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NONDIFFERENTIABILITY OF CURVES ON THE BROWNIAN SHEET

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For a Brownian sheet on the nonnegative quadrant, we show that any nontrivial curve in the quadrant with the property that the Brownian sheet restricted to the curve gives rise to a differentiable function cannot be differentiable at any point. This result has several implications for level sets of the Brownian sheet. In particular, any Jordan arc contained in a level set must be nowhere differentiable.

1. Introduction. In this paper, we consider the geometry of the level sets of the Brownian sheet $W\colon \mathbb{R}_+^2 \to \mathbb{R}$. This work can be thought of as a continuation of the investigation of path properties of the Brownian sheet, motivated in part by analogous questions for linear Brownian motion.

For linear Brownian motion, the structure of the level sets is well understood. Indeed, the zero level set is the range of a stable $\frac{1}{2}$ subordinator ([13], Chapter 10, Exercise (1.11)). The excursions away from 0 can be thought of as a point process of excursions indexed by local time. Given this representation, it is immediate that there cannot be a random time point on this level set which is the endpoint of both a negative and a positive excursion. The celebrated result of Dvoretsky, Erdös and Kakutani [8] can be thought of as saying that, further, there is no random level x and no random point in the level set of x which is the endpoint of both a positive and a negative excursion from x. This result is usually expressed as "Brownian motion has no point of increase" (see [10], Chapter 5.6) or

> there does not exist a continuous, increasing and injective function $f\colon [-1,1] \to \mathbb{R}_+$ such that $B(f(t)) < B(f(0))$ if $t < 0$ and $B(f(t)) > B(f(0))$ if $t > 0$.

These results motivate similar questions for the Brownian sheet. In [11], Mountford showed that even for a fixed level, say 0, there exist points in $\mathbb{R}_+^2$ which are boundary points of both a positive and of a negative excursion. Here, an *excursion* away from 0, also called a *Brownian bubble* [5], is the restriction of a sample path of the Brownian sheet to a connected component of the random set $\{t \in \mathbb{R}_+^2\colon\ W(t) \neq 0\}$. However, as the geometry of open sets in $\mathbb{R}^2$ can be quite complicated and as this is the case of Brownian bubbles

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[5, 6], this result does not imply that

> there exists a continuous, increasing and injective function f: $[-1,1] \to \mathbb{R}_+^2$ such that $W(f(t)) < W(f(0))$ if $t < 0$ and $W(f(t)) > W(f(0))$ if $t > 0$

(here, "increasing" means with respect to the natural partial order). The existence of such curves was established by Dalang and Mountford [3].

Given that the natural Brownian sheet analog of this Brownian path property does not hold, it is natural to consider other weaker path properties. Since Brownian motion has no points of increase, it cannot have a nonzero derivative at any point (in fact, it is also proved in [8] that the Brownian path does not possess finite one-sided derivatives at any point). In particular, Brownian motion cannot be constant on a nondegenerate interval.

The analog of this last statement for the Brownian sheet would be that level sets of the Brownian sheet contain curves. The question of whether or not this occurs is particularly interesting in view of the fact observed by Kendall [9] that, for purely topological reasons, the level set $L(x)$ at level $x \in \mathbb{R}$ of the Brownian sheet must contain nontrivial connected components, but is also "totally disconnected at almost all of its points."

Csörgő and Révész [2] proved a statement to the effect that "the Brownian sheet is nowhere differentiable in any direction." Their result implies that level sets of the Brownian sheet contain no straight-line segments. We shall prove a stronger result.

Recall that a Jordan arc $\Gamma \subset \mathbb{R}^2$ is any continuous one-to-one image of the unit interval. A Jordan arc Γ is *differentiable* at $t \in \Gamma$ if there exists a continuous and one-to-one function γ: $[0,1] \to \Gamma$ and $x \in (0,1)$ such that $\gamma(x) = t$, γ is differentiable at x and $\gamma'(x) \neq (0,0)$ [here, $\gamma'(x)$ is the gradient of γ at x]. If there is no $t \in \Gamma$ such that Γ is differentiable at t, then we say that Γ is *nowhere differentiable.*

THEOREM 1. *Almost surely, for all $x \in \mathbb{R}$, if $\Gamma \subset L(x) \cap (0,\infty)^2$ is a Jordan arc, then Γ is nowhere differentiable.*

An extension of the result of Csörgő and Révész mentioned above that clearly does imply Theorem 1 is the following.

THEOREM 2. *The event*

> *"there exist a continuous injective function γ: $[0,1] \to (0,\infty)^2$ and $x \in (0,1)$ such that $\gamma(\cdot)$ has a nonzero gradient at x and $W(\gamma(\cdot))$ is differentiable at x"*

has probability 0.

Theorem 2 may be seen as an appropriate analog to the result of Paley, Wiener and Zygmund [12] on the nondifferentiability of Brownian paths. The proof of Theorem 2 will be accomplished by showing that Theorem 2 can be successively reduced to simpler propositions. Further comments concerning level sets of the Brownian sheet are given in Section 5.

2. The basic estimate. Recall that a *standard Brownian sheet* is a mean-zero continuous Gaussian process $W = (W(t),\ t \in \mathbb{R}_+^2)$, defined on some probability space $(\Omega, \mathscr{F}, P)$, with the covariance

$$E(W(s)W(t)) = \min(s_1, t_1)\min(s_2, t_2)$$

for all $s = (s_1, s_2)$ and $t = (t_1, t_2)$ in $\mathbb{R}_+^2$. It is well known [14] that the restriction of W to horizontal or vertical lines yields a Brownian motion. More precisely, $W(t_1, \cdot)$ [resp. $W(\cdot, t_2)$] is a Brownian motion with speed t_1 (resp. t_2). In this paper, we use the term *Brownian motion* to refer to any Brownian motion with speed between $\frac{1}{2}$ and 3. Recall also that *white noise* is the vector measure W defined on the bounded Borel sets of $\mathbb{R}_+^2$ with values in $L^2(\Omega, \mathscr{F}, P)$ such that $W([0, t_1] \times [0, t_2]) = W(t_1, t_2)$, for all $(t_1, t_2) \in \mathbb{R}_+^2$. A basic property of white noise is that $W(A)$ and $W(B)$ are independent if A and B are disjoint bounded Borel sets.

We now assemble some definitions. The argument of a vector in $\mathbb{R}^2$ is the angle the vector makes with the x-axis, measured in the counterclockwise direction. An interval of arguments will be an interval of $[0, 2\pi]$ with the endpoints 0 and 2π identified. Given $t_0 \in (0, \infty)^2$ and $I \subset [0, 2\pi]$, $C(t_0, I)$ denotes the set of points $t \in \mathbb{R}^2$ such that the argument of $t - t_0$ belongs to I. As usual, given $t \in \mathbb{R}^2$ and $\varepsilon > 0$, $B(t, \varepsilon)$ will denote the set $\{s \in \mathbb{R}^2 \colon |t - s| < \varepsilon\}$, while $\delta B(t, \varepsilon)$ is the set $\{s \colon |t - s| = \varepsilon\}$ ($|t|$ is the Euclidean norm of t). Lebesgue measure in $\mathbb{R}^2$ is denoted m.

Let $\mathscr{D}^n$ be the set of all squares with sides of length 2^{-n} and with vertices which have dyadic coordinates of order n. Most of this paper is devoted to proving the following proposition from which Theorem 2 follows easily.

PROPOSITION 1. *Let $n \in \mathbb{N}$ and, for $D \in \mathscr{D}^n$, $\theta \in [0, 2\pi]$, $\delta > 0$, $K > 0$ and $\varepsilon > 0$, let $F(\theta, \delta, K, \varepsilon, D)$ be the event* (*see Figure* 1)

> "*there exist $t \in D$ and a Jordan arc $\Gamma \subset C(t, (\theta - \delta, \theta + \delta))$ that connects t to $\delta B(t, \varepsilon)$ such that $|W(s) - W(t)| \le K|s - t|$ for all $s \in \Gamma$.*"

Given $\theta \in [0, 2\pi]$, there exists $\delta(\theta) > 0$ such that, *for all $K > 0$ and $\varepsilon > 0$*, *for all sufficiently large $n \in \mathbb{N}$ and $D \in \mathscr{D}^n$ with $D \subset [1, 2]^2$, $P(F(\theta, \delta(\theta), K, \varepsilon, D)) \le 2^{-3n}$.*

PROOF OF THEOREM 2. By scaling considerations, the square $[1, 2]^2$ in Proposition 1 can be replaced by any closed square in $(0, \infty)^2$. Let R be any such square with vertices with dyadic coordinates. We begin by deducing from Proposition 1 that

> for each $\theta \in [0, 2\pi]$, there is $\delta(\theta) > 0$ such that, for all $K > 0$ and $\varepsilon > 0$,

$$P(F(\theta, \delta(\theta), K, \varepsilon, R)) = 0. \tag{1}$$

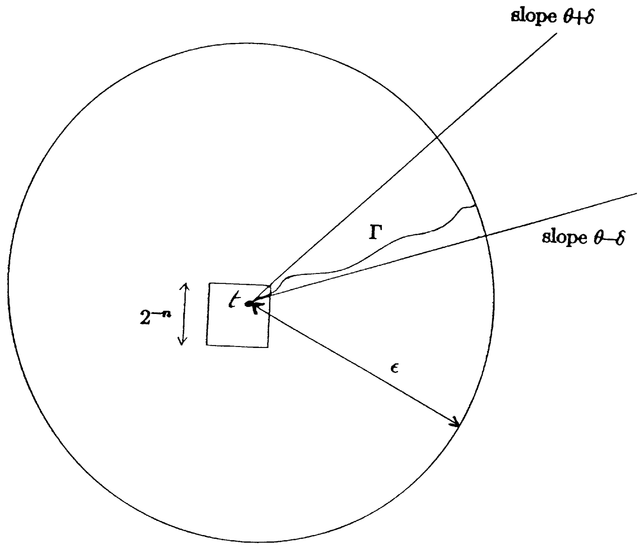

FIG. 1. *Illustration of the event* $F(\theta, \delta, K, \varepsilon, D)$.

Indeed, let $\delta(\theta)$ be as in Proposition 1. Since for sufficiently large n,

$$F(\theta, \delta(\theta), K, \varepsilon, R) \subset \bigcup_{D \in \mathscr{D}^n,\, D \subset R} F(\theta, \delta(\theta), K, \varepsilon, D)$$

and there are $m(R)2^{2n}$ elements of $\mathscr{D}^n$ contained in R, we conclude from Proposition 1 (with $[1,2]^2$ replaced by R) that, for all $K > 0$ and $\varepsilon > 0$,

$$P(F(\theta, \delta(\theta), K, \varepsilon, R)) \le m(R)\, 2^{2n}\, 2^{-3n} = m(R)\, 2^{-n}$$

for sufficiently large $n \in \mathbb{N}$. This proves (1).

We now prove the statement in Theorem 2. Notice that the event described in the statement of this theorem, with $(0,\infty)^2$ replaced by an arbitrary closed square $R \subset (0,\infty)^2$, is contained in the event

> "there exists an angle $\theta \in [0, 2\pi]$ such that, for all $\delta > 0$, there is $\varepsilon > 0$ and a Jordan arc $\Gamma \subset C(t, (\theta - \delta, \theta + \delta))$, with one extremity $t \in R$, the other ε units away, such that
>
> $$\sup_{s \in \Gamma} \frac{|W(s) - W(t)|}{|s - t|} < +\infty."$$

In other words, the event described in the statement of the theorem is contained in

$$\bigcup_{\theta \in [0,2\pi]} \bigcap_{\delta > 0} \bigcup_{\varepsilon > 0} \bigcup_{K > 0} F(\theta, \delta, K, \varepsilon, R). \tag{2}$$

Since $\{(\theta - \delta(\theta), \theta + \delta(\theta)),\ \theta \in [0, 2\pi]\}$ is an open cover of $[0, 2\pi]$, there is a finite subcover $\{(\theta_i - \delta(\theta_i), \theta_i + \delta(\theta_i)),\ i = 1, \ldots, n\}$. Observe that the event in (2) is contained in

$$\bigcup_{i=1}^{n} \bigcup_{\varepsilon>0} \bigcup_{K>0} F(\theta_i, \delta(\theta_i), K, \varepsilon, R).$$

For each i, the unions over ε and K only need to be taken over rational values. As each term of this union is a null set by (1), this proves Theorem 2 since $(0, \infty)^2$ is the union of an increasing sequence of closed squares with vertices with dyadic coordinates. $\square$

Using reflection about the line $t_2 = t_1$ and time inversion (which does not affect differentiability properties), it suffices to prove Proposition 1 for $\theta \in [0, \pi/4]$. Since for $a > 0$ the transformation $(t_1, t_2) \mapsto (t_1/\sqrt{a}, t_2\sqrt{a})$ preserves the Brownian sheet, transforms straight lines into straight lines and, in particular, lines with slope 1 into lines with slope a, it suffices to prove Proposition 1 for $\theta = 0$ and $\theta = \pi/4$. We begin with the latter case.

3. The case $\boldsymbol{\theta = \pi/4}$. Let $\mathscr{D}_0^n$ be the subset of those squares in $\mathscr{D}^n$ with all four vertices in $[1, 2]^2$. Given $n \in \mathbb{N}$ and $D \in \mathscr{D}_0^n$, let $t = (t_1, t_2)$ be such that $D = [t_1, t_1 + 2^{-n}] \times [t_2, t_2 + 2^{-n}]$. For $\alpha > 0$ and $x \in \mathbb{R}$, let

$$f_\alpha^D(x) = t_2 - 2^{-n}(1 - \alpha) + (1 - \alpha)(x - t_1),$$
$$g_\alpha^D(x) = t_2 + 2^{-n} + (1 + \alpha)(x - t_1).$$

Notice that $y = f_\alpha^D(x)$ [resp. $y = g_\alpha^D(x)$] is the equation of the straight line which passes through $(t_1 + 2^{-n}, t_2)$ [resp. $(t_1, t_2 + 2^{-n})$] with slope $1 - \alpha$ (resp. $1+\alpha$). Call these lines L_α^D and U_α^D, respectively. For $x \geq 0$, consider the vertical segment

$$V_\alpha^D(x) = \{t_1 + x\} \times [f_\alpha^D(t_1 + x), g_\alpha^D(t_1 + x)]$$

(see Figure 2).

For $K > 0$ and $\alpha > 0$, let $G(\alpha, K, D)$ be the event

"there exists a Jordan arc Γ with one extremity t_Γ on $V_\alpha^D(2^{-n})$, the other on $V_\alpha^D(2^{-n/10})$, which lies between L_α^D and U_α^D, such that

$$|W(s) - W(t_\Gamma)| \leq K|s - t_\Gamma| \quad \text{for all } s \in \Gamma." \tag{3}$$

Clearly, if $2^{-n/10} < \varepsilon$ and $|1 - \tan(\pi/4 \pm \delta)| < \alpha$, then

$$F(\pi/4, \delta, K, \varepsilon, D) \subset G(\alpha, K, D).$$

Therefore, Proposition 1 with $\theta = \pi/4$ is a consequence of the following.

PROPOSITION 2. *There exists $\alpha > 0$ such that for all $K > 0$, for all large $n \in \mathbb{N}$ and $D \in \mathscr{D}_0^n$, $P(G(\alpha, K, D)) \leq 2^{-3n}$.*

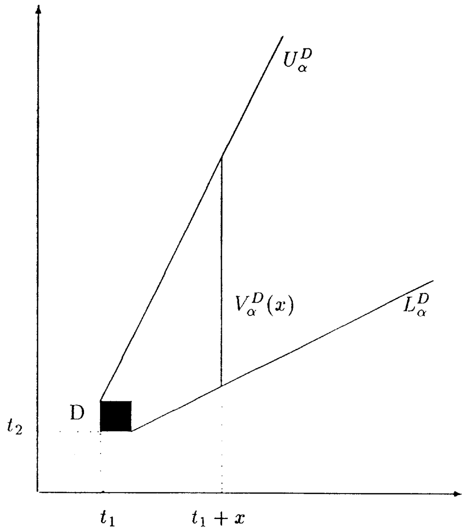

FIG. 2. *The lines L_α^D, U_α^D and $V_\alpha^D(x)$.*

For $n \in \mathbb{N}$, $D \in \mathscr{D}_0^n$ and t as above, let $x_i = 2^{3i-n}$ for $i \in \mathbb{N}$, and set

$$H_\alpha^D(i) = \{\inf |W(u) - W(s)| \leq 3K|x_{2i+1} - x_{2i}|\}, \tag{4}$$

where the infimum is taken over $s \in V_\alpha^D(x_{2i})$ and $u \in V_\alpha^D(x_{2i+1})$.

LEMMA 1. (a) *For all large n, for all $i \leq n/5$ and all $\alpha \in (0, \frac{1}{2}]$, $V_\alpha^D(x_i) \subset B(t, 2^{-n/10})$.*

(b) *For all large n and all $\alpha \in (0, \frac{1}{2}]$, $G(\alpha, K, D) \subset \bigcap_{0 \leq i \leq n/10} H_\alpha^D(i)$.*

(c) *For all $n \in \mathbb{N}$ and all $\alpha \in (0, \frac{1}{2}]$, the events $H_\alpha^D(i)$, $0 \leq i \leq n/10$, are independent.*

PROOF. (a) It suffices (see Figure 2) to show that the distance between the point $(t_1 + 2^{3n/5-n}, t_2 + g_{1/2}^D(t_1 + 2^{3n/5-n}))$ and (t_1, t_2) is less than or equal to $2^{-n/10}$. This elementary calculation is left to the reader.

(b) If $G(\alpha, K, D)$ occurs, then by (a) the Jordan arc whose existence is asserted must intersect each of the segments $V_\alpha^D(x_i)$, $0 \leq i \leq n/5$. The bound in (3), the equality $x_{2i} + x_{2i+1} = \frac{9}{7}(x_{2i+1} - x_{2i})$ and the triangle inequality now imply that $H_\alpha^D(i)$ occurs for $0 \leq i \leq n/10$.

(c) Because the events $H_\alpha^D(i)$ only involve increments of the Brownian sheet, independence will follow if these increments only depend on white noise in nonoverlapping regions. This is the case provided that, for $0 < \alpha \leq \frac{1}{2}$, large n and $0 \leq i \leq n/5$, $g_\alpha^D(t_1 + x_i) < f_\alpha^D(t_1 + x_{i+1})$. This elementary calculation is also left to the reader. □

LEMMA 2. $\lim_{\alpha \downarrow 0} \sup_{K>0} \lim_{n\to\infty} \sup_{D \in \mathscr{D}_0^n} \sup_{n/20 \leq i \leq n/10} P(H_\alpha^D(i)) = 0.$

PROOF. The main idea is to express the event $H_\alpha^D(i)$ as an event concerning two independent Brownian motions and an independent Gaussian random variable, which is (nearly) subject to a scaling argument.

Notice that the length of the segment $V_\alpha^D(x)$ is $l(n,\alpha,x) = 2^{-n}(2-\alpha)+2\alpha x$. Let $u_\alpha^D(i)$ and $v_\alpha^D(i)$ be, respectively, the lower and upper endpoints of $V_\alpha^D(x_i)$, and set $Z_i = W(v_\alpha^D(2i)) - W(u_\alpha^D(2i+1))$,

$$W_1^i(u) = W(u) - W(u_\alpha^D(2i+1)) \quad \text{and} \quad W_2^i(s) = W(v_\alpha^D(2i)) - W(s).$$

Then $W(u)-W(s) = W_1^i(u) - Z_i + W_2^i(s)$. Moreover, if $\alpha \in (0, \frac{1}{2}]$ and n is large, the observation made in the proof of Lemma 1(c) implies that, for any $D \in \mathscr{D}_0^n$, the vector

$$\big((W_1^i(u),\ u \in V_\alpha^D(x_{2i+1})),\ Z_i,\ (W_2^i(s),\ s \in V_\alpha^D(x_{2i}))\big)$$

has the same law as

$$\big(l(n,\alpha,x_{2i+1})^{1/2} B_1,\ E(Z_i^2)^{1/2} Z,\ l(n,\alpha,x_{2i})^{1/2} B_2\big),$$

where $B_1 = (B_1(x),\ 0 \le x \le 1)$ and $B_2 = (B_2(z),\ 0 \le z \le 1)$ are Brownian motions, Z is an $N(0,1)$ random variable and B_1, B_2 and Z are independent. Since $x_{2i+1} - x_{2i} = 7x_{2i}$, it follows that $P(H_\alpha^D(i))$ is equal to

$$P\Big\{\inf_{0\le x\le 1,\, 0\le z\le 1} |l(n,\alpha,x_{2i+1})^{1/2} B_1(x) + l(n,\alpha,x_{2i})^{1/2} B_2(z) - E(Z_i^2)^{1/2} Z| \le 21 K x_{2i}\Big\}.$$

The event on the right-hand side occurs if $E(Z_i^2)^{1/2} Z$ is within $21Kx_{2i}$ of the range of the map $(x,z) \mapsto l(n,\alpha,x_{2i+1})^{1/2} B_1(x) + l(n,\alpha,x_{2i})^{1/2} B_2(z)$, which is an interval containing $0 = E(Z)$, and therefore the probability of this event is a nondecreasing function of the coefficient of $B_j(\cdot)$, $j=1,2$, and a nonincreasing function of the coefficient of Z. Since $D \subset [1,2]^2$, for $i \le n/10$,

$$l(n,\alpha,x_{2i+1}) = (16\alpha + (2-\alpha)2^{-6i})2^{6i-n} \quad \text{and} \quad E(Z_i^2) \ge x_{2i+1} - x_{2i} \ge 2^{6i-n},$$

and therefore, for any $K > 0$, $P(H_\alpha^D(i))$ is bounded above by

$$P\Big\{\inf_{0\le x\le 1,\, 0\le z\le 1} |(16\alpha + (2-\alpha)2^{-6i})^{1/2}\, 2^{3i-n/2}(B_1(x)+B_2(z)) - 2^{3i-n/2} Z| \le 21K2^{6i-n}\Big\}.$$

Dividing both sides of this inequality by $2^{3i-n/2}$, we see that, for $n/20 \le i \le n/10$, the probability above is bounded by

$$P\Big\{\inf_{0\le x\le 1,\, 0\le z\le 1} |(16\alpha + (2-\alpha)2^{-6n/20})^{1/2}(B_1(x)+B_2(z)) - Z| \le 21K2^{-n/5}\Big\}.$$

For $\alpha > 0$ and $K > 0$, this last probability, which does not depend on D or i, converges as $n \to \infty$ to

$$P\Big\{\inf_{0\le x\le 1,\, 0\le z\le 1} |4\sqrt{\alpha}\,(B_1(x)+B_2(z)) - Z| \le 0\Big\},$$

which does not depend on K. As $\alpha \downarrow 0$, this probability converges to $P\{|Z| \le 0\} = 0$. This completes the proof. □

PROOF OF PROPOSITION 2. Fix $\varepsilon > 0$ such that $\varepsilon^{1/20} \le 2^{-3}$. By Lemma 2, there is $\alpha > 0$ such that, for all $K > 0$, for all sufficiently large n, $D \in \mathscr{D}_0^n$ and all $i \in [n/20, n/10]$, $P(H_\alpha^D(i)) \le \varepsilon$. By Lemma 1(b) and (c), for this α, for all $K > 0$, for all sufficiently large n and $D \in \mathscr{D}_0^n$,

$$P(G(\alpha, K, D)) \le \prod_{n/20 \le i \le n/10} P(H_\alpha^D(i)) \le \varepsilon^{n/20} \le 2^{-3n}.$$

This completes the proof of Proposition 2, and therefore of Proposition 1 in the case $\theta = \pi/4$. □

4. The case $\theta = 0$. It remains to prove Proposition 1 in the case $\theta = 0$. Given $n \in \mathbb{N}$ and $D \in \mathscr{D}_0^n$, let $t = (t_1, t_2)$ be such that $D = [t_1, t_1 + 2^{-n}] \times [t_2, t_2 + 2^{-n}]$. For $\alpha > 0$, let L_α^D [resp. U_α^D] be the line with slope α [resp. $-\alpha$] which passes through t [resp. $(t_1, t_2 + 2^{-n})$]. For $x > 0$, let $V_\alpha^D(x)$ be the vertical segment of $\{t_1 + x\} \times \mathbb{R}_+$ with extremities on L_α^D and U_α^D. With these new definitions of the lines L_α^D, U_α^D and $V_\alpha^D(x)$ (see Figure 3), for $K > 0$ and $\alpha > 0$, let $G_0(\alpha, K, D)$ be the event

> "there exists a Jordan arc Γ with one extremity t_Γ on $V_\alpha^D(2^{-n})$, the other on $V_\alpha^D(2^{-n/10})$, which lies between L_α^D and U_α^D, such that $|W(s) - W(t_\Gamma)| \le K|s - t_\Gamma|$ for all $s \in \Gamma$."

Clearly, if $2^{-n/10} < \varepsilon$ and $|\tan \delta| < \alpha$, then $F(0, \delta, K, \varepsilon, D) \subset G_0(\alpha, K, D)$. Therefore, Proposition 1 with $\theta = 0$ is a consequence of the following.

PROPOSITION 3. *There exists $\alpha > 0$ such that, for all $K > 0$, for all large $n \in \mathbb{N}$ and $D \in \mathscr{D}_0^n$, $P(G_0(\alpha, K, D)) \le 2^{-3n}$.*

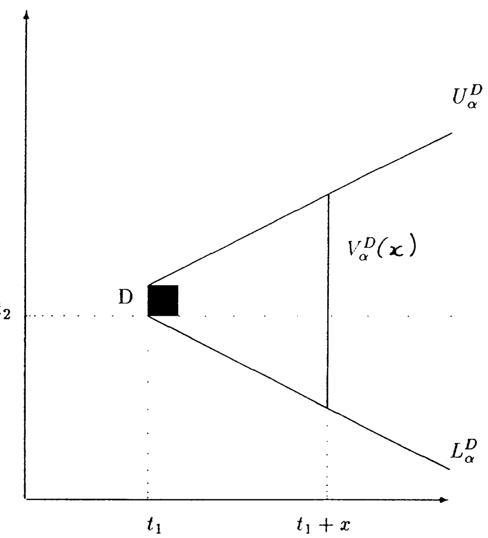

FIG. 3. *The new definitions of L_α^D, U_α^D and $V_\alpha^D(x)$.*

In the previous section, where θ was equal to $\pi/4$, it sufficed to define x_i in such a way that x_i and x_{i+1} were far enough apart, for the events $H_\alpha^D(i)$ to be independent [see Lemma 1(c)]. This was used in a crucial way in the proof of Proposition 2 to get the bound 2^{-3n}. Here, independence of these events cannot be obtained, so we define x_i differently.

For $n \in \mathbb{N}$, $D \in \mathscr{D}_0^n$ and t as above, for $i \in \mathbb{N}$, let $x_i = 2^{i-n}$ and define $H_\alpha^D(i)$ in a manner analogous to (4):

$$H_\alpha^D(i) = \{\inf |W(u) - W(s)| \le 6K|x_{i+1} - x_i|\}, \tag{5}$$

where the infimum is taken over $s \in V_\alpha^D(x_i)$ and $u \in V_\alpha^D(x_{i+1})$.

As in Lemma 1(a) and (b), but using the equality $x_{i+1}+x_i = 3(x_{i+1}-x_i)$, one checks that, for all large n and all $\alpha \in (0, \frac{1}{2}]$, $G_0(\alpha, K, D) \subset \bigcap_{0\le i\le 2n/3} H_\alpha^D(i)$. However, observe that the events $H_\alpha^D(i)$ are *not* independent. This is the main difference between the case $\theta = 0$ discussed here and the case $\theta = \pi/4$ examined in the previous section.

In order to compensate for the lack of independence, we make the following observation: $H_\alpha^D(i)$ can occur either because

(a) the ranges of W on $V_\alpha^D(x_i)$ and $V_\alpha^D(x_{i+1})$ are large,

or because

(b) the increment $W(t_1 + x_{i+1}, t_2) - W(t_1 + x_i, t_2)$ is small.

Since the Brownian sheet has independent increments, the events in (b) are independent, even though those in (a) are not. The strategy will be to show that, with high probability, sufficiently many events of type (b) occur for the conclusion of Proposition 3 to hold.

For $x \ge 0$ and $s \in V_\alpha^D(x)$, set

$$X_x^D(s) = W(s) - W(t_1 + x, t_2),$$

and for α, $\lambda > 0$, consider the event

$$A_{\alpha,\lambda}^D(x) = \Big\{ \sup_{s\in V_\alpha^D(x)} |X_x^D(s)| \ge \lambda\sqrt{x} \Big\}.$$

LEMMA 3. *Let $N_{\alpha,\lambda}^D$ be the number of integers $i \in [n/3, 2n/3]$ for which the event $A_{\alpha,\lambda}^D(x_i)$ occurs. For all $\lambda > 0$, there is $\alpha > 0$ such that, for all large n and $D \in \mathscr{D}_0^n$, $P\{N_{\alpha,\lambda}^D \ge n/12\} \le 2^{-4n}$.*

Using this lemma, we now prove Proposition 3.

PROOF OF PROPOSITION 3. Fix $0 < \lambda < 2^{-28}$ and, by Lemma 3, let $\alpha > 0$ be such that, for all large n and $D \in \mathscr{D}_0^n$, $P\{N_{\alpha,\lambda}^D \ge n/12\} \le 2^{-4n}$. Notice that

$$G_0(\alpha, K, D) \subset \{N_{\alpha,\lambda}^D \ge n/12\} \cup \left(\bigcap_{n/3\le i\le 2n/3} H_\alpha^D(i) \cap \{N_{\alpha,\lambda}^D < n/12\} \right). \tag{6}$$

If $H^D_\alpha(i)$ occurs but neither $A^D_{\alpha,\lambda}(x_i)$ nor $A^D_{\alpha,\lambda}(x_{i+1})$ do, then, by the definition of $H^D_\alpha(i)$,

$$|W(t_1+x_{i+1},t_2)-W(t_1+x_i,t_2)|\le 6K(x_{i+1}-x_i)+\lambda\sqrt{x_i}+\lambda\sqrt{x_{i+1}},$$

and, in particular, $W(t_1+x_{i+1},t_2)-W(t_1+x_i,t_2)$, or, equivalently, $W(t_1+x_{i+1},\frac{1}{2})-W(t_1+x_i,\frac{1}{2})$, is contained in an interval of length no greater than

$$12K(x_{i+1}-x_i)+2\lambda\sqrt{x_i}+2\lambda\sqrt{x_{i+1}}\le 12Kx_i+6\lambda\sqrt{x_i}.$$

This last random variable is independent of $N^D_{\alpha,\lambda}$, $A^D_{\alpha,\lambda}(x_i)$ and $H^D_\alpha(i)$ for $D\in\mathscr{D}^n_0$, and the probability that it is contained in an interval of the specified length is less than or equal to $24K\sqrt{x_i}+12\lambda$. Given $K>0$, for all sufficiently large n and $i\in[n/3,2n/3]$, this probability is less than or equal to $16\lambda\le 2^{-24}$.

By definition, on $\{N^D_{\alpha,\lambda}<n/12\}$, there are at least $n/3-2n/12=n/6$ distinct integer values of $i\in[n/3,2n/3]$ for which neither $A^D_{\alpha,\lambda}(x_i)$ nor $A^D_{\alpha,\lambda}(x_{i+1})$ occurs. From (6), we conclude that

$$P(G_0(\alpha,K,D))\le 2^{-4n}+(2^{-24})^{n/6}<2^{-3n}.$$

This proves Proposition 3, and therefore Proposition 1 in the case $\theta=0$. □

It remains to prove Lemma 3. For $0\le z\le\xi$ and $s=(s_1,s_2)\in V^D_\alpha(\xi)$, set

$$Y^D_{z,\xi}(s)=\begin{cases} X^D_\xi(s)-X^D_z(t_1+z,s_2), & \text{if } (t_1+z,s_2)\in V^D_\alpha(z),\\ X^D_\xi(s)-X^D_z(t_1+z,t_2+2^{-n}+\alpha z), & \text{if } s_2\ge 2^{-n}+\alpha z,\\ X^D_\xi(s)-X^D_z(t_1+z,t_2-\alpha z), & \text{if } s_2\le -\alpha z.\end{cases}$$

The quantities on the right-hand side represent the white noise measure of the three sets shown in Figure 4.

LEMMA 4. *For $D\in\mathscr{D}^n_0$, $x>0$ and $\lambda>0$, let*

$$G^D_x(\lambda)=\left\{\sup_{0\le z\le\xi\le x}\ \sup_{s\in V^D_\alpha(\xi)}|Y^D_{z,\xi}(s)|\ge\frac{\lambda}{4}\sqrt{x}\right\}.$$

Then $\lim_{\alpha\downarrow 0}\lim_{n\to\infty}\sup_{D\in\mathscr{D}^n_0}\sup_{2^{-2n/3}\le x\le 2^{-n/3}}P(G^D_x(\lambda))=0.$

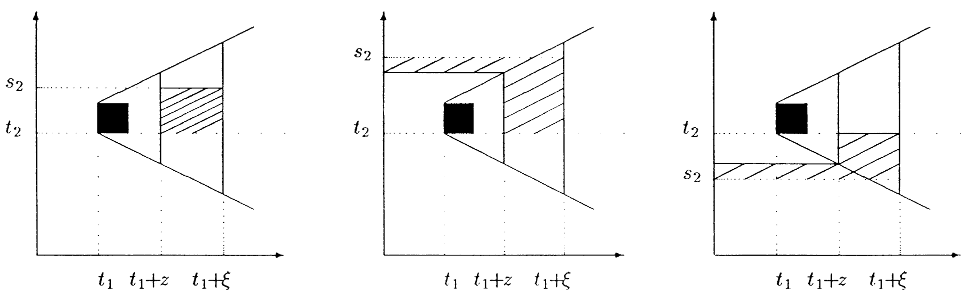

FIG. 4. *The sets used in the definition of $Y^D_{z,\xi}(s)$.*

PROOF. This proof uses a scaling argument as in the proof of Lemma 2. Assume $D = [t_1, t_1 + 2^{-n}] \times [t_2, t_2 + 2^{-n}]$ and let

$$J_\alpha^D(x) = \Big\{ \sup_{0 \le z \le x} \sup_{s \in V_\alpha^D(z)} |X_z^D(s)| \ge 2^{-4}\lambda\sqrt{x} \Big\}.$$

From the definition of $Y_{z,\xi}^D(s)$, observe that $P(G_\lambda^D(x)) \le 4P(J_\alpha^D(x))$. Let $B^D(y) = W(t_1, t_2 + y) - W(t_1, t_2)$. Then $(B^D(y),\ y \ge 0)$ is a Brownian motion (with speed t_1), and, for $(t_1 + z, t_2 + y) \in V_\alpha^D(z)$, $X_z^D(t_1 + z, t_2 + y) = B^D(y) + \varepsilon^D(z, y)$, where $(\varepsilon^D(z, y),\ (z, y) \in \mathbb{R}_+^2)$ is a standard Brownian sheet. It follows that $P(J_\alpha^D(x))$ is bounded by twice

$$P\Big\{ \sup_{0 \le y \le 2^{-n} + \alpha x} |B^D(y)| \ge 2^{-5}\lambda\sqrt{x} \Big\} + P\Big\{ \sup_{0 \le z \le x} \sup_{0 \le y \le 2^{-n} + \alpha z} |\varepsilon^D(z, y)| \ge 2^{-5}\lambda\sqrt{x} \Big\}.$$

By the scaling properties of Brownian motion and of the Brownian sheet, this sum is bounded by

$$P\Big\{ \sup_{0 \le y \le 1} |B(y)| \ge 2^{-6}\lambda(\alpha + 2^{-n}/x)^{-1/2} \Big\} + P\Big\{ \sup_{0 \le z \le 1} \sup_{0 \le y \le 2^{-n} + \alpha x z} |W(z, y)| \ge 2^{-5}\lambda \Big\},$$

where B is a standard Brownian motion and W is a standard Brownian sheet. For $2^{-2n/3} \le x \le 2^{-n/3}$, we can bound this sum by

$$P\Big\{ \sup_{0 \le y \le 1} |B(y)| \ge 2^{-6}\lambda(\alpha + 2^{-n/3})^{-1/2} \Big\} + P\Big\{ \sup_{0 \le z \le 1} \sup_{0 \le y \le 2^{-n} + \alpha 2^{-n/3} z} |W(z, y)| \ge 2^{-5}\lambda \Big\}.$$

This expression no longer depends on x or D. As $n \to \infty$, it converges to

$$P\Big\{ \sup_{0 \le y \le 1} |B(y)| \ge 2^{-6}\lambda\alpha^{-1/2} \Big\}.$$

As $\alpha \downarrow 0$, this probability converges to 0. □

PROOF OF LEMMA 3. Let $p > 0$ be such that, for all large n, $P\{B(n/3, p) \ge n/12\} < 2^{-4n-1}$. Here (and subsequently), $B(m, p)$ denotes a binomial random variable with parameters m and p. The existence of such a p follows from elementary bounds on binomial tails (see, e.g., Cramér's theorem and Exercise 2.2.23 in [7]).

Fix $\lambda > 0$. By Lemma 4, let $\alpha > 0$ be such that, for all large n, $D \in \mathscr{D}_0^n$ and $2^{-2n/3} \le x \le 2^{-n/3}$,

$$P(G_x^D(\lambda)) < p. \tag{7}$$

For large n and $D \in \mathscr{D}_0^n$, assume, as before, that $D = [t_1, t_1+2^{-n}] \times [t_2, t_2+2^{-n}]$ and consider the event

$$B_i = \left\{ \sup_{0 \le x \le x_i} \sup_{s \in V_\alpha^D(x)} |X_x^D(s)| \ge \lambda\sqrt{x_i} \right\}.$$

Observe that $A_{\alpha,\lambda}^D(x_i) \subset B_i$, and therefore it suffices to prove that $P\{N \ge n/12\} \le 2^{-4n}$, where N is the number of integers $i \in [n/3, 2n/3]$ for which B_i occurs.

For $x \ge 0$, consider the σ-field

$$\mathscr{G}_x = \sigma\{W(u_1, u_2)\colon\ t_1 \le u_1 \le t_1 + x,\ t_2 - \alpha x \le u_2 \le t_2 + 2^{-n} + \alpha x\}.$$

Then $(\mathscr{G}_x,\ x \ge 0)$ is a filtration and, for $i \in [n/3, 2n/3]$, the random variable

$$\chi_i = \inf\left\{ x \ge 0 : \sup_{s \in V_\alpha^D(x)} |X_x^D(s)| \ge \lambda\sqrt{x_i} \right\}$$

is a stopping time relative to this filtration. Let $\eta_i = x_i \wedge \chi_i$ and note that $\eta_i \le \eta_{i+1}$. Set

$$H = \left\{ \sup_{s \in V_\alpha^D(0)} |X_0^D(s)| < \lambda 2^{-n/3} \right\}$$

and observe that $H \cap B_i \in \mathscr{G}_{\eta_i}$, and, on H, $\sup_{s \in V_\alpha^D(\eta_i)} |X_{\eta_i}^D(s)| \le \lambda\sqrt{x_i}$ by continuity. Moreover,

$$B_{i+1} \subset \left\{ \sup_{\eta_i \le \xi \le x_{i+1}} \sup_{s \in V_\alpha^D(\xi)} |Y_{\eta_i,\xi}^D(s)| \ge \lambda\sqrt{x_{i+1}} - \lambda\sqrt{x_i} \right\}.$$

Since $\lambda\sqrt{x_{i+1}} - \lambda\sqrt{x_i} \ge (\lambda/4)\sqrt{x_{i+1}}$ and the event on the right-hand side is independent of $\mathscr{G}_{\eta_i}$, we can bound $P(B_{i+1} \mid \mathscr{G}_{\eta_i})$ by the unconditional probability $P(G_{x_{i+1}}^D(\lambda))$, which, by (7), is less than p for $i \in [n/3, 2n/3]$. Therefore, for large n,

$$\begin{aligned} P\{N \ge n/12\} &\le P(H^c) + P\{B(n/3, p) \ge n/12\} \\ &\le e^{-2^{n/6}/2} + 2^{-4n-1} \le 2^{-4n}. \end{aligned}$$

This proves Lemma 3. □

5. Consequences for level sets. The (uniform) Hausdorff dimension (equal to 3/2) of level sets of the standard Brownian sheet was obtained by Adler [1]. Kendall [9] showed that almost all points on the level set at level x, denoted $L(x)$, of the Brownian sheet are points of disconnection; that is, for almost all $t \in L(x)$, the connected component of $L(x)$ which contains t is equal to $\{t\}$. Here, "almost all" refers to the measure on $L(x)$ induced by local time. As mentioned in the Introduction, further geometric and topological properties of the level sets and of Brownian bubbles were obtained in [5], [6] and [11].

The observation by Kendall [9] that $L(x)$ must contain nontrivial connected components motivates attempts for understanding the nature of these components. According to Theorem 1, if such a component is a Jordan arc, then it must be nowhere differentiable. However, the proof of Theorem 2, and therefore the proof of Theorem 1, only makes use of certain of the properties of Jordan arcs and, in particular, of the property of being "connected," rather than "arc-connected," and in fact, the same proof yields a slightly stronger statement concerning level sets.

If Γ is a connected subset of $\mathbb{R}^2$, we say that Γ has a *tangent* at $t \in \Gamma$ in direction $\theta \in [0, 2\pi]$ if $\Gamma \setminus \{t\} \neq \varnothing$ and if, for all $\varepsilon > 0$, there is $\delta > 0$ such that $\Gamma \cap B(t, \delta) \subset C(t, (\theta - \varepsilon, \theta + \varepsilon))$.

THEOREM 3. *The event*

> *"there exist $t \in (0, \infty)^2$ and a connected set $\Gamma \subset \mathbb{R}_+^2$ with a tangent at t such that Γ is contained in the level set $L(W(t))$"*

has probability 0.

PROOF. The proof proceeds along the same lines as the proof of Theorem 2. A key element of this proof is the observation that $G(\alpha, K, D) \subset H_\alpha^D(i)$ for $0 \leq i \leq n/10$. This property remains valid even if Γ is connected, rather than arc-connected. Similarly, the property $G_0(\alpha, K, D) \subset H_\alpha^D(i)$ for $0 \leq i \leq 2n/3$ also remains valid. The remaining estimates still hold. $\square$

This result tells us that nontrivial connected components of level sets of the Brownian sheet must be extremely irregular. The question of existence of any Jordan arc in a level set remains open. A partial result in this direction, which indicates that such curves may exist, can be found in [4].

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