The Annals of Probability
2002, Vol. 30, No. 3, 1266–1312

ON SOME RANDOM WALKS ON $\mathbb{Z}$ IN RANDOM MEDIUM

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We consider random walks on $\mathbb{Z}$ in a stationary random medium, defined by an ergodic dynamical system, in the case when the possible jumps are $\{-L, \dots, -1, 0, +1\}$ for some fixed integer L. We provide a recurrence criterion expressed in terms of the sign of the maximal Liapounov exponent of a certain random matrix and give an algorithm of calculation of that exponent. Next, we characterize the existence of the absolutely continuous invariant measure for the Markov chain of "the environments viewed from the particle" and also characterize, in the transient cases, the existence of a nonzero drift. To study the validity of the central limit theorem, we consider the notion of harmonic coordinates introduced by Kozlov. We characterize the existence of both the invariant measure and the harmonic coordinates and show in the recurrent case that the existence of those two objects is equivalent to the validity of an invariance principle. We give sufficient conditions for the validity of the central limit theorem in the transient cases. Finally, we consider the previous results in the context of a random medium defined by an irrational rotation on the circle and study their realization in terms of regularity and Diophantine approximation.

1. Introduction. One-dimensional random walks in a random environment have appeared in many fields of probability theory and theoretical physics, for instance, in crystallography. We refer to [4] for an introduction. Such random walks can be defined as a class of Markov chains on $\mathbb{Z}$ whose transition laws are statistically homogeneous. For clarity of the exposition, we present a general model with bounded steps, whereas in this paper we will only consider a particular submodel. The homogeneity property can be expressed as follows.

1.1. *The model.* Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible dynamical system, that is, a probability space $(\Omega, \mathcal{F}, \mu)$ with an invertible transformation T, measurable as well as its inverse and preserving μ. We suppose that T is ergodic with respect to μ. The space Ω is interpreted as the space of environments.

Let $L \geq 1$ and $R \geq 1$ be two fixed integers and introduce the set of consecutive integers $\Lambda = \{-L, \dots, R\}$ which will be the set of possible jumps of the random walks. Let $(p_z)_{z\in\Lambda}$ be a collection of positive random variables on $(\Omega, \mathcal{F})$ satisfying $\sum_{z\in\Lambda} p_z(\omega) = 1$, μ-a.e., and an ellipticity condition:

$$\exists\, \varepsilon > 0,\ \forall\, z \in \Lambda \text{ and } z \neq 0, \qquad (p_z/p_R) \geq \varepsilon,\ \mu\text{-a.e.}$$

Received July 2000; revised November 2001.
AMS 2000 subject classifications. 60J10, 60K37.
Key words and phrases. Random walk in a random environment, Markov chain, positive random matrices, Liapounov exponents, invariant measure, harmonic coordinates, central limit theorem.

For a fixed medium ω, let $(\xi_n(\omega))_{n\geq 0}$ be the Markov chain on $\mathbb{Z}$ defined by $\xi_0(\omega) = 0$ and the transition probabilities

$$(1) \qquad \forall\, x \in \mathbb{Z}, \forall\, z \in \Lambda, \qquad \mathcal{P}_\omega\big(\xi_{n+1}(\omega) = x + z | \xi_n(\omega) = x\big) := p_z(T^x\omega).$$

As introduced in (1), let $\mathcal{P}_\omega$ be the measure induced by $(\xi_n(\omega))_{n\geq 0}$ on the space of jumps $\Lambda^{\mathbb{N}}$. We aim at studying the behavior of that random walk with probability 1 with respect to $\mathcal{P}_\omega$ for a given ω, μ-a.e. This point of view is called the "quenched," in contrast to the "annealed," point of view which consists of expressing results under the probability $\int_\Omega \mathcal{P}_\omega \, d\mu(\omega)$. We will use the latter only when considering the validity of the central limit theorem.

We present a few conventions. In the rest of the paper the dependence on ω will always be implicit. Any expression of the form $f(T^k\omega)$ will simply be denoted by $T^k f$ or $f(k)$. For the random walk $(\xi_n(\omega))$, we write P_k for the probability starting at k, $k \in \mathbb{Z}$, and E_k for the corresponding expectation.

1.2. *General overview and content of the article.*

Recurrence properties. In the independent case and with $L = R = 1$, Solomon [28] showed that the random walk is recurrent or transient whether $\int \log(p_{-1}/p_1)\, d\mu$ is equal to 0 or not. This result can be extended to any ergodic environment; for example, see Alili [1]. For the general model with any fixed L and R but in the independent case, Key [18] proved a criterion expressed in terms of the sign of two intermediate Liapounov exponents of a random matrix built with the $(p_z)_{z\in\Lambda}$. After a slight change in the proof of Theorem (17), page 539, of [18] using conditional expectation, Key's proof can also be extended to any ergodic environment. The form of Key's theorem can be simplified (see [22] for instance) and one then obtains a criterion involving only one Liapounov exponent of a matrix that has one dimension less in comparison with Key's. This new matrix will be written M in this paper. We will consider here the case when $R = 1$ and $L \geq 1$ is fixed. Whereas the understanding of the general model requires a deep comprehension of Oseledet's vectors of M with respect to T (see [22] for the general model and [7] for $L = R = 2$), when $R = 1$ this is not completely necessary, simply due to the fact that in this situation M has nonnegative entries. Such matrices have in general directional contraction properties. We refer to [15] for a detailed explanation and references. Using this property of the matrix M, we will give a rather simple proof of a recurrence criterion involving its maximal Liapounov exponent $\gamma(M, T)$ with respect to T. Similarly, we will deduce an algorithm of calculation of $\gamma(M, T)$, giving theoretically an easier access to it. This generalizes results of Letchikov [21] and Derriennic [12] in the ergodic context with $(L, R) = (2, 1)$, where $\gamma(M, T)$ can be shown to be equal to $\int \log \varphi \, d\mu$ with some random continued fraction φ.

Further behavior. Random walks on $\mathbb{Z}$ in a random environment are known to present highly diversified behaviors. The case $R = L = 1$ with an independent environment has been intensely studied. Solomon [28] has given a criterion under which $\xi_n(\omega)/n \to 0$. It is remarkable that this condition may be fulfilled whereas the random walk is transient. In this situation, Kesten, Kozlov and Spitzer [17] have shown that the behavior is nonclassical, more precisely, that the law of $\xi_n(\omega)/n^t$ for some $t \in (0, 1)$ and under the annealed probability converges to a nondegenerated distribution. In the recurrent case, still in the independent context and with $L = R = 1$, the typical growth is even much slower. Sinaï [27] has shown that $|\xi_n(\omega)/\log^2 n - m_n(\omega)|$ converges to 0 under the annealed probability, where $m_n(\omega)$ is some random point depending on the environment and whose law, up to a normalizing constant, converges to the law of some functional of the standard Brownian motion. The validity of this theorem relies only on the existence of an invariance principle for the ergodic sums associated to $\log(p_{-1}/p_1)$ and the proof reveals that the behavior of the random walk is intimately governed by the pointwise behavior of these sums. A study in this direction has been developed in [16], where it is shown, for example, that $\limsup \xi_n(\omega)/(\log^2 n \log_3 n)$ is equal to a finite constant > 0, a.e. We will deal with Sinaï's theorem in a forthcoming work in the context of Gibbs measures (see [6] and a remark in Section 4). Note also that large deviations techniques have been developed by Greven and Hollander [14] and that a survey on the question can be found in [13].

On the law of large numbers and the central limit theorem. Besides this, one can find conditions under which a classical behavior is valid. In the case $L = R = 1$, two different approaches have been considered to get a law of large numbers and both will be presented in Section 3. The first one is based on "hitting times;" see [1, 22] and the Saint-Flour course of Zeitouni [29]. The other one, which fully exploits the stationarity of the transition laws, relies on the study of the Markov chain of the "environments viewed from the particle." It was introduced first by Kozlov [19]. The law of large numbers is then a consequence of the existence of a probability measure on Ω equivalent to μ and invariant by the corresponding transition operator. As such a measure is unique when it exists, we will refer to it as the "invariant measure" without mentioning the fact that it is abolutely continuous. In the ergodic context and with $L = R = 1$, the existence of such an invariant measure has been characterized by Conze and Guivarc'h [11]. We will extend their result to the case $R = 1$ and $L \geq 1$. We will then generalize the approach relying on hitting times in order to give convenient formulas for the drift and to characterize the existence of a nonzero drift in the transient cases.

Going further in the study, we will consider the central limit theorem. An "annealed" CLT in the transient cases and for $L = R = 1$ can be found in [17] in the independent context and in [29] for any ergodic environment. For less stochastic media, such as an irrational rotation on the circle, a method is developed in [1] which also produces a quenched CLT. In the recurrent case with $R = L = 1$ and for an ergodic environment, we find a quenched CLT in [24]. To abort this question, we present here the notion of harmonic coordinates introduced by Kozlov [19] and which already requires the existence of the invariant measure. This corresponds to the existence of a decomposition of the function $(\omega, z) \mapsto z$ according to certain subspaces of $L^2(\Omega \times \Lambda)$ and this allows us to introduce an appropriate martingale. The other methods developed, for example, in [1] and in [29], are rather similar. However, we focus here on harmonic coordinates and we will characterize their existence. The reason is that in the ergodic context with $L = R = 1$, in the recurrent case and under the hypothesis that $\log(p_{-1}/p_1)$ is an additive coboundary, Letchikov [23] has shown that the existence of harmonic coordinates and the invariant measure is equivalent to the validity of a functional quenched CLT. We will extend this result to the case $R = 1$ and $L \geq 1$ and suppress the coboundary hypothesis. In the transient cases, we will give sufficient conditions for the validity of the quenched CLT, improving the results of [1], and for the validity of the annealed CLT, following [29]. We will extend the results of Zeitouni, himself inspired by Kozlov [19]. We will then study the example of a random medium defined by an irrational rotation on the circle when the transition laws are regular. We will extend the results of [1] and show that under Diophantine conditions on the rotation angle and regularity assumptions on the p_i's, a law of large numbers and a quenched CLT occur.

We now assume for the whole paper that $R = 1$ and that $L \geq 1$ is some fixed integer. In this situation the ellipticity condition introduced in the presentation of the model can be written as

$$\exists\, \varepsilon > 0, \forall\, z \in \Lambda \text{ and } z \neq 0, \qquad (p_z/p_1) \geq \varepsilon,\ \mu\text{-a.e.} \tag{2}$$

1.3. *Contraction property of nonnegative matrices.* This section relates known results on the directional contraction properties in the positive cone of $\mathbb{R}^L$ for matrices with nonnegative entries. For a complete and detailed survey and for a proof of the two following propositions, we refer the reader to [15]. Let S be the set of invertible matrices of size $L \times L$ with entries greater than or equal to 0 and let $\tilde{S}$ be the subset of those matrices with entries greater than 0. Consider $\mathbb{R}^L$ with its canonical basis and the 1-norm, that is, $\|x\| = \sum_{i=1}^{L} |x_i|$, for $x = (x_i)_{1 \leq i \leq n}$. Introduce the cone $C = \{x \in \mathbb{R}^L, x_i > 0\}$ and its intersection with the sphere: $B = C \cap \{x \in \mathbb{R}^L, \|x\| = 1\}$. We write $\bar{B}$ and $\bar{C}$ for the respective closure of these sets. We will define a distance on $\bar{B}$ for which the elements of S act on $\bar{B}$ as a contraction. For any $(x, y) \in \bar{B} \times \bar{B}$, set first

$$m(x, y) = \sup\{s \in \mathbb{R}_+ \mid s y_i \leq x_i,\ 1 \leq i \leq L\}.$$

Introducing the function $\varphi(t) := (1 - t)/(1 + t)$ which is a bijection of $[0, 1]$, we finally define

$$d(x, y) = \varphi\big(m(x, y)m(y, x)\big).$$

A matrix $M \in S$ induces a transformation of $\bar{B}$. The image of $x \in \bar{B}$ is written as $M.x$ and is defined by $M.x = Mx/\|Mx\|$. We then have the following result.

PROPOSITION 1.1. (i) *The map* $d : \bar{B} \times \bar{B} \mapsto [0, 1]$ *defined above is a distance on* $\bar{B}$ *which satisfies*

$$\forall\, (x, y) \in \bar{B}, \qquad \|x - y\| \le 2d(x, y).$$

(ii) *For every* $M = (M_{i,j}) \in S$, *there exists a constant* $c(M) \le 1$ *such that*

$$\forall\, x, y \in \bar{B}, \qquad d(M.x, M.y) \le c(M) d(x, y).$$

(iii) *The value of* $c(M)$ *is*

$$c(M) = \max_{i,j,k,l=1,\dots,L} \frac{|M_{ki} M_{lj} - M_{kj} M_{li}|}{M_{ki} M_{lj} + M_{kj} M_{li}}.$$

We now consider the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ introduced previously and a random matrix M defined on this system and with values in S. The following proposition is proved in [15] by using Proposition 1.1 and Kingman's subadditive ergodic theorem.

PROPOSITION 1.2. *Assume that* $\mu\{\exists\, n \ge 1, M_n \in \tilde{S}\} > 0$. *Then*, μ-*a.e.*, *the decreasing sequence of compact sets*

$$K_n(\omega) = M(\omega) \cdots M(T^{n-1}\omega).(\bar{B})$$

is contracted to one point $V(\omega)$. *Moreover*, *this convergence holds at an exponential rate for the distance* d. *The random variable* $V(\omega)$ *verifies the relation* $M.TV = V$ *and is the unique random variable with values in* B *having this property*.

1.4. *Definitions.* For $1 \le i \le L$, introduce the quantities $a_i = (p_{-i} + \cdots + p_{-L})/p_1$. We define the following invertible random matrices of size $L \times L$ that will play a central role in this paper:

$$M := \begin{pmatrix} a_1 & \cdots & a_{L-1} & a_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} a_1 & \cdots & T^{L-2}a_{L-1} & T^{L-1}a_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

For $k \ge l$, set $M(k, l) = M(k) \cdots M(l)$ and for $k \le l$, $N(k, l) = N(k) \cdots N(l)$. We write $(e_i)_{1 \le i \le L}$ for the canonical basis of $\mathbb{R}^L$. Introduce the notation

$$\forall\, l \in \mathbb{Z},\ \delta(l, l+1) = 1 \text{ and } \forall\, k \ge l, \qquad \delta(k, l) = \langle e_1, M(k) \cdots M(l) e_1 \rangle,$$

with the standard scalar product on $\mathbb{R}^L$. By convention, any sum $\sum_r^s$, where $s < r$, of any quantities will be 0. From condition (2) and the fact that the form of M^{-1} and N^{-1} is "similar" to that of M, we observe that Oseledet's multiplicative ergodic theorem (see [25] and [26]) can be applied to M and to N. We write

$\gamma_1(M,T) \geq \gamma_2(M,T) \geq \cdots \geq \gamma_L(M,T)$ for the Liapounov exponents of M with respect to $(\Omega, \mathcal{F}, \mu, T)$. These quantities can be defined recursively by

$$\sum_{l=1}^{i} \gamma_l(M,T) = \lim_{n\to+\infty} \frac{1}{n} \log \|\Lambda_i M(n-1,0)\|, \qquad \mu\text{-a.e., } 1 \leq i \leq L.$$

We will write $\gamma(M,T)$ for $\gamma_1(M,T)$. Due to positivity, we have

$$\gamma(M,T) = \lim_{n\to+\infty} \frac{1}{n} \log \|M(n-1,0)x\|, \qquad \forall\, x \in B,\ \mu\text{-a.e.}$$

We note that the product of any L matrices of the same form as M or N has strictly positive entries. We can then apply Proposition 1.2 to (M, T^{-1}) and (N, T), respectively. We denote by V and λ the unique random vector in B and the unique scalar map such that $MV = \lambda TV$ and similarly by W and ρ the unique random vector in B and the unique scalar map such that $NW = \rho T^{-1} W$. Under condition (2), we observe that $\log(\lambda)$ and $\log(\rho)$ are bounded maps. Set also $v = \langle V, e_1\rangle$ and $w = \langle W, e_1\rangle$. From the equality $M(n-1,0)V = (T^{n-1}\lambda \cdots \lambda) T^n V$ and Birkhoff's ergodic theorem, using that $\|V\| = 1$, we deduce that

$$\gamma(M,T) = \int \log(\lambda)\, d\mu. \tag{3}$$

We will express the results of the following sections in terms of the matrix M. This way, we exhibit the following link between the matrices M and N:

$${}^tN = T\Phi M \Phi^{-1}, \tag{4}$$

with

$$\Phi := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \dfrac{p_{-2}+\cdots+p_{-L}}{p_1} & \dfrac{p_{-3}+\cdots+p_{-L}}{p_1} & \cdots & \dfrac{p_{-L+1}+p_{-L}}{p_1} & \dfrac{p_{-L}}{p_1} \\ 0 & T\Big(\dfrac{p_{-3}+\cdots+p_{-L}}{p_1}\Big) & T\Big(\dfrac{p_{-4}+\cdots+p_{-L}}{p_1}\Big) & \cdots & T\Big(\dfrac{p_{-L+1}+p_{-L}}{p_1}\Big) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & T^{L-2}\Big(\dfrac{p_{-L}}{p_1}\Big) & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

The matrix M will appear rather soon in the calculations and will be some kind of transition matrix, whereas the matrix N will appear when we search for the invariant measure. Therefore, as in the classic theory, this study involves the matrix ${}^tM$. From condition (2), we see that Φ and Φ^{-1} are bounded matrices. Therefore, (4) gives $\gamma_i(M,T) = \gamma_i(N,T^{-1})$ for all $1 \leq i \leq L$. In particular, we have $\int \log(\lambda)\, d\mu = \int \log(\rho)\, d\mu$, but a more precise result is true:

LEMMA 1.3. *Set* $\theta = \langle T^{-1}W, \Phi V\rangle$. *Then* $\rho = \lambda \frac{T\theta}{\theta}$.

PROOF. One has $\langle NW, \Phi V\rangle = \rho\langle T^{-1}W, \Phi V\rangle$, but using relation (4), it is also equal to $\langle W, {}^tN\Phi V\rangle = \lambda\langle W, T\Phi T V\rangle$. $\square$

One observes in the previous lemma that θ and $1/\theta$ are bounded maps. This remark will often be implicitly used in the rest of the paper, as well as the following lemma. Similarly, the value of the constants appearing in the majorations, as in the next lemma, will not matter and we will often use "generic" constants. We write $\preceq$ for the partial order on $\mathbb{R}^L$ coordinate by coordinate.

LEMMA 1.4. (i) *There exists a constant* $C > 0$ *such that*

$$(1/C)M(L-1, 0)e_1 \preceq M(L-1, 0)V \preceq CM(L-1, 0)e_1.$$

(ii) *The vector* V *is strictly in the interior of the positive cone of* $\mathbb{R}^L$, *uniformly on* ω; *that is*, *there exists a constant* $\delta > 0$ *such that* $\langle e_i, V\rangle \geq \delta$ *for* $1 \leq i \leq L$. *Consequently*, *there exists a constant* $C > 0$ *such that*, *for all* $k \geq l$,

$$(1/C)(T^k\lambda \cdots T^l\lambda) \leq \delta(k, l) \leq C(T^k\lambda \cdots T^l\lambda).$$

PROOF. The first point derives from condition (2). For the second one, we have

$$MT^{-1}M \cdots T^{-L+1}M.(T^{-L+1}V) = TV.$$

We set $TV = {}^t(v_1, \dots, v_L)$ and $MT^{-1}M \cdots T^{-L+1}M = (a_{ij}(\omega))$. From condition (2), there exist constants δ_0 and δ_1 such that $\delta_0 \geq a_{ij}(\omega) \geq \delta_1 > 0$. Therefore,

$$\frac{\delta_1}{\delta_0} \leq \min_{1\leq k\leq L}\left\{\frac{a_{ik}}{a_{jk}}\right\} \leq \frac{v_i}{v_j} \leq \max_{1\leq k\leq L}\left\{\frac{a_{ik}}{a_{jk}}\right\} \leq \frac{\delta_0}{\delta_1}.$$

The conclusion follows from the fact that $\|V\| = 1$. $\square$

2. Recurrence and transience of the random walk. We will first compute exactly the probabilities of leaving a given interval by one side or the other and the expectation of the exit time. A recurrence criterion depending on $\gamma(M, T)$ will follow. We will then give a way to compute that exponent.

2.1. *Exit probabilities and a recurrence criterion.* The following quantities were first introduced by Sinaï [27]. They appear to be decisive in quantifying the evolution of the random walk. Consider integers (a, b, k) with $a < b$ and define

$$P_k\{a, b, -\} := P_k\{\text{reach }]-\infty, a] \text{ before } [b, +\infty[\}$$

and similarly $P_k\{a, b, +\} := P_k\{\text{reach } [b, +\infty[\text{ before }]-\infty, a]\} = 1 - P_k\{a, b, -\}$. For $0 \leq j \leq L-1$, set also $P_k\{a, b, a-j\} = P_k\{\text{reach }]-\infty, a] \text{ before } [b, +\infty[\text{ and at the point } a-j\}$. We have the following result.

LEMMA 2.1. (i) *If* $a < k < b$, *then*

$$P_k\{a, b, -\} = \frac{\sum_{j=k}^{b-1} \delta(j, a+1)}{\sum_{j=a}^{b-1} \delta(j, a+1)}.$$

(ii) *For* $0 \le j \le L-1$ *and setting* $e_{L+1} = 0$,

$$P_{a+1}\{a, b, a-j\} = \frac{\sum_{l=a+1}^{b-1} \langle e_1, M(l, a+1)(e_{j+1} - e_{j+2})\rangle}{\sum_{j=a}^{b-1} \delta(j, a+1)}.$$

PROOF. For $0 \le j \le L-1$, set $f^j(k) = P_k\{a, b, a-j\}$ and $f(k) := P_k\{a, b, -\}$. For k such that $a < k < b$, using the Markov property, we get a harmonic-type recurrence equation:

$$f^j(k) = p_0(k) f^j(k) + p_1(k) f^j(k+1) + \sum_{i=1}^{L} p_{-i}(k) f^j(k-i).$$

Setting $g^j(k) := f^j(k) - f^j(k+1)$ in order to suppress $p_0(k)$ and $U^j(k) := {}^t(g^j(k), \dots, g^j(k-L+1))$, we obtain the relations

$$\begin{aligned} U^j(k) &= M(k) U^j(k-1) \\ &= M(k, a+1) U^j(a), \end{aligned}$$

with $U^j(a) = (e_{j+1} - e_{j+2} - f^j(a+1)e_1)$. Therefore, summing from $a+1$ to $b-1$, we deduce that

$$f^j(a+1) = \sum_{l=a+1}^{b-1} {}^t e_1 M(l, a+1)\bigl(e_{j+1} - e_{j+2} - f^j(a+1)e_1\bigr). \tag{5}$$

The formula in part (ii) then follows. Summing (5) for $0 \le j \le L-1$, we get that

$$f(a+1) = \frac{\sum_{j=a+1}^{b-1} \delta(j, a+1)}{\sum_{j=a}^{b-1} \delta(j, a+1)}. \tag{6}$$

Setting $g(k) := f(k) - f(k+1)$ and $U(k) := {}^t(g(k), \dots, g(k-L+1))$, we also obtain

$$f(k) = \sum_{l=k}^{b-1} \delta(l, a+1)\bigl(1 - f(a+1)\bigr). \tag{7}$$

From (6) and (7), we finally obtain the value of $f(k)$. $\square$

REMARK. One can give a general formula for $P_k\{a, b, a-j\}$, but it is rather heavy and it will not be required in the rest of the paper. We will now let a or b tend to $-\infty$ or $+\infty$ and express the result according to the sign of $\gamma(M, T)$. We naturally extend the previous notations to infinite intervals.

PROPOSITION 2.2. (i) *If* $\gamma(M,T) < 0$, *then* $P_0\{-\infty, 1, +\} = 1$ *and* $P_0\{-1, +\infty, -\} = 1 - 1/\sum_{k=-1}^{+\infty} \delta(k, 0)$.

(ii) *If* $\gamma(M,T) = 0$, *then* $P_0\{-1, +\infty, -\} = 1$ *and* $P_0\{-\infty, 1, +\} = 1$.

(iii) *If* $\gamma(M,T) > 0$, *then*

$$P_0\{-\infty, 1, +\} = 1 - \frac{Tv}{Tv + \sum_{k=-\infty}^{0} (\lambda \cdots T^k\lambda)^{-1}\, T^k v}.$$

We have $P_0\{-1, +\infty, -\} = 1$, *which is the sum on* j *for* $1 \le j \le L$ *of*

$$P_0\{-1, +\infty, -j\} = \sum_{t=0}^{L-j} T^t \left(\frac{p_{-t-j}}{p_1}\right) \frac{T^t w}{(\rho \cdots T^t\rho) T^{-1} w}.$$

PROOF. We first observe that

$$\begin{aligned} &P_0\{-1, n, -\} \\ &\quad = P_0\{-1, +\infty, -\} - P_0\{\text{reach } [n, +\infty[\text{ before coming into }]-\infty, -1]\}. \end{aligned}$$

As the sequence of events $(\{\text{reach } [n, +\infty[\text{ before coming into }]-\infty, -1]\})_{n \ge 0}$ decreases to the null set, we get $P_0\{-1, n, -\} \longrightarrow P_0\{-1, +\infty, -\}$ as $n \to +\infty$. Using Lemma 2.1, we have

$$P_0\{-1, n, -\} = 1 - \frac{1}{\sum_{k=-1}^{n-1} \delta(k, 0)}.$$

Since $\gamma(M,T) = \int \log(\lambda)\, d\mu$, we get from Lemma 1.4 that if $\gamma(M,T) < 0$, then the sum $\sum_{k=-1}^{n-1} \delta(k, 0)$ converges. If $\gamma(M,T) > 0$, then it diverges, as in the case when $\gamma(M,T) = 0$, using Lemma 1.4 and the following classical lemma (Lemma 2.3) about the recurrence of the ergodic sums in the critical case. We now evaluate $P_0\{-1, +\infty, -1-j\}$ for $0 \le j \le L-1$ in the case when $\gamma(M,T) > 0$. Using Lemma 2.1, we have

$$P_0\{-1, n, -1-j\} = \frac{\sum_{l=0}^{n-1} {}^t e_1 M(l, 0)(e_{j+1} - e_{j+2})}{\sum_{j=-1}^{n-1} \delta(j, 0)} =: \frac{A(n)}{B(n)}.$$

We then have, for $l \ge L - j$,

$$\begin{aligned} M(l,0)(e_{j+1} - e_{j+2}) &= M(l,1)\big[(a_{j+1} - a_{j+2})(0) e_1 + e_{j+2} - e_{j+3}\big] \\ &= \sum_{t=1}^{L-j} M(l,t) e_1 \left(\frac{p_{-t-j}}{p_1}\right)(t-1). \end{aligned}$$

We thus obtain

$$A(n) \sim \sum_{t=1}^{L-j} \left(\frac{p_{-t-j}}{p_1}\right)(t-1) \sum_{l=t}^{n-1} \delta(l, t) \quad \text{and} \quad B(n) \sim \sum_{l=t}^{n-1} \delta(l, 0).$$

Therefore,

$$\frac{\sum_{l=t}^{n-1}\delta(l,t)}{\sum_{l=t}^{n-1}\delta(l,0)} = \frac{{}^t e_1 \sum_{l=t}^{n-1} N(t,l)e_1}{{}^t e_1 N(0,t-1)\sum_{l=t}^{n-1} N(t,l)e_1},$$

which is dominated by $C/(\rho(0)\cdots\rho(t-1))$ for some universal constant C. From the directional contraction property, we deduce that

$$\frac{\sum_{l=t}^{n-1}\delta(l,t)}{\sum_{l=t}^{n-1}\delta(l,0)} \to \frac{{}^t e_1 W(t-1)}{{}^t e_1 N(0,t-1)W(t-1)} \qquad \text{as } n \to +\infty,$$

which gives the announced formula. The fact that the sum on j from 0 to $L-1$ is equal to 1 is equivalent to the identity $NW = \rho T^{-1} W$. We now consider $P_0\{-\infty, 1, +\}$ which is the limit of $P_0\{-n, 1, +\}$ as $n \to +\infty$. Using Lemma 2.1,

$$P_0\{-n,1,+\} = 1 - \frac{\delta(0,-n+1)}{\sum_{k=-n}^{0}\delta(k,-n+1)}.$$

As $\delta(0,-n+1) = {}^t e_1 N(-n+1,0)e_1$, if $\gamma(M,T)$, which is equal to $\gamma(N,T^{-1})$, is less than 0, then $P_0\{-\infty,1,+\} = 1$. If $\gamma(M,T) \geq 0$, write

$$\begin{aligned}&\frac{\delta(k,-n+1)}{\delta(0,-n+1)}\\ &\qquad = \frac{{}^t e_1 M(k,-n+1)e_1}{{}^t e_1 M(0,k+1)M(k,-n+1)e_1} \to \frac{{}^t e_1 V(k+1)}{{}^t e_1 M(0,k+1)V(k+1)}\end{aligned}$$

as $n \to +\infty$, and up to universal constants, has order $1/(\lambda(0)\cdots\lambda(k+1))$. If $\gamma(M,T) > 0$, the sum of such terms converges and we obtain the result. If $\gamma(M,T) = 0$, using again Lemma 2.3, we get that the sum diverges and the conclusion follows. □

LEMMA 2.3 (Atkinson and Kesten [3]). *Let (Ω, T, μ) be an ergodic dynamical system and $f \in L^1(\mu)$. If $\int f\,d\mu = 0$, then, μ-a.e.*,

$$\exists\,(n_i(\omega)) \to +\infty, \qquad \sum_{k=0}^{n_i(\omega)-1} f(T^k\omega) \to 0 \textit{ as } i \to +\infty.$$

As a consequence of Proposition 2.2 and traditional computations with the Borel–Cantelli lemma, we obtain:

THEOREM 2.4. *The asymptotic behavior of the random walk is the following*:

(i) *If $\gamma(M,T) < 0$, then $\xi_n(\omega) \to +\infty$, $\mathcal{P}_\omega$-a.e., μ-a.e.*
(ii) *If $\gamma(M,T) = 0$, then $-\infty = \liminf \xi_n(\omega) < \limsup \xi_n(\omega) = +\infty$, $\mathcal{P}_\omega$-a.e., μ-a.e.*
(iii) *If $\gamma(M,T) > 0$, then $\xi_n(\omega) \to -\infty$, $\mathcal{P}_\omega$-a.e., μ-a.e.*

REMARK. It can be proved (see [22]) that Theorem 2.4 is equivalent to Key's theorem [18]. We note that only $\gamma(M,T)$ is concerned in the criterion. The next lemma says that all the other exponents $\gamma_i(M,T)$ for $i \geq 2$ are less than 0. The proof is a consequence of the structure of the random walk and relies on a idea of Key [18].

LEMMA 2.5. *We have $\gamma(M,T) > \gamma_2(M,T)$ and $\gamma_2(M,T) < 0$.*

PROOF. The first point is a direct application of Corollary 2, page 1548, of [15]. Consider now, for $0 \leq j \leq L-1$, $f^j(k) := P_k\{-1, +\infty, -1-j\}$. As in the proof of Lemma 2.1, setting $g^j(k) := f^j(k) - f^j(k+1)$ and $U^j(k) := {}^t(g^j(k), \dots, g^j(k-L+1))$, we obtain

$$U^j(k) = M(k,0)U^j(-1), \qquad k \geq 0,$$

with $U^j(-1) = (e_{j+1} - e_{j+2} - f^j(0)e_1)$. Thus, the rank of $(U^j(-1))_{0 \leq j \leq L-1}$ is greater than or equal to $L-1$. Since, for all $0 \leq j \leq L-1$, $U^j(k)$ remains bounded as $k \to +\infty$, we deduce that $\gamma_2(M,T) \leq 0$. If $\gamma(M,T) > 0$, then the previous rank is exactly $L-1$ due to $\sum_{0 \leq j \leq L-1} U^j(-1) = (1 - P_0\{-1, +\infty, -\})e_1 = 0$ in that case. For some constant $r > 0$, we now introduce $K_r = \operatorname{diag}(1, r, r^2, \dots, r^{L-1})$. We have

$$K_r M K_r^{-1} = r \begin{pmatrix} \dfrac{p_{-1} + \cdots + p_{-L}}{p_1 r} & \cdots & \dfrac{p_{-L+1} + p_{-L}}{p_1 r^{L-1}} & \dfrac{p_{-L}}{p_1 r^L} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} =: r\tilde{M}.$$

For $r < 1$ close to 1, using condition (2), there exist random transition probabilities $(p'_z)_{z \in \Lambda}$ such that the corresponding matrix is $\tilde{M}$. As the exponents of (M,T) are deduced from the exponents of $(\tilde{M},T)$ by translation by $\log(r)$, we deduce at last that $\gamma_2(M,T) < 0$. $\square$

2.2. *Expectation of the exit time of some interval.* We now introduce for some fixed interval $[a,b]$ the time $\tau_{a,b}$ to reach $]-\infty, a] \cup [b, +\infty[$ and write $E_k\{a,b\}$ for $E_k[\tau_{a,b}]$. Similarly, for any constant $A > 0$, we introduce $E_k^A\{a,b\} = E_k[\tau_{a,b} \wedge A]$. We will need an exact evaluation of such quantities.

LEMMA 2.6. *Let $a < k < b$. Then*

$$E_k^A\{a,b\} = P_k\{a,b,+\} \sum_{k \leq l' \leq l \leq b-1} \frac{P_{l'}\{\tau_{a,b} \leq A\}}{p_1(l')} \delta(l, l'+1)$$
$$+ \sum_{l'=a+1}^{k-1} \frac{P_{l'}\{\tau_{a,b} \leq A\}}{p_1(l')} \big(P_k\{l', b, -\} - P_k\{a, b, -\}\big) \sum_{l=l'}^{b-1} \delta(l, l'+1).$$

PROOF. We first estimate $f(k) := E_k[\tau_{a,b}]$. For $a < k < b$, one has a recursive relation

$$f(k) = p_0(k)f(k) + p_1(k)f(k+1) + \sum_{i=1}^{L} p_{-i}(k)f(k-i) + 1.$$

As in Lemma 2.1, set $g(k) := f(k) - f(k+1)$ and $U(k) := {}^t(g(k), \dots, g(k-L+1))$. We then have

$$\begin{aligned} U(k) &= M(k)U(k-1) + e_1/p_1(k) \\ &= M(k, a+1)U(a) + \sum_{l=a+1}^{k} M(k, l+1)/p_1(l). \end{aligned} \tag{8}$$

We note that $U(a) = -f(a+1)e_1$. Taking the scalar product with e_1 and summing from $a+1$ to $b-1$, we obtain

$$f(a+1) = \sum_{l=a+1}^{b-1} \delta(l, a+1)\big(-f(a+1)\big) + \sum_{l=a+1}^{b-1} \sum_{l'=a+1}^{l} \frac{\delta(l, l'+1)}{p_1(l')}.$$

Therefore,

$$f(a+1) = \frac{\sum_{l=a+1}^{b-1} \sum_{l'=a+1}^{l} \delta(l, l'+1)/p_1(l')}{\sum_{l=a}^{b-1} \delta(l, a+1)}.$$

We now sum from k to $b-1$ in (8) after taking the scalar product with e_1. Using Lemma 2.1, we get the equalities

$$\begin{aligned} f(k) &= \sum_{l=k}^{b-1} \delta(l, a+1)\big(-f(a+1)\big) + \sum_{l=k}^{b-1} \sum_{l'=a+1}^{l} \frac{\delta(l, l'+1)}{p_1(l')} \\ &= P_k\{a, b, +\} \sum_{l=k}^{b-1} \sum_{l'=a+1}^{l} \frac{\delta(l, l'+1)}{p_1(l')} - P_k\{a, b, -\} \sum_{l=a+1}^{k-1} \sum_{l'=a+1}^{l} \frac{\delta(l, l'+1)}{p_1(l')} \\ &= P_k\{a, b, +\} \sum_{l'=a+1}^{k-1} \frac{1}{p_1(l')} \sum_{l=k}^{b-1} \delta(l, l'+1) \\ &\quad + P_k\{a, b, +\} \sum_{l=k}^{b-1} \sum_{l'=k}^{l} \frac{\delta(l, l'+1)}{p_1(l')} \\ &\quad - P_k\{a, b, -\} \sum_{l'=a+1}^{k-1} \frac{1}{p_1(l')} \sum_{l=l'}^{k-1} \delta(l, l'+1). \end{aligned}$$

Then

$$f(k) = P_k\{a, b, +\} \sum_{k \le l' \le l \le b-1} \frac{\delta(l, l'+1)}{p_1(l')}$$
$$+ \sum_{l'=a+1}^{k-1} \frac{1}{p_1(l')} \left[P_k\{a, b, +\} \sum_{l=k}^{b-1} \delta(l, l'+1) - P_k\{a, b, -\} \sum_{l=l'}^{k-1} \delta(l, l'+1) \right].$$

Next, set $B := P_k\{a, b, +\} \sum_{l=k}^{b-1} \delta(l, l'+1) - P_k\{a, b, -\} \sum_{l=l'}^{k-1} \delta(l, l'+1)$. We have

$$B = \sum_{l=k}^{b-1} \delta(l, l'+1) - P_k\{a, b, -\} \sum_{l=l'}^{b-1} \delta(l, l'+1)$$
$$= \big(P_k\{l', b, -\} - P_k\{a, b, -\}\big) \sum_{l=l'}^{b-1} \delta(l, l'+1).$$

Consequently,

$$E_k\{a, b\} = P_k\{a, b, +\} \sum_{k \le l' \le l \le b-1} p_1(l')\delta(l, l'+1)$$
$$+ \sum_{l'=a+1}^{k-1} \frac{(P_k\{l', b, -\} - P_k\{a, b, -\})}{p_1(l')} \sum_{l=l'}^{b-1} \delta(l, l'+1).$$

For any fixed integer $A > 0$, consider $E_k^A\{a, b\}$. This also verifies a recursive relation for $a < k < b$:

$$E_k^A\{a, b\} = p_0(k) E_k^A\{a, b\} + p_1(k) E_{k+1}^A\{a, b\}$$
$$+ \sum_{i=1}^{L} p_{-i}(k) E_{k-i}^A\{a, b\} + P_k\{\tau_{a,b} \le A\}.$$

In the previous calculations, $1/p_1(k)$ will simply be changed into $P_k\{\tau_{a,b} \le A\}/p_1(k)$ and we finally get the announced formula. □

We now proceed as in the previous section by letting a or b become infinite. The following proposition will be used later in the evaluation of the mean drift when considering the law of large numbers.

PROPOSITION 2.7. (i) *If* $\gamma(M, T) \le 0$, *then* $E_0\{-1, +\infty\} = +\infty$. *If* $\gamma(M, T) > 0$, *then*

$$E_0\{-1, +\infty\} = \sum_{l=0}^{+\infty} \frac{1}{T^l p_1} \frac{T^l w}{(\rho \cdots T^l \rho) w}.$$

(ii) *If* $\gamma(M,T) \geq 0$, *then* $E_0\{-\infty, 1\} = +\infty$. *If* $\gamma(M,T) < 0$, *then*

$$E_0\{-\infty, 1\} = \sum_{l=-\infty}^{0} \frac{1}{T^l p_1} \delta(0, l+1).$$

PROOF. From Lemma 2.6, taking $A = +\infty$, $k = 0$, $a = -1$ and $b = n$ and using Lemma 2.1, we have

$$\begin{aligned} E_0\{-1, n\} &= P_0\{-1, n, +\} \sum_{0 \leq l' \leq l \leq n-1} \frac{\delta(l, l'+1)}{p_1(l')} \\ &= \sum_{l'=0}^{n-1} \frac{1}{p_1(l')} \biggl(\frac{\sum_{l=l'}^{n-1} {}^t e_1 N(l'+1, l) e_1}{\sum_{l=-1}^{n-1} {}^t e_1 N(0, l) e_1} \biggr). \end{aligned}$$

The same reasoning as in Proposition 2.2 gives that the quantity in the parentheses tends to $\langle e_1, W(l)\rangle / (\rho(0) \cdots \rho(l) \langle e_1, W(0)\rangle)$, staying dominated by $C/(\rho(0) \cdots \rho(l))$. If $\gamma(M,T) > 0$, the formula follows from the dominated convergence theorem. Otherwise, the limit is $+\infty$. Similarly, concerning the second point, we have

$$\begin{aligned} E_0\{-n, 1\} &= \frac{P_0\{-n, 1, +\}}{p_1(0)} \\ &\quad + \sum_{l'=-n+1}^{-1} \biggl(\frac{P_0\{l', 1, -\} - P_0\{-n, 1, -\}}{p_1(l')} \biggr) \sum_{l=l'}^{0} \delta(l, l'+1). \end{aligned}$$

We then note that

$$P_0\{l', 1, -\} - P_0\{-n, 1, -\} = \sum_{t=0}^{L-1} P_0\{l', b, l'-t\} P_{l'-t}\{-n, 1, +\}. \tag{9}$$

In view of Lemma 2.1, if $\gamma(M,T) < 0$, it is exponentially decreasing and we obtain

$$\begin{aligned} E_0\{-\infty, 1\} &= \frac{1}{p_1(0)} + \sum_{l'=-\infty}^{-1} \sum_{t=0}^{L-1} \frac{P_0\{l', 1, l'-t\}}{p_1(l')} \sum_{l=l'}^{0} \delta(l, l'+1) \\ &= \sum_{l'=-\infty}^{0} \frac{\delta(0, l'+1)}{p_1(l')}. \end{aligned}$$

For $\gamma(M,T) \geq 0$, the evaluation of $P_0\{l', 1, -\}$ given by Lemma 2.1 gives that $E_0\{-\infty, 1\} = +\infty$. $\square$

2.3. *Computation of* $\gamma(M,T)$. This section tries to give an easier access to the Liapounov exponent $\gamma(M,T)$ that intervenes in the recurrence criterion (Theorem 2.4). Studies in this direction were initiated by Letchikov [21] and improved by Derriennic [12] when $L=2$. The latter, using the theory of representation of a Markov chain by cycles and weights, has given when $L=2$ and $p_{-1}=0$ a recurrence criterion depending on the sign of $\int \log\varphi\, d\mu$ with some random continued fraction φ. A posteriori, this quantity appears to be $\gamma(M,T)$. In the model we consider, we will use the directional contraction properties of the matrix M in order to give a theoretical algorithm of calculation of $\gamma(M,T)$ of a similar type.

PROPOSITION 2.8. *Let* $(f_n)_{n\geq 0}$ *be the sequence of random variables defined by* $f_0=\cdots=f_{L-2}=1$ *and by*

$$f_n = a_1 + \frac{a_2}{T^{-1}f_{n-1}} + \cdots + \frac{a_L}{T^{-1}f_{n-1}\cdots T^{-L+1}f_{n-L+1}} \quad \text{for } n\geq L-1. \tag{10}$$

Then $(f_n(\omega))$ *converges, uniformly in* ω *at an exponential rate, to a function* $f(\omega)$ *which is the unique positive and measurable solution of the equation*

$$f = a_1 + \frac{a_2}{T^{-1}f} + \cdots + \frac{a_L}{T^{-1}f\cdots T^{-L+1}f}. \tag{11}$$

Furthermore, $T^{-L+1}\log(f)$ *is equal to* $\log(\lambda)$ *up to a bounded coboundary. Thus*,

$$\gamma(M,T) = \int \log(f)\, d\mu.$$

PROOF. For $n\geq L-1$, the relation (10) can be written in the following way, involving the matrix M:

$$(T^{-L+1}f_{n-L+1})G_n = MT^{-1}G_{n-1}, \tag{12}$$

with

$$G_n := {}^t(f_nT^{-1}f_{n-1}\cdots T^{-L+2}f_{n-L+2},\\ T^{-1}f_{n-1}\cdots T^{-L+2}f_{n-L+2},\ldots,T^{-L+2}f_{n-L+2},1).$$

Thus, for some function $\rho_n>0$,

$$\rho_nG_n = MT^{-1}M\cdots T^{-n-L+1}MT^{-n-L+2}G_{L-2}.$$

Applying Proposition 1.2, we have that $X(n) := G_n/\|G_n\|$ converges in direction at an exponential rate to the direction of TV. According to the contraction coefficient of Proposition 1.1 for the matrix $M(L-1,0)$ which has strictly positive entries and condition (2), the previous convergence is uniform in ω. Therefore, setting $X(n) = (x_i(n))_{1\le i\le L}$ and $TV = (v_i)_{1\le i\le L}$, there exist positive constants C and $\delta < 1$, such that

$$\|X(n) - TV\| \le C\delta^n.$$

Since $G_n = X(n)/x_L(n)$, we have

$$G_n - \frac{TV}{v_L} = \frac{X(n) - TV}{v_L} + X(n)\left(\frac{1}{x_L(n)} - \frac{1}{v_L}\right).$$

Using Lemma 1.4, we get $v_L \ge \delta > 0$. From uniform convergence, there exists a constant $A > 0$ such that, for n large enough, $1/x_L(n) \le A$, uniformly in ω. Thus, uniformly in ω at an exponential rate, (G_n) converges to TV/v_L. From (12), one deduces that $(T^{-L+1} f_{n-L+1})_{n\ge L}$ converges to some $T^{-L+1}f$ such that

$$T^{-L+1} f = \lambda\left(\frac{v_L}{T^{-1}v_L}\right). \tag{13}$$

Since v_L and v_L^{-1} are bounded functions, we obtain that $\int \log(f)\,d\mu = \int \log(\lambda)\,d\mu$. The unicity of a positive measurable solution of (11) is a consequence of the unicity in the direction of TV, since it gives an equation of the same kind as (12). $\square$

REMARK. With the previous notation, any initial value of G_{L-2} in the positive cone of $\mathbb{R}^L$ is convenient. When $L = 1$, we have $f = p_{-1}/p_1$ and we obtain Solomon's criterion [28] with $\int \log(p_1/p_{-1})\,d\mu$. If $L = 2$, then f can be expressed as a continued fraction and we get the Derriennic–Letchikov criterion ([12] and [21]). In the last section, we will consider the case of an environment defined by an irrational rotation on the circle S^1. If M is regular, the previous construction will show that λ is also regular with the same regularity.

3. Law of large numbers. We first present the theory relative to the invariant measure as developed in [19]. We will then characterize its existence. Next, we will generalize an approach based on hitting times in order to give rather explicit expressions for the drift. This method was introduced by Solomon [28] in the independent case for $L = 1$ and extended in [1] and [29] in the ergodic context and $L = 1$. We will then characterize the existence of a nonzero drift.

3.1. *Auxiliary Markov chains, invariant measure, general results.* We now introduce two fundamental objects for further study. Fixing ω, let $(\omega_n)_{n\ge 0}$, with $\omega_n := T^{\xi_n(\omega)}\omega$, be the sequence of the environments from the point of view of a moving particle: after n steps, the particle seems to be at 0 in the medium ω_n. Collecting all the chains $(\xi_n(\omega))$, we observe that (ω_n) is a Markov chain on Ω whose transition operator, written P in the rest of the paper is

$$Pf(\omega) = \sum_{z\in\Lambda} p_z(\omega) f(T^z\omega).$$

To recover the trajectories of all the chains $(\xi_n(\omega))$, we consider the Markov chain (x_n) on $(\Omega \times \Lambda)$ defined by $x_k := (\omega_k, z_k)$, where $z_k = \xi_{k+1}(\omega) - \xi_k(\omega)$. One then has $\xi_n(\omega) = \sum_{k=0}^{n-1} z_k$. The corresponding transition operator, written as $\tilde{P}$, is

$$\tilde{P} g(\omega, z) = \sum_{z' \in \Lambda} p_{z'}(T^z \omega) g(T^z \omega, z'). \tag{14}$$

From the following results, it appears that, in order to study (x_n), we only need to consider (ω_n). We now discuss, for the chain (ω_n), the existence of a P-invariant initial distribution on Ω, absolutely continuous with respect to μ. We denote by (IM) the existence of such a density $\pi(\omega)$; that is,

$$\begin{aligned} &\exists\, \pi \geq 0, \text{ measurable}, \\ &\int \pi \, d\mu = 1 \quad \text{and} \quad \pi(\omega) = P^* \pi(\omega) = \sum_{z \in \Lambda} p_z(T^{-z}\omega) \pi(T^{-z}\omega). \end{aligned}$$

The next results were developed by Kozlov and can be proved for a general model of random walk in $\mathbb{Z}^d$. For complete details, see [19].

THEOREM 3.1. *If π satisfies (IM), then*

(i) $\pi > 0$, *μ-a.e., and π is unique in $L^1(\mu)$.*

(ii) *The chain (ω_k) is stationary and ergodic for the initial distribution $\pi \, d\mu$ on Ω.*

(iii) *The chain (x_k) is stationary and ergodic for the initial distribution on $(\Omega \times \Lambda)$ of density $\tilde{\pi}(\omega, z) := p_z(\omega)\pi(\omega)$ with respect to $\mu \otimes$ (counting measure).*

Let $\mathcal{P}'_\omega$ be the measure on the space of trajectories $(\Omega \times \Lambda)^{\mathbb{N}}$ of the chain (x_k) corresponding to the initial distribution $\{\omega, z(\omega)\}$, where ω is fixed and $z(\omega)$ has the distribution $\mathcal{P}'_\omega\{z(\omega) = z\} = p_z(\omega)$. Using Birkhoff's ergodic theorem and the fact, from Theorem 3.1, that $\pi > 0$, μ-a.e., we deduce the following law of large numbers:

THEOREM 3.2. *Under (IM), let $\varphi : (\Omega \times \Lambda) \mapsto \mathbb{R}$ be such that, $\forall\, z \in \Lambda$, $\varphi(\omega, z) \in L^1(\Omega, \pi\mu)$. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(x_k) = \sum_{z \in \Lambda} \int p_z(\omega)\pi(\omega)\varphi(\omega, z) \, d\mu(\omega), \qquad \mathcal{P}'_\omega\text{-a.e., } \mu\text{-a.e.}$$

COROLLARY 3.3. *Under (IM), the following law of large number holds:*

$$\lim_{n \to \infty} \frac{1}{n} \xi_n(\omega) = \sum_{z \in \Lambda} \int p_z(\omega)\pi(\omega) z \, d\mu(\omega), \qquad \mathcal{P}_\omega\text{-a.e., } \mu\text{-a.e.}$$

REMARK. In the case when (IM) is satisfied, the previous average quantity will be called the "drift" and will be denoted by $c := \sum_{z \in \Lambda} \int p_z(\omega)\pi(\omega) z \, d\mu(\omega)$.

3.2. *Characterization of the existence of the invariant measure.* We begin by analyzing the condition (IM). The equation $\pi = P^*\pi$ can be written as

$$\pi = p_0\pi + T^{-1}p_1T^{-1}\pi + Tp_{-1}T\pi + \cdots + T^L p_{-L}T^L\pi. \tag{15}$$

Setting $w_1 = p_1\pi$, we obtain

$$(1-p_0)\frac{w_1}{p_1} = T^{-1}w_1 + T\left(\frac{p_{-1}}{p_1}\right)Tw_1 + \cdots + T^L\left(\frac{p_{-L}}{p_1}\right)T^L w_1. \tag{16}$$

With $\varphi := T^{-1}w_1 - [(p_{-1}+\cdots+p_{-L})/p_1]w_1 - \cdots - T^{L-1}(p_{-L}/p_1)T^{L-1}w_1$, we observe that (16) is equivalent to $T\varphi = \varphi$. As T is ergodic with respect to μ, there exists a constant c' such that $\varphi = c'$, that is,

$$\begin{pmatrix} T^{-1}w_1 \\ \vdots \\ \vdots \\ T^{L-2}w_1 \end{pmatrix} - \begin{pmatrix} \frac{p_{-1}+\cdots+p_{-L}}{p_1} & \cdots & T^{L-2}\left(\frac{p_{-L+1}+p_{-L}}{p_1}\right) & T^{L-1}\left(\frac{p_{-L}}{p_1}\right) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} w_1 \\ \vdots \\ \vdots \\ T^{L-1}w_1 \end{pmatrix} = \begin{pmatrix} c' \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Setting $W_1 = {}^t(w_1, \ldots, T^{L-1}w_1)$ and recognizing the matrix N introduced in Section 1, the previous equation can be written as $T^{-1}W_1 = NW_1 + c'e_1$. Reciprocally, a strictly positive solution to this equation leads back to (15). In this case, one has

$$\begin{aligned} c' = \int \varphi\, d\mu &= \int w_1 - [(p_{-1}+\cdots+p_{-L})/p_1]w_1 - \cdots - (p_{-L}/p_1)w_1\, d\mu \\ &= \int \left(p_1 + \sum_{j=1}^{L}(-j)p_{-j}\right)(w_1/p_1)\, d\mu = c\int (w_1/p_1)\, d\mu \end{aligned}$$

as $\pi = (w_1/p_1)/\int(w_1/p_1)\,d\mu$. These results are summarized in the following proposition.

PROPOSITION 3.4. *Condition* (*IM*) *is satisfied if and only if there exist a vector* W_1 *in* $L^1(\mu)$ *with strictly positive components and a constant* c' *such that* $T^{-1}W_1 = NW_1 + c'e_1$. *In this case, writing* $W_1 = {}^t(w_1, \ldots, T^{L-1}w_1)$, *the density is* $\pi = (w_1/p_1)/\int(w_1/p_1)\,d\mu$ *and the drift is* $c = c'/\int(w_1/p_1)\,d\mu$.

Using Proposition 3.4, we now come to the main result of this section.

THEOREM 3.5. (i) *If $\gamma(M,T)=0$, then (IM) is equivalent to the existence of a measurable $\varphi>0$ with $\varphi\in L^1(\mu)$ such that $\lambda=\varphi/T\varphi$. In this case, $\pi=wT\varphi/(T\theta p_1)/(\int wT\varphi/(T\theta p_1)\,d\mu)$ and the drift c is* 0.

(ii) *If $\gamma(M,T)<0$, then (IM) is equivalent to $\int\sum_{n=0}^{+\infty}(T^{n-1}\lambda\cdots\lambda)\,d\mu<+\infty$. In this case,*

$$\pi=\frac{(1/p_1)\sum_{n=0}^{+\infty}\delta(n,1)}{\int(1/p_1)\sum_{n=0}^{+\infty}\delta(n,1)\,d\mu}$$

and the drift is

$$c=\frac{1}{\int(1/p_1)\sum_{n=0}^{+\infty}\delta(n,1)\,d\mu}.$$

(iii) *If $\gamma(M,T)>0$, then (IM) is equivalent to $\int\sum_{n=1}^{+\infty}(T^{-n}\lambda\cdots T^{-1}\lambda)^{-1}\,d\mu<+\infty$. In this case, $\pi=(w_1/p_1)/\int(w_1/p_1)\,d\mu$ and the drift is $c=-1/\int(w_1/p_1)\,d\mu$, where*

$$w_1=w\sum_{n=0}^{+\infty}\frac{T^{-n}(v/\theta)}{(T^{-n}\rho\cdots\rho)}-Z_1$$

and

$$Z_1=\big(1-T(v/\theta)\big)+\sum_{n=1}^{+\infty}\big[\delta(n,1)-T^{n+1}(v/\theta)T^n\rho\cdots T\rho\big]\text{ is a bounded map.}$$

PROOF. (i) Assume that $\gamma(M,T)=0$. If $\lambda=\varphi/T\varphi$ with $\varphi>0$ and in $L^1(\mu)$, one has $\rho=(\varphi/\theta)/(T\varphi/T\theta)$. Thus, $N(W(T\varphi/T\theta))=T^{-1}(W(T\varphi/T\theta))$ and the conclusion follows from Proposition 3.4. Reciprocally, if (IM) is satisfied, then, using again Proposition 3.4, there exist $W_1>0$ in $L^1(\mu)$ and a constant c' such that $T^{-1}W_1=NW_1+c'e_1$. From Corollary 3.3, a law of large numbers occurs and c', up to a strictly positive constant, is the drift of the random walk. Since we are in the recurrent case, we get $c'=0$, that is, $T^{-1}W_1=NW_1$. As W is unique in direction and as its components are uniformly minored by a strictly positive constant, there exists $\varphi>0$ in $L^1(\mu)$ such that $W_1=\varphi W$. We then get $\rho=T^{-1}\varphi/\varphi$ and $\lambda=(\theta T^{-1}\varphi)/(T\theta\varphi)$ with $(\theta T^{-1}\varphi)\in L^1(\mu)$.

(ii) Assume that $\gamma(M,T)<0$. In the case when (IM) is satisfied, there exist $W_1>0$ in $L^1(\mu)$ and a constant c' such that $T^{-1}W_1=NW_1+c'e_1$. As above, we get that c' is the drift up to some strictly positive constant. Since the random walk is transient to the right, we get $c'\geq 0$. If $c'=0$, using Lemma 2.3 on the recurrence of the ergodic sums in Theorem 3.2 and Fubini's theorem, since $\pi>0$, μ-a.e., we would deduce that μ-a.e. the random walk $(\xi_n(\omega))$ is recurrent. As it is not the case, we obtain $c'>0$. We have $c'{}^te_1\Phi+{}^tW_1T\Phi M={}^tT^{-1}W_1\Phi$. Multiplying both sides by V and since ${}^t\Phi e_1 = e_1$, we get $c'v + \langle W_1, T\Phi T V\rangle\lambda = \langle T^{-1}W_1, \Phi V\rangle$. As v is bounded and $\int \log(\lambda)\,d\mu < 0$, we deduce from the Poincaré recurrence theorem that

$$\langle T^{-1}W_1, \Phi V\rangle = c'\left[v + \sum_{n=1}^{+\infty}(T^{n-1}\lambda\cdots\lambda)T^n v\right].$$

The left-hand side is in $L^1(\mu)$ and from Lemma 1.4, we have $v \geq \delta > 0$. Then the integrability condition follows. Reciprocally, if it is satisfied, we set $W_1 = e_1 + \sum_{n=1}^{+\infty} N(1,n)e_1$, which satisfies $T^{-1}W_1 = NW_1 + e_1$. The function W_1 is in $L^1(\mu)$, since there exists a constant $C > 0$ such that

$$\|N(1,n)e_1\| \leq C\|TN\cdots T^n N T^n W\| = CT\rho\cdots T^n\rho.$$

(iii) Assume that $\gamma(M,T) > 0$. If (IM) is realized, then there exist $W_1 > 0$ in $L^1(\mu)$ and a constant c' such that $T^{-1}W_1 + c'e_1 = NW_1$. As above, we get $c' > 0$ and $\lambda\langle W_1, T\Phi T V\rangle = c'v + \langle T^{-1}W_1, \Phi V\rangle$, which gives similarly

$$\langle W_1, T\Phi T V\rangle = c'\left[\sum_{n=0}^{+\infty}(T^{-n}\lambda\cdots\lambda)^{-1}T^{-n}v\right]. \tag{17}$$

Thus, the integrability condition follows. Reciprocally, if it is satisfied, we will find some positive W_1 such that $NW_1 = T^{-1}W_1 + e_1$. In view of Lemma 2.5, let $\mathcal{H}(N, T^{-1})$ be the $(L-1)$-dimensional subspace of $\mathbb{R}^L$ corresponding to the strictly negative exponents of (N, T^{-1}). Then write $W_1 = \alpha W + R$ with $R \in \mathcal{H}(N, T^{-1})$ and $e_1 = sT^{-1}W + S$ with $S \in T^{-1}\mathcal{H}(N, T^{-1})$. We search for α and R. We note first that, since W is strictly in the positive cone uniformly in ω, s and S are bounded functions. Inserting these decompositions in the equation $NW_1 = T^{-1}W_1 + e_1$, we then get $(T^{-1}\alpha + s - \alpha\rho) = 0$ and $T^{-1}R + S - NR = 0$. Thus, we define

$$\alpha := \sum_{n=0}^{+\infty}\frac{T^{-n}s}{(T^{-n}\rho\cdots\rho)} \quad\text{and}\quad R := -\left[TS + \sum_{n=1}^{+\infty}(TN\cdots T^n N)T^{n+1}S\right]. \tag{18}$$

Let us show that R is well defined. Consider $\mathcal{H}({}^tN, T)$ the $(L-1)$-dimensional subspace of $\mathbb{R}^L$ corresponding to the strictly negative exponents of $({}^tN, T)$ and let $(g_1, \ldots, g_{L-1})$ be a random orthonormal basis of $\mathcal{H}({}^tN, T)$. From Lemma 2.5, we have $\gamma_1({}^tN, T) > \gamma_2({}^tN, T)$, giving $(T^{-1}W)^\perp = \mathcal{H}({}^tN, T)$. Since W is strictly in the positive cone uniformly in ω, there exists a constant $C > 0$ such that, if we write p_r for the orthogonal projector on $\mathcal{H}({}^tN, T)$, then, for every $a \in \mathcal{H}(N, T^{-1})$, $\|a\| \leq C\|p_r(a)\|$. However, for $1 \leq i \leq L-1$, one has $\langle g_i, NTN\cdots T^n N T^{n+1}S\rangle = \langle T^{n+1}S, T^n({}^tN)\cdots({}^tN)g_i\rangle$, which decreases exponentially fast as S is bounded. Thus, R is well defined and we have found a measurable solution to the equation $NW_1 = T^{-1}W_1 + e_1$. We will show in Section 4.3 that R is bounded but it is not required for the rest of the proof.

Returning to (15), we obtain some π, with $\mu\{\pi \neq 0\} > 0$ and such that $P^*\pi = \pi$. Thus, $|\pi| \leq P^*|\pi|$. Setting as before $w_1' = p_1|\pi|$ and $\varphi := T^{-1}w_1' - [(p_{-1} + \cdots + p_{-L})/p_1]w_1' - \cdots - T^{L-1}(p_{-L}/p_1)T^{L-1}w_1'$, we get $T\varphi \leq \varphi$. Therefore, $T\varphi = \varphi$, μ-a.e., and then $|\pi| = P^*|\pi|$. We now choose a map h among π^+ and π^- such that $h = P^*h$ and $\mu\{h > 0\} > 0$. We thus have proved the existence of a positive W_2 solution of $NW_2 = T^{-1}W_2 + e_1$. We obtain as above $\lambda\langle W_2, T\Phi TV\rangle = v + \langle T^{-1}W_2, \Phi V\rangle$. Using the Poincaré recurrence theorem, we deduce that

$$\langle W_2, T\Phi TV\rangle = \sum_{n=0}^{+\infty} (T^{-n}\lambda \cdots \lambda)^{-1} T^{-n} v.$$

Therefore, W_2 belongs to $L^1(\mu)$, which concludes the proof, using Proposition 3.4. The formula for the drift comes from the fact that ${}^tN\Phi V = \lambda T\Phi TV$, which gives $(\Phi V)^\perp = T^{-1}\mathcal{H}(N, T^{-1})$ and, consequently, $\langle \Phi V, e_1\rangle = s\langle T^{-1}W, \Phi V\rangle$. $\square$

REMARK. The formula for the drift in the case $\gamma(M, T) > 0$ is not really efficient. We will give another expression for it in the next section, where we will relate the existence of a nonzero drift with the finiteness of the expectation of the exit time of some intervals. In Theorem 3.5, consider now the case when $\gamma(M, T) < 0$ for an independent environment. In this situation, the drift has the following form:

$$1/c = \left(\int 1/p_1\, d\mu\right) \sum_{n=0}^{+\infty} {}^te_1 \left(\int M\, d\mu\right)^n e_1.$$

Writing $u_n := {}^te_1(\int M\ d\mu)^n e_1$ and $r_i := \int (p_{-i}/p_1)\, d\mu$ for $1 \leq i \leq L$, one has for $n \geq 0$ the relation $u_{n+L} = (r_1 + \cdots + r_L)u_{n+L-1} + \cdots + r_L u_n$. Setting $S = \sum_{n\geq 0} u_n$, we therefore get

$$\begin{aligned} S - (u_0 + \cdots + u_{L-1}) &= (r_1 + \cdots + r_L)(S - u_0 - \cdots - u_{L-2}) \\ &\quad + (r_{L-1} + r_L)(S - u_0) + r_L S. \end{aligned}$$

Thus, $S = R/(1 - \sum_{i=1}^L i r_i)$, where $R = (u_0 + \cdots + u_{L-1}) - (r_1 + \cdots + r_L) \times (u_0 + \cdots + u_{L-2}) - (r_{L-1} + r_L)u_0$. As for $1 \leq t \leq L - 1$, one has $u_{L-t} = {}^te_1(\int M\, d\mu)^{L-t} e_1 = \sum_{i=1}^{L-t} (r_i + \cdots + r_L)u_{L-1-t}$, summing on t from 1 to $L - 1$ gives that $R = u_0 = 1$ and then, finally,

$$c = \frac{1 - \int (1/p_1) \sum_{i=1}^L i p_{-i}\, d\mu}{\int (1/p_1)\, d\mu}.$$

This formula is an extension of expressions contained in [1], [28] and [29], which had been obtained with the method that we will now develop.

3.3. *Characterization of a nonzero drift.*

PROPOSITION 3.6. (i) *If* $\int E_0\{-1,+\infty\}\,d\mu = +\infty$, *then* $\liminf \xi_n(\omega)/n \geq 0$, $\mathcal{P}_\omega$-*a.e.*, μ-*a.e. If* $\int E_0\{-1,+\infty\}\,d\mu < +\infty$, *then the law of large numbers occurs with a drift*:

$$c = -\frac{\int (\sum_{l=1}^{L} l P_0\{-1,+\infty,-l\})\pi_1\,d\mu}{\int E_0\{-1,+\infty\}\pi_1\,d\mu}, \tag{19}$$

where $\pi_1 = \langle \Pi_1, e_1\rangle$ *and* Π_1 *is the positive vector such that* $T^{-1}\Pi_1 = \tilde{N}\Pi_1$ *and* $\|\Pi_1\| = 1$ *with*

$$\tilde{N} := \begin{pmatrix} u_1 & 1 & \cdots & 0 \\ u_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ u_L & \cdots & \cdots & 0 \end{pmatrix} \quad \textit{and} \quad u_i = P_0\{-1,+\infty,-i\} \qquad \textit{for } 1 \leq i \leq L.$$

(ii) *If* $\int E_0\{-\infty,1\}\,d\mu = +\infty$, *then* $\limsup \xi_n(\omega)/n \leq 0$, $\mathcal{P}_\omega$-*a.e.*, μ-*a.e. If* $\int E_0\{-\infty,1\}\,d\mu < +\infty$, *then the law of large numbers occurs with a drift*:

$$c = \frac{1}{\int E_0\{-\infty,1\}\,d\mu}. \tag{20}$$

PROOF. We consider case (i) and assume that $\gamma(M,T) \geq 0$; otherwise, the conclusion is obvious since the random walk is transient to the right and Proposition 2.7 gives that the expectation of $E_0\{-1,+\infty\}$ is infinite. We now copy the traditional construction of the invariant measure as introduced in the beginning of this section and developed in [19]. Consider the first time τ_1 when the random walk comes to $]-\infty,-1]$ and write l_1 for the point of entrance between $-L$ and -1 in this interval and also $\tilde{\omega}_1 = \omega_{\tau_1}$. Then set recursively, for $n \geq 2$, $\tau_n(\omega) = \tau_1(\tilde{\omega}_{n-1})$, $l_n(\omega) = l_1(\tilde{\omega}_{n-1})$ and $\tilde{\omega}_n = \omega_{\sum_{k=1}^n \tau_k}$. We observe that $(\tilde{\omega}_n, l_n)_{n\geq 0}$ with $(\tilde{\omega}_0, l_0) = (\omega, 0)$ is a Markov chain on $\Omega \times \{-L,\dots,-1\}$ with transition operator

$$(\tilde{\mathcal{P}}g)(\omega,l) = \sum_{t=-L}^{-1} u_{-t}(T^l\omega)g(T^l\omega,t) \qquad \text{for any measurable } g \geq 0.$$

As detailed in [19], there will exist a $\tilde{\mathcal{P}}$-invariant measure on $\Omega \times \{-L,\dots,-1\}$ absolutely continuous with respect to $\mu \otimes$ counting measure if one can find some $\pi_1 > 0$ with $\int \pi_1\,d\mu = 1$ and satisfying the following condition similar to (IM):

$$\pi_1 = \sum_{l=-L}^{-1} u_{-l}(T^{-l}\omega)\pi_1(T^{-l}\omega). \tag{21}$$

The Markov chains $(\tilde{\omega}_n)_{n\geq 0}$ and $(\tilde{\omega}_n, l_n)_{n\geq 0}$ will then be stationary and ergodic, respectively, with the initial distribution of densities π_1 on Ω and $u_{-l}(\omega)\pi_1(\omega)$ on $\Omega \times \{-L, \dots, -1\}$. Equation (21) can then be rewritten in the form $\Pi_1 = N_1 T \Pi_1$ with $\Pi_1 = {}^t(\pi_1, \dots, T^{L-1}\pi_1)$ and

$$N_1 := \begin{pmatrix} Tu_1 & \cdots & T^{L-1}u_{L-1} & T^L u_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

However, one can observe that ${}^t N_1 = T\Psi T^t \tilde{N} \Psi^{-1}$, where

$$\Psi := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & Tu_{-2} & Tu_{-3} & \cdots & Tu_{-L} \\ 0 & T^2u_{-3} & T^2u_{-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & T^{L-1}u_{-L} & 0 & \cdots & 0 \end{pmatrix}.$$

To solve (21), we now apply Proposition 1.2 to $(\tilde{N}, T^{-1})$. Then let U_1 be the unique vector with positive components and α the unique random scalar such that $\|U_1\| = 1$ and $\tilde{N} T U_1 = \alpha U_1$. Taking the scalar product with the vector ${}^t(1, \dots, 1)$, one gets $\alpha = 1$. We therefore set $\Pi_1 = {}^t\Psi^{-1} T U_1$ which satisfies $\Pi_1 = N_1 T \Pi_1$ and finally we have $\pi_1 = \langle e_1, T U_1 \rangle$. We note that, as a consequence of condition (2) and the formulas in Proposition 2.2, π_1 is contained between two strictly positive constants. As in the proof of the law of large numbers in [19], we finally get

$$\left(\frac{l_1 + \cdots + l_n}{n}\right) \to -\left(\int \pi_1 \, d\mu\right)^{-1} \int \left(\sum_{l=1}^{L} l P_0\{-1, +\infty, -l\}\right) \pi_1 \, d\mu, \qquad \mathcal{P}_\omega\text{-a.e., } \mu\text{-a.e.}$$

Consider now the sequence $(\tau_n)_{n\geq 1}$. We observe that this sequence is stationary under the probability $(\int \pi_1 \, d\mu)^{-1} \int_\Omega \mathcal{P}_\omega \pi_1(\omega) \, d\mu(\omega)$. With the same astute proof as in [29], page 12, using the ergodicity of $(\tilde{\omega}_n)_{n\geq 0}$ in the final argument, we obtain that this sequence is ergodic. Therefore, we deduce the following convergence:

$$\left(\frac{\tau_1 + \cdots + \tau_n}{n}\right) \to \left(\int \pi_1 \, d\mu\right)^{-1} \int (E_0\{-1, +\infty\}) \pi_1 \, d\mu, \qquad \mathcal{P}_\omega\text{-a.e., } \mu\text{-a.e.}$$

Introducing $s(n) = \sum_{k=1}^{n} \tau_k$, we can write

$$\frac{\xi_{s(n)}(\omega)}{s(n)} = \left(\frac{l_1 + \cdots + l_n}{n}\right)\left(\frac{n}{\tau_1 + \cdots + \tau_n}\right).$$

Consider now any instant $n \geq 1$ and let k be such that $s(k) \leq n < s(k+1)$. One then has

$$\frac{\xi_{s(k)}(\omega) + (n - s(k))}{n} \geq \frac{\xi_n(\omega)}{n} \geq \frac{\xi_{s(k+1)}(\omega)}{n} \geq \frac{\xi_{s(k+1)}(\omega)}{s(k+1)}.$$

If $\int E_0\{-1, +\infty\}\pi_1 \, d\mu = +\infty$, that is, equivalently, if $\int E_0\{-1, +\infty\} \, d\mu = +\infty$, then $\xi_{s(k)}(\omega)/s(k)$ tends to 0 and we get the result. If $\int E_0\{-1, +\infty\} \, d\mu < +\infty$, then $n \sim s(k)$ and $\xi_n(\omega)/n$ has the same limit as $\xi_{s(k)}(\omega)/s(k)$ and the conclusion follows. Case (ii) with $E_0\{-\infty, 1\}$ can be treated similarly. $\square$

REMARK. In the case $\gamma(M, T) > 0$, it is not obvious that the formulas for the drift given in Theorem 3.5 and Proposition 3.6 are the same. The expression given in Proposition 3.6 suggests that, with the notations of the proof of Theorem 3.5, it is possible to reveal the form of R in the decomposition of $W_1 = \alpha W + R$. In the case $\gamma(M, T) < 0$, it is directly seen that the formulas are the same. We finish this section by giving a classification of the possible behaviors with respect to speed, using all the previous results.

THEOREM 3.7. (1) *The following statements are equivalent*:

(i) *there is an LLN with a drift* $c > 0$;
(ii) $\gamma(M, T) < 0$ *and* (*IM*) *is verified*;
(iii) $\int \sum_{n=0}^{+\infty} (T^{n-1}\lambda \cdots \lambda) \, d\mu < +\infty$;
(iv) $\int E_0\{-\infty, 1\} \, d\mu < +\infty$.

Then the drift is given by formula (19).

(2) *The following statements are equivalent*:

(i) *there is an LLN with a drift* $c < 0$;
(ii) $\gamma(M, T) > 0$ *and* (*IM*) *is verified*;
(iii) $\int \sum_{n=1}^{+\infty} (T^{-n}\lambda \cdots T^{-1}\lambda)^{-1} \, d\mu < +\infty$;
(iv) $\int E_0\{-1, +\infty\} \, d\mu < +\infty$.

Then the drift is given by formula (20).

(3) *In the other cases*,

(i) $\gamma(M, T) < 0$ *and* $\int \sum_{n=0}^{+\infty} T^{n-1}\lambda \cdots \lambda \, d\mu = +\infty$;
(ii) $\gamma(M, T) > 0$ *and* $\int \sum_{n=1}^{+\infty} (T^{-n}\lambda \cdots T^{-1}\lambda) \, d\mu = +\infty$;
(iii) $\gamma(M, T) = 0$.

We have an LLN with zero drift: $\xi_n(\omega)/n \to 0$, $\mathcal{P}_\omega$-*a.e.*, μ-*a.e.*

PROOF. Consider (1). If (i) is verified, then Proposition 3.6 implies that $\int E_0\{-\infty, 1\} \, d\mu < +\infty$, that is, (iv). If (iv) is true, then Proposition 2.7 implies that $\gamma(M, T) < 0$ and (iii). Then (iii) implies (ii) by Theorem 3.5. Finally, Corollary 3.3 implies that there is a law of large numbers and the first argument in the proof of the second case of Theorem 3.5 says that the drift is strictly positive. Thus (ii) implies (i). Point (2) can be treated similarly, whereas (3) follows from Propositions 3.6 and 2.7. □

REMARK. This classification, except what concerns the invariant measure, goes back to Solomon [28] for an independent medium and $L = 1$. It was extended in [1] and [29] for any ergodic environment and $L = 1$. In the same context, the equivalence between (ii) and (iii) is due to Conze and Guivarc'h [11]. We finish this section by a remark concerning the recurrent case. In that situation there is always a zero drift, but the existence of a solution to (IM) is not always true. An example is Sinaï's random walk, that is, the case $L = 1$ in an independent medium with $\int (\log(p_{-1}/p_1))^2 \, d\mu > 0$. Then $\log(p_{-1}/p_1)$ is not a coboundary with the integrability condition stated in Theorem 3.5 since an invariance principle holds.

4. Central limit theorem. In this section we consider the validity of the central limit theorem. We will keep the quenched point of view in the recurrent case, while in the transient case we will also consider the "annealed" point of view. We present an approach to this problem due to Kozlov [19] and that relies on the decomposition of the function $(\omega, z) \longmapsto z$ according to certain subspaces of $L^2(\Omega \times \Lambda)$. It will be presented just below. We will characterize the existence of such a decomposition. Using this tool and extending a method due to Letchikov [23], we will characterize the validity of an invariance principle in the recurrent case. In the transient cases, we will give sufficient conditions for the realization of the central limit theorem. In this section it will be required that condition (IM) is fulfilled. Note that, in the case when the random walk is transient and (IM) is not satisfied, Kesten, Kozlov and Spitzer [17] for $L = 1$ and an independent medium have given a complete answer to the question of finding quantities a_n and b_n such that the law of $(\xi_n(\omega) - a_n)/b_n$ under the annealed probability converges to a nondegenerated limit law.

4.1. *Harmonic coordinates, general results.* The general results that will be recalled now are available for a general model of random walks on $\mathbb{Z}^d$. We assume the existence of an invariant measure absolutely continuous with respect to μ with density π realizing condition (IM). We consider $L^2(\Omega \times \Lambda)$ with the norm

$$\|\varphi\|_{L^2(\Omega\times\Lambda)} = \left(\sum_{z\in\Lambda} \int |\varphi(\omega, z)|^2 p_z(\omega)\pi(\omega)\mu(d\omega) \right)^{1/2}.$$

We introduce the set $H = \{\varphi \in L^2(\Omega \times \Lambda), \tilde{P}\varphi = 0\}$ of $\tilde{P}$-harmonic functions, where $\tilde{P}$ was defined in (14). The relation $\tilde{P}\varphi = 0$ is equivalent to the equation

$$\sum_{z\in\Lambda} p_z(\omega)\varphi(\omega, z) = 0, \qquad \mu\text{-a.e.} \tag{22}$$

The set H is a closed subspace in $L^2(\Omega \times \Lambda)$ which can be identified with the orthogonal of $L^2(\Omega)$ in $L^2(\Omega \times \Lambda)$. We now recall the definition of a cocycle. A function $E(\omega, z): (\Omega \times \mathbb{Z}) \to \mathbb{R}$ is the (u, T)-cocycle if $E(\omega, 0) = 0$ and if there exists a measurable function u such that

$$E(\omega, z) = \begin{cases} \displaystyle\sum_{n=0}^{z-1} u(T^n\omega), & \text{for } z \geq 1, \\ \displaystyle -\sum_{n=z}^{-1} u(T^n\omega), & \text{for } z \leq -1. \end{cases} \tag{23}$$

We denote by "harmonic coordinates" and write (HC) in the rest of the paper the existence of a function $x(\omega, z)$ in H and a (u, T)-cocycle $E(\omega, z)$, with $u \in L^1(\mu)$ and $\int u \, d\mu = 0$, such that

$$\forall z \in \Lambda, \qquad z = x(\omega, z) + E(\omega, z) + c,$$

where c is the drift. If (HC) is true, then one can write

$$\forall n \geq 0, \qquad \xi_n(\omega) = \sum_{k=0}^{n-1} x(\omega_k, z_k) + E(\omega, \xi_n(\omega)) + nc. \tag{24}$$

Under condition (HC), the first term on the right-hand side of (24) will be a martingale with respect to its own filtration for a fixed ω. Relying on Brown's theorem on the functional CLT, the following results can be found in [19].

THEOREM 4.1. *Assume (IM) and let $x(\omega, z) \in H$. Then, μ-a.e., under $\mathcal{P}'_\omega$,*

$$\left(\frac{\sum_{k=0}^{[nt]-1} x(\omega_k, z_k)}{\sqrt{n}}\right)_{t \in [0,1]} \rightharpoonup \mathcal{W}(0, \sigma^2) \qquad \text{where } \sigma^2 = \|x\|^2_{L^2(\Omega \times \Lambda)}.$$

COROLLARY 4.2. *Assume (IM), (HC) and that $c = 0$. Then the functional quenched CLT holds; that is, μ-a.e., under the measure $\mathcal{P}_\omega$,*

$$\left(\frac{\xi_{[nt]}(\omega)}{\sqrt{n}}\right)_{t \in [0,1]} \rightharpoonup \mathcal{W}(0, \sigma^2) \qquad \text{where } \sigma^2 = \|x\|^2_{L^2(\Omega \times \Lambda)} > 0.$$

REMARK. The fact that $\sigma > 0$ follows from $\int E(\omega, y) \, d\mu(\omega) = 0$. More generally, if (HC) is true, then $\|x\|_{L^2(\Omega \times \Lambda)} > 0$. Indeed, if $\|x\|_{L^2(\Omega \times \Lambda)} = 0$, we get $z = c + E(\omega, z)$ and, integrating with respect to μ, we obtain $z = c$, which is impossible. If (IM) and (HC) are verified, then π and u determine all the quantities involved by the validity of (IM) and (HC). Precisely, one has that $E(\omega, z)$ is the (u, T)-cocycle and that $x(\omega, z) = z - c - E(\omega, z)$. Therefore, when talking about (IM) and (HC), we will mention only the quantities (π, u). We also note that it will be a corollary of the proofs of the next results that if the harmonic decomposition (HC) exists then it is unique.

We consider now the case of a nonzero mean drift, that is, $c \neq 0$. The idea of the first part of the following result is taken from [29]. It can be proved by following exactly the end of the proof on page 24 of the same reference. The second part is a consequence of Theorem 4.1. We write $E_\mu[\ |\mathcal{F}_k]$ for the conditional expectation with respect to μ and a sub-σ-algebra $\mathcal{F}_k$.

THEOREM 4.3. *Assume that $\gamma(M,T) \neq 0$ and that (IM) and (HC) are verified with quantities (π, u). Introduce the σ-algebras $\mathcal{F}_k = \sigma\{T^l u, l \in \mathbb{Z}, l \leq k\}$ for $k \in \mathbb{Z}$.*

(i) *If*

$$\sum_{n\geq 1}\left(\int \left(E[u|\mathcal{F}_{-n}]\right)^2 d\mu\right)^{1/2} < +\infty, \tag{25}$$

then the annealed CLT holds; that is, under the measure $\int_\Omega \mathcal{P}_\omega \, d\mu(\omega)$,

$$\left(\frac{\xi_n(\omega) - nc}{\sqrt{n}}\right) \rightharpoonup \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \|x\|^2_{L^2(\Omega\times\Lambda)} + \left(\int u^2 \, d\mu + 2\sum_{n=1}^{+\infty}\int u T^n u \, d\mu\right) > 0. \tag{26}$$

(ii) *If $u = g - Tg$ with a bounded g, then the functional quenched CLT holds; that is, μ-a.e., under the measure $\mathcal{P}_\omega$,*

$$\left(\frac{\xi_{[nt]}(\omega) - c[nt]}{\sqrt{n}}\right)_{t\in[0,1]} \rightharpoonup \mathcal{W}(0,\sigma^2) \qquad \text{where } \sigma^2 = \|x\|^2_{L^2(\Omega\times\Lambda)} > 0.$$

REMARK. We will give expressions for u in Section 4.3. In case (i), the convergence may be functional. The hypotheses of the two cases are opposite. The first case concerns rather "independent" environments, whereas the second one deals essentially with less stochastic media. We will see it in the last section when considering an irrational rotation on the circle.

We now consider condition (HC). With the same notation as above, we investigate u by studying the equation $\tilde{P}x = 0$. It can be written as

$$\begin{aligned} &p_1(1-u) + p_{-1}(-1 + T^{-1}u) + \cdots \\ &\qquad + p_{-L}(-L + T^{-1}u + \cdots + T^{-L}u) = c. \end{aligned} \tag{27}$$

Setting $w_2 := 1 - u$, we obtain $w_2 = (p_{-1} + \cdots + p_{-L})/p_1 T^{-1}w_2 + \cdots + (p_{-L})/p_1 T^{-L}w_2 + c/p_1$. Introducing $W_2 := {}^t(w_2, T^{-1}w_2, \ldots, T^{-L+1}w_2)$, we get the relation $W_2 = MT^{-1}W_2 + (c/p_1)e_1$. Therefore, we obtain the following result.

PROPOSITION 4.4. *Under (IM) with a drift equal to c, condition (HC) is satisfied if and only if there exists a vector $W_2 := {}^t(w_2, T^{-1}w_2, \dots, T^{-L+1}w_2)$ such that $W_2 = MT^{-1}W_2 + (c/p_1)e_1$ and $u := 1 - w_2$ is in $L^1(\mu)$, $\int u\,d\mu = 0$ and $x(\omega, z) := z - E(\omega, z) - c$ is in $L^2(\Omega \times \Lambda)$, where E is the (u, T)-cocycle.*

4.2. *Characterization of the functional CLT in the recurrent case.*

THEOREM 4.5. *Assume that $\gamma(M, T) = 0$. The following assertions are equivalent*:

(i) *For μ-a.e. medium ω and under $\mathcal{P}_\omega$, $(n^{-1/2}\xi_{[nt]}(\omega))_{t\in[0,1]} \rightharpoonup \mathcal{W}(0, \sigma^2(\omega))$, with $\sigma^2(\omega) > 0$.*

(ii) *Conditions (IM) and (HC) are satisfied.*

(iii) *There exists a measurable $\varphi > 0$ such that φ and $1/\varphi$ are in $L^1(\mu)$ and $\lambda = \varphi/T\varphi$.*

In this situation, the diffusion coefficient is a constant σ^2 equal to

$$\sigma^2 = \frac{\int \langle PV, UPV\rangle w/(T\theta\lambda\varphi)\,d\mu}{\left[\int (v/\varphi)\,d\mu\right]^2 \int wT\varphi/(T\theta p_1)\,d\mu}, \tag{28}$$

where P and U are the following matrices of size $L \times L$:

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$$

and

$$U = \begin{pmatrix} \frac{p_{-1}}{p_1}\left(1 + \frac{p_{-1}}{p_1}\right) & \cdots & \cdots & \cdots \\ \vdots & \ddots & \frac{p_{-i}p_{-j}}{p_1^2} & \vdots \\ \vdots & \frac{p_{-i}p_{-j}}{p_1^2} & \ddots & \vdots \\ \vdots & \cdots & \cdots & \frac{p_{-L}}{p_1}\left(1 + \frac{p_{-L}}{p_1}\right) \end{pmatrix}.$$

The fact that (ii) implies (i) is Corollary 4.2. In the case when $L = 1$ and under the assumption that (p_1/p_{-1}) is a multiplicative coboundary, this theorem has been proved by Letchikov [23]. We will adopt the same strategy.

REMARK. In the case when $L = 1$, the formula (28) gives $1/\sigma^2 = (\int \varphi\, d\mu) \times (\int 1/\varphi\, d\mu)$. In the context of an environment defined by a two-sided subshift of finite type equipped with a Gibbs measure and if M is Hölder continuous, one can show the following alternative (see [6]): if $\gamma(M, T) = 0$, either condition (iii) of Theorem 4.5 is realized and then a functional CLT occurs or the conditions of Sinaï's theorem [27] are true, implying a behavior for the random walk with typical deviations of order $\log^2 n$.

PROOF OF (iii) $\Rightarrow$ (ii). Assume that $\lambda = \varphi/T\varphi$. From Theorem 3.5, condition (IM) is satisfied and the density π is

$$\pi = \frac{wT\varphi/(T\theta p_1)}{\int wT\varphi/(T\theta p_1)\, d\mu}.$$

We now search for harmonic coordinates. As $c = 0$, using Proposition 4.4, we have to solve $MT^{-1}W_2 = W_2$ with $W_2 = {}^t(w_2, T^{-1}w_2, \ldots, T^{-L+1}w_2)$. As $M(V/\varphi) = (TV/T\varphi)$, we choose $w_2 = (TV/T\varphi)/\int(v/\varphi)\, d\mu$ since $1/\varphi \in L^1(\mu)$. We have $\int w_2\; d\mu = 1$ and, taking the notation of Proposition 4.4, $u \in L^1(\mu)$ and $\int u\, d\mu = 0$. Then let $E(\omega, z)$ be the (u, T)-cocycle and $x(\omega, z) := z - E(\omega, z)$. The function x already verifies $\tilde{P}x = 0$. We now check that it belongs to $L^2(\Omega \times \Lambda)$:

$$\|x\|^2_{L^2(\Omega\times\Lambda)} = \int \left(p_1 w_2^2 + \sum_{i=1}^{L} p_{-i}(T^{-1}w_2 + \cdots + T^{-i}w_2)^2\right)\pi\, d\mu.$$

However, $w_2 = \sum_{i=1}^{L} p_{-i}/p_1(T^{-1}w_2 + \cdots + T^{-i}w_2) = Z_1 PT^{-1}W_2$, with $Z_1 = (p_{-1}/p_1, \ldots, p_{-L}/p_1)$. Therefore, with $Z_2 = \operatorname{diag}(p_{-1}, \ldots, p_{-L})$, we obtain

$$\begin{aligned}\|x\|^2_{L^2(\Omega\times\Lambda)} &= \int \left[{}^tT^{-1}W_2{}^tP(p_1{}^tZ_1Z_1 + Z_2)PT^{-1}W_2\right]\pi\, d\mu \\ &= \frac{\int {}^tV{}^tPUPVw/(T\theta\lambda\varphi)\, d\mu}{\left[\int(v/\varphi)\, d\mu\right]^2 \int wT\varphi/(T\theta p_1)\, d\mu} < +\infty\end{aligned}$$

as $1/\varphi \in L^1(\mu)$, which concludes the proof. $\square$

PROOF OF (i) $\Rightarrow$ (iii). We now assume that μ-a.e. the random walk $(\xi_n(\omega))_{n\geq 0}$ satisfies a nondegenerate functional CLT. A first lemma will use the properties of the Brownian motion. The second point has been proved in [9] by Bulycheva and Molchanov. For completeness, we give a proof. For any interval $[a, b]$ and any discrete-time or continuous-time process X, let $\tau_{a,b}(X)$ be the time to reach $]-\infty, a] \cup [b, +\infty[$ and $\tau_b^+(X)$ the time to reach $[b, +\infty[$ and similarly $\tau_a^-(X)$ the time to reach $]-\infty, a]$. For $\sigma > 0$, we write $\mathcal{W}^\sigma$ for the Brownian motion with variance σ^2 and $P_{\mathcal{W}^\sigma}$ for the corresponding Wiener measure on $D(0, 1)$.

LEMMA 4.6. (i) *We have* $\sigma^2(\omega) = \sigma^2 > 0$, μ-*a.e.*

(ii) *The following limit holds*: $P_0\{-N, N, +\} \to 1/2$ *as* $N \to +\infty$, μ-*a.e.*

(iii) *For any constant* $A > 0$,

$$P_0\{\tau_{-N,N} \le N^2 A\} \to P_{\mathcal{W}^\sigma}\big[\tilde{\tau}_{-1,1}(\mathcal{W}^\sigma) \le A\big] > 0 \qquad \text{as } N \to +\infty,\ \mu\text{-a.e.}$$

PROOF. Point (i) comes from the fact that $(\xi_n(T\omega) + 1)_{n\ge 0}$ is a Markov chain on $\mathbb{Z}$ with the same transition laws as $(\xi_n(\omega))_{n\ge 0}$ but with a variance equal to $\sigma(T\omega)$. As T is ergodic with respect to μ, the conclusion follows. Consider now (ii) and simplify $\tau_N^+((\xi_n(\omega)))$ and $\tau_N^-((\xi_n(\omega)))$ into τ_N^+ and τ_N^-. As these quantities are finite $\mathcal{P}_\omega$-a.e., μ-a.e., we have

$$P_0\{-N, N, +\} = \sum_{k=1}^{+\infty} P_0\{\tau_N^+ < \tau_{-N}^-, (k-1)^2 N^2 < \tau_{-N}^- \le k^2 N^2\}.$$

For $t \in [0, 1]$, define $W_{t,n}(\omega) := \xi_{[nt]}(\omega)/\sqrt{n}$. We observe that $\tau_{1/k}^+(W_{t,k^2N^2}) = \tau_N^+/(k^2N^2)$. Now set $P_0^{(k)} := \lim_{N\infty} P_0\{\tau_N^+ < \tau_{-N}^-, (k-1)^2 N^2 < \tau_{-N}^- \le k^2 N^2\}$. We will prove below that it is well defined. It can be written as

$$P_0^{(k)} = \lim_{N\infty} P_0\bigg\{\tau_{1/k}^+(W_{t,k^2N^2}) < \tau_{-1/k}^-(W_{t,k^2N^2}),\\ \left(\frac{k-1}{k}\right)^2 < \tau_{-1/k}^-(W_{t,k^2N^2}) \le 1\bigg\}.$$

Then set

$$A(k) := \left\{\tau_{1/k}^+(\mathcal{W}^\sigma) < \tau_{-1/k}^-(\mathcal{W}^\sigma), \left(\frac{k-1}{k}\right)^2 < \tau_{-1/k}^-(\mathcal{W}^\sigma) \le 1\right\}.$$

As $P_{\mathcal{W}^\sigma}(\partial A(k)) = 0$, using weak convergence to Wiener measure $P_{\mathcal{W}^\sigma}$ on $D(0, 1)$, we obtain that $P_0^{(k)}$ is well defined and that $P_0^{(k)} = P_{\mathcal{W}^\sigma}(A(k))$. From the self-similarity of the Brownian motion, we now get

$$\begin{aligned} P_0^{(k)} &= P_{\mathcal{W}^\sigma,\ \mathcal{C}[0,1]}(A(k)) = P_{\mathcal{W}^\sigma,\ \mathcal{C}(\mathbb{R}^+)}(A(k)) \\ &= P_{\mathcal{W}^\sigma,\ \mathcal{C}(\mathbb{R}^+)}\{\tau_1^+(\mathcal{W}^\sigma) < \tau_{-1}^-(\mathcal{W}^\sigma), (k-1)^2 < \tau_{-1}^-(\mathcal{W}^\sigma) \le k^2\}. \end{aligned}$$

Finally, summing on k from 1 to $+\infty$, we obtain, with Fatou's lemma,

$$\liminf P_0\{-N, N, +\} \ge P_{\mathcal{W}^\sigma,\ \mathcal{C}(\mathbb{R}^+)}\{\tau_1^+(\mathcal{W}^\sigma) < \tau_{-1}^-(\mathcal{W}^\sigma)\} = 1/2.$$

As the same treatment can be applied to $P_0\{-N, N, -\}$ and as $P_0\{-N, N, +\} + P_0\{-N, N, -\} = 1$, the conclusion follows. Point (iii) follows directly from the convergence to Wiener measure $P_{\mathcal{W}^\sigma}$. $\square$

PROOF OF (i) $\Rightarrow$ (iii). Let $A > 0$ be fixed and apply Lemma 2.6 with $k = 0$, $a = -N$, $b = N$ and A changed into $N^2 A$. We obtain, with a constant $C > 0$ that will become generic in the rest of the paper,

$$N^2A \geq CP_0\{-N,N,+\} \sum_{0\leq j\leq p\leq N-1} \delta(p,j+1)P_j\{\tau_{-N,N} \leq N^2A\}$$
$$+ C\sum_{j=-N+1}^{-1} P_0\{j,N,-\}P_j\{-N,N,+\}$$
$$\times\ P_j\{\tau_{-N,N} \leq N^2A\}\sum_{p=j}^{N-1}\delta(p,j+1),$$

as one can write, using the same decomposition as (9),

$$P_0\{j,N,-\} - P_0\{-N,N,-\} \geq P_0\{j,N,-\}\min_{0\leq t\leq L-1} P_{j-t}\{-N,N,+\}$$
$$\geq CP_0\{j,N,-\}P_j\{-N,N,+\}.$$

Therefore, using Lemmas 2.1 and 1.4,

$$A \geq (C/N^2)P_0\{-N,N,+\} \times \sum_{0\leq j\leq p\leq N-1}(T^p\lambda\cdots T^j\lambda)P_j\{\tau_{-N,N}\leq N^2A\} \tag{29}$$

$$+ (C/N^2)P_0\{-N,N,-\} \times \sum_{-N+1\leq p\leq j\leq -1}(T^j\lambda\cdots T^p\lambda)^{-1}P_j\{\tau_{-N,N}\leq N^2A\}. \tag{30}$$

Next, we use the fact that, for $-N \leq j \leq N$,

$$P_j\{\tau_{-N,N}\leq N^2A\}(\omega)$$
$$= P_0\{\tau_{-N-j,N-j}\leq N^2A\}(T^j\omega) \geq P_0\{\tau_{-2N,2N}\leq N^2A\}(T^j\omega).$$

Consider now (29) and, for any constant $D > 0$, integrate it with respect to μ:

$$A \geq C\int P_0\{-N,N,+\}\frac{1}{N^2}\sum_{0\leq j\leq p\leq N-1}(T^p\lambda\cdots T^j\lambda\wedge D) \times T^jP_0\{\tau_{-2N,2N}\leq N^2A\}\,d\mu. \tag{31}$$

Using Lemma 4.6(ii), we obtain

$$A \geq C\limsup\int\frac{1}{N^2}\Bigg[\sum_{0\leq k\leq N-1}(N-k)(T^{k-1}\lambda\cdots\lambda\wedge D)\Bigg]$$
$$\times\ P_0[\tau_{-2N,2N}\leq N^2A]\,d\mu.$$

Using Lemma 4.6(iii) in the same way and restricting the sum to the indices $0 \le k \le N/2$, we get, for a constant $C(A)$ depending only on A,

$$C(A) \ge \limsup \int \frac{1}{N} \sum_{0 \le k \le N-1} (T^{k-1}\lambda \cdots \lambda \wedge D)\, d\mu \tag{32}$$

$$\ge D \limsup \int \frac{1}{N} \sum_{0 \le k \le N-1} \mathbb{1}_{\{S_k(g) \ge \log(D)\}}\, d\mu, \tag{33}$$

with $g := \log \lambda$ and $S_k(g) = \sum_{l=0}^{k-1} T^l g$. Since the same reasoning can be used with (30), we finally obtain, for a constant $C(A)$ depending only on A and any $D > 0$,

$$C(A) \ge D \limsup \int \frac{1}{N} \sum_{0 \le k \le N-1} \mathbb{1}_{\{|S_k(g)| \ge \log(D)\}}\, d\mu.$$

From the following lemma, if g is not a measurable coboundary, then, for any $D > 0$,

$$\frac{1}{N} \sum_{0 \le k \le N-1} \mathbb{1}_{\{|S_k(g)| \ge \log(D)\}} \to 1, \qquad \mu\text{-a.e.}$$

We would therefore get $C(A) \ge D$ for any $D > 0$, which is a contradiction. Thus, there exists a measurable $\varphi > 0$ such that $\lambda = \varphi / T\varphi$. From (32), one then gets that, for any $D > 0$,

$$\begin{aligned} C(A) &\ge \limsup \int \frac{1}{N} \sum_{0 \le k \le N-1} \bigl((\varphi / T^k \varphi) \wedge D\bigr)\, d\mu \\ &\ge \limsup \int (\varphi \wedge \sqrt{D}) \frac{1}{N} \sum_{0 \le k \le N-1} \bigl((1/T^k\varphi) \wedge \sqrt{D}\bigr)\, d\mu \\ &\ge \biggl(\int \varphi \wedge \sqrt{D}\, d\mu\biggr)\biggl(\int (1/\varphi) \wedge \sqrt{D}\, d\mu\biggr). \end{aligned}$$

Now letting $D \to +\infty$, we get that φ and $(1/\varphi)$ belong to $L^1(\mu)$, which completes the proof. $\square$

The next result is proved in [10] in a slightly different form. It says that if g is not a measurable coboundary, then the density of the return times of $(S_n(g))$ in a bounded interval tends to 0.

LEMMA 4.7. *If g is not a measurable coboundary, then*

$$\forall\, A > 0, \qquad \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_{[-A,A]}\bigl(S_n(g)\bigr) \to 0, \qquad \mu\text{-a.e.}$$

PROOF. We consider the extended dynamical system $(\Omega \times \mathbb{R}, \mu \otimes \tilde{\lambda}, T_g)$, where $\tilde{\lambda}$ is Lebesgue measure and $T_g(\omega, y) = (T\omega, y + g(\omega))$. Therefore, we have $T_g^n(\omega, y) = (T^n\omega, y + S_n(g)(\omega))$. For $A > 0$, the function $\mathbb{1}_{[-A,A]}(y)$ is in $L^1(\Omega \times \mathbb{R}, \mu \otimes \tilde{\lambda})$. Fixing $A > 0$ and using the ergodic theorem (in infinite measure), we obtain the existence of a positive and measurable function $h = h(\omega, y)$ such that

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathbb{1}_{[-A,A]}\big(S_n(g)(\omega) + y\big) \to h, \qquad \mu \otimes \tilde{\lambda}\text{-a.e.}$$

The function h is T_g-invariant and belongs to $L^1(\Omega \times \mathbb{R}, \mu \otimes \tilde{\lambda})$, as by Fatou's lemma $\int h \le 2A$. We will prove the lemma by showing that $h = 0$, $(\mu \otimes \tilde{\lambda})$-a.e. If $(\mu \otimes \lambda)\{h > 0\} > 0$, as $h \in L^1(\Omega \times \mathbb{R}, \mu \otimes \lambda)$, using Fubini's theorem, we get $\int |h(\omega, y)|\,dy < +\infty$, μ-a.e. Thus, we define μ-a.e. a function $\theta(t, \omega)$, continuous in $t \in [0, 1]$, by

$$\theta(t, \omega) := \int h(\omega, y) e^{-2i\pi t y}\,dy.$$

As h is T_g-invariant, we have

$$\theta(t, \omega) = \left(\int h\big(T\omega, y + g(\omega)\big) e^{-2i\pi t(y+g(\omega))}\,dy\right) e^{2i\pi t g(\omega)} = e^{2i\pi t g(\omega)}\theta(t, T\omega).$$

From the ergodicity of T with respect to μ and the continuity in t, there exists a function $\alpha(t)$ such that $\forall\, t \in [0, 1]$, $|\theta(t, \omega)| = \alpha(t)$. Since h is not 0, $\mu \otimes \tilde{\lambda}$-a.e., using again the ergodicity of T with respect to μ, there exist $\delta_0 > 0$ and $\delta_1 > 0$ such that, μ-a.e.,

$$|t| \le \delta_0 \Rightarrow |\theta(t, \omega)| > \delta_1.$$

For $|t| \le \delta_0$, we set $\eta(t, \omega) := \theta(t, \omega)/|\theta(t, \omega)|$ which also satisfies $\eta(t, \omega)/\eta(t, T\omega) = e^{2i\pi t g(\omega)}$. Then μ-a.e., there exists a function $\psi(t, \omega)$ continuous in t and with $\psi(0, \omega) = 0$ such that $\eta_t(\omega) = \exp(2i\pi\psi(t, \omega))$. Therefore, if $|t| \le \delta$,

$$t g(\omega) = \psi(t, \omega) - \psi(t, T\omega) \qquad \mathrm{mod}(1).$$

As $t g(\omega) - \psi(t, \omega) - \psi(t, T\omega)$ belongs to $\mathbb{Z}$, is continuous in t and tends to 0 as $t \to 0$, it is equal to 0. We finally deduce that g is a measurable coboundary:

$$g(\omega) = \frac{1}{\delta_0}\psi(t, \omega) - \frac{1}{\delta_0}\psi(t, T\omega). \qquad \square$$

4.3. *Characterization of (IM) and (HC) in the transient cases.*

PROPOSITION 4.8. (i) *If $\gamma(M,T) < 0$, then conditions* (*IM*) *and* (*HC*) *are equivalent to*

$$\int \Biggl(\sum_{n=1}^{+\infty}(T^{-n}\lambda\cdots T^{-1}\lambda)\Biggr)^2 \Biggl(\sum_{p=0}^{+\infty}(T^{p-1}\lambda\cdots\lambda)\Biggr)\,d\mu < +\infty. \tag{34}$$

Then, *in the harmonic decomposition*, *u is given by*

$$u = 1 - c\sum_{n=0}^{+\infty}\delta(n-1,0)/T^{-n}p_1.$$

(ii) *If $\gamma(M,T) > 0$*, *then conditions* (*IM*) *and* (*HC*) *are equivalent to*

$$\int \Biggl(\sum_{n=0}^{+\infty}(T^{n}\lambda\cdots\lambda)^{-1}\Biggr)^2 \Biggl(\sum_{p=1}^{+\infty}(T^{-p}\lambda\cdots T^{-1}\lambda)^{-1}\Biggr)\,d\mu < +\infty. \tag{35}$$

Then, *in the harmonic decomposition*, *u is given by*

$$u = 1 + cTv\sum_{n=1}^{+\infty}\frac{T^n(w/T\theta)}{T^n p_1(T\lambda\cdots T^n\lambda)} - Z_2, \tag{36}$$

where Z_2 is a bounded map equal to

$$Z_2 = c\frac{1-wTv}{p_1T\theta} + c\sum_{n=1}^{+\infty}\bigl[\delta(0,-n+1) - (\lambda\cdots T^{-n+1}\lambda)T^{-n+1}vT^{-n}(w/p_1T\theta)\bigr].$$

PROOF. (i) Assume that $\gamma(M,T) < 0$ and (34) is satisfied. From Theorem 3.5, (IM) is true and the invariant density π is given by $\pi = cw_1/p_1$, where c is the drift, $w_1 = \langle e_1, W_1\rangle$ and, using Proposition 3.4, W_1 satisfies $T^{-1}W_1 = NW_1 + e_1$. Looking for harmonic coordinates, using Proposition 4.4, we have to solve the equation $W_2 = MT^{-1}W_2 + (c/p_1)e_1$, whose solution is

$$W_2 = c\Biggl[\biggl(\frac{1}{p_1}\biggr)e_1 + \sum_{n=0}^{+\infty}MT^{-1}M\cdots T^{-n}M\frac{e_1}{T^{-n-1}p_1}\Biggr]. \tag{37}$$

Setting $w_2 = \langle e_1, W_2\rangle = c\sum_{n=0}^{+\infty}\delta(n-1,0)/T^{-n}p_1$, we have the equalities

$$\begin{aligned}\langle NW_1, \Phi T^{-1}W_2\rangle &= \langle T^{-1}W_1, \Phi T^{-1}W_2\rangle - T^{-1}w_2\\ &= \langle W_1, T\Phi MT^{-1}W_2\rangle\\ &= \langle W_1, T\Phi W_2\rangle - cw_1/p_1.\end{aligned} \tag{38}$$

Therefore, $T^{-1}w_2 = cw_1/p_1 + \langle T^{-1}W_1, \Phi T^{-1}W_2\rangle - \langle W_1, T\Phi W_2\rangle$. Now, from (34), $\langle W_1, T\Phi W_2\rangle$ belongs to $L^1(\mu)$ and then $\int w_2\,d\mu = 1$. With the notation of Proposition 4.4, we obtain that the map u defining the cocycle E will be in $L^1(\mu)$ and that $\int u\,d\mu = 0$. As Λ is bounded, the condition (34) ensures that the harmonic part $x(\omega, z) = z - E(\omega, z) - c$ in the harmonic decomposition belongs to $L^2(\Omega \times \Lambda)$. Reciprocally, suppose that (IM) and (HC) are true. Then from Theorem 3.5, the invariant density is defined by $\pi = (c/p_1)\sum_{n=0}^{+\infty} \delta(n, 1)$. Similarly, since (HC) is true, there is a random vector W_2 such that $W_2 = MT^{-1}W_2 + (c/p_1)e_1$. Using the fact that $\gamma(M, T) < 0$ and Poincaré's recurrence theorem, we know that W_2 is defined by (37) and is then strictly positive. The integrability condition (34) follows.

(ii) Assume that $\gamma(M, T) > 0$ and that (35) is true. Then from Theorem 3.5 and Proposition 3.4, there exists an invariant density $\pi = (w_1/p_1)/(\int w_1/p_1\,d\mu)$, where W_1 has strictly positive components, belongs to $L^1(\mu)$ and verifies $NW_1 = T^{-1}W_1 + e_1$. One could also write $W_1 = \alpha W + R$, with α and R given by (18). Consider now the equation for harmonic coordinates $MT^{-1}W_2 = W_2 - (c/p_1)e_1$, where c is the drift. Let $\mathcal{H}(M, T)$ be the $(L-1)$-dimensional subspace of $\mathbb{R}^L$ corresponding to the strictly negative exponents of (M, T) and write $W_2 = \beta TV + X$, with $X \in T\mathcal{H}(M, T)$, looking for β and X. Decompose also $e_1 = tTV + Y$, $Y \in T\mathcal{H}(M, T)$. As in the proof of Theorem 3.5, t and Y are bounded maps and we get the equations $T^{-1}\beta\lambda - \beta - ct/p_1 = 0$ and $MT^{-1}X = X - (c/p_1)Y$. We then set

$$
\begin{aligned}
\beta &= -c\sum_{n=1}^{+\infty} \frac{T^n t}{T^n p_1 T\lambda \cdots T^n\lambda},\\
X &= c\left[\frac{Y}{p_1} + \sum_{n=1}^{+\infty} MT^{-1}M \cdots T^{-n+1}MT^{-n}\frac{Y}{p_1}\right].
\end{aligned}
\tag{39}
$$

As in Theorem 3.5, (X) is well defined. Let us show that it is bounded. From Lemma 2.5 and taking the same notation, we know that the vectors $(U^j(-1))_{1\le j\le L-1}$ form a basis of $\mathcal{H}(M, T)$. As in Lemma 2.5, choose $r < 1$ close to 1 such that the matrix $\tilde{M} := r^{-1}K_r M K_r^{-1}$, with $K_r = \mathrm{diag}(1, r, \ldots, r^{L-1})$, corresponds to another random walk and that $\gamma(\tilde{M}, T) > 0$. Similarly, we write $(\tilde{U}^j(-1))_{1\le j\le L-1}$ for the corresponding basis of $\mathcal{H}(\tilde{M}, T)$. Thus, with $V^j = K_r^{-1}\tilde{U}j(-1)$, we get that $(V^j)_{1\le j\le L-1}$ is also a basis of $\mathcal{H}(M, T)$ but the following relation holds:

$$
M(k)\cdots M(0)\ K_r^{-1}V^j = r^{k+1}K_r^{-1}\tilde{U}^j(k).
$$

We then decompose $T^{-1}Y$ in the basis $(V^j)_{1\le j\le L-1}$ in the form $T^{-1}Y = \sum_{j=1}^{L-1}\alpha_j V^j$. Now, as Y is bounded and from the form of the $(V^j)_{1\le j\le L-1}$ given in Lemma 2.5, we get from Cramér's formulas that the (α_j)'s are bounded maps and then

$$
MT^{-1}M\cdots T^{-n+1}MT^{-n}Y = r^{n+1}T^{-n+1}\left(\sum_{j=1}^{L-1} K_r^{-1}\alpha_j\tilde{U}^j(n-1)\right).
$$

Therefore, X is bounded. In fact, the same argument used in the proof of Theorem 3.5 when considering the projection on $\mathcal{H}({}^tN, T)$ for the decomposition of $W_1 = \alpha W + R$ shows that R is also bounded. Using the same trick as in (38), we get that

$$T^{-1}w_2 - c(w_1/p_1) = \langle W_1, T\Phi W_2\rangle - \langle T^{-1}W_1, \Phi T^{-1}W_2\rangle.$$

From the decompositions (18) and (39) and condition (35), we get that $\langle W_1, T\Phi W_2\rangle \in L^1(\mu)$ and then that $\int w_2\, d\mu = c\int w_1/p_1\, d\mu = 1$. Consequently, taking the notation of Proposition 4.4, the map u will be in $L^1(\mu)$ and $\int u\, d\mu = 0$. As in the previous case, (35) and the fact that Λ is bounded imply that the harmonic part $x(\omega, z) = z - E(\omega, z) - c$ belongs to $L^2(\Omega \times \Lambda)$.

Reciprocally, if (IM) and (HC) are true, then the equations $NW_1 = T^{-1}W_1 + e_1$ and $MT^{-1}W_2 = W_2 - (c/p_1)e_1$ have unique solutions of the form $W_1 = \alpha W + R$ and $W_2 = \beta TV + X$ with (α, R) and (β, X) given by (18) and (39). Moreover, the reasoning developed above says that R and X are bounded maps. One now observes that ${}^tM({}^tT\Phi W) = \rho({}^t\Phi T^{-1}W)$. As the dominant exponent $\gamma({}^tM, T^{-1})$ is simple, we deduce that ${}^tT\Phi W \perp Y$ and then $\langle {}^tT\Phi W, e_1\rangle = t\langle TV, {}^tT\Phi W\rangle$, giving $t = w/T\theta$. Similarly, we know from the proof of Theorem 3.5 that s is minored by a strictly positive constant. Then set $A := \sum_{n=0}^{+\infty}(T^n\lambda\cdots\lambda)^{-1}$ and $B := \sum_{p=1}^{+\infty}(T^{-p}\lambda\cdots T^{-1}\lambda)^{-1}$. From the finiteness of $\int u^2\pi\, d\mu$ and $\int B\, d\mu$, a consequence of Theorem 3.5, we get $\int A^2\pi\, d\mu < +\infty$, that is, $\int A^2 w_1\, d\mu < +\infty$. We now note that there is a constant $C > 0$ such that, for all $1 \le i \le L-1$, $A \le CT^iA$. Using the T-invariance of the measure μ, we get $\int A^2\|W_1\|\, d\mu < +\infty$. Finally, from (17), we obtain that $\int A^2B\, d\mu < +\infty$. $\square$

REMARK. Combining Proposition 4.8 with Theorem 4.3, we obtain sufficient conditions for the existence of the CLT in the transient cases. Concerning the annealed CLT in Theorem 4.3, our hypothesis (34) is a little weaker than in [29] and the other one (25) comes precisely from that paper. In the independent case with $L = 1$, those hypotheses are realized under reasonable assumptions of integrability and one can then check that the second term in the expression for the variance (26) given in Theorem 4.3 is in general not 0. Concerning the quenched CLT in Theorem 4.3, we have exactly the same hypotheses as in [1] but here the functional CLT holds for μ-a.e. medium and not under the annealed probability. However, we cannot express the value of the variance in such a convenient way as in [1], that is, in terms of the variance of some exit time.

5. An example: the circle case. We assume throughout this section that Ω is the circle S^1 identified with $[0, 1[$, that $T = T_\alpha$ is an irrational rotation with angle α and that μ is Lebesgue measure.

5.1. *Diophantine approximation and additive coboundary equation.* The irrational numbers of $]0,1[$ are usually classified according to the behavior of the sequence $(d(q\alpha,\mathbb{Z}))_{q\geq 1}$, where $d(x,\mathbb{Z})$ is the distance from a real x to $\mathbb{Z}$. An irrational α of $]0,1[$ is said to be of type η if

$$\eta = \sup\{t > 0,\ \liminf q^t d(q\alpha,\mathbb{Z}) = 0\}.$$

From Dirichlet's principle, we always have $\eta \geq 1$. For the Lebesgue measure, almost all numbers are of type 1. For instance, this is the case for every irrational algebraic on $\mathbb{Q}$ or having a bounded expansion in continued fraction.

Let $m \geq 0$ be an integer and let δ be such that $0 \leq \delta < 1$. We will say that a real-valued or vector-valued function defined on S^1 is $C^{m+\delta}$ if it is m times differentiable and if its mth derivative is δ-Hölder continuous. The following result on sufficient conditions for solving a coboundary equation can be found in [2] in a slightly different form.

LEMMA 5.1. *Let f be a $C^{m+\delta}$ real-valued function and let α be of finite type η. If $\int f\,d\mu = 0$ and $m+\delta > \eta$, then there exists a continuous real-valued function g such that $f = g - Tg$.*

PROOF. Consider the Fourier expansion of f, $(c_n(f))_{n\in\mathbb{Z}}$. From the hypothesis on the regularity of f, we have $c_n(f) = O(n)^{-(m+\delta)}$. We now look for the expansion of some function g such that $f = g - Tg$. For $n \neq 0$, we formally obtain coefficients $(c_n(g))$, whose absolute value satisfies $|c_n(g)| = |c_n(f)|/(2|\sin(\pi n\alpha)|)$. We only need to prove that

$$\sum_{n=1}^{+\infty} \frac{1}{n^{m+\delta}\, d(n\alpha,\mathbb{Z})} < +\infty. \tag{40}$$

Using now the definition of the type η, for all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$d(q\alpha,\mathbb{Z}) \geq C(\varepsilon) q^{-(\eta+\varepsilon)} \qquad \text{for } q \geq 1.$$

We fix $i \geq 0$ and consider the strictly increasing subsequence of indices $(n_k^{(i)})_{k\geq 1}$ such that

$$2^{-(i+1)} \leq d\big(n_k^{(i)}\alpha,\mathbb{Z}\big) < 2^{-i}.$$

Write S_i for the sum restricted to these indices. To prove (40), we only need to show that $\sum_{i=1}^{+\infty} S_i$ is finite. We first have

$$d\big((n_{k+1}^{(i)} - n_k^{(i)})\alpha,\mathbb{Z}\big) < 2^{-(i-1)}.$$

Consequently,

$$\frac{1}{2^{i-1}} > \frac{C(\varepsilon)}{N_i^{\eta+\varepsilon}} \qquad \text{where } N_i = \min\{n_1^{(i)}, n_{k+1}^{(i)} - n_k^{(i)}, k > 0\}.$$

Therefore, $N_i > (2^{i-1}C(\varepsilon))^{1/(\eta+\varepsilon)}$. As $n_1^{(i)} \geq N_i$ and $n_k^{(i)} \geq kN_i$, using the previous results and observing that $m+\delta > \eta \geq 1$, for some constant $D(\varepsilon) > 0$ we have

$$S_i \leq \sum_{k=1}^{+\infty} \frac{2^{i+1}}{(kN_i)^{m+\delta}} = \frac{2^{i+1}}{N_i^{m+\delta}}\left(\sum_{k=1}^{+\infty}\frac{1}{k^{m+\delta}}\right) \leq D(\varepsilon)2^{i(1-(m+\delta)/(\eta+\varepsilon))}.$$

We finally choose ε such that $\eta+\varepsilon < m+\delta$ to ensure summability. $\square$

5.2. *On the regularity of V and λ.*

THEOREM 5.2. *Assume that M is $\mathcal{C}^{m+\delta}$. Then (V, λ) and (W, ρ) are also $\mathcal{C}^{m+\delta}$.*

We begin with a lemma showing that the study of the regularity of V and λ can be proved by studying the function built in Proposition 2.8.

LEMMA 5.3. *Let f be the function built in Proposition* 2.8. *Then V and λ are $\mathcal{C}^{m+\delta}$ if and only if f is $\mathcal{C}^{m+\delta}$.*

PROOF. From Proposition 2.8, we deduce the relation $T^{-L+1}fG = MT^{-1}G$, with

$$G := {}^t(fT^{-1}f\cdots T^{-L+2}f, T^{-1}f\cdots T^{-L+2}f, \ldots, T^{-L+2}f, 1).$$

Therefore, $V = T^{-1}G/\|T^{-1}G\|$ and $\lambda = T^{-L+1}f\|G\|/\|T^{-1}G\|$. The proof of the inverse statement is similar. $\square$

Recall now that the function f was the uniform limit of the sequence of random variables $(f_n)_{n\geq 0}$ defined by $f_p = 1$ for $0 \leq p \leq L-2$ and for $n \geq L-1$ by the relation

$$f_n = \sum_{i=1}^{L}\frac{a_i}{\prod_{k=1}^{i-1}T^{-k}f_{n-k}} \qquad \text{convening that } \prod_{k=1}^{0} = 1. \tag{41}$$

We begin by evaluating precisely the quantities $f_n(\omega_1) - f_n(\omega_2)$ for $(\omega_1, \omega_2) \in S^1 \times S^1$.

LEMMA 5.4. *Let $(\omega_1, \omega_2) \in S^1 \times S^1$. Then, for $n \geq L-1$,*

$$\begin{aligned}1 - \frac{f_n(\omega_2)}{f_n(\omega_1)} &= \sum_{i=1}^{L}\frac{a_i(\omega_1) - a_i(\omega_2)}{\prod_{k=0}^{i-1}T^{-k}f_{n-k}(\omega_1)} \\ &\quad + \sum_{k=1}^{L-1}\left[\prod_{l=0}^{k-1}\frac{T^{-l}f_{n-l}(\omega_2)}{T^{-l}f_{n-l}(\omega_1)}\right]h_{n,k}(\omega_2)\left[\frac{T^{-k}f_{n-k}(\omega_2)}{T^{-k}f_{n-k}(\omega_1)} - 1\right],\end{aligned}$$

where we set

$$h_{n,k}(\omega_2) := \left[1 - \sum_{i=1}^{k} \frac{a_i(\omega_2)}{\prod_{p=0}^{i-1} T^{-p} f_{n-p}(\omega_2)}\right].$$

PROOF. For $i \geq 2$, consider first the quantity

$$D_i := \left(\frac{a_i}{\prod_{k=1}^{i-1} T^{-k} f_{n-k}}\right)(\omega_1) - \left(\frac{a_i}{\prod_{k=1}^{i-1} T^{-k} f_{n-k}}\right)(\omega_2).$$

Expressing D_i in successive differences, we have

$$\begin{aligned} D_i &= \frac{a_i(\omega_1) - a_i(\omega_2)}{\prod_{k=1}^{i-1} T^{-k} f_{n-k}(\omega_1)} \\ &\quad + \sum_{l=0}^{i-2} \frac{a_i(\omega_2)}{\prod_{p=1}^{l} T^{-p} f_{n-l}(\omega_1)} \left[\frac{1}{T^{-l-1} f_{n-l-1}(\omega_1)} - \frac{1}{T^{-l-1} f_{n-l-1}(\omega_2)}\right] \\ &\qquad \times \frac{1}{\prod_{p=l+2}^{i-1} T^{-p} f_{n-p}(\omega_2)}. \end{aligned}$$

Therefore, summing the D_i's from $i = 2$ to L and adding $(a_1(\omega_1) - a_1(\omega_2))$, we deduce that

$$\begin{aligned} f_n(\omega_1) - f_n(\omega_2) &= \sum_{i=1}^{L} \frac{a_i(\omega_1) - a_i(\omega_2)}{\prod_{k=1}^{i-1} T^{-k} f_{n-k}(\omega_1)} \\ &\quad + \sum_{l=1}^{L-1} \left[\frac{1}{T^{-l} f_{n-l}(\omega_1)} - \frac{1}{T^{-l} f_{n-l}(\omega_2)}\right] \\ &\qquad \times \frac{1}{\prod_{p=1}^{l-1} T^{-p} f_{n-p}(\omega_1)} \sum_{i=l+1}^{L} \frac{a_i(\omega_2)}{\prod_{p=l+1}^{i-1} T^{-p} f_{n-p}(\omega_2)}. \end{aligned} \tag{42}$$

Next, from (41) we have

$$\begin{aligned} &\sum_{i=l+1}^{L} \frac{a_i(\omega_2)}{\prod_{p=l+1}^{i-1} T^{-p} f_{n-p}(\omega_2)} \\ &\quad = \left[\prod_{r=0}^{l} T^{-r} f_{n-r}(\omega_2)\right] \sum_{i=l+1}^{L} \frac{a_i(\omega_2)}{\prod_{s=0}^{i-1} T^{-s} f_{n-s}(\omega_2)} \\ &\quad = \left[\prod_{r=0}^{l} T^{-r} f_{n-r}(\omega_2)\right] h_{n,l}(\omega_2). \end{aligned} \tag{43}$$

Finally, dividing each side of (42) by $f_n(\omega_1)$ and using (43), we get the result. □

Recall now that we write $(e_i)_{1\le i\le L}$ for the canonical basis of $\mathbb{R}^L$. We introduce the notation $(\zeta_i)_{1\le i\le L-1}$ for the canonical basis of $\mathbb{R}^{L-1}$. The following proposition computes exactly the differences $f_n(\omega_1) - f_n(\omega_2)$ for $(\omega_1,\omega_2) \in S^1 \times S^1$ in a convenient way.

PROPOSITION 5.5. *There exists $\tau > 0$ such that if $|\omega_1 - \omega_2| \le \tau$, then*

$$
\begin{aligned}
&1 - \frac{f(\omega_2)}{f(\omega_1)} \\
&\quad = \sum_{k=0}^{+\infty} h(T^{-k}\omega_1, T^{-k}\omega_2)\langle e_1, (Q-I)T^{-1}Q\cdots T^{-k+1}Qe_1\rangle(\omega_1,\omega_2),
\end{aligned}
\tag{44}
$$

where

$$
h(\omega_1,\omega_2) := \sum_{i=1}^{L} \frac{a_i(\omega_1) - a_i(\omega_2)}{\prod_{k=0}^{i-1} T^{-k} f(\omega_1)}
$$

and $Q(\omega_1,\omega_2)$ is the following matrix of size $L \times L$:

$$
Q(\omega_1,\omega_2) := \begin{pmatrix}
1-\rho^1 & \rho^1 - \rho^2 & \cdots & \rho^{L-2} - \rho^{L-1} & \rho^{L-1} \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 1 & 0
\end{pmatrix},
$$

where, for $1 \le k \le L-1$,

$$
\rho^k(\omega_1,\omega_2) := \left[\prod_{l=0}^{k-1} T^{-l}\left(\frac{f(\omega_2)}{f(\omega_1)}\right)\right]\left[1 - \sum_{i=1}^{k} \frac{a_i(\omega_2)}{\prod_{p=0}^{i-1} T^{-p} f(\omega_2)}\right].
$$

PROOF. For $n \ge 0$, introduce

$$
g_n(\omega_1,\omega_2) := 1 - \frac{f_n(\omega_2)}{f_n(\omega_1)}, \qquad h_n(\omega_1,\omega_2) := \sum_{i=1}^{L} \frac{a_i(\omega_1) - a_i(\omega_2)}{\prod_{k=0}^{i-1} T^{-k} f_{n-k}(\omega_1)}
$$

and

$$
\rho_n^k(\omega_1,\omega_2) := \left[\prod_{l=0}^{k-1} \frac{T^{-l} f_{n-l}(\omega_2)}{T^{-l} f_{n-l}(\omega_1)}\right]\left[1 - \sum_{i=1}^{k} \frac{a_i(\omega_2)}{\prod_{p=0}^{i-1} T^{-p} f_{n-p}(\omega_2)}\right].
$$

With $G_n(\omega_1,\omega_2) := {}^t(g_n(\omega_1,\omega_2), T^{-1}g_{n-1}(\omega_1,\omega_2), \dots, T^{-L+2}g_{n-L+2}(\omega_1,\omega_2))$ and the following matrix P_n of size $(L-1)\times(L-1)$:

$$P_n(\omega_1,\omega_2) := \begin{pmatrix} -\rho_n^1(\omega_1,\omega_2) & -\rho_n^2(\omega_1,\omega_2) & \cdots & -\rho_n^{L-1}(\omega_1,\omega_2) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

we deduce from Lemma 5.4 that

$$G_n(\omega_1,\omega_2) = P_n(\omega_1,\omega_2)T^{-1}G_{n-1}(\omega_1,\omega_2) + h_n(\omega_1,\omega_2)\zeta_1. \tag{45}$$

As the sequence (f_n) converges uniformly to f and since condition (2) holds, there exist constants $\tau > 0$ and $\delta_0 > 0$ such that, for $|w_1 - w_2| \le \tau$ and for all $n \ge L-1$,

$$\begin{aligned} 1 &\ge \rho_n^1(\omega_1,\omega_2) + \delta_0 \ge \rho_n^2(\omega_1,\omega_2) + 2\delta_0 \ge \cdots \\ &\ge \rho_n^L(\omega_1,\omega_2) + L\delta_0 \ge (L+1)\delta_0. \end{aligned} \tag{46}$$

Omitting the dependence on (ω_1,ω_2), using (45) and the fact that $G_{L-2} = 0$, a consequence of the fact that the initial values $(f_p)_{0\le p\le L-2}$ all equal 1, we have

$$G_n = h_n\zeta_1 + P_n\zeta_1T^{-1}h_{n-1} + \cdots + (P_n\cdots T^{-n+L}P_L)\zeta_1T^{-n+L-1}h_{L-1}.$$

Consequently, taking the scalar product with ζ_1, we obtain

$$\begin{aligned} g_n = h_n &+ \langle\zeta_1, P_n\zeta_1\rangle T^{-1}h_{n-1} + \cdots \\ &+ \langle\zeta_1, P_n\cdots T^{-n+L}P_L\zeta_1\rangle T^{-n+L-1}h_{L-1}. \end{aligned} \tag{47}$$

Consider now the following matrix $Q_n = Q_n(\omega_1,\omega_2)$ of size $(L\times L)$:

$$Q_n := \begin{pmatrix} 1-\rho_n^1 & \rho_n^1-\rho_n^2 & \cdots & \rho_n^{L-2}-\rho_n^{L-1} & \rho_n^{L-1} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

The next lemma says that, for any $l \ge 0$,

$$\langle\zeta_1, P_n\cdots T^{-l}P_{n-l}\zeta_1\rangle = \langle e_1, (Q_n - I)T^{-1}Q_{n-1}\cdots T^{-l}Q_{n-l}e_1\rangle.$$

Replacing in (47), we therefore obtain

$$\begin{aligned} g_n = h_n &+ \langle\zeta_1, (Q_n - I)\zeta_1\rangle T^{-1}h_{n-1} + \cdots \\ &+ \langle\zeta_1, (Q_n - I)T^{-1}Q_{n-1}\cdots T^{-n+L}Q_L\zeta_1\rangle\ T^{-n+L-1}h_{L-1}. \end{aligned} \tag{48}$$

We now observe that $Q_n(\omega_1,\omega_2)$ is a stochastic matrix when $|\omega_1-\omega_2| \le \tau$. Using the explicit value for the contraction coefficient in Proposition 1.1 for any matrix of the type $T^{-1}Q_{i_1}\cdots T^{-L}Q_{i_L}$, which has strictly positive entries, and the existence of δ_0 defined in (46), we deduce that $T^{-1}Q_{n-1}\cdots T^{-k+1}Q_{n-k+1}x_1$ converges in direction as $k\to+\infty$ at an exponential rate uniformly in ω to the direction of ${}^t(1,\dots,1)$, independently on n. Since this quantity remains bounded, as the matrices are stochastic, we finally obtain the result. $\square$

LEMMA 5.6. *Let $Q_n = Q_n(\omega_1,\omega_2)$ be defined as above. Then, for any $l\ge 0$,*

$$\langle \zeta_1, P_n\cdots T^{-l}P_{n-l}\zeta_1\rangle = \langle e_1, (Q_n - I)T^{-1}Q_{n-1}\cdots T^{-l}Q_{n-l}e_1\rangle.$$

PROOF. We first check that

$$P_n = {}^t\big[({}^tQ_n) \text{ restricted to } \big({}^t(1,\dots,1)\big)^{\perp} \text{ in the basis } (e_i - e_{i+1})_{1\le i\le L-1}\big]. \tag{49}$$

This way we introduce $i_j = e_j - e_{j+1}$ for $1\le j\le L-1$. We first have ${}^tQ_n i_j = i_{j-1}$ for $j\ge 2$. Furthermore,

$$\begin{aligned}{}^tQ_n i_1 &= -\rho_n^1 e_1 + (\rho_n^1 - \rho_n^2)e_2 + \cdots + (\rho_n^{L-2} - \rho_n^{L-1})e_{L-1} + \rho_n^{L-1}e_L\\ &= -\rho_n^1 i_1 - \cdots - \rho_n^{L-1} i_{L-1},\end{aligned}$$

which proves the previous claim. Then, using the fact that $({}^tQ_n - I)e_1 = {}^tQ_n i_1$, we have

$$\begin{aligned}&\langle e_1, (Q_n - I)T^{-1}Q_{n-1}\cdots T^{-l}Q_{n-l}e_1\rangle\\ &\quad= \langle e_1, T^{-l}({}^tQ_{n-l})\cdots T^{-1}({}^tQ_{n-1})({}^tQ_n - I)e_1\rangle\\ &\quad= \langle \zeta_1, T^{-l}({}^tP_{n-l})\cdots({}^tP_n)\zeta_1\rangle\\ &\quad= \langle \zeta_1, P_n\cdots T^{-l}P_{n-l}\zeta_1\rangle.\end{aligned}$$

$\square$

PROOF OF THEOREM 5.2. The case $m = 0$ is an application of Proposition 5.5. Consider now the situation when $m\ge 1$ and $0\le\delta<1$. We begin by computing the first derivative. We will then see that for the next derivatives the form of the calculus will be preserved. Using relation (45), we have

$$J_n(\omega) = R_n(\omega)\; T^{-1}J_{n-1}(\omega) + k_n(\omega)\zeta_1, \tag{50}$$

where we define $J_n := {}^t(j_n, T^{-1}j_{n-1},\dots,T^{-L+2}j_{n-L+2})$ with

$$j_n = \frac{f_n'}{f_n} \quad\text{and}\quad k_n = \frac{1}{f_n}\sum_{i=1}^{L}\frac{a_i'}{\prod_{k=1}^{i-1}T^{-k}f_{n-k}} \tag{51}$$

and the following random matrix R_n:

$$R_n = \begin{pmatrix} -\theta_n^1 & -\theta_n^2 & \cdots & -\theta_n^{L-1}\\ 1 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 1 & 0\end{pmatrix},$$

with

$$\theta_n^k = 1 - \sum_{i=1}^{k} \frac{a_i}{\prod_{p=0}^{i-1} T^{-p} f_{n-p}} \qquad \text{for } 1 \le k \le L-1.$$

Using (48), we get

$$\begin{aligned} j_n &= k_n + \langle e_1, (S_n - I)e_1 \rangle T^{-1} k_{n-1} + \cdots \\ &\quad + \langle e_1, (S_n - I) T^{-1} S_{n-1} \cdots T^{-n+L} S_L e_1 \rangle \, T^{-n+L-1} k_{L-1}, \end{aligned} \tag{52}$$

where we introduce the random matrix S_n:

$$S_n = \begin{pmatrix} 1-\theta_n^1 & \theta_n^1 - \theta_n^2 & \cdots & \theta_n^{L-2} - \theta_n^{L-1} & \theta_n^{L-1} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

As before, there exists a constant $\delta_1 > 0$ such that

$$\forall\, n \ge L-1, \qquad 1 \ge \theta_n^1 + \delta_1 \ge \theta_n^2 + 2\delta_1 \ge \cdots \ge \theta_n^L + L\delta_1 \ge (L+1)\delta_1.$$

Then S_n is a stochastic matrix. Using the previous argument of uniform directional contraction, we conclude that the sequence (j_n) converges uniformly with respect to ω. Therefore, f is C^1. Assume now that $m \ge 2$. We consider the next derivatives up to the $(m-1)$th. From (50), for any $1 \le k \le m-1$, we have

$$J_n^{(k)} = R_n T^{-1} J_{n-1}^{(k)} + \left(\sum_{l=0}^{k-1} C_k^l R_n^{(k-l)} T^{-1} J_{n-1}^{(l)} + k_n^{(k)} \zeta_1 \right). \tag{53}$$

We now observe that all the lines of R_n are constant except the first one. Consequently, there exists a scalar t_n such that

$$t_n \zeta_1 = \sum_{l=0}^{k-1} C_k^l R_n^{(k-l)} T J_{n-1}^{(l)} + k_n^{(k)} \zeta_1.$$

Hence,

$$\begin{aligned} j_n^{(k)} &= t_n + \langle \zeta_1, (S_n - I)\zeta_1 \rangle T^{-1} t_{n-1} + \cdots \\ &\quad + \langle \zeta_1, (S_n - I) T^{-1} S_{n-1} \cdots T^{-n+L} S_L \zeta_1 \rangle T^{-n+L-1} t_{L-1}. \end{aligned} \tag{54}$$

Next, if, for $0 \le l \le k$, the sequences of derivatives $(f_n^{(l)})_{n \ge 0}$ converge uniformly, we deduce that the sequence of scalars (t_n) converges also uniformly. As before, from relation (54), we conclude that the sequence $(f_n^{(k+1)})_{n \ge 0}$ also converges uniformly. In every case, if $m = 1$ or if $m \ge 2$, we arrive at the equality

$$J_n^{(m-1)} = R_n T^{-1} J_{n-1}^{(m-1)} + u_n \zeta_1,$$

where the sequence of scalars (u_n) converges uniformly. Hence, we deduce that, for any (ω_1, ω_2),

$$\begin{aligned} J_n^{(m-1)}(\omega_1) - J_n^{(m-1)}(\omega_2) = R_n(\omega_2)\big[T^{-1}J_{n-1}^{(m-1)}(\omega_1) - T^{-1}J_{n-1}^{(m-1)}(\omega_2)\big] \\ + \big[R_n(\omega_1) - R_n(\omega_2)\big]T^{-1}J_{n-1}^{(m-1)}(\omega_1) \\ + \big(u_n(\omega_1) - u_n(\omega_2)\big)\zeta_1. \end{aligned}$$

Now set

$$v_n(\omega_1, \omega_2)\zeta_1 := \big[R_n(\omega_1) - R_n(\omega_2)\big]T^{-1}J_{n-1}^{(m-1)}(\omega_1) + \big(u_n(\omega_1) - u_n(\omega_2)\big)\zeta_1.$$

Therefore,

$$\begin{aligned} & j_n^{(m-1)}(\omega_1) - j_n^{(m-1)}(\omega_2) \\ &\quad = v_n + \langle e_1, (S_n - I)e_1\rangle T^{-1}v_{n-1} + \cdots \\ &\qquad + \langle e_1, (S_n - I)T^{-1}S_{n-1}\cdots T^{-n+L}S_L e_1\rangle T^{-n+L-1}v_{L-1}. \end{aligned}$$

Letting $n \to +\infty$, using the directional contraction as before and writing $j^{(m-1)}$, u and S for the respective limits of $j_n^{(m-1)}$, u_n and S_n, we obtain

$$\begin{aligned} & j^{(m-1)}(\omega_1) - j^{(m-1)}(\omega_2) \\ &\quad = \sum_{k=0}^{+\infty} T^{-k}v(\omega_1, \omega_2)\langle e_1, (S - I)T^{-1}S(\omega_2)\cdots T^{-k+1}Se_1\rangle(\omega_2). \end{aligned} \tag{55}$$

From (55), we deduce that $j^{(m-1)}$ is δ-Hölder continuous. Indeed, we have

$$v(\omega_1, \omega_2) = \big\langle \zeta_1, \big[R(\omega_1) - R(\omega_2)\big]T^{-1}J^{(m-1)}(\omega_2)\big\rangle + u(\omega_1) - u(\omega_2).$$

As R is differentiable, we only need to consider $u(\omega_1) - u(\omega_2)$. However, from (53), we have

$$u = \sum_{l=0}^{m-2} C_{m-1}^{l}\langle \zeta_1, R^{(m-1-l)}T^{-1}J^{(l)}\rangle + k^{(m-1)}.$$

Since $\sum_{l=0}^{m-2} C_{m-1}^{l}\langle \zeta_1, R^{(m-1-l)}T^{-1}J^{(l)}\rangle$ is differentiable, we observe that we are in the same situation as in the case when $m = 1$; that is, we only need to consider $k^{(m-1)}$. However, $k^{(m-1)}$ is δ-Hölder continuous from the definition of k given in (51). Consequently, $j^{(m-1)}$ is δ-Hölder continuous and therefore also $f^{(m)}$.

From Lemma 5.3, we deduce that V and λ are $\mathcal{C}^{m+\delta}$. Concerning W and ρ, the proofs of Proposition 2.8 and of the present theorem can be extended, changing only $(a_i)_{1\le i\le L}$ into $(T^{i-1}a_i)_{1\le i\le L}$ and T into T^{-1}. $\square$

5.3. *Application to the central limit theorem.*

THEOREM 5.7. *Let $\Omega = S^1$, let $T = T_\alpha$ be an irrational rotation and let μ be Lebesgue measure. Assume that α is of finite type η and that M is $\mathcal{C}^{m+\delta}$ with $m+\delta > \eta$.*

(i) *If $\gamma(M,T) = 0$, then $(\xi_n(\omega))$ is recurrent, $\mathcal{P}_\omega$-a.e., μ-a.e., and the functional quenched CLT of Corollary* 4.2 *holds.*

(ii) *If $\gamma(M,T) < 0$, then $n^{-1}\xi_n(\omega) \to c > 0$, $\mathcal{P}_\omega$-a.e., μ-a.e., and the functional quenched CLT of Theorem* 4.3 *holds.*

(iii) *If $\gamma(M,T) > 0$, then $n^{-1}\xi_n(\omega) \to c < 0$, $\mathcal{P}_\omega$-a.e., μ-a.e., and the functional quenched CLT of Theorem* 4.3 *holds.*

PROOF. Case (i) follows from Theorems 4.5 and 5.2. Concerning (ii), condition (34) is satisfied as λ is continuous and the dynamical system (S^1, T_α, μ) is uniquely ergodic, giving that the convergence in (34) is uniform. Taking the notation we employed for harmonic coordinates, the expression of u given in Proposition 4.8 says that u is $\mathcal{C}^{m+\delta}$. From Lemma 5.1, we get that u is a continuous coboundary and we can then apply the second point of Theorem 4.3. Consider now (iii). Condition (35) is satisfied for the same reason as above. We now show that the u involved in the harmonic decomposition is $\mathcal{C}^{m+\delta}$. In formula (36), the first term is $\mathcal{C}^{m+\delta}$, as w, λ and θ are $\mathcal{C}^{m+\delta}$. With the notation of Section 4.3, we consider now Z_2. From the defining relation (39), we have $Z_2 = \langle e_1, X\rangle$ and we note that $Y = e_1 - tTV$ with $t = w/T\theta$ is also $\mathcal{C}^{m+\delta}$. Using the method of projection on the subspace of $\mathcal{H}({}^tM, T^{-1})$, defined as the $(L-1)$-dimensional subspace of $\mathbb{R}^L$ of vectors that have a strictly negative exponent with respect to $({}^tM, T^{-1})$, as in the proof of the third point of Theorem 3.5, we finally obtain that Z_2 is $\mathcal{C}^{m+\delta}$ and then also u. $\square$

Acknowledgments. I am grateful to my Ph.D. advisor J.-P. Conze for his perceptive guidance and attention, and to Y. Guivarc'h, H. Hennion and A. Raugi for useful discussions. I also thank the referee for essential remarks and for pointing out missing references.

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