [4], condition (3.1) does not imply $E(R_n) \to 0$ as $n \to \infty$. For each $n \geq 1$, we consider two possibilities: $E(R_n) < t_0/10$ or $E(R_n) \geq t_0/10$. In the first case, choose an integer m_n such that $t_0/4 \leq m_n E(R_n) \leq t_0/2$, and then proceed as in [4]. When $E(R_n) \geq t_0/10$, we note from (4.5) that

$$\begin{aligned} P(H_n) &\geq P(F(0, n)) = P\big(T(1, n) = t_0\big) = P(R_n \geq t_0) \\ &\geq c(d, \alpha, W, t_0) E(R_n) \geq c'(d, \alpha, W, t_0). \end{aligned}$$

Let

$$H = \limsup_{n \to \infty} H_n$$

and

$$A^0 = \big\{\omega \in \Omega : \exists\, s = s(\omega) \in [0, t_0) \\ \text{such that } X(t, \omega) \in X(s, \omega) + \overline{W} \text{ for all } t \in [s, s + t_0)\big\}.$$

In view of Lemma 2 and the fact that H_n's are independent, to prove the theorem, it is sufficient to check $H \subseteq A^0$.

Assume that $\omega \in H$. Then there exist sequences $\{j_k\}$ and $\{n_k\}$ (depending on ω) so that $n_k \uparrow \infty$, $\omega \in F(j_k, n_k)$, and $s_k \equiv T(j_k, n_k)$ converges to some $s \in [0, t_0]$. The crucial step in proving $\omega \in A^0$ is to verify the continuity of X at s

$$\lim_{t \to s} X(t) = X(s). \tag{4.6}$$

After that, $\omega \in A^0$ follows easily.

To this end, we may assume that $\{s_k\}$ is monotone and consider only the case when $\{s_k\}$ is strictly increasing; the decreasing case is analogous and simpler. Since X is right continuous and has left limits, both $X(s) = \lim_{t\downarrow s} X(t)$ and $X(s-) = \lim_{t\uparrow s} X(t)$ exist.

Assume that $X(s) \neq X(s-)$, and choose m so that

$$2^{-a_m} < |X(s) - X(s-)|/8.$$

Choose $\delta \in (0, t_0/2)$ so that

$$|X(t) - X(s-)| < 2^{-a_{m+1}-3} \qquad \forall t \in (s-\delta, s)$$

and choose k_0 so that if $k > k_0$ then $s_k \in (s-\delta, s)$; thus

$$|X(s_k) - X(s-)| < 2^{-a_{m+1}-3}.$$

Fix an integer $k > k_0$, with $n_k > m+2$. Since $\omega \in F(j_k, n_k)$, it follows that for $t \in [s_k, s_k + t_0)$,

$$X(t) \in X(s_k) - y_{n_k} + W$$

and that if $X(t)$ leaves $X(s_k) - y_{n_k} + W[p]$ $(1 \leq p \leq n_k)$, then it goes to $X(s_k) - y_{n_k} + W^*[p-1]$.

Consider $t \in [s_k, s)$; then t is in $(s-\delta, s) \cap [s_k, s_k + t_0)$; therefore

$$|X(t) - X(s_k)| \leq 2^{-a_{m+1}-2}$$

and

$$X(t) \in X(s_k) - y_{n_k} + W.$$

Hence

$$X(t) \in X(s_k) - y_{n_k} + W[m+1] \qquad \forall t \in [s_k, s),$$

which implies that

$$X(s) \in X(s_k) - y_{n_k} + W[m].$$

Consequently,

$$\begin{aligned} |X(s) - X(s-)| &\leq |X(s) - X(s_k)| + |X(s_k) - X(s-)| \\ &\leq 2^{-a_m+1} < |X(s) - X(s-)|/2, \end{aligned}$$

which is impossible. Therefore $X(s) = X(s-)$ and the continuity (4.6) follows. This completes the proof of Theorem 1. $\square$

5. Proof of Theorem 2. Again we follow the structure of the proof of Theorem A(ii) in [4]. The key is Lemma 3; very roughly, it says that when W is thin at 0, the probability of the process starting in $W \cap \{|x| < \varepsilon\}$ ($\varepsilon > 0$ small), making at least m "forward landings" in $W \cap \{\varepsilon \le |x| \le \frac{1}{8}\}$ before leaving W, goes down geometrically with respect to m. Methods of estimating harmonic measures for thorns do not apply; we use (2.7) repeatedly. Because W does not point in any specific direction, we need to put a band around W to contain paths with small shifts.

Given $i_0 > 1$ and $X(0) = x \in W(i_0)$, define a sequence of stopping times $S(m)$ as follows. Let $S(0) = 0$ and

$$S(m+1) = \begin{cases} \tau_{W(i_m - 1)}, & \text{if } i_m > 1, \\ S(m), & \text{if } i_m = 0, \end{cases}$$

where $i_m, m \ge 1$, is the integer > 1 such that $X(S(m)) \in W^*(i_m)$ if it exists, and $i_m = 0$ otherwise. While $i_m, m \ge 1$, is uniquely determined by induction, the choice of i_0 is not; the specific value of i_0 is important in defining $\{S(m)\}$. Note that $i_{m+1} < i_m - 1$, $0 < S(1) < S(2) < \cdots < S(m)$, and that $\{i_1, i_2, \ldots, i_m\}$ records the forward landings according to the rules given.

For $i < k$, $m \ge 1$ and $x \in W(i)$, define

$$\begin{aligned} &H(k, i, m, x, W) \\ &\quad = \{\omega \in \Omega : i_0 = i, X(0) = x, S(m-1) < S(m), X(S(m)) \in W^*(k)\} \end{aligned}$$

to be the collection of paths that start at x, with $i_0 = i$, and end in $W^*(k)$ at time $S(m)$.

LEMMA 3. *There exists $C_0 > 0$ so that for $m \ge 1$, $i > k > K$ and $x \in W(i)$, if*

$$\sum_{j=K}^{i-2} \Lambda(W, j)(2^{-j})^{-d-\alpha} |W^*(j)| < C_0^{-1} \tag{5.1}$$

then

$$P^x\big(H(k, i, m, x, W)\big) \le C_0 2^{-m} \lambda^x(W, i)(2^{-k})^{-d-\alpha} |W^*(k)|. \tag{5.2}$$

PROOF. We write

$$P^x\big(H(k, i, m, x, W)\big) = P^x\big(S(m-1) < S(m), X(S(m)) \in W^*(k)\big).$$

In the case $i = k + 1$, $X(S(1)) \in W(k)^c$; and (5.2) holds trivially.

Assume from now on $i \ge k + 2$ and $|W^*(k)| > 0$. We shall prove (5.2) by induction on m.

When $m = 1$ and $i = k + 2$, note from (3.3) that

$$\begin{aligned} P^x\big(S(0) < S(1), X(S(1)) \in W^*(k)\big) &= \mu^x\big(W^*(i-2), W(i-1)\big) \\ &= 2^{1+2(d+\alpha)} 2^{-1} \mu^x\big(W^*(i-2), W(i-1)\big) \\ &\quad \times 2^{-i(d+\alpha)} |W^*(i-2)|^{-1} 2^{k(d+\alpha)} |W^*(k)| \\ &= 2^{1+2(d+\alpha)} 2^{-1} \lambda^x(W, i) 2^{k(d+\alpha)} |W^*(k)|. \end{aligned}$$

When $m = 1$ and $i > k + 2$, in view of (2.7),

$$\begin{aligned} &P^x\big(S(0) < S(1), X(S(1)) \in W^*(k)\big) \\ &\quad = \mu^x\big(W^*(k), W(i-1)\big) \cong E^x\big(\tau_{W(i-1)}\big)(2^{-k})^{-d-\alpha} |W^*(k)|. \end{aligned}$$

Since $E^x(\tau_{W(i-1)}) \lesssim \lambda^x(W, i)$,

$$\begin{aligned} &P^x\big(S(0) < S(1), X(S(1)) \in W^*(k)\big) \\ &\quad \le C_5 2^{-1} \lambda^x(W, i)(2^{-k})^{-d-\alpha} |W^*(k)| \end{aligned}$$

for some $C_5 > 0$. Let

$$C_0 = \max\{2^{1+2(d+\alpha)}, C_5\}$$

then (5.2) holds for $m = 1$.

Assume that (5.2) has been proved for some $m \ge 1$ and all $i > k > K$ and $x \in W(i)$. Given $i \ge k + 2$ and $x \in W(i)$, we have

$$\begin{aligned} &P^x\big(S(m) < S(m+1), X\big(S(m+1)\big) \in W^*(k)\big) \\ &\quad = \sum_{j=k+2}^{i-2} E^x\big(S(m-1) < S(m), X(S(m)) \in W^*(j)\big), \\ &\qquad\qquad P^{X(S(m))}\big(X_{\tau_{W(j-1)}} \in W^*(k)\big) \\ &\quad \le \sum_{j=k+2}^{i-2} P^x\big(S(m-1) < S(m), X(S(m)) \in W^*(j)\big) \\ &\qquad\qquad \times \sup_{y \in W^*(j)} P^y\big(X_{\tau_{W(j-1)}} \in W^*(k)\big) \\ &\quad = \sum_{j=k+2}^{i-2} P^x\big(H(j, i, m, x, W)\big) \sup_{y \in W^*(j)} P^y\big(H(k, j, 1, y, W)\big). \end{aligned}$$

(Note that when $j = k + 1$ or $i - 1$, the events are void.)

The induction hypothesis yields that

$$\begin{aligned} &P^x\big(S(m) < S(m+1), X\big(S(m+1)\big) \in W^*(k)\big) \\ &\quad \le \sum_{j=k+2}^{i-2} C_0 2^{-m} \lambda^x(W,i) 2^{j(d+\alpha)} |W^*(j)| C_0 2^{-1} \Lambda(W,j) 2^{k(d+\alpha)} |W^*(k)| \\ &\quad \le C_0^2 2^{-m-1} \lambda^x(W,i) 2^{k(d+\alpha)} |W^*(k)| \sum_{j=K}^{i-2} \Lambda(W,j) 2^{j(d+\alpha)} |W^*(j)| \\ &\quad = C_0 2^{-m-1} \lambda^x(W,i) 2^{k(d+\alpha)} |W^*(k)|. \end{aligned}$$

Now (5.2) has been proved for all $m \ge 1$.

For each $n > 0$, we define a sequence of stopping times $\{T(j,n)\}$ modeled on those in [4] by letting $T(0,n) = 0$ and

$$T(j+1,n) = \inf\{s > T(j,n) : X(s) \notin B\big(X(T(j,n)), 2^{-n}\big)\} \qquad \text{for } j \ge 0.$$

Since $\{\tau_{B(0,2^{-n})} \circ \theta_{T(j,n)}\}$ are independent and identically distributed, the proof of Lemma 4.7 in [4] yields

$$\sum_{j=0}^{\infty} P\big(T(j,n) \le N\big) \le c(d,\alpha) N / E\big(\tau_{B(0,2^{-n})}\big) \cong N 2^{n\alpha}, \tag{5.3}$$

which in turn implies that $P(\{\lim_{j\to\infty} T(j,n) < \infty\}) = 0$.

We assume as we may that all sample paths $t \to X(t,\omega)$, are right continuous with left limits that for all $n > 0$,

$$\lim_{j\to\infty} T(j,n) = \infty$$

and that ω does not belong to the following set:

$$\begin{aligned} \Omega_1 = \big\{&\omega \in \Omega : \exists s = s(\omega) \ge 0, \\ &\exists a = a(\omega) > 0 \ni X(t,\omega) = X(s,\omega)\ \forall t \in [s, s+a)\big\}. \end{aligned}$$

Let $Q(s,n) = \inf\{t > s, X(t) \notin B(X(s), 2^{-n})\}$. Then for all s, $Q(s,n,\omega) > s$, $\lim_{n\to\infty} Q(s,n,\omega) = s$ and $\lim_{n\to\infty} X(Q(s,n,\omega)) = X(s,\omega)$ by the right continuity of the process. For $a > 0$, let

$$\begin{aligned} Z(s,a,\omega) = \big\{&\ell \ge 1 : \exists q \ge 1 \text{ such that } Q(s,q,\omega) \in (s, s+a) \\ &\text{and } X(Q(s,q,\omega)) \in B\big(X(s,\omega), 2^{-\ell}\big) \setminus B\big(X(s,\omega), 2^{-\ell-1}\big)\big\}, \end{aligned}$$

which represents another way to record forward landings. Since $\omega \notin \Omega_1$, $Z(s,a,\omega)$ is an infinite set. For integers $i > k$, let

$$Z(s,a,k,i,\omega) = Z(s,a,\omega) \cap [k,i].$$

For $\Gamma \subseteq [0,\infty)$ and $k>0$, let $A(\Gamma,k)=\{\omega\in\Omega : \exists\, s=s(\omega)\in\Gamma$ and $a=a(\omega)>0$ such that $X(t,\omega)\in X(s,\omega)+\overline{W}$ $\forall t\in[s,s+a)$ and $\sup_{t\in[s,s+a)}|X(t,\omega)-X(s,\omega)|\in[2^{-k-1},2^{-k})\}$.

To show $P(A(W))=0$, it suffices to prove

$$P\big(A([0,N],k)\big)=0 \qquad \forall N,k>0. \tag{5.4}$$

Fix N and k from now on. For $m\ge 1$ and $i>k$, let

$$A(\Gamma,k,i,m)=\{\omega\in A(\Gamma,k):\#Z\big(s(\omega),a(\omega),k,i\big)\ge m\}.$$

Because $\#Z(s,a,\omega)=\infty$,

$$A([0,N],k)=\bigcup_{i=k+1}^{\infty}A([0,N],k,i,m)$$

for all $m\ge 1$. Since $A([0,N],k,i,m)$ increases as i increases, in order to prove (5.4) it suffices to show that

$$P\big(A([0,N],k,i,6m)\big)\le c(k)N2^{-m} \tag{5.5}$$

for all $m\ge 1$ and all pairs $(n,i)\in\mathcal{A}$ with $i>k>K$ for some $K>0$.

Fix $(n,i)\in\mathcal{A}$ with $i>k$, then

$$\begin{aligned}&P\big(A([0,N],k,i,6m)\big)\\ &\quad=\bigcup_{j=0}^{\infty}P\big(A\big([0,N]\cap[T(j,n),T(j+1,n)],k,i,6m\big)\big).\end{aligned} \tag{5.6}$$

Suppose

$$\omega\in A\big([0,N]\cap[T(j,n),T(j+1,n)],k,i,6m\big), \tag{5.7}$$

then:

(a) $T(j,n)\le N$;

(b) there exist $s=s(\omega)\in[T(j,n),T(j+1,n))$, and $a=a(\omega)>0$ such that $X(t,\omega)\in X(s)+\overline{W(k)}$ for all $t\in[s,s+a)$;

(c) $\sup\{|X(t)-X(s)|:s\le t<s+a\}\in[2^{-k-1},2^{-k})$; and

(d) $\#Z(s(\omega),a(\omega),k,i)\ge 6m$.

Since $|X(s)-X(T(j,n))|<2^{-n}$, inequalities $2^{-j-1}<|x-X(s)|<2^{-j}$, $j\le n-2$, imply $2^{-j-2}<|x-T(j,n)|<2^{-j+1}$. We shift the reference point from $X(s)$ to $X(T(j,n))$, then the path of ω is contained in the enlarged set $\overline{W_n}$ with respect to $X(T(j,n))$. Consequently:

(b′) $X(t)\in B(X(T(j,n)),2^{-n})+\overline{W(k)}\subseteq X(T(j,n))+\overline{W_n(k)}$ for all $t\in[T(j,n),s+a)$;

(c′) $\sup\{|X(t)-X(T(j,n))|:T(j,n)\le t<s+a\}\in[2^{-k-2},2^{-k+1})$; and

(d′) $\#Z(T(j,n), s(\omega)+a(\omega)-T(j,n), k, i) \geq 2m$.

The decrease from $6m$ in (d) to $2m$ in (d′) is due to the shift from $X(s)$ to $X(T(j,n))$. Therefore it follows from (a) and (b′)–(d′) that

$$\omega \in \{T(j,n) \leq N\} \cap \theta^{-1}_{T(j,n)}\left(\bigcup_{m'=m}^{\infty} \bigcup_{k'=k-1}^{k+1} H(k', i+2, m', 0, W_n)\right). \tag{5.8}$$

The reason for the decrease from $2m$ in (d) to m in (5.8) is the following. In defining $S(m)$, the set $\{i_0, i_1, \ldots, i_m\}$ that records the forward landings does not contain consecutive integers; on the other hand, $Z(T(j,n,\omega), s(\omega)+a(\omega)-T(j,n,\omega), k, i)$ may contain blocks of consecutive integers. The change from i in (d′) to $i+2$ in (5.8) is for convenience when quoting Lemma 2; the change is insignificant because m is large. From (5.6)–(5.8) and the strong Markov property, it follows that

$$\begin{aligned} &P\big(A([0,N], k, i, 6m)\big) \\ &\quad \leq \sum_{j=0}^{\infty} P\big(T(j,n) \leq N\big)\left(\sum_{m'=m}^{\infty} \sum_{k'=k-1}^{k+1} P^0\big(H(k', i+2, m', 0, W_n)\big)\right). \end{aligned}$$

Applying Lemma 3 to W_n and using (3.8) in place of (5.1), we obtain for $k > K$ (some $K > 0$),

$$P\big(H(k', i+2, m', 0, W_n)\big) \leq C_0 2^{-m'} \lambda^0(W_n, i+2)\big(2^{-k'}\big)^{-d-\alpha} |W_n^*(k')|.$$

It has been stated in (5.3) that $\sum_{j=0}^{\infty} P(T(j,n) \leq N) \lesssim N2^{n\alpha}$. Therefore for $k > K$,

$$P\big(A([0,N], k, i, 6m)\big) \leq c(k) N 2^{n\alpha} 2^{-m} \lambda^0(W_n, i+2).$$

Recall from (3.3) that

$$\lambda^0(W_n, i+2) = \mu^0\big(W_n^*(i), W(i+1)\big) 2^{-i(d+\alpha)} |W_n^*(i)|^{-1}.$$

Finally, condition (3.7) and harmonic measure estimate (2.7) yield

$$\begin{aligned} \lambda^0(W_n, i+2) &\cong \mu^0\big(W_n^*(i), B(0, 2^{-n}) 2^{-i(d+\alpha)} |W^*(i)|^{-1}\big) \\ &\cong E^0\big(\tau_{B(0,2^{-n})}\big) \cong 2^{-n\alpha}. \end{aligned}$$

Finally $P(A([0,N], k, i, 6m)) \leq c(k) N 2^{-m}$ for $k > K$, which is (5.5). This proves $P(A(W)) = 0$. □

6. On examples. First we verify Example 2. The following lemma on expected life time should be known.

LEMMA 4. *Let* $S = (0,1) \times (-\infty, \infty)^{d-1}$. *Then* $\sup_{x \in S} E^x(\tau_S) < \infty$.

PROOF. Let $T = (-1, 1) \times (-\infty, \infty)^{d-1}$. Then

$$a \equiv \sup_{x\in S} P^x\big(X(t) \in S\ \forall 0 \le t \le 1\big) \le P^0\big(X(t) \in T\ \forall 0 \le t \le 1\big) < 1$$

and $P^x(X(t) \in S\ \forall 0 \le t \le N) \le a^N$ (N positive integer) for all $x \in S$. From this, it follows that $E^x(\tau_S) \le (1-a)^{-2}$ for all $x \in S$. □

LEMMA 5. *Let* $0 < \delta < 1$, *m an integer in* $[1, d]$ *and* $Q = (0, \delta)^m \times (0, 1)^{d-m}$. *Then for any* $x \in (\frac{\delta}{4}, \frac{3\delta}{4})^m \times (\frac{1}{4}, \frac{3}{4})^{d-m}$,

$$E^x(\tau_Q) \cong \sup_{x\in Q} E^x(\tau_Q) \cong \delta^\alpha.$$

PROOF. Let $T_m = (-1, 1)^m \times (-\infty, \infty)^{d-m}$; note from Lemma 4 and the monotonicity that $C_6 \equiv \max_{1\le m\le d} \sup_{x\in T_m} E^x(\tau_{T_m})$ is finite. Again by monotonicity and scaling note that $\sup_{x\in Q} E^x(\tau_Q) \lesssim C_6\delta^\alpha$. The fact that $E^x(\tau_Q) \gtrsim \delta^\alpha$ for all $x \in (\frac{\delta}{4}, \frac{3\delta}{4})^m \times (\frac{1}{4}, \frac{3}{4})^{d-m}$ follows from (2.3). This completes the proof. □

To check Example 2, we note from Lemma 5 and scaling that

$$\sup_{x\in W(i)} E^x\big(\tau_{W(i)}\big) \gtrsim \delta_i^\alpha 2^{-i\alpha}.$$

Therefore $\int_W E^x(\tau_W)|x|^{-d-\alpha}\,dx \gtrsim \sum \delta_i^{\alpha+m(i)}$; assertion (i) in Example 2 follows from Theorem 1.

Assume that $\delta_i \ne 0$ for infinitely many i's; otherwise (ii) is trivial. Consider only pairs (n, i) satisfying $\delta_i > 0$ and $\delta_i 2^{-i-1} \le 2^{-n} < \delta_i 2^{-i}$. We claim that

$$E^x\big(\tau_{W_n(i)}\big) \lesssim 2^{-n\alpha} \qquad \forall x \in W_n(i).$$

Since $E^x(\tau_{W_n(i)})$ is continuous in $W_n(i)$ and goes to 0 as x approaches $\partial W_n(i)$, $\sup\{E^x(\tau_{W_n(i)}) : x \in W_n(i)\}$ is attained at some point $z \in W_n(i)$. Assume that $z \in W_n^*(j)$ for some $j \in [i, n]$. Then

$$\begin{aligned} E^z\big(\tau_{W_n(i)}\big) &= E^z\big(\tau_{W_n^*(j)}\big) + \int_{W_n(i)\setminus W_n^*(j)} E^y\big(\tau_{W_n(i)}\big)\,d\mu^z\big(y, W_n^*(j)\big) \\ &\le E^z\big(\tau_{W_n^*(j)}\big) + E^z\big(\tau_{W_n(i)}\big)\mu^z\big(W_n(i)\setminus W_n^*(j), W_n^*(j)\big). \end{aligned}$$

Note from the definition of W that $W_n(i)^c$ contains some ball of diameter 2^{-j-1} within a distance 2^{-j+1} from $W_n^*(j)$. Calculations using (2.7) and the monotonicity yield

$$\mu^z\big(W_n(i)^c, W_n^*(j)\big) > C_7 > 0,$$

and by Lemma 5,

$$E^z\big(\tau_{W_n(i)}\big) \le C_7^{-1} E^z\big(\tau_{W_n^*(j)}\big) \lesssim (\delta_j 2^{-j})^\alpha \lesssim 2^{-n\alpha}.$$

This proves the claim.

From the harmonic measure estimate (2.7) and the claim, it follows that

$$\begin{aligned}\mu^0\big(W_n^*(i), W_n(i+1)\big) &\cong E^0\big(\tau_{W_n(i+1)}\big)(2^{-i})^{-\alpha-d}|W_n^*(i)| \\ &\lesssim 2^{-n\alpha}(2^{-i})^{-\alpha-d}|W_n^*(i)| \cong \mu^0\big(W_n^*(i), B(0,2^{-n})\big).\end{aligned}$$

This proves (3.7) in Theorem 2.

Note from (1.1), (3.6) and Lemma 5 that for $x \in W_n(j)$ and $j \geq i$,

$$\lambda^x(W_n, j) \cong E^x(\tau_{W_n(j-1)}) \lesssim (\delta_j^\alpha + \delta_{j-1}^\alpha)2^{-j\alpha}$$

(the sum $\delta_j^\alpha + \delta_{j-1}^\alpha$ is needed since δ_{j-1} may be zero), and that

$$\sum_{j=1}^{i} \Lambda(W_n, j)(2^{-j})^{-\alpha-d}|W_n^*(j)| \lesssim \sum_{j=1}^{i} \delta_j^{\alpha+m(j)}.$$

This proves (3.8) in Theorem 2 and thus assertion (ii) in Example 2.

REMARK 2. In Example 2, the requirement in keeping Q_j's uniformly apart is for the convenience of the proof. The conclusions remain if Q_j's are allowed to stay in $\{2^{-j-1} < |x| < 2^{-j}\}$, or are replaced by bilipschitz images of Q_j's with uniformly bounded bilipschitz constants.

Example 1 is a variation of Example 2 in the case $m(j) = 1$ for all j. It is especially interesting to note that $P(A(W)) = 1$ as long as $\limsup \delta_j > 0$; in particular, W can be very lacunary.

In Example 3, the set is scattered, and we need some harmonic measure estimates. For $x \in \mathbb{R}^d$, let

$$\|x\| = \max\{|x_j| : 1 \leq j \leq d\}.$$

LEMMA 6. *Let $0 < \varepsilon < \frac{1}{10}$, $r > 0$, $\mathcal{L}$ be the set of lattice points in $\mathbb{R}^d$, $W = \bigcup_{x\in\mathcal{L}} B(x,\varepsilon)$ and $W^r = W \cap \{\|x\| < r + \frac{1}{4}\}$. Then*

$$\mu^{x_0}\big(W \setminus B(x_0,\varepsilon), B(x_0,\varepsilon)\big) \cong \varepsilon^{\alpha+d} \qquad \forall x_0 \in \mathcal{L}. \tag{6.1}$$

Suppose $\varepsilon^{\alpha+d} < N^{-\alpha}$ and $N > 10$, then

$$\mu^x(W \setminus W^N, W^N) \lesssim \varepsilon^{\alpha+d}N^{-\alpha} \qquad \forall x \in W^{N/2}, \tag{6.2}$$

$$\mu^{x_0}(W \setminus W^N, W^N) \cong \mu^{x_0}\big(W^{2N} \setminus W^N, B(x_0,\varepsilon)\big) \cong \varepsilon^{\alpha+d}N^{-\alpha} \qquad \forall x_0 \in \mathcal{L} \quad \textit{with } \|x\| \leq \frac{N}{2} \tag{6.3}$$

and there exists $C_8 > 0$ so that if $0 < \varepsilon < C_8$ then

$$E^x(\tau_W) \lesssim \varepsilon^\alpha \qquad \forall x \in W. \tag{6.4}$$

PROOF. It follows from (2.1) that

$$\mu^0\big(W \setminus B(0,\varepsilon), B(0,\varepsilon)\big) \cong E^0\big(\tau_{B(0,\varepsilon)}\big) \int_1^\infty t^{-d-\alpha} \varepsilon^d t^{d-1}\,dt \cong \varepsilon^{\alpha+d}$$

and (6.1) follows by translation.

Monotonicity and calculation as above yield that if $x \in B(x_0,\varepsilon) \subseteq W^N$ then

$$\begin{aligned}\mu^x(W \setminus W^N, W^N) &\le \mu^x\big(W \setminus B(x_0,\varepsilon), B(x_0,\varepsilon)\big)\\ &\le \mu^x\big(W \setminus B(x_0,\varepsilon), B(x,2\varepsilon)\big) \cong \varepsilon^{\alpha+d}.\end{aligned} \tag{6.5}$$

If $x \in B(x_0,\varepsilon) \subseteq W^{N/2}$, then (2.1), (2.2), (2.5) and monotonicity yield

$$\begin{aligned}\mu^x\big(W \setminus W^N, B(x_0,\varepsilon)\big) &\le \mu^x\big(W \setminus W^N, B(x,2\varepsilon)\big)\\ &\cong E^x\big(\tau_{B(x,2\varepsilon)}\big) \int_{N/2}^\infty t^{-d-\alpha} \varepsilon^d t^{d-1}\,dt\\ &\cong \varepsilon^{\alpha+d} N^{-\alpha}\\ &\cong \mu^{x_0}\big(W^{2N} \setminus W^N, B(x_0,\varepsilon)\big).\end{aligned} \tag{6.6}$$

Now let $x \in B(x_0,\varepsilon) \subseteq W^{N/2}$. Then from the Markov property, (6.5), (6.6) and the assumption $\varepsilon^{\alpha+d} < N^{-\alpha}$, it follows that

$$\begin{aligned}\mu^x(W \setminus W^N, W^N) &= \mu^x\big(W \setminus W^N, B(x_0,\varepsilon)\big)\\ &\quad + \int_{W^N \setminus B(x_0,\varepsilon)} \mu^y(W \setminus W^N, W^N)\, d\mu^x\big(y, B(x_0,\varepsilon)\big)\\ &\lesssim \varepsilon^{\alpha+d} N^{-\alpha} + \varepsilon^{2(\alpha+d)} \lesssim \varepsilon^{\alpha+d} N^{-\alpha}.\end{aligned}$$

This gives (6.2).

The estimate in (6.3) follows from (6.2), (6.6) and the fact that $\mu^{x_0}(W \setminus W^N, W^N) \ge \mu^{x_0}(W \setminus W^N, B(x_0,\varepsilon))$.

It is easy to see from the geometry of the set W that $\inf_{x\in W} P^x(X(1) \in W^c) > 0$. Arguing as in Lemma 4 we obtain $\sup_{x\in W} E^x(\tau_W) < \infty$. Since $E^x(\tau_W)$ is continuous in W and approaches 0 uniformly on ∂W, $\sup_{x\in W} E^x(\tau_W)$ is attained in W. Since W is translation invariant we may choose $z \in B(0,\varepsilon)$ so that $E^z(\tau_W) = \sup_{x\in W} E^x(\tau_W)$. By Markov property, monotonicity and (6.5),

$$\begin{aligned}E^z(\tau_W) &= E^z\big(\tau_{B(0,\varepsilon)}\big) + \int_{W \setminus B(0,\varepsilon)} E^x(\tau_W)\, d\mu^z\big(x, B(0,\varepsilon)\big)\\ &\le E^z\big(\tau_{B(0,\varepsilon)}\big) + E^z(\tau_W)\mu^z\big(W \setminus B(0,\varepsilon), B(0,\varepsilon)\big)\\ &\le E^z\big(\tau_{B(0,\varepsilon)}\big) + C_9 E^z(\tau_W)\varepsilon^{\alpha+d}.\end{aligned}$$

Now if $\varepsilon^{\alpha+d} < (2C_9)^{-1}$, then

$$E^z(\tau_W) \le 2E^z\big(\tau_{B(0,\varepsilon)}\big) \lesssim \varepsilon^\alpha,$$

which gives (6.4). $\square$

To verify Example 3, we apply Theorems 1 and 2 in the rectangular settings, that is, in the definitions of $W(j)$, $W^*(j)$ and $W_n(j)$ and $W_n^*(j)$, we use $\|\cdot\|$ instead of $|\cdot|$, for example, $W(j) = W \cap \{\|x\| < 2^{-j}\}$.

Assume $\sum \varepsilon_k^{\alpha+d} = \infty$. Using (2.1) and (2.2), we obtain for $x \in \frac{1}{2}Q \in \mathcal{C}_k'$, $E^x(\tau_W) \gtrsim \varepsilon_k^\alpha r_k^\alpha$ and

$$\int_{\bigcup_{\mathcal{C}_k'} Q} |x|^{-d-\alpha}\, dx \cong \int_{r_k}^{\varepsilon_{k-1} r_{k-1}} t^{-d-\alpha} \varepsilon_k^d t^{d-1}\, dt \cong \varepsilon_k^d r_k^{-\alpha}.$$

Therefore $\int_W E^x(\tau_W)|x|^{-d-\alpha}\, dx \gtrsim \sum \varepsilon_k^{d+\alpha} = \infty$; the conclusion $P(A(W)) = 1$ follows from Theorem 1.

Next we verify part (ii), and let $n(k)$ be the integer satisfying $2^{-n(k)} = \varepsilon_k r_k$, $i(k) = n(k-1)$, and $m(k)$ be the smallest integer such that $2^{-m(k)-1} \le r_k - \varepsilon_k r_k$; in other words, $\{\|x\| < 2^{-m(k)-1}\}$ is the largest cube of the form $\{\|x\| < 2^{-j}\}$ that does not meet $\bigcup\{x + Q_k : x \in Q \in \mathcal{C}_k'\}$. Note that $2^{-m(k)} \cong r_k,\ n(k) > m(k) > i(k)$ and that

$$\bigcup\{x + Q_k : x \in Q \in \mathcal{C}_k'\} \subseteq \{2^{-m(k)-1} < \|x\| < 2^{-i(k)}\}$$

and

$$W_{n(k)} \subseteq \{\|x\| < 2^{-n(k)}\} \cup \bigcup_{\ell=1}^{k} \{2^{-m(\ell)-1} < |x| < 2^{-i(\ell)}\}$$

for each $k \ge 1$.

We shall check (3.7) and (3.8) for pairs $(n(k),\ i(k)),\ k \ge 1$.

Note from monotonicity, assumption $\varepsilon_k^{\alpha+d} < N_k^{-\alpha}$ and a scaled version of (6.3) that

$$\mu^0\big(W_{n(k)}^*(i(k)), W_{n(k)}\big(i(k)+1\big)\big) \cong \mu^0\big(W_{n(k)}^*(i(k)), Q_k\big) \cong \varepsilon_k^{\alpha+d} N_k^{-\alpha}.$$

This gives (3.7).

To check (3.8), we fix $k \ge 1$ and for simplicity, we use (n, i), W_n for $(n(k), i(k))$ and $W_{n(k)}$ and use $p(j)$ for $\max\{i : i \le j-2 : W_n^*(i) \ne \phi\}$. We then proceed to estimate $\mu^x(W_n^*(p(j)), W_n(j-1))$ and $\Lambda(W_n, j)$ for $j \in \bigcup_{\ell=1}^k [i(\ell), m(\ell)]$ and $x \in W_n^*(j)$.

Let $\ell \in [1, k]$ and consider first $j \in [i(\ell)+2, m(\ell)]$; in this case $p(j) = j-2$, $|W_n^*(p(j))| \cong |W_n^*(j)|$ and there are $\mathcal{N}(k, \ell, j) \cong 2^{-jd} r_\ell^{-d}$ cubes in $\mathcal{C}_k'$ that meet $W_n(j-2)$. Therefore monotonicity and a scaled version of (6.3) imply that for $x \in W_n^*(j)$,

$$\begin{aligned}\mu^x\big(W_n^*(p(j)), W_n(j-1)\big) &= \mu^x\big(W_n^*(j-2), W_n(j-1)\big)\\ &\cong \varepsilon_\ell^{d+\alpha}\big(\mathcal{N}(k,\ell,j)^{1/d}\big)^{-\alpha}\\ &\cong \varepsilon_\ell^{d+\alpha} r_\ell^\alpha 2^{j\alpha}.\end{aligned}$$

Consequently, it follows from (3.3) and (3.4) that

$$
\begin{aligned}
&\sum_{j=i(\ell)+2}^{m(\ell)} \Lambda(W,j)(2^{-j})^{-d-\alpha}|W_n^*(j)| \\
&\qquad \cong \sum_{j=i(\ell)+2}^{m(\ell)} \varepsilon_\ell^{d+\alpha} r_\ell^{\alpha} 2^{j\alpha} \\
&\qquad \cong \varepsilon_\ell^{\alpha+d} r_\ell^{\alpha} 2^{m(\ell)\alpha} \\
&\qquad \cong \varepsilon_\ell^{\alpha+d}.
\end{aligned}
\tag{6.7}
$$

For $\ell \in [1,k]$ and $j = i(\ell)$ or $i(\ell)+1$, we have $p(j) = m(\ell-1)$ and $2^{-p(j)} \cong r_{\ell-1}$, and have $W(j-1) = W(i(\ell)) \subseteq \bigcup_{\mathcal{C}_\ell} Q$, $2^{-j} = \varepsilon_{\ell-1} r_{\ell-1}$ and $|W_n^*(i(\ell))| \cong |W_n^*(i(\ell)+1)| \cong (\varepsilon_{\ell-1} r_{\ell-1})^d \varepsilon_\ell^d$. Because there is a thick ring separating $W_n(j-1)$ from $W_n(p(j))$, it follows from (3.6) that

$$
\lambda^x(W_n, j) \cong E^x(\tau_{W_n(j-1)}) = E^x(\tau_{W_n(i(\ell))}) \qquad \forall x \in W_n^*(j).
$$

A scaled version of (6.4) shows that

$$
E^x(\tau_{W_n(i(\ell))}) \lesssim \varepsilon_\ell^{\alpha} r_\ell^{\alpha} \qquad \forall x \in W_n^*(j).
$$

Therefore when $j = i(\ell)$ or $i(\ell)+1$,

$$
\Lambda(W_n, j)(2^{-j})^{-d-\alpha}|W_n^*(j)| \lesssim \varepsilon_\ell^{\alpha} r_\ell^{\alpha} \varepsilon_{\ell-1}^{-d-\alpha} r_{\ell-1}^{-d-\alpha} (\varepsilon_{\ell-1} r_{\ell-1})^d \varepsilon_\ell^d \lesssim \varepsilon_\ell^{\alpha+d}.
\tag{6.8}
$$

With $k \ge 1$ still fixed, we obtain from (6.7) and (6.8)

$$
\begin{aligned}
&\sum_{j=1}^{i(k)} \Lambda(W_{n(k)}(j))|W_n^*(j)|2^{j(d+\alpha)} \\
&\qquad \le \sum_{\ell=1}^{k} \sum_{j=i(\ell)}^{m(\ell)} \Lambda(W_{n(k)}(j))|W_n^*(j)|2^{j(d+\alpha)} \\
&\qquad \lesssim \sum_{\ell=1}^{k} \varepsilon_\ell^{\alpha+d}.
\end{aligned}
$$

Since $\sum_{\ell=1}^{\infty} \varepsilon_\ell^{d+\alpha} < \infty$, it is clear that there exists K so that condition (3.8) is satisfied for all pairs $(n(k), i(k))$; assertion (ii) in Example 3 follows from Theorem 2.

REMARK 3. In part (ii) of Example 3, $\varepsilon_k^{\alpha+k} < N_k^{-\alpha}$ is used to obtain (3.7) and $\sum \varepsilon_\ell^{d+\alpha} < \infty$ is used to obtain (3.8).

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