The Annals of Probability
1997, Vol. 25, No. 2, 662–701

NONLINEAR PARABOLIC P.D.E. AND ADDITIVE FUNCTIONALS OF SUPERDIFFUSIONS[1]

BY E. B. DYNKIN AND S. E. KUZNETSOV

Cornell University, and
Russian Academy of Sciences and Cornell University

Suppose that E is an arbitrary domain in $\mathbb{R}^d$, L is a second order elliptic differential operator in $S = \mathbb{R}_+ \times E$ and S^e is the extremal part of the Martin boundary for the corresponding diffusion ξ. Let $1 < \alpha \le 2$. We investigate a boundary value problem

$$(*)\qquad \begin{aligned} \frac{\partial u}{\partial r} + Lu - u^\alpha &= -\eta \quad \text{in } S,\\ u &= \nu \quad \text{on } S^e,\\ u &= 0 \quad \text{on } \{\infty\} \times E \end{aligned}$$

involving two measures η and ν. For the existence of a solution, we give sufficient conditions in terms of a Martin capacity and necessary conditions in terms of hitting probabilities for an (L, α)-superdiffusion X. If a solution exists, then it can be expressed by an explicit formula through an additive functional A of X.

An (L, α)-superdiffusion is a branching measure-valued process. A natural linear additive (NLA) functional A of X is determined uniquely by its potential h defined by the formula $P_\mu A(0, \infty) = \int h(r, x)\mu(dr, dx)$ for all $\mu \in \mathscr{M}^*$ (the determining set of A). Every potential h is an exit rule for ξ and it has a unique decomposition into extremal exit rules. If η and ν are measures which appear in this decomposition, then $(*)$ can be replaced by an integral equation

$$(**)\qquad u(r, x) + \int p(r, x; t, dy)u(t, y)^\alpha \, ds = h(r, x),$$

where $p(r, x; t, dy)$ is the transition function of ξ. We prove that h is the potential of a NLA functional if and only if $(**)$ has a solution u. Moreover,

$$u(r, x) = -\log P_{r,x} e^{-A(0,\infty)}.$$

By applying these results to homogeneous functionals of time-homogeneous superdiffusions, we get a stronger version of theorems proved in an earlier publication. The foundation for our present investigation is laid by a general theory developed in the accompanying paper.

0. Introduction.

0.1. *Linear equation.* Suppose that L is a second order elliptic operator in $\mathbb{R}^d$, D is a bounded domain with smooth boundary ∂D, $\rho \ge 0$ is a Hölder

Received September 1995; revised October 1996.

[1]Partially supported by NSF Grant DMS-93-01315.

AMS 1991 *subject classifications*. Primary 60J60, 35J65; secondary 60J80, 31C15, 60J25, 60J55, 31C45, 35J60.

Key words and phrases. Measure-valued processes, branching, natural linear additive functionals, probabilistic solutions of PDEs, boundary value problems involving measures, potentials, log-potentials, capacities, polar sets.

continuous function in D and $\varphi \geq 0$ is a bounded continuous function on ∂D. The solution of the boundary value problem

$$
\begin{aligned}
Lh &= -\rho \quad \text{in } D, \\
h &= \varphi \quad \text{on } \partial D,
\end{aligned}
\tag{0.1}
$$

can be expressed by formula

$$
h(x) = \int_D g(x, y)\rho(y)\,dy + \int_{\partial D} k(x, y)\varphi(y)\sigma(dy),
\tag{0.2}
$$

where $g(x, y)$ is the Green's function of L in D, $k(x, y)$ is the Poisson kernel and σ is the surface area on ∂D. For arbitrary finite measures η on D and ν on ∂D, the function

$$
h(x) = \int_D g(x, y)\eta(dy) + \int_{\partial D} k(x, y)\nu(dy)
\tag{0.3}
$$

can be considered as a "mild" solution of a boundary value problem with measures

$$
\begin{aligned}
Lh &= -\eta \quad \text{in } D, \\
h &= \nu \quad \text{on } \partial D.
\end{aligned}
\tag{0.4}
$$

Formula (0.2) can be replaced by a probabilistic formula

$$
h(x) = \Pi_x\left[\int_0^\tau \rho(\xi_s)\,ds + \varphi(\xi_\tau)\right],
\tag{0.5}
$$

where $\xi = (\xi_s, \Pi_x)$ is the diffusion with generator L stopped at the first exit time $\tau = \inf\{t\colon \xi_t \notin D\}$ from D. We can also write

$$
h(x) = \Pi_x A(0, \infty),
\tag{0.6}
$$

where A is a random measure on $(0, \infty)$ concentrated on $(0, \tau]$, equal to $\rho(\xi_s)\,ds$ on $(0, \tau)$ and charging the point τ by mass $\varphi(\xi_\tau)$ if $\tau < \infty$. This is an example of an additive functional of ξ. (In the Introduction we consider only homogeneous additive functionals.) For certain classes of measures η and ν, problem (0.4) can also be solved by (0.6) with $A = A_\eta + A_\nu$, where A_η and A_ν are additive functionals of ξ. We get A_η by the formula

$$
A_\eta(0, t] = \lim_{\lambda \to \infty} \int_0^{t \wedge \tau} \rho_\lambda(\xi_s)\,ds,
\tag{0.7}
$$

where

$$
\eta = \lim_{\lambda \to \infty} \rho_\lambda
\tag{0.8}
$$

(we postpone an explanation of the exact meaning of "lim" in these formulas). To define A_ν, we consider a sequence of domains D_n such that $\bar{D}_n \subset D_{n+1}$ and $D_n \uparrow D$. We introduce A_ν as a measure concentrated at τ and charging τ by

$$
\lim_{n \to \infty} f(\xi_{\tau_n}),
\tag{0.9}
$$

where

$$f(x) = \int_{\partial D} k(x, y)\nu(dy) \tag{0.10}$$

and τ_n is the first exit time from D_n. This approach works only if η does not charge sets of capacity 0 and if ν is absolutely continuous with respect to σ.

0.2. *Nonlinear equation.* Consider a boundary value problem involving a nonlinear operator $Lu - u^\alpha$ with $\alpha > 1$:

$$\begin{aligned} Lu - u^\alpha &= -\rho \quad \text{in } D, \\ u &= \varphi \quad \text{on } \partial D. \end{aligned} \tag{0.11}$$

It is equivalent to the integral equation

$$u(x) + \int_D g(x, y)u(y)^\alpha \, dy = h(x), \tag{0.12}$$

where h is given by (0.2) (or (0.5)).

There is no way to express u by an explicit formula through diffusion ξ. However, if $\alpha \le 2$, it is possible to get an expression in terms of a superdiffusion X.

A superdiffusion describes an evolution of a random cloud. It can be obtained by a passage to the limit from a system of indistinguishable particles which move according to the law of ξ. Suppose that each particle is frozen at the first exit time from D. The state at time t is a finite measure X_t on $\mathbb{R}^d$. The restriction $\tilde{X}_t$ of X_t to D describes the mass distribution of particles which are still in D at time t. We call $\tilde{X}$ the part of X in D. Denote by X'_t the mass distribution of particles which are frozen during the time interval $[0, t]$. We call X' the absorption process on D^c.

Under the assumptions on D, ρ and φ stated in Section 0.1, the solution of (0.11) can be obtained by the formula

$$u(x) = -\log P_x e^{-A(0,\infty)}, \tag{0.13}$$

where P_x is the law of the process started from Dirac's measure δ_x and A is given by the formula

$$A(0, t] = \int_0^t \langle \rho, \tilde{X}_s \rangle \, ds + \langle \varphi, X'_t \rangle. \tag{0.14}$$

The boundary value problem with measures [similar to (0.4)]

$$\begin{aligned} Lu - u^\alpha &= -\eta \quad \text{in } D, \\ u &= \nu \quad \text{on } \partial D \end{aligned} \tag{0.15}$$

is equivalent to an integral equation (0.12) with h given by (0.3). We prove that, if a solution exists, then it can be expressed by (0.13) with $A = A_\eta + A_\nu$, where

$$A_\eta(0, t] = \lim_{\lambda \to \infty} \int_0^t \langle \rho_\lambda, X_s \rangle \, ds \tag{0.16}$$

and

$$A_\nu(0,t] = \lim_{n\to\infty} \langle f, (X^n)'_t \rangle. \tag{0.17}$$

Here $(X^n)'_t$ is the absorption process on D_n^c (D_n, ρ_λ and f are the same as in Section 0.1).

It is remarkable that, in contrast to the linear case, the probabilistic formula (0.13) works always when a solution exists.

0.3. *Natural linear additive functionals.* A random measure $A(dt)$ is called an additive functional of a superdiffusion X if the value $A(I)$ on an open interval I is determined by events observable during this interval. We assume that A is homogeneous and natural. The potential of A is defined by the formula

$$h(x) = P_x A(0, \infty). \tag{0.18}$$

We say that A is linear if

$$P_\mu A(0,\infty) = \langle h, \mu \rangle < \infty \tag{0.19}$$

for a sufficiently large set $\mathscr{M}^*$ of measures μ. Put

$$u(x) = -\log P_x e^{-A(0,\infty)}. \tag{0.20}$$

According to [17], (0.19) implies that, if $\mu \in \mathscr{M}^*$ and if

$$\int \mu(dx) g(x,y) h(y)^\alpha \, dy < \infty,$$

then

$$P_\mu e^{-A(0,\infty)} = e^{-\langle u, \mu \rangle}$$

and u satisfies, μ-a.e., equation (0.12). Another implication of [17] is that all natural linear additive functionals are continuous.

We characterize measures η and ν for which problem (0.15) has a solution, both probabilistically (in terms of the range of process X) and analytically (in terms of capacities associated with Green's and Poisson's kernels).

Substantial part of our results are extended to arbitrary domains D (with the geometric boundary ∂D replaced by the Martin boundary associated with L).

In the main part of the article, we consider diffusions and superdiffusions in a time-inhomogeneous setting and we investigate related problems for parabolic PDE's. The results on elliptic PDE's are implications of this more general theory.

1. Statement and discussion of principal results.

1.1. *General definition of additive functionals.* A *filtration* $\mathfrak{F}$ of a measurable space $(\Omega, \mathscr{F})$ is a family of σ-algebras $\mathscr{F}(I) \subset \mathscr{F}$ indexed by open intervals $I \subset \mathbb{R}_+$ with the properties: $\mathscr{F}(I) \subset \mathscr{F}(\tilde{I})$ for $I \subset \tilde{I}$ and $\mathscr{F}(I) = \bigvee \mathscr{F}(I_n)$

as $I_n \uparrow I$. Let $A(\omega, \cdot)$ be a measure on $(0, \infty)$ which depends on parameter $\omega \in \Omega$. Suppose that $\mathfrak{F}$ is a filtration of $(\Omega, \mathscr{F})$ and $\mathbb{P}$ is a class of probability measures on $\mathscr{F}$. We say that A is an *additive functional of* $(\mathfrak{F}, \mathbb{P})$ if it satisfies the following conditions.

1.1.A. For every interval I, $A(I)$ is measurable relative to the universal completion of $\mathscr{F}$.

1.1.B. For every open interval I and every $P \in \mathbb{P}$, $A(I)$ is measurable relative to the P-completion of $\mathscr{F}(I)$.

An additive functional A is *continuous* if we have the following.

1.1.C. There exists a $\mathbb{P}$-negligible set Ω' (i.e., $P(\Omega') = 0$ for all $P \in \mathbb{P}$) such that, for every $\omega \notin \Omega'$, the measure $A(\omega, \cdot)$ is diffuse (i.e., it does not charge single points).

We say an additive functional A *has only fixed discontinuities* under the following conditions.

1.1.D. There exists a $\mathbb{P}$-negligible set Ω' and a set Λ, at most countable and independent of ω, such that $A(\omega, \{t\}) = 0$ for all $\omega \notin \Omega'$ and all $t \notin \Lambda$.

Denote by $\mathscr{P}_r$ the σ-algebra in $(r, \infty) \times \Omega$ generated by functions $F(t, \omega)$ which are left continuous in t and adapted to $\mathscr{F}(r, t)$. An additive functional A is called *natural* if, for every r and every $P \in \mathbb{P}$, the function $A(r, t], r < t$ is P-indistinguishable from a $\mathscr{P}_r$-measurable function.

1.2. *Additive functionals of a diffusion.* Let ξ be a diffusion in a domain $E \subset \mathbb{R}^d$. For every interval $I \subset \mathbb{R}_+$, we denote by $\mathscr{F}^0(I)$ the σ-algebra generated by ξ_s, $s \in I$. For every finite measure μ on $S = \mathbb{R}_+ \times E$, we set

$$\Pi_\mu = \int_S \Pi_{r,x} \mu(dr, dx)$$

[(ξ_t, Π_μ) is a stochastic process with a random birth time β and μ is the joint distribution of the birth time and birth place]. We define an additive functional of ξ as an additive functional of $(\mathfrak{F}^0, \mathbb{P}^0)$ where $\mathfrak{F}^0 = \{\mathscr{F}^0(I)\}$ and $\mathbb{P}^0 = \{\Pi_\mu\}$. A simple example of an additive functional is given by the formula

$$A(I) = \int_I \rho^s(\xi_s)\lambda(ds), \tag{1.1}$$

where $\rho^s(x) = \rho(s, x)$ is a positive Borel function on S and λ is a σ-finite measure on $(0, \infty)$. The functional A can have only fixed discontinuities and it is continuous if λ is diffuse.

Suppose that Q is a finely open set [that is, for every $(r, x) \in Q$, there exists, $\Pi_{r,x}$-a.s., $t > r$ such that $(s, \xi_s) \in Q$ for all $s \in (r, t)$]. Let ξ be a diffusion frozen at time $\tau = \inf\{t: (t, \xi_t) \notin Q\}$. [In the time inhomogeneous setting, it is natural to stop keeping time after τ and not to consider combinations (t, ξ_τ) for $t > \tau$.] For every positive Borel function φ on S, formula

$$A(I) = 1_I(\tau)\varphi(\tau, \xi_\tau)$$

defines an additive functional of ξ.

1.3. *Additive functionals of a superdiffusion.* A system $\mathfrak{F}^0$ can be defined for an arbitrary Markov process. A superdiffusion X is such a process but it can also be viewed as a collection of exit measures (X_Q, P_μ) from p-open subsets of S. [The class of p-open sets described in Section 2.2 is an intermediate class between the class of open sets and the class of finely open sets.] All exit measures are defined on the same space $(\Omega, \mathscr{F})$. We denote by $\mathscr{M}(E)$ the space of all finite measures on a measurable space E. For every $\mu \in \mathscr{M} = \mathscr{M}(S)$, P_μ is a probability measure describing the evolution with initial time-space distribution μ. We write $P_\mu = P_{r,x}$ if $\mu = \delta_{(r,x)}$ is the unit measure concentrated at point (r, x). We deal with a special class of subsets $\mathbb{P}$ of the set $\{P_\mu\}$. We say that a set $\mathscr{M}^* \subset \mathscr{M}$ is *total* if the following hold.

1.3.A. If $\mu \in \mathscr{M}^*$ and if $\tilde{\mu} \le \mu$, then $\tilde{\mu} \in \mathscr{M}^*$.

1.3.B. For every $\mu \in \mathscr{M}^*$ and for an arbitrary Q, $P_\mu\{X_Q \in \mathscr{M}^*\} = 1$. Moreover, $P_\mu\{X_t \text{ and } X_{t-} \in \mathscr{M}^* \text{ for all } t\} = 1$.

1.3.C. The set $S^* = \{(r, x): \delta_{(r,x)} \in \mathscr{M}^*\}$ is the complement of a ξ-polar set. (A set $\tilde{S} \subset S$ is called ξ-polar if $\Pi_{r,x}\{\xi_t \in \tilde{S} \text{ for some } t > r\} = 0$ for all r, x.)

1.3.D. Every $\mu \in \mathscr{M}^*$ is concentrated on S^*.

Clearly, the intersection of any countable family of total sets is a total set.

Consider the Markov semigroup $T^r_s h^s(x) = \Pi_{r,x} h^s(\xi_s)$ corresponding to ξ. We say that h is an *exit rule* for ξ if h is a positive Borel function on S such that

$$T^r_s h^s \le h^r \quad \text{and} \quad T^r_s h^s \to h^r \quad \text{as } s \downarrow r.$$

We say that h is a *pure exit rule* if, in addition, $T^r_s h^s \downarrow 0$ as $s \to \infty$. (Every exit rule is a sum of a pure exit rule and an exit rule with the property $T^r_s h^s = h^r$ for all $r < s$.) Denote by H the set of all pure exit rules h which are finite a.e. [Writing "a.e." means "outside a set of Lebesgue measure 0". We use the notation m for the Lebesgue measure $dr\,dx$ on S.] To every $h \in H$ there corresponds a total set $\mathscr{M}(h) = \{\mu \in \mathscr{M}: \langle h, \mu\rangle < \infty\}$.

Additive functionals of a superdiffusion X with determining (*total*) *set* $\mathscr{M}^*$ are defined as additive functionals of $(\mathfrak{F}^0, \mathbb{P})$ where $\mathbb{P} = \{P_\mu: \mu \in \mathscr{M}^*\}$. If ρ is a positive Borel function and λ is a measure on $(0, \infty)$, then the formula

$$A(I) = \int_I \langle \rho^s, X_{s-}\rangle \lambda(ds) \tag{1.2}$$

defines an additive functional with determining set $\mathscr{M}^* = \mathscr{M}$. [Formula (1.2) with X_{s-} replaced by X_s also defines an additive functional. The difference between these two functionals is a deterministic measure (see Section 2.3).]

Let A and $\tilde{A}$ be two additive functionals of X with determining sets $\mathscr{M}^*$ and $\tilde{\mathscr{M}}^*$. We say that A and $\tilde{A}$ are *equivalent* if they are P_μ-indistinguishable for all $\mu \in \mathscr{M}^* \cap \tilde{\mathscr{M}}^*$.

1.4. *NLA functionals.* Let $h \in H$. We say that A is a *linear additive functional with potential* h if A is an additive functional with determining set $\mathscr{M}^* \subset \mathscr{M}(h)$ and if, for all $\mu \in \mathscr{M}^*$,

(1.3) $$P_\mu A(0, \infty) = \langle h, \mu \rangle$$

and

(1.4) $$P_\mu\{A(0, r] = 0\} = 1 \quad \text{if } \mu(S_{<r}) = 0.$$

[Here $S_{<r} = [0, r) \times E$. Notation $S_{>t}, S_{\ge t}, \dots$ has a similar meaning.]

We use an abbreviation NLA for natural linear additive functionals. It follows from [5], VII.8 and VII.21, that NLA functionals with equal potentials are equivalent.

The *log-potential of* A is defined by formula

(1.5) $$u^r(x) = -\log P_{r,x} e^{-A(0,\infty)} \quad \text{for } (r, x) \in S^*$$

(the set S^* is defined in 1.3.C). By Jensen's inequality,

(1.6) $$u^r(x) \le h^r(x) \quad \text{on } S^*.$$

1.5. *Operator $\mathscr{E}$ and $\mathscr{E}$-equation.* The fundamental role in the theory of superdiffusion is played by an operator which acts on positive Borel functions on S by the formula

(1.7) $$\mathscr{E}(u)(r, x) = \Pi_{r,x} \int_r^\infty u(s, \xi_s)^\alpha \, ds = \int_r^\infty ds \int_E p(r, x; s, y) u(s, y)^\alpha \, dy,$$

where $p(r, x; s, y)$ is the transition density of ξ. The expression $\mathscr{E}(u, \mu) = \langle \mathscr{E}(u), \mu \rangle$ can be considered as a generalized energy integral. [A similar generalization is introduced in nonlinear potential theory (see, e.g., [1], Section 2.2, especially (2.2.6)).]

Let $h \in H$. We call

(1.8) $$u + \mathscr{E}(u) = h$$

the $\mathscr{E}$-equation. We use the notation

(1.9) $$S_{\mathscr{E}}(h) = \{(r, x) \colon (h + \mathscr{E}(h))(r, x) < \infty\}$$

and

(1.10) $$\mathscr{M}_{\mathscr{E}}(h) = \{\mu \colon \langle h + \mathscr{E}(h), \mu \rangle < \infty\}.$$

For every total set $\mathscr{M}^*$, we put $\mathscr{M}^*_{\mathscr{E}}(h) = \mathscr{M}^* \cap \mathscr{M}_{\mathscr{E}}(h)$ and $S^*_{\mathscr{E}}(h) = S^* \cap S_{\mathscr{E}}(h)$. All measures $\mu \in \mathscr{M}^*_{\mathscr{E}}(h)$ are concentrated on $S^*_{\mathscr{E}}(h)$.

The following results on NLA functionals have been proved in [17] for a wide class of superprocesses which contains (L, α)-superdiffusions (see Theorems 1.2, 1.4, 4.1 and 1.7 there):

1.5.A. Let A be an NLA functional with potential h and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. Then A can have only fixed discontinuities.

1.5.B. If A is an NLA functional with potential h, log-potential u and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$, then

$$P_\mu e^{-A(0,\infty)} = e^{-\langle u, \mu\rangle} \quad \text{for all } \mu \in \mathscr{M}^* \tag{1.11}$$

and u satisfies the $\mathscr{E}$-equation (1.8) on S^*.

1.5.C. Put $X_{<t} = X_{S_{<t}}$. Let h be a pure exit rule for ξ and let $\mathscr{M}^*$ be a total subset of $\mathscr{M}(h)$. The following condition is necessary and sufficient for the existence of an NLA functional A with potential h and determining set $\mathscr{M}^*$: for every $\mu \in \mathscr{M}^*$, the stochastic process $(\langle h, X_{<t}\rangle, P_\mu)$ belongs to class (D).

1.5.D. If $h + \mathscr{E}(h) \in H$, then there exists an NLA functional A with potential h and determining set $\mathscr{M}_{\mathscr{E}}(h)$.

The following uniqueness result will be established in Section 3.

THEOREM 1.1. *If $u, \hat{u}$ satisfy the $\mathscr{E}$-equation on a set $B \subset \{h < \infty\}$ and if $m(B^c) = 0$, then $\hat{u} = u$ on B.*

Put $h \in H^*$ if $h \in H$ and if the $\mathscr{E}$-equation holds everywhere for some u.

Note that $h \in H^*$ if $\mathscr{E}$-equation (1.8) holds a.e. for some u. Indeed, then $\mathscr{E}(u) \le h$ a.e. and therefore $\mathscr{E}(u) \le h$ everywhere because $\mathscr{E}(u)$ and h are exit rules. The function

$$\tilde{u} = \begin{cases} h - \mathscr{E}(u), & \text{on } \{h < \infty\}, \\ \infty, & \text{on } \{h = \infty\} \end{cases} \tag{1.12}$$

is positive and it satisfies (1.8) everywhere.

We say that $h \in H$ is a *NLA-potential* and we write $h \in H^p$ if h is the potential of a NLA functional A. We write $h \in H^{p*}$ if, in addition,

$$u + \mathscr{E}(u) = h \quad \text{on } S^*, \tag{1.13}$$

where u is the log-potential of A. Clearly, $H^{p*} \subset H^* \cap H^p$. It follows from 1.5.B that H^{p*} contains all $h \in H^p$ such that $\mathscr{E}(h) \in H$. A much stronger result is proved in Section 5.

THEOREM 1.2. *The three classes H^*, H^p and H^{p*} coincide.*

REMARK. Clearly, H^p is a convex cone (i.d., $c_1 h_1 + c_2 h_2 \in H^p$ if $h_1, h_2 \in H^p$ and $c_1, c_2 \ge 0$). It follows from 1.5.C that H^p is a face of cone H (i.e., if

$h_1, h_2 \in H$ and $h_1 + h_2 \in H^p$, then $h_1, h_2 \in H^p$). Theorem 1.2 implies that the classes H^* and H^{p*} have the same properties, which is difficult to see from their definitions.

1.6. *Spectral measures of NLA-potentials.* We use the Martin representation of an exit rule h as an integral over the exit space of ξ. A construction of the exit space in a very general setting is given in [7]. To apply the general theory to our case, we choose a reference point $c \in E$ and we put

$$k(r, x; s, y) = p(r, x; s, y)/p(0, c; s, y). \tag{1.14}$$

There exist a continuous injective mapping from S to a compact metrizable space $\bar{S}$ and an extension of $k(r, x; s, y)$ to $S \times \bar{S}$ such that (1) for every $z \in S$, $k(z, w) \to k(z, \tilde{w})$ as $w \to \tilde{w} \in \bar{S} \setminus S$ and (2) if $k(\cdot, w_1) = k(\cdot, w_2)$, then $w_1 = w_2$.

We call $\bar{S}$ the *exit space of* ξ. The set $\partial S = \bar{S} \setminus S$ is called the *Martin exit boundary*. For every $(s, y) \in S$, $h^r(x) = p(r, x; s, y)$ is an extremal element of H (which means if $h = h_1 + h_2$ and if $h_1, h_2 \in H$, then h_1, h_2 are proportional to h). We denote by S^e the set of all $w \in \partial S$ such that $h^r(x) = k(r, x; w)$ is an extremal element of H (this is a Borel subset of ∂S.)

We use the name *parabolic functions* for solutions of the equation

$$\dot{h} + Lh = 0 \quad \text{in } S.$$

Every positive parabolic function f has a unique representation

$$f = \int_{S^e} k(r, x; w)\nu(dw), \tag{1.15}$$

where ν is a finite measure.

For every measure η on S, we put

$$G\eta(r, x) = \int_S p(r, x; s, y)\eta(ds, dy). \tag{1.16}$$

An arbitrary element h of H can be represented uniquely in the form

$$h = G\eta + f, \tag{1.17}$$

where η is a measure on S and f is a positive parabolic function. Formula (1.17) can be rewritten in the form

$$h^r(x) = \int_{S \cup S^e} k(r, x; w)\gamma(dw), \tag{1.18}$$

where $\gamma = \nu$ on S^e and $d\gamma = q\,d\eta$ on S with $q(s, y) = p(0, c; s, y)$. Measure γ is determined uniquely by h and we call it the *spectral measure of* h.

The Martin capacity CM is defined on compact subsets of $\bar{S}$ by the formula

$$CM(\Gamma) = \sup\left\{\gamma(\Gamma) \colon \int_S p(0, c; r, x)\, dr\, dx \left[\int_\Gamma k(r, x; w)\gamma(dw)\right]^\alpha \le 1\right\}. \tag{1.19}$$

The *graph* $\mathscr{G}$ of a superdiffusion is the minimal closed subset of $\bar{S}$ which contains the support of the exit measure X_Q, P_μ-a.s., for every Q. A set $\Gamma \subset \bar{S}$

is called *$\mathscr{G}$-polar* if it does not contain any set $S_{<t}$ and if $P_{r,x}\{\mathscr{G} \cap \Gamma = \varnothing\} = 1$ for all $(r, x) \notin \Gamma$.

Before we prove Theorem 1.2, we establish in Section 4 the following result.

THEOREM 1.3. *Let γ be the spectral measure of $h \in H$. If $\gamma(\Gamma) = 0$ for all compact sets Γ with $CM(\Gamma) = 0$, then $h \in H^{p*}$. If $h \in H^p \cup H^*$, then γ does not charge any $\mathscr{G}$-polar set.*

To prove Theorem 1.3, we use a result established in [12], Theorem 3.2 (see also [14], Theorem 7.2).

1.6.A. A set $\Gamma \subset S$ is $\mathscr{G}$-polar if and only if $CM(\Gamma) = 0$.

1.7. *Discrete approximation of NLA functionals.* An arbitrary NLA functional A can be approximated by linear combinations of functionals

$$A(I) = 1_I(t)\langle \rho, X_{t-}\rangle,$$

which charge a single point of $(0, \infty)$. To formulate a precise result, we need a little preparation. Consider a set $\Lambda = \{0 = t_0 < t_1 < \cdots < t_k\}$. If $a(\Lambda)$ is a real-valued function of Λ, then writing $\lim_\Lambda a(\Lambda) = a$ means that $a(\Lambda_n) \to a$ for every increasing sequence of sets Λ_n whose union is everywhere dense in $[0, \infty)$ (we call such a sequence *standard*). To every $h \in H$ there corresponds a positive function of interval:

$$h_\Delta(x) = \begin{cases} h^s(x) - T_t^s h^t(x), & \text{for } \Delta = (s, t], \\ h^s(x), & \text{for } \Delta = (s, \infty). \end{cases} \tag{1.20}$$

The following approximation result was deduced in [17] from 1.5.A and a general theorem on compensators of local supermartingales.

1.7.A. Let A be a NLA functional with potential h and determining set $\mathscr{M}^*$. Then, for every $\mu \in \mathscr{M}^*$ and all $0 \le r < t \le \infty$,

$$A(r, t] = \lim_\Lambda A_\Lambda(r, t] \quad \text{weakly in } L^1(P_\mu), \tag{1.21}$$

where

$$A_\Lambda(ds) = \sum_1^n \delta_{t_k}(ds)\langle h_{\Delta_k}, X_{t_k-}\rangle. \tag{1.22}$$

(Here $\Delta_1 = (t_1, t_2], \ldots, \Delta_{n-1} = (t_{n-1}, t_n]$, $\Delta_n = (t_n, \infty)$ for $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$.) If $\mu \in \mathscr{M}_{\mathscr{E}}^*(h)$, then (1.21) holds with strong convergence in $L^1(\mu)$.

1.8. *New approximation results.* These results allow us to construct all NLA functionals starting from functionals of the form (1.2) with $\lambda(ds) = ds$ and from absorption processes. (Absorption processes were introduced heuristically in Section 0.2. A rigorous definition is given in Section 2.4.)

We say that a domain D is smooth if ∂D belongs to class $C^{2,\lambda}$.

THEOREM 1.4. *Let D be a bounded smooth domain. Suppose X is a superdiffusion in cylinder $Q = [0, b) \times D$ corresponding to an elliptic differential operator L which satisfies conditions* 2.1.A–C *in $\bar{Q}$. Let A be an NLA functional of X with potential $h = G\eta$ and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. Put*

$$A_\lambda(I) = \int_I \langle \rho_\lambda^s, X_s \rangle \, ds, \tag{1.23}$$

where

$$\rho_\lambda^s(x) = \lambda \int_Q e^{-\lambda(s-r)} p(r, x; s, y) \eta(ds, dy). \tag{1.24}$$

Then for every $\mu \in \mathscr{M}^*$ and for all $0 \le r < t \le b$,

$$A(r, t] = \lim_{\lambda \to \infty} A_\lambda(r, t] \quad \textit{in } P_\mu\textit{-probability}. \tag{1.25}$$

THEOREM 1.5. *Let A be an NLA functional with potential $h = G\eta$. There exist NLA functionals A_n such that $A_n \uparrow A$ and potentials h_n of A_n satisfy condition $\mathscr{E}(h_n) \in H$.*

We say that a sequence of bounded p-open sets Q_n is a *standard approximating sequence for S* if $\bar{Q}_n \subset Q_{n+1}$ and $Q_n \uparrow S$.

THEOREM 1.6. *Let Q_n be a standard approximating sequence for S. Suppose X is a superdiffusion in S, $(X^n)'$ is the absorption process on Q_n^c and A is an NLA functional of X with parabolic potential h and determining set $\mathscr{M}^*$. We have*

$$A(0, t] = \lim_{n \to \infty} \langle h, (X^n)_t' \rangle \quad P_\mu\text{-a.s.} \tag{1.26}$$

for every $t \in (0, \infty]$ and for every $\mu \in \mathscr{M}^*$.

REMARK. Function $F_n(t) = \langle h, (X^n)_t' \rangle$ is monotone increasing in t for every ω. It follows from (1.26) that measures corresponding to F_n converge weakly to measure $A(dt)$ for almost all ω [relative to all measures P_μ with $\mu \in \mathscr{M}_{\mathscr{E}}^*(h)$].

We prove Theorem 1.5 in Section 4. Theorems 1.4 and 1.6 will be proved in Section 6.

1.9. *Homogeneous additive functionals.* Processes ξ and X are defined on two unrelated sample spaces Ω_0 and Ω. If diffusion ξ has a stationary transition density $p(r, x; s, y) = p_{s-r}(x, y)$, then it is possible to choose Ω_0 and to define a semigroup of transformations θ_t: $\Omega_0 \to \Omega_0$ in such a way that

$$\xi_s(\theta_t \omega) = \xi_{s+t}(\omega), \qquad \Pi_{r+t, x}(\theta_t C) = \Pi_{r, x}(C). \tag{1.27}$$

Put $\varkappa_s(r, x) = (r + s, x)$. Conditions (1.27) imply: for every Q,

$$\theta_t(\tau, \xi_\tau) = \varkappa_{-t}(\tau_t, \xi_{\tau_t}) \tag{1.28}$$

where τ is the first exit time from Q and τ_t is the first exit time from $Q^t = S_{<t} \cup \varkappa_t Q$.

The superdiffusion X corresponding to ξ has a stationary transition function and it can be defined in a space Ω with a semigroup of transformations Θ_t subject to the conditions (see [11, Section 1.12]): for every Q,

$$X_Q(\Theta_t \omega, B) = X_{Q^t}(\omega, \varkappa_t B), \qquad P_{\mu_t}(\Theta_t C) = P_\mu(C), \tag{1.29}$$

where μ_t is the image of μ under $\varkappa_t$. The filtration $\mathfrak{F}$ generated by X has the property

$$\Theta_t \mathscr{F}(I) = \mathscr{F}(I + t). \tag{1.30}$$

An additive functional A of X is called *homogeneous* if we have the following.

1.9.A. The determining set $\mathscr{M}^*$ is invariant with respect to $\varkappa_t$.

1.9.B. There exists a set Ω^* such that $A(\Theta_t \omega, I) = A(\omega, I + t)$ for all I and all $\omega \in \Omega^*$ and $P_\mu(\Omega^*) = 1$ for all $\mu \in \mathscr{M}^*$.

A class of equivalent natural linear additive functionals of X contains a homogeneous functional if and only if it contains a functional A which satisfies 1.9.A and the following condition.

1.9.A*. The potential $h(r, x)$ of A does not depend on r.

Proposition 1.5.A implies that all homogeneous NLA functionals with $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$ are continuous. Since the countable sum of continuous functionals is continuous, Lemma 3.3 in [18] and Theorems 1.1 and 1.2 in [19] imply that all homogeneous NLA functionals are continuous if E is a bounded smooth domain.

By applying Theorems 1.1–1.6 to homogeneous functionals and to time-independent exit rules (which are the same as excessive functions for ξ), we get a stronger version of results proved in [18] and [19]. In Section 7 we describe the relation between [18] and [19] and the present paper in more detail.

1.10. *Boundary value problems with measures.* Put

$$Gf(r, x) = \Pi_{r,x} \int_0^\infty f(s, \xi_s)\, ds = \int_0^\infty ds \int_E p(r, x; s, y) f(s, y)\, dy. \tag{1.31}$$

Note that $\mathscr{E}(f) = G(f^\alpha)$ and that $Gf = G\eta$ for $\eta(dr, dx) = f(r, x)\, dr\, dx$.

Let E be a bounded domain with smooth boundary and let $h \in H$ have the form

$$h(r, x) = G\rho(r, x) + \Pi_{r,x} 1_{\zeta < \infty} \sigma(\zeta, \xi_{\zeta-})$$

where ζ is the lifetime of ξ, $\rho \geq 0$ is a bounded function of class $C^1(S)$ and $\sigma \geq 0$ is a bounded continuous function on $\partial' S = \mathbb{R}_+ \times \partial E$. Then u is a solution of (1.8) if and only if it is a solution of the boundary value problem

$$\begin{aligned} \dot{u} + Lu - u^\alpha &= -\rho \quad \text{in } S, \\ u &= \sigma \quad \text{on } \partial' S, \\ u &= 0 \quad \text{on } \{\infty\} \times E \end{aligned} \tag{1.32}$$

where a second-order elliptic operator L is the generator of ξ. The solution of problem (1.32) can be expressed by a probabilistic formula

$$u(r, x) = -\log P_{r,x} e^{-A(r,\infty)} \tag{1.33}$$

where A is an NLA functional with potential h. For a general $h \in H$ given by (1.17) and (1.15), (1.33) describes a "mild" solution of the boundary value problem

$$\begin{aligned} \dot{u} + Lu - u^\alpha &= -\eta \quad \text{in } S, \\ u &= \nu \quad \text{on } \partial' S, \\ u &= 0 \quad \text{on } \{\infty\} \times E. \end{aligned} \tag{1.34}$$

Therefore the results stated above can be interpreted as propositions on the boundary value problem (1.34). [For an arbitrary domain E, $\partial' S$ in (1.34) should be replaced by S^e.]

In a time-homogeneous setting, formula (1.33) (with a homogeneous A) solves the problem (0.15).

2. Superdiffusion.

2.1. *Diffusion.* We start from a differential operator

$$Lu = \sum_{i,j} a_{ij} \nabla_i \nabla_j u + \sum_i b_i \nabla_i u \tag{2.1}$$

(∇_i stands for the partial derivative with respect to x_i) in a cylinder $S = \mathbb{R}_+ \times E$ where E is an arbitrary domain in $\mathbb{R}^d$. The coefficients a_{ij} and b_i satisfy the following conditions:

2.1.A. for every nonzero vector $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$ and for all $(r, x) \in S$,

$$\sum_{i,j} a_{ij}(r, x) \lambda_i \lambda_j > 0;$$

2.1.B. a_{ij} and b_i are locally Hölder continuous;

2.1.C. $\nabla_i \nabla_j a_{ij}$ and $\nabla_i b_i$ are continuous.

Under conditions 2.1.A, and 2.1.B, there exists a function p from $S \times S$ to $\mathbb{R}_+$ with the following properties.

(i) For all $0 \le r < s < t$, $x, z \in E$,

$$\int_E p(r,x;s,y)\,dy\,p(s,y;t,z) = p(r,x;t,z). \tag{2.2}$$

(ii) If f is a continuous function on S with compact support, then, for every t,

$$u(r,x) = \int_E p(r,x;t,y) f(t,y)\,dy, \tag{2.3}$$

satisfies the conditions

$$\frac{\partial u(r,x)}{\partial r} = Lu(r,x) \quad \text{in } S_{<t} = [0,t) \times E, \tag{2.4}$$

and, for every $y \in E$,

$$u(r,y) \to f(t,y) \quad \text{as } r \uparrow t. \tag{2.5}$$

Moreover, there exists a minimal function with properties (i) and (ii) and, for it, we have the following conditions.

(iii) For all $0 \le r < s$, $x \in D$,

$$\int_{Q_s} p(r,x;s,y)\,dy \le 1. \tag{2.6}$$

(iv) $p(r,x;s,y) = 0$ for $r \ge s$.

By (i) and (iii), $p(r,x;t,dy) = p(r,x;t,y)\,dy$ is a Markov transition function in E. This is the transition function of a continuous Markov process $\xi = (\xi_t, \Pi_{r,x})$ on a random time interval $[\beta, \zeta)$. (See, e.g., [6].) We call it an *L-diffusion in S.* (For every r, x, $\beta = r$, $\xi_\beta = x$ and $\{\zeta < \infty\} \subset \{\xi_{\zeta-} \in \partial E\}$ $\Pi_{r,x}$-a.s.)

2.2. *Parts of diffusion.* We define a *simple cylinder* as a set $[a,b) \times D$ where $0 \le a < b$, D is open and $\bar{D} \subset E$. We say that a set Q is *p-open* if it is open in the topology of S determined by simple cylinders. The boundary of Q in this topology is denoted by ∂Q. (If $Q = [a,b) \times D$, then $\partial Q = ([a,b] \times \partial D) \cup (\{b\} \times D)$.)

Let Q be a p-open set. The *part $\tilde{\xi}$ of ξ in Q* is obtained by restricting ξ_t to interval $[\beta, \tau)$ where

$$\tau = \inf\{t \colon t \ge \beta, (t, \xi_t) \notin Q\} \tag{2.7}$$

is *the first exit time of ξ from Q* [if $(t,\xi_t) \in Q$ for all $t \in [0,\zeta)$, then we set $\tau = \zeta$]. The transition density of $\tilde{\xi}$ is defined by the formula

$$\tilde{p}(r,x;t,y) = p(r,x;t,y) - \Pi_{r,x} p(\tau, \xi_\tau; t, y) \quad \text{for } (r,x), (t,y) \in Q. \tag{2.8}$$

We set $\tilde{p}(r,x;t,y) = 0$ for $r \ge t$ and also if (r,x) or (t,y) is not in Q.

For every measure η on S, we put

$$G_Q \eta(r,x) = \int_Q \tilde{p}(r,x;s,y)\eta(ds,dy) \quad \text{for } (r,x) \in Q. \tag{2.9}$$

Formula (2.9) coincides with (1.16) if $Q = S$.

We write $G_Q\rho$ for $G_Q\eta$ with $\eta(ds, dx) = \rho(s, x)\, ds\, dx$. We call G_Q *Green's operator of* ξ in Q.

For every positive Borel function φ, we set

$$K_Q\varphi(r, x) = \Pi_{r,x}\varphi(\tau, \xi_\tau)1_{\tau<\zeta}. \tag{2.10}$$

For $(r, x) \notin Q$, $\Pi_{r,x}\{\tau = r\} = 1$ and therefore $K_Q\varphi(r, x) = \varphi(r, x)$.

2.3. *Superdiffusions.* Let $\xi = (\xi_t, \Pi_{r,x})$ be a Markov process in a measurable space $(E, \mathscr{B})$. A (ξ, α)-*superprocess* is a Markov process $X = (X_t, P_{r,\mu})$ in $\mathscr{M} = \mathscr{M}(E)$ which satisfies the condition: for every $\mu \in \mathscr{M}$, every positive $\mathscr{B}$-measurable function f and for all $r < t \in \mathbb{R}_+$,

$$\begin{gathered} P_{r,\mu}\exp\langle -f, X_t\rangle = \exp\langle -u^r, \mu\rangle, \\ u^r(x) + \Pi_{r,x}\int_r^t u^s(\xi_s)^\alpha\, ds = \Pi_{r,x}f(\xi_t). \end{gathered} \tag{2.11}$$

Suppose that ξ is an L-diffusion in S. Then, for every $1 < \alpha \le 2$, there exists a right (ξ, α)-superprocess X (see [10] or [14, 15]). We call it an (L, α)-*superdiffusion in* S.

For an arbitrary p-open set $U \subset S$ and an arbitrary finite measure μ on S, we introduce a random measure (X_U, P_μ) which we call *the exit measure from* U. Its probability distribution is given by formulas similar to (2.11):

$$\begin{gathered} P_\mu \exp\langle -\varphi, X_U\rangle = \exp\langle -u, \mu\rangle, \\ u + \mathscr{E}_U(u) = K_U\varphi, \end{gathered} \tag{2.12}$$

where $\mathscr{E}_U$ is given by (1.7) with ξ replaced by its part in U and K_U is defined by (2.10) with Q replaced by U. Note that $P_\mu\{X_U(U) = 0\} = 1$ for all μ and, if μ is concentrated on U, then X_U is concentrated on ∂U.

[The measures $P_{r,\mu}$ can be considered as a particular case of the measures P_μ if we interpret a measure $\mu \in \mathscr{M}(E)$ as a measure on S concentrated on $S_r = \{r\} \times E$.]

Formulas (2.12) imply

$$P_\mu\langle \varphi, X_U\rangle = \langle K_U\varphi, \mu\rangle. \tag{2.13}$$

Note that, for every $t > r$ and every x, $P_{r,x}\{X_t = X_{t-}\} = 1$. [This follows, for instance, from the fact that X is a Hunt process (see [20]).] Therefore, for every $\mu \in \mathscr{M}$,

$$P_\mu\{X_t = X_{t-} + \mu_{\{t\}}\} = 1,$$

where $\mu_{\{t\}}$ is the restriction of μ to $\{t\} \times E$.

Formula (2.13) implies

$$P_\mu\langle f, X_{t-}\rangle = \int_S \mu(dr, dx)\int_E p(r, x; t, y)f(y)\, dy. \tag{2.14}$$

The joint probability distribution of $X_{U_1}, \dots, X_{U_n}$ is determined by (2.12) and by this property: for every positive $\mathscr{F}_{\supset U}$-measurable Y,

$$P_\mu\{Y \big| \mathscr{F}_{\subset U}\} = P_{X_U} Y, \tag{2.15}$$

where $\mathscr{F}_{\subset U}$ is the σ-algebra generated by $X_{U'}$ with $U' \subset U$ and $\mathscr{F}_{\supset U}$ is the σ-algebra generated by $X_{U''}$ with $U'' \supset U$.

The existence of a family (X_U, P_μ) subject to conditions (2.12) and (2.15) is proved in [14].

We state a result which is an immediate implication of Theorem I.1.8 in [14].

Let ρ be a positive Borel function on S and let $\lambda(ds)$ be a measure on $\mathbb{R}_+$. Then, for every $\mu \in \mathscr{M}(S)$,

$$P_\mu \exp\left\{-\int_0^\infty \langle \rho^s, X_s\rangle \lambda(ds)\right\} = e^{-\langle u, \mu\rangle}, \tag{2.16}$$

where u satisfies everywhere the $\mathscr{E}$-equation (1.8) with

$$h = G\eta, \qquad \eta(ds, dx) = \rho^s(x)\lambda(ds)\, dx. \tag{2.17}$$

2.4. *Part of X in Q. Absorption process.* Let Q be a p-open subset of S. Put $Q_{<t} = \{(r, x) \in Q\colon r < t\} = Q \cap S_{<t}$ and denote by Q_t the t-section of Q.

Consider the restriction $\tilde{X}_t$ of $X_{Q_{<t}}$ to Q_t. Note that $(\tilde{X}_t, P_{r,\mu})$ with $\mu \in \mathscr{M}(Q_r)$ is an $(\tilde{L}, \alpha)$-superdiffusion where $\tilde{L}$ is the restriction of L to Q. (The state space Q_r of $\tilde{X}$ is, in general, variable. It is constant if $Q = \mathbb{R}_+ \times D$). We call $\tilde{X}$ the *part of X in Q*.

The restriction X'_t of X_Q to $S_{\le t}$ is called the *absorption process on Q^c*. If φ is a positive Borel function, then, for all ω, $\langle \varphi, X'_t\rangle$ is a monotone increasing function in t. It is bounded and right continuous if $\langle \varphi, X_Q(\omega)\rangle < \infty$. We have

$$X_{Q_{<t}} = \tilde{X}_t + X'_t \quad P_\mu\text{-a.s. for every } \mu \in \mathscr{M}(Q_{<t}). \tag{2.18}$$

2.5. Let Q_n be a standard sequence approximating S and let $p^n(r, x; t, y)$ be the transition density of the part of L-diffusion ξ in Q_n. The sequence p^n is monotone increasing and its limit p is the transition density of an L-diffusion in S.

Let X^n_t be the part of (L, α)-superdiffusion in Q_n. Then, for every μ,

$$X^n_t \le X^{n+1}_t \quad P_\mu\text{-a.s.} \tag{2.19}$$

and

$$X^n_t \uparrow X_t \quad P_\mu\text{-a.s.} \tag{2.20}$$

Formula (2.19) follows from [16, Lemma 4.1]. By (2.14),

$$P_\mu X^n_t(\Gamma) = \int_S \mu(dr, dx) p^n(r, x; t, \Gamma) \uparrow \int_S \mu(dr, dx) p(r, x; t, \Gamma) = P_\mu X_t(\Gamma),$$

which implies (2.20).

3. Monotonicity properties of the $\mathscr{E}$-equation.

3.1. We prove a stronger version of Theorem 1.1.

THEOREM 3.1. *Suppose that η is a measure on S, $u, \hat{u} \geq 0$ and*

$$\hat{u} + \mathscr{E}(\hat{u}) = u + \mathscr{E}(u) + G\eta < \infty \quad \text{on } B. \tag{3.1}$$

If $m(B^c) = 0$, then $\hat{u} \geq u$ on B.

3.2. We use, as a tool, the processes $(\xi_s, \Pi_{r,x}^{t,y})$ where $s \in [r, t]$. Their finite-dimensional distributions are given by the formula

$$\Pi_{r,x}^{t,y}\{\xi_{t_1} \in dy_1, \dots, \xi_{t_n} \in dy_n\} = p(r, x; t_1, dy_1)p(t_1, y_1; t_2, dy_2)\dots \\ p(t_{n-1}, y_{n-1}; t_n, dy_n)p(t_n, y_n; t, y) \tag{3.2}$$

for all $r < t_1 < \dots < t_n < t$. (The normalized measure $\Pi_{r,x}^{t,y}$ can be obtained by conditioning the diffusion ξ to begin at time r at the point x and to end at time t at the point y.)

Let f be a positive Borel function. The formula

$$p^f(r, x; t, y) = \Pi_{r,x}^{t,y}\left\{\exp\left\{-\int_r^t f(s, \xi_s)\, ds\right\}\right\} \tag{3.3}$$

defines the transition density of a Markov process obtained from ξ by killing with rate $f(s, x)$ at the point (s, x).

Denote by G_f the operator corresponding to p^f by formula (1.31).

LEMMA 3.1. *If γ is a signed measure on S, then*

$$G\gamma - G_f\gamma = G_f(fG\gamma) \tag{3.4}$$

on the set $\{G|\gamma| < \infty\}$.

PROOF. First, we prove the formula

$$p(r, x; t, y) - p^f(r, x; t, y) = \int_r^t \int_E ds\, dz\, p^f(r, x; s, z)f(s, z)p(s, z; t, y), \tag{3.5}$$

which is a particular case of (3.4) for $\gamma = \delta_{(t,y)}$. Equation (3.5) follows from (3.3), (3.2), the Markov property of ξ, Fubini's theorem and relation

$$\int_r^t ds\, Y_s \exp\left\{-\int_r^s Y_u\, du\right\} = 1 - \exp\left\{-\int_r^t Y_u\, du\right\},$$

which we apply to $Y_u = f(u, \xi_u)$.

To get (3.4), we integrate both parts of (3.5) with respect to measure γ. $\square$

LEMMA 3.2. *Suppose that* $f \geq 0$ *and that*

$$G\eta + G|fw| < \infty \quad \text{on } B \tag{3.6}$$

and

$$w + G(fw) = G\eta \quad \text{on } B. \tag{3.7}$$

If $m(B^c) = 0$, *then*

$$w = G_f \eta \quad \text{on } B. \tag{3.8}$$

PROOF. By (3.7),

$$G_f(fw) + G_f[fG(fw)] = G_f(fG\eta). \tag{3.9}$$

By (3.4), the left-hand side in (3.9) is equal to $G(fw)$ and the right-hand side is equal to $G\eta - G_f\eta$ on B. Therefore $G(fw) = G\eta - G_f\eta$ and (3.8) follows from (3.7). □

PROOF OF THEOREM 3.1. Put $w = \hat{u} - u$ on B and $w = 0$ on B^c. There exists a function $f \geq 0$ such that $\hat{u}^\alpha - u^\alpha = fw$ a.e. Equation (3.1) implies (3.7). Since $G|fw| \leq \mathscr{E}(u) + \mathscr{E}(\hat{u}) < \infty$ on B, Theorem 3.1 follows from Lemma 3.2. □

REMARK. Theorem 3.1 can also be deduced from the domination principle of potential theory ([5], XII.27).

3.3. We also use another monotonicity property of the $\mathscr{E}$-equation.

THEOREM 3.2. *Suppose that* Q *is a* p*-open subset of* S *and* $\varphi \geq 0$. *Let* $B \subset Q$ *and* $m(Q \setminus B) = 0$. *If*

$$\hat{u} + \mathscr{E}_Q(\hat{u}) = u + \mathscr{E}_Q(u) + K_Q\varphi < \infty \quad \text{on } B, \tag{3.10}$$

then $\hat{u} \geq u$ *on* B.

The proof is similar to the proof of Theorem 3.1 but, instead of ξ, we use its part in Q, the corresponding operators G_Q^f and operators

$$K_Q^f\varphi(r, x) = \Pi_{r,x}\varphi(\tau, \xi_\tau)1_{\tau<\zeta} \exp\left\{-\int_r^\tau f(s, \xi_s)\, ds\right\}$$

and, instead of (3.4) we prove that

$$K_Q\varphi - K_Q^f\varphi = G_Q^f(fK_Q\varphi)$$

on the set $\{K_Q\varphi < \infty\}$.

4. Spectral measures of NLA functionals.

4.1. Let $h \in H$ and let γ be the spectral measure of h. First, we prove three theorems which, obviously, imply Theorem 1.3.

THEOREM 4.1. *If $\gamma(\Gamma) = 0$ for all compact sets Γ with $CM(\Gamma) = 0$, then $h \in H^{p*}$.*

THEOREM 4.2. *If $h \in H^*$, then γ does not charge $\mathscr{G}$-polar sets.*

THEOREM 4.3. *If $h \in H^p$, then γ does not charge $\mathscr{G}$-polar sets.*

Theorem 4.1 is a direct implication of 1.5.C and the following two propositions.

4.1.A. Let A_n be an NLA functional with potential h_n and determining set $\mathscr{M}_n^*$. If $h = \sum h_n \in H$, then $A = \sum A_n$ is an NLA functional with potential h and determining set $\mathscr{M}(h) \cap \mathscr{M}_1^* \cap \cdots \cap \mathscr{M}_n^* \cap \cdots$. If, in addition, $\mathscr{E}(h_n) \in H$, then $h \in H^{p*}$.

4.1.B. If the spectral measure γ of h does not charge compact sets Γ with $CM(\Gamma) = 0$, then

$$\gamma = \gamma_1 + \cdots + \gamma_n + \cdots, \qquad h = h_1 + \cdots + h_n + \cdots, \tag{4.1}$$

where γ_n is the spectral measure of h_n and

$$\mathscr{E}(h_n) \in H. \tag{4.2}$$

PROOF OF 4.1.A. The first statement is obvious. To prove the second statement, we note that, by Jensen's inequality, $[(a + b)/2]^\alpha \le (a^\alpha + b^\alpha)/2$ for all $a, b \ge 0$, $\alpha > 1$ and therefore

$$\mathscr{E}(h_1 + h_2) \le 2^{\alpha-1}[\mathscr{E}(h_1) + \mathscr{E}(h_2)] \tag{4.3}$$

for all h_1, h_2. Put $h^n = h_1 + \cdots + h_n$. If $\mathscr{E}(h_n) \in H$ for all n, then $\mathscr{E}(h^n) \in H$ for all n. By 1.5.B, the $\mathscr{E}$-equation (1.13) holds for h^n and the log-potential u_n of $A_1 + \cdots + A_n$. By using the monotone convergence theorem, we get that $h \in H^{p*}$. $\square$

PROOF OF 4.1.B. The proposition holds by [3], Lemma 5.2, if γ is concentrated on S and it holds by [18], Theorem 2.2, if γ is concentrated on S^e. Clearly, it is valid for γ if it holds for the restrictions of γ to S and to S^e. $\square$

4.2. To prove Theorem 4.2, we need some preparations.

Let $C_0^\infty(Q)$ be the set of all infinitely differentiable functions, with compact supports, on an open set $Q \subset S$. Put

$$\|f\|_{1,2;\alpha'} = \|f\|_{\alpha'} + \|\dot{f}\|_{\alpha'} + \sum_i \|\nabla^i f\|_{\alpha'} + \sum_{i,j} \|\nabla^i \nabla^j f\|_{\alpha'},$$

where $\|\cdot\|_\alpha$ is the norm in $L^\alpha(Q)$ and $\alpha' = \alpha/(\alpha - 1)$.

Suppose that $\Gamma \subset Q$. Then $CM(\Gamma) = 0$ if and only if $CM_Q(\Gamma) = 0$ where CM_Q is defined by (1.19) with p replaced be the transition density of the part of ξ in Q. We will drop the superscript Q dealing with capacities of $\Gamma \subset Q$.

LEMMA 4.1. *Suppose that η is a signed measure on Q. If*

$$\int_Q f(r,x)\eta(dr,dx) \le \text{const.}\, \|f\|_{1,2;\alpha'} \tag{4.4}$$

for all $f \in C_0^\infty(Q)$, then $\eta(\Gamma) = 0$ for all $\Gamma \subset Q$ with $CM(\Gamma) = 0$.

PROOF (cf. proof of Proposition 4.1 in [3]). If $CM(\Gamma) = 0$, then, by Proposition 3.2 in [3], there exists a sequence $f_n \in C_0^\infty(Q)$ such that $0 \le f_n \le 1$, each $f_n = 1$ in some neighborhood of Γ and $\|f_n\|_{1,2;\alpha'} \to 0$.

There exist Borel sets Q_+ and Q_- such that $Q_+ \cup Q_- = Q$ and $\eta_+(B) = \eta(B \cap Q_+) \ge 0$, $\eta_-(B) = -\eta(B \cap Q_-) \ge 0$ for all Borel $B \subset Q$. Suppose $\Gamma \subset Q_+$ is compact. Since $\eta_-(\Gamma) = 0$, there exists, for every $\varepsilon > 0$, a neighborhood U of Γ such that $U \subset Q$ and $\eta_-(U) < \varepsilon$. We have

$$\eta(\Gamma) = \eta_+(\Gamma) \le \int_U f_n\, d\eta_+ = \int_U f_n\, d\eta + \int_U f_n\, d\eta_- \le \int_U f_n\, d\eta + \varepsilon. \tag{4.5}$$

It follows from (4.4) and (4.5) that $\eta(\Gamma) = 0$. The case of $\Gamma \subset Q_-$ can be reduced to the case $\Gamma \subset Q_+$ by replacing η, f by $-\eta, -f$ which also satisfy (4.4). □

LEMMA 4.2. *Let $f \ge 0$ be parabolic and let $h = G\eta + f \in H$. Suppose that*

$$u + \mathscr{E}(u) = h \quad \text{on } B \subset \{h < \infty\} \tag{4.6}$$

and that Q is a bounded p-open set such that $\bar{Q} \subset S$. Then

$$u + \tilde{\mathscr{E}}(u) = \tilde{G}\eta + \tilde{K}u \quad \text{on } B \cap Q, \tag{4.7}$$

where $\tilde{\mathscr{E}}$ is defined by (1.7) with ξ replaced by its part in Q and $\tilde{G}, \tilde{K}$ have an analogous meaning.

PROOF. We note that

$$G = \tilde{G} + \tilde{K}G, \qquad \mathscr{E} = \tilde{\mathscr{E}} + \tilde{K}\mathscr{E} \tag{4.8}$$

by (2.8) and $\tilde{K}f = f$ by the mean value property of parabolic functions (see, e.g., [14], Theorem II.1.4). Therefore

$$\begin{aligned} u + \tilde{\mathscr{E}}(u) + \tilde{K}\mathscr{E}(u) &= u + \mathscr{E}(u) = h = G\eta + f \\ &= \tilde{G}\eta + \tilde{K}G\eta + \tilde{K}f = \tilde{G}\eta + \tilde{K}h \\ &= \tilde{G}\eta + \tilde{K}u + \tilde{K}\mathscr{E}(u) \quad \text{on } B \cap Q. \end{aligned}$$

Since $\tilde{K}\mathscr{E}(u) \le \mathscr{E}(u) \le h < \infty$ on B, this implies (4.7). □

REMARK. If h is parabolic (that is, if $\eta = 0$), then (4.6) implies

$$u + \tilde{\mathscr{E}}(u) = \tilde{K}u \quad \text{on } B \cap Q. \tag{4.9}$$

LEMMA 4.3. *Let $f \ge 0$ be parabolic and let $h = G\eta + f \in H^*$. Then f also belongs to H^*.*

PROOF. Cf. [16, Lemma 4.1]. Let Q_n be a standard approximating sequence for S and let $\mathscr{E}_n$ and K_n be the operators (1.7) and (2.10) for the part of ξ in Q_n. By Lemma 4.2, the $\mathscr{E}$-equation (1.8) implies

$$u + \mathscr{E}_n(u) = G_n\eta + K_n u \quad \text{on } B_n, \tag{4.10}$$

where $B_n = \{h < \infty\} \cap Q_n$. By (2.12),

$$u_n(r, x) = -\log P_{r,x} \exp(-\langle u, X_{Q_n}\rangle)$$

satisfies the equation

$$u_n + \mathscr{E}_n(u_n) = K_n u \quad \text{on } Q_n. \tag{4.11}$$

Let $m > n$. Note that $K_m u$ is parabolic in Q_m. By the Remark to Lemma 4.2, the equation

$$u_m + \mathscr{E}_m(u_m) = K_m u \quad \text{on } B_m \tag{4.12}$$

implies

$$u_m + \mathscr{E}_n(u_m) = K_n u_m \quad \text{on } B_n. \tag{4.13}$$

We apply Theorem 3.1 to get from (4.10) and (4.11) that $u_n \le u$ on B_n. Then we apply Theorem 3.2 to get from (4.11) and (4.13) that $u_m \le u_n$ on B_n. Therefore there exists a limit

$$v = \lim_{n\to\infty} u_n \quad \text{on } \{h < \infty\}.$$

By the monotone convergence theorem, we get from (4.10) that

$$u + \mathscr{E}(u) = G\eta + \lim K_n u \quad \text{on } \{h < \infty\}.$$

In combination with (1.8), this yields $\lim K_n u = f$ on $\{h < \infty\}$. By (1.8), $\mathscr{E}(u) < \infty$ on $\{h < \infty\}$ and, by the dominated convergence theorem, $\lim \mathscr{E}_n(u_n) = \mathscr{E}(v)$. Therefore (4.11) implies that (1.8), with u, h replaced by v, f, holds a.e. Hence $f \in H^*$. $\square$

PROOF OF THEOREM 4.2. The main steps are the same as in the proof of Theorem 2.2 in [16].

1°. By Lemma 4.3, the parabolic part f of h belongs to H^* and, by Theorem 3.1 in [18], the spectral measure of f does not charge $\mathscr{G}$-polar sets. Theorem 3.1 in [18] is proved for time-homogeneous processes, but only minor modifications are needed in the time-inhomogeneous setting.

2°. It remains to show that η does not charge $\mathscr{G}$-polar sets Γ. By 1.6.A, it is sufficient to show that $\eta(\Gamma) = 0$ if $CM(\Gamma) = 0$. We can assume, in addition, that Γ is compact. Choose a bounded open set Q such that $\Gamma \subset Q \subset \bar{Q} \subset S$ and let $\rho > 0$ be a Borel function such that

$$J = \int_S \rho(r, x) h(r, x)\, dr\, dx < \infty.$$

The function

$$\tilde\rho(s, y) = \int_S dr\, dx \rho(r, x) p(r, x; s, y)$$

is strictly positive and lower semicontinuous. Equation (1.8) implies $\mathscr{E}(u) \le h$ and therefore

$$\int_S \tilde\rho(s, y) u(s, y)^\alpha\, ds\, dy \le J. \tag{4.14}$$

Since $\inf_Q \tilde\rho > 0$, we conclude from (4.14) that

$$\int_Q u(s, y)^\alpha\, ds\, dy < \infty.$$

By Lemma 4.2, u satisfies (4.7). By Lemma 4.3, there exists a function $v \ge 0$ such that

$$v + \tilde{\mathscr{E}}(v) = \tilde K u. \tag{4.15}$$

By Theorem 3.1, $w = u - v \ge 0$ on $\{h < \infty\}$. There exists a function $\varphi \ge 0$ such that $u^\alpha - v^\alpha = \varphi w$ a.e. By (4.7) and (4.15), $w = \tilde G \tilde\eta$ where $\tilde\eta(dr, dx) = \eta(dr, dx) - (\varphi w)(r, x)\, dr\, dx$. Suppose that $f \in C_0^\infty(Q)$. Let $\psi = -\dot f - L^* f$ where L^* is the formal adjoint for L. Then

$$f(s, y) = \int_Q dr\, dx\, \psi(r, x) p(r, x; s, y),$$

where p is the transition density of the part of ξ in Q (see [21], Section I.8). Therefore

$$\begin{aligned}\int_S f(s, y)\tilde\eta(ds, dy) &= \int_Q w(r, x)\psi(r, x)\, dr\, dx \\ &\le \|w\|_\alpha \|\psi\|_{\alpha'} \le \text{const.}\, \|w\|_\alpha \|f\|_{1,2;\alpha'}.\end{aligned} \tag{4.16}$$

Since $\|w\|_\alpha \le \|u\|_\alpha < \infty$, we conclude from Lemma 4.1 that $\tilde\eta(\Gamma) = 0$ for all sets Γ with $CM(\Gamma) = 0$. Clearly, $m(\Gamma) = 0$ and therefore $\eta(\Gamma) = 0$. $\square$

4.3. *Measures* Π_μ^h. A one-parameter family of measures η_s on E is called an *entrance rule* if

$$\eta_s T_t^s \le \eta_t, \qquad \eta_s T_t^s \to \eta_t \quad \text{as } s \uparrow t.$$

By [24], to every pair (entrance rule η, exit rule h) such that $\eta_s\{h^s = \infty\} = 0$ there corresponds a stochastic process (ξ_t, Π) on a random time interval (β, ζ) with finite-dimensional distributions

$$\begin{aligned}&\Pi\{\beta < t_1,\ \xi_{t_1} \in dy_1, \dots, \xi_{t_n} \in dy_n,\ t_n < \zeta\} \\ &\quad = \eta_{t_1}(dy_1) p(t_1, y_1; t_2, dy_2) \cdots p(t_{n-1}, y_{n-1}; t_n, dy_n) h(t_n, y_n)\end{aligned} \tag{4.17}$$

for $0 < t_1 < \cdots < t_n$. [The measure $\Pi_{r,x}^{t,y}$ used in Section 3.1 is a particular case corresponding to $\eta_s(B) = p(r, x; s, B)$, $h^r(x) = p(r, x; t, y)$.] We denote by Π_μ^h the measure Π, corresponding to entrance rule

$$\eta_s(B) = \int \mu(dr, dx) 1_{r<s} p(r, x; s, B)$$

and to exit rule h. Suppose that h is parabolic with the spectral measure ν. Then the total mass of Π_μ^h is equal to $\langle h, \mu \rangle$. If $\mu \in \mathscr{M}(h)$, then, Π_μ^h-a.s., (s, ξ_s) tends to a limit $\Xi \in S^e$ (in the topology of the exit space $\bar{S}$) as $s \uparrow \zeta$ and

$$(4.18) \qquad \begin{aligned} &\Pi_\mu^h\{\beta < t < \zeta,\ \Xi \in B\} \\ &\quad = \int_{S_{<t}} \mu(dr, dx) \int_E p(r, x; t, dy) \int_B k(t, y; w) \nu(dw). \end{aligned}$$

In particular,

$$(4.19) \qquad \Pi_{r,x}^h\{\Xi \in B\} = \int_B k(r, x; w) \nu(dw)$$

and

$$(4.20) \qquad \Pi_{r,x}^h\{t < \zeta\} = \int_E p(r, x; t, dy) h(t, y) \quad \text{for } r < t.$$

4.4. *Localization.* Let

$$h(r, x) = \int_{S^e} k(r, x; w) \nu(dw).$$

To every positive bounded Borel function φ on the exit space $\bar{S}$ there corresponds a parabolic function

$$(4.21) \qquad h^\varphi(r, x) = \int_{S^e} k(r, x; w) \varphi(w) \nu(dw).$$

If h is the potential of an NLA functional A, then, by 1.5.C, h^φ is also the potential of an NLA functional A^φ. We call A^φ the φ-*localization* of A. Recall that $h_\Delta(x) = h^s(x) - T_t^s h^t(x)$ for $\Delta = (s, t]$. For every measure μ, set $\mu^\varphi(dr, dx) = \varphi(r, x) \mu(dr, dx)$.

LEMMA 4.4. *Suppose that A is an NLA functional with parabolic potential h and that A^φ is its φ-localization with continuous φ. For every $\mu \in \mathscr{M}^*$,*

$$(4.22) \qquad A^\varphi(0, \infty) = \lim_\Lambda B_\Lambda^\varphi \quad \textit{weakly in } L^1(P_\mu),$$

where

$$(4.23) \qquad B_\Lambda^\varphi = \sum_k \langle h_{\Delta_k}, X_{t_k}^\varphi \rangle$$

for $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$.

PROOF. Let

$$A_\Lambda^\varphi = \sum_k \langle h_{\Delta_k}^\varphi, X_{t_k} \rangle,$$

where

$$h^{\varphi}_{\Delta}(x) = h^{\varphi}(s,x) - \int p(s,x;t,dy)h^{\varphi}(t,y) \quad \text{for } \Delta = (s,t].$$

By 1.7.A, $\lim_{\Lambda} A^{\varphi}_{\Lambda} = A^{\varphi}(0,\infty)$ weakly in $L^1(\mu)$. Put $J_{\Lambda} = A^{\varphi}_{\Lambda} - B^{\varphi}_{\Lambda}$. To prove (4.22), it is sufficient to show that $\lim_{\Lambda} J_{\Lambda} = 0$. Note that

$$J_{\Lambda} = \sum_k \langle f_{\Delta_k}, X_{t_k}\rangle,$$

where $f_{\Delta} = h^{\varphi}_{\Delta} - \varphi h_{\Delta}$. It follows from (4.18) that

$$f_{\Delta}(x) = \Pi^h_{s,x}[\varphi(\Xi) - \varphi(s,\xi_s)]1_{\zeta \le t} \quad \text{for } \Delta = (s,t].$$

By (2.14) and the Markov property of ξ,

$$P_{\mu}\langle |f_{\Delta}|, X_t\rangle = \Pi_{\mu}|f_{\Delta}(t,\xi_t)| \le \Pi^h_{\mu}|\varphi(\Xi) - \varphi(s,\xi_s)|1_{\zeta \le t}$$

and

$$P_{\mu}J_{\Lambda} \le \Pi^h_{\mu}|\varphi(\Xi) - \varphi(\varkappa(\zeta), \xi_{\varkappa(\zeta)})|,$$

where

$$\varkappa(t) = t_k \quad \text{for } t_k < t \le t_{k+1}.$$

Clearly, $P_{\mu}J_{\Lambda_n} \to 0$. □

4.5. *Proof of Theorem* 4.3. It is similar to that of Theorem 3.1 in [18]. Let $\Gamma \subset \bar{S}$ be a compact $\mathscr{G}$-polar set. Put

$$Q_n = \left\{(r,x) \in S\colon d(r,x;\Gamma) > \frac{1}{n}\right\},$$

where d is the distance in the exit space $\bar{S}$. The bounded positive continuous functions

$$\varphi_n(r,x) = (1 - nd(r,x;\Gamma))_+$$

vanish on Q_n. Consider the corresponding localizations A^{φ_n}. It follows from Lemma 4.4 that, for every $\mu \in \mathscr{M}^*$,

$$A \ge A^{\varphi_1} \ge \cdots \ge A^{\varphi_n} \ge \cdots \quad P_{\mu}\text{-a.s.}$$

and

$$\{\mathscr{G} \subset Q_n\} \subset \{A^{\varphi_n}_{\infty} = 0\} \quad P_{\mu}\text{-a.s.}$$

Let $\mu \in \mathscr{M}(h)$ and $\mu(\Gamma) = 0$. Since Γ is $\mathscr{G}$-polar, $1_{\mathscr{G} \subset Q_n} \uparrow 1$ P_{μ}-a.s. and therefore $A^{\varphi_n}_{\infty} \to 0$ P_{μ}-a.s. By the dominated convergence theorem,

$$\lim P_{\mu}A^{\varphi_n}_{\infty} = 0. \tag{4.24}$$

On the other hand, by (1.3) and (4.21),

$$P_{\mu}A^{\varphi_n}_{\infty} = \int \mu(dr,dx)\int_{\bar{S}} k(r,x;z)\varphi_n(z)\gamma(dz) \downarrow \int \mu(dr,dx)\int_{\Gamma} k(r,x;z)\gamma(dz).$$

By (4.24), $\int_{\Gamma} k(r,x;z)\gamma(dz) = 0$ on $\Gamma^c \cap \{h < \infty\}$ which implies $\gamma(\Gamma) = 0$. □

4.6. *Proof of Theorem* 1.5. If $G\eta$ is the potential of an NLA functional A, then, by Theorem 4.3, η does not charge $\mathscr{G}$-polar sets. By 1.6.A, it does not charge Γ with $CM(\Gamma) = 0$ and Theorem 1.5 follows from 4.1.B and 1.5.D. □

5. The identity $H^* = H^p = H^{p*}$.

5.1. We start from a general lemma applicable to all exit rules h.

LEMMA 5.1. *Let $h \in H$ and let Q_n be an arbitrary monotone increasing sequence of p-open sets. Then, for every $\mu \in \mathscr{M}(h)$, there exists, P_μ-a.s., a finite limit*

$$Z = \lim_{n\to\infty} \langle h, X_{Q_n} \rangle \tag{5.1}$$

and

$$h(r, x) \ge P_{r,x} Z. \tag{5.2}$$

If $Q_n \uparrow S$ and if $h = G_\eta$, then, for every $\mu \in \mathscr{M}(h)$, $Z = 0$ P_μ-a.s.

PROOF. Put $G_n = G_{Q_n}$, $K_n = K_{Q_n}$. The limit of $Z_n = \langle h, X_{Q_n} \rangle$ exists and is finite because Z_n is a positive supermartingale relative to $(\mathscr{F}_{\subset Q_n}, P_\mu)$. Indeed, by (2.15) and (2.13),

$$P_\mu\{Z_n | \mathscr{F}_{\subset Q_i}\} = P_{X_{Q_i}} Z_n = \langle K_n h, X_{Q_i} \rangle \le Z_i \quad P_\mu\text{-a.s.}$$

for all $i < n$ since $K_n h \le h$ on Q_n. By (2.15), $\Pi_{r,x} Z_n = K_n h(r, x) \le h(r, x)$ which implies, by Fatou's lemma, (5.2). If $h = G\eta$, then $K_n h \downarrow 0$ on S^* and $P_\mu Z = 0$ by Fatou's lemma if $\langle h, \mu \rangle < \infty$. □

REMARK. In proving the existence of the limit (5.1) we used only the property $K_Q h \le h$ for all p-open sets Q. Note that this property holds for $h 1_{S_{\le t}}$ if it holds for h.

5.2. Let $\mathscr{O}$ stand for the collection of all bounded p-open sets Q with $\bar{Q} \subset S$. Denote by $\mathscr{F}_e$ the σ-algebra generated by X_Q, $Q \in \mathscr{O}$ and let $\mathscr{F}^\mu_{\supset Q}$, $\mathscr{F}^\mu_e$ stand for the completions of $\mathscr{F}_{\supset Q}$, $\mathscr{F}_e$ relative to P_μ.

LEMMA 5.2. *Let A be an NLA functional with potential h and determining set $\mathscr{M}^*$ and let $\mu \in \mathscr{M}^*$. For every I, $A(I)$ is $\mathscr{F}^\mu_e$-measurable. If h is parabolic, then $A(I)$ is $\mathscr{F}^\mu_{\supset Q}$-measurable for every $Q \in \mathscr{O}$.*

PROOF. 1°. Since $X_t(B) = \langle 1_B 1_{\{t\}}, X_{<t} \rangle$, the first statement will be proved if we show that $\langle f, X_{<t} \rangle$ is $\mathscr{F}^\mu_e$-measurable for every $t \in \mathbb{R}_+$ and every bounded continuous f. Consider a sequence $Q_k \in \mathscr{O}$ such that $Q_k \uparrow S_{<t}$ and let τ_k be the first exit time from Q_k. Clearly, $f(\tau_n, \xi_{\tau_n}) \to f(\tau, \xi_\tau)$ Π_μ-a.s. where τ is the first exit time from $S_{<t}$. By Theorem 4.1 in [13], this implies $\langle f, X_{Q_k} \rangle \to \langle f, X_{<t} \rangle$ P_μ-a.s.

2°. Let φ be a continuous function on $\bar{S}$ which is equal to 1 on ∂S and vanishes on Q. Then $h^\varphi = h$ and therefore $A^\varphi = A$. By (4.22) and (4.23), $A(0,\infty)$ is measurable with respect to the P_μ-completion of the σ-algebra generated by measures X_t^φ, $t > 0$. The set $\tilde{Q}_t = Q \cup S_{<t}$ contains Q and $P_\mu\{X^\varphi_{\tilde{Q}_t} = X_t^\varphi + \mu_t\} = 1$ where μ_t is the restriction of μ^φ to $S_{>t}$. Hence X_t^φ is $\mathscr{F}^\mu_{\supset Q}$-measurable. □

5.3.

THEOREM 5.1. *Let A be an NLA functional with parabolic potential h and determining set $\mathscr{M}^*$. Suppose that Q_n is a standard sequence approximating S and let Z be given by (5.1). Then*

$$P_\mu\{A(0,\infty) = Z\} = 1 \quad \textit{for all } \mu \in \mathscr{M}^*. \tag{5.3}$$

PROOF. Note that the minimal σ-algebra which contains all $\mathscr{F}_{\subset Q_n}$ coincides with $\mathscr{F}_e$. By Lemma 5.2, the Markov property (2.15) and (1.3),

$$P_\mu\{A(0,\infty)|\mathscr{F}_{\subset Q_n}\} = P_{X_{Q_n}} A(0,\infty) = \langle h, X_{Q_n}\rangle \tag{5.4}$$

and therefore

$$P_\mu\{A(0,\infty)|\mathscr{F}_e^\mu\} = Z \quad P_\mu\text{-a.s.} \tag{5.5}$$

Formula (5.3) follows from (5.5) and Lemma 5.2. □

5.4. By (1.17), every $h \in H$ has a unique representation $h = G\eta + f$ where f is parabolic. Let Q_n be a standard approximating sequence for S and let τ_n be the first exit time of ξ from Q_n. For every $h \in H$, $\Pi_{r,x} h(\tau_n, \xi_{\tau_n}) \downarrow f(r,x)$ on S^* and therefore f is the maximal parabolic minorant of h.

5.4.A. A positive solution of the differential equation

$$\dot{u} + Lu = u^\alpha \tag{5.6}$$

is dominated by $h \in H$ if and only if it is dominated by the maximal parabolic minorant f of h.

Indeed, it follows from (5.6) that $\dot{u} + Lu \geq 0$ and therefore $u(r,x) \leq \Pi_{r,x} u(\tau_n, \xi_{\tau_n})$ in Q_n, which implies that $u \leq f$.

By 5.4.A, the class U^* of positive solutions of (5.6) dominated by functions of class H coincides with the class of solutions dominated by parabolic functions. Time-homogeneous processes are considered in [18] but the same arguments work in the time-inhomogeneous setting. The results proved in [18] (see Theorems 1.1–1.3, 1.5 and Lemma 1.1 there) in combination with 5.4.A imply the following.

5.4.B. For every $h \in H$, there exists a maximal solution of (5.6) dominated by h. It satisfies the condition

$$\langle u, \mu\rangle = -\log P_\mu e^{-Z} \quad \text{for all } \mu \in \mathscr{M}(h), \tag{5.7}$$

where Z is given by (5.1).

[This was proved in [18] for parabolic h. If $h = G\eta + f$ with parabolic f, then, by 5.4.A, the maximal solution of (5.6) dominated by f is, at the same time, the maximal solution dominated by h. On the other hand, if $\mu \in \mathscr{M}(h)$, then, by Lemma 5.1, Z coincides P_μ-a.s. with Z^* corresponding to f by formula (5.1).]

5.4.C. If $u \in U^*$, then $h = u + \mathscr{E}(u)$ is the minimal parabolic majorant of u, and u is the maximal solution u of (5.6) dominated by h.

5.4.D. If $h \in H^*$ is parabolic, then

$$\langle h, \mu\rangle = P_\mu Z \quad \text{for all } \mu \in \mathscr{M}(h). \tag{5.8}$$

To simplify notation we write "a.s." instead of "$P_{r,x}$-a.s. for m-almost all (r, x)". Analogously, "a.s. on Q" means "$P_{r,x}$-a.s. for m-almost all $(r, x) \in Q$." Note that $\mathscr{A}$ holds a.s. if it holds P_μ-a.s. for all μ in a total set $\mathscr{M}^*$.

LEMMA 5.3. *Let $h \in H$ and let Z be given by (5.1). Suppose that*

$$h(r, x) = P_{r,x} Z \quad \text{a.e.} \tag{5.9}$$

Then $h \in H^*$. Moreover, $h \in H^{p*}$ if $h \in H^p$.

PROOF. Consider the maximal solution u of (5.6) dominated by h and denote by h^* its minimal parabolic majorant. Let f be the maximal parabolic minorant of h. By 5.4.A, $u \le f$ and therefore $h^* \le f \le h$. By 5.4.C, $h^* = u + \mathscr{E}(u)$ and u is the maximal solution dominated by h^*. By 5.4.C, for every $\mu \in \mathscr{M}(h)$,

$$P_\mu e^{-Z} = P_\mu e^{-Z^*} = e^{-\langle u, \mu\rangle}, \tag{5.10}$$

where Z^* corresponds to h^* by Lemma 5.1. Clearly, $P_\mu\{Z^* \le Z\} = 1$ and (5.10) implies $P_\mu\{Z^* = Z\} = 1$. By (5.9) and (5.2),

$$h(r, x) = P_{r,x} Z = P_{r,x} Z^* \le h^*(r, x) \quad \text{a.s.},$$

which implies $h = h^*$ a.s. Since h and h^* are both exit rules, $h = h^*$ everywhere, which proves the first part of the lemma.

If h is the potential of a NLA functional A, then, by Theorem 5.1,

$$P_{r,x} e^{-A(0,\infty)} = P_{r,x} e^{-Z} \quad \text{on } S^*.$$

By 5.4.B, the log-potential v of A coincides a.e. with u. Hence $v + \mathscr{E}(v) = u + \mathscr{E}(u) = h^* = h$ a.e. and $h \in H^{p*}$. □

5.5.

THEOREM 5.2. *The three classes H^*, H^p and H^{p*} have the same intersection with the class of parabolic functions.*

PROOF. Denote the three intersections by $\tilde{H}^*$, $\tilde{H}^p$ and $\tilde{H}^{p*}$.

1°. $\tilde{H}^* \subset \tilde{H}^p$. To prove the existence of an NLA functional with potential $h \in \tilde{H}^*$, we apply the criterion 1.5.C. Put $Y_t = \langle h, X_t\rangle$ and note that, P_μ-a.s.,

$\langle h, X_{<t}\rangle = Y_t + \langle h, \mu_t\rangle$ where μ_t is the restriction of μ to $S_{>t}$. If $\mu \in \mathscr{M}(h)$, then the second term is a bounded deterministic process. Therefore it is sufficient to show that (Y_t, P_μ) belongs to class (D). Let T be an arbitrary stopping time with respect to the filtration $\mathscr{F}_t^0$. By Theorem 4.1 in [18],

$$P_\mu\{Z|\mathscr{F}_T^0\} \geq F(T, X_T), \tag{5.11}$$

where $F(t,\nu) = P_{t,\nu}Z$. If $\mu \in \mathscr{M}(h)$, then, by 1.3.B, $X_T \in \mathscr{M}(h)$ P_μ-a.s. and, by (5.11) and 5.4.D,

$$P_\mu\{Z|\mathscr{F}_T^0\} \geq Y_T.$$

Hence, the family $\{Y_T\}$ is uniformly integrable with respect to P_μ and (Y_t, P_μ) belongs to class (D).

2°. $\tilde{H}^p \subset \tilde{H}^{p*}$. Indeed, if h is the potential of A, then (5.9) holds by Theorem 5.1 and (1.3), and $h \in H^{p*}$ by Lemma 5.3.

3°. The inclusion $\tilde{H}^{p*} \subset \tilde{H}^*$ is obvious. □

5.6.

LEMMA 5.4. *If $h = G\eta + f \in H^*$, then $G\eta$ and f belong to H^{p*}.*

PROOF. By Theorem 4.3, η does not charge $\mathscr{G}$-polar sets. By 1.6.A and Theorem 4.1, $G\eta \in H^{p*}$. By Lemma 4.3, $f \in H^*$ and $f \in H^{p*}$ by Theorem 5.2. □

LEMMA 5.5. *If $h = G\eta + f \in H^p$, then f and $G\eta$ belong to H^{p*}.*

PROOF. By 1.5.C, $f, G\eta \in H^p$. By Theorem 5.2, $f \in H^{p*}$. By Theorem 4.3, η does not charge $\mathscr{G}$-polar sets and, by 1.6.A and Theorem 4.1, $G\eta \in H^{p*}$. □

5.7. In the next lemma the superscripts Q indicate that we consider operators and classes of functions associated with the part of ξ in Q.

LEMMA 5.6. *Let Q be a bounded p-open set. Suppose that η does not charge compact sets Γ with $CM(\Gamma) = 0$ and φ is a positive a.e. finite Borel function. Then $G_Q\eta$, $K_Q\varphi$ and $G_Q\eta + K_Q\varphi$ belong to H_Q^{p*}.*

PROOF. By 4.1.B, there exist measures η_n such that $\eta = \sum \eta_n$ and $\mathscr{E}_Q(G_Q\eta_n) \in H_Q$. Let τ be the first exit time from Q. If $Q \subset S_{<b}$, then $\tau \leq r \vee b$ $P_{r,x}$-a.s. for all r, x. Put

$$f_n = K_Q[\varphi 1_{n\leq\varphi<n+1}].$$

Note that

$$\Pi_{r,x} f_n(t,\xi_t) 1_{t<\tau} = 0 \quad \text{for } t > r \vee b$$

and

$$\mathscr{E}_Q(f_n)(r,x) = \Pi_{r,x}\int_r^{\tau\wedge r} f_n(s,\xi_s)^\alpha\, ds \leq (n+1)^\alpha b.$$

Since $G_Q\eta = \sum G_Q\eta_n$ and $K_Q\eta = \sum f_n$, Lemma 5.6 follows from 4.1.A and 1.5.D. □

LEMMA 5.7. *Let Q and φ be as in Lemma 5.6 and let A be an NLA functional of the part of X in Q with potential $h = K_Q\varphi$. Then*

$$A(0,\infty) = \langle \varphi, X_Q\rangle \quad P_\mu\text{-a.s.} \tag{5.12}$$

for every μ in the determining set $\mathscr{M}^*$ of A.

PROOF. Let Q_n be a monotone increasing sequence of p-open sets such that $\bar{Q}_n \subset Q_{n+1}$ and the union of Q_n is equal to Q. Function h is parabolic in Q and we apply Theorem 5.1 to h, to the part $\tilde{X}$ of X in Q and to the sequence Q_n. By (5.3),

$$P_\mu\{A(0,\infty) = Z\} = 1 \quad \text{for all } \mu \in \mathscr{M}^*,$$

where Z is given by (5.1). To get (5.12), it is sufficient to prove that

$$\langle h, X_{Q_n}\rangle \to \langle \varphi, X_Q\rangle \quad P_\mu\text{-a.s.} \tag{5.13}$$

for $\mu \in \mathscr{M}^*$. By Theorem 4.1 in [13], (5.13) will follow if we prove that $h(\tau_n, \xi_{\tau_n}) \to \varphi(\tau, \xi_\tau)$ Π_μ-a.s. where τ_n is the first exit time of ξ from Q_n and τ is the first exit time from Q. To get this relation, we note that

$$\Pi_\mu\{\varphi(\tau, \xi_\tau)|\mathscr{F}_{\tau_n}\} = h(\tau_n, \xi_{\tau_n}) \quad P_\mu\text{-a.s.}$$

and that

$$\Pi_\mu\{\varphi(\tau, \xi_\tau)|\mathscr{F}_{\tau-}\} = \varphi(\tau, \xi_\tau) \quad P_\mu\text{-a.s.}$$

because $\varphi(\tau, \xi_\tau) = \varphi(\tau, \xi_{\tau-})$ is measurable with respect to $\mathscr{F}_{\tau-} = \vee\mathscr{F}_{\tau_n}$. □

LEMMA 5.8. *Suppose that Q_n is a standard approximating sequence for S, X^n is the part of X in Q_n and G_n is Green's operator in Q_n. If $G\eta \in H^p$, then $G_n\eta \in H^p_{Q_n}$. If A is an NLA functional of X with potential $h = G\eta$ and A^n is an NLA functional of X^N with the potential $G_n\eta$, then*

$$A^n(0,\infty) \uparrow A(0,\infty) \quad \text{a.s.} \tag{5.14}$$

PROOF. If $G\eta \in H^p$, then, by Theorem 4.3, η does not charge $\mathscr{G}$-polar sets. Its restriction η_n to Q_n does not charge $\mathscr{G}$-polar sets for X^n and, by 1.6.A, it does not charge sets of CM-capacity 0. By Theorem 4.1, $G_n\eta = G_n\eta_n$ is the potential of an NLA functional A^n of X^n. It follows from 1.7.A that, for almost all (r, x),

$$A(0,\infty) = \lim_\Lambda A_\Lambda(0,\infty) \quad \text{weakly in } L^1(P_{r,x}),$$

where

$$A_\Lambda(0,\infty) = \sum \langle h_{\Delta_i}, X_{t_i-}\rangle.$$

Analogously, for almost all $(r, x) \in Q_n$,

$$A^n(0,\infty) = \lim_\Lambda A^n_\Lambda(0,\infty) \quad \text{weakly in } L^1(P_{r,x}), \tag{5.15}$$

where

$$A^n_\Lambda(0,\infty)=\sum\langle h^n_{\Delta_i}, X^n_{t_i-}\rangle. \tag{5.16}$$

If $\Delta=(s,t]$ or $\Delta=(s,\infty)$, then

$$h_\Delta(x)=\int_{S(\Delta)} p(s,x;u,z)\eta(du,dz)$$

and

$$h^n_\Delta(x)=\int_{Q_n(\Delta)} p^n(s,x;u,z)\eta(du,dz),$$

where $S(\Delta)=\Delta\times E$, $Q_n(\Delta)=S(\Delta)\cap Q_n$.

Hence, $h^n_{\Delta_i}\le h^{n+1}_{\Delta_i}\le h_{\Delta_i}$ and

$$A^1(0,\infty)\le\cdots\le A^n(0,\infty)\le\cdots\le A(0,\infty)\quad\text{a.s.}$$

This implies (5.14) because $P_{r,x}A^n(0,\infty)=G_n\eta\to G\eta=P_{r,x}A(0,\infty)$. □

LEMMA 5.9. *If*

$$v+\mathscr{E}_Q(v)=u'+u''+\mathscr{E}_Q(u')+\mathscr{E}_Q(u'')<\infty\quad\text{on } B \tag{5.17}$$

and if $m(Q\setminus B)=0$, then $v\le u'+u''$ on B.

PROOF. We have $\mathscr{E}_Q(u'+u'')-\mathscr{E}_Q(u')-\mathscr{E}_Q(u'')=G_Q\rho$ where $\rho=(u'+u'')^\alpha-(u')^\alpha-(u'')^\alpha\ge 0$. It follows from (5.17) that

$$v+\mathscr{E}_Q(v)+G_Q\rho=u'+u''+\mathscr{E}_Q(u'+u'')\quad\text{on } B \tag{5.18}$$

and $v\le u'+u''$ by Theorem 3.1. □

5.8.

LEMMA 5.10. *If H^{p*} contains $G\eta$ and a parabolic function f, then it contains $h=G\eta+f$.*

PROOF. 1°. Consider NLA functionals A_η and A_f with potentials $G\eta$ and f and denote by u_η and u_f their log-potentials. By definition of H^{p*},

$$u_\eta+\mathscr{E}(u_\eta)=G\eta\quad\text{a.e.},\qquad u_f+\mathscr{E}(u_f)=f\quad\text{a.e.} \tag{5.19}$$

Our objective is to prove that the log-potential v of the NLA functional $A_\eta+A_f$ satisfies

$$v+\mathscr{E}(v)=h\quad\text{a.e.} \tag{5.20}$$

Consider a standard approximating sequence Q_n for S. By Theorem 4.2 and 1.6.A, η does not charge sets Γ with $CM(\Gamma)=0$. By Lemma 4.2,

$$\begin{aligned}&u_\eta+\mathscr{E}_n(u_\eta)=G_n\eta+K_nu_\eta\quad\text{a.e. on } Q_n,\\&u_f+\mathscr{E}_n(u_f)=K_nu_f\quad\text{a.e. on } Q_n\end{aligned} \tag{5.21}$$

and therefore

$$u_\eta + u_f + \mathscr{E}_n(u_\eta) + \mathscr{E}_n(u_f) = h_n \quad \text{a.e. on } Q_n, \tag{5.22}$$

where

$$h_n = G_n\eta + K_n(u_\eta + u_f).$$

2°. Let X^n be the part of X in Q_n. By Lemma 5.6, $G_n\eta$, $K_Q(u_\eta + u_f)$ are the potentials of NLA functionals A^n_η, A^n_f of X^n. Their sum $A^n_\eta + A^n_f$ is an NLA functional with potential h_n and therefore the corresponding log-potential

$$v_n(r, x) = -\log P_{r,x} \exp(-A^n_\eta(0, \infty) - A^n_f(0, \infty)) \quad \text{a.e. on } Q_n \tag{5.23}$$

satisfies the equation

$$v_n + \mathscr{E}_n(v_n) = h_n \quad \text{a.e. on } Q_n. \tag{5.24}$$

By (5.22) and (5.24),

$$v_n + \mathscr{E}_n(v_n) = u_\eta + u_f + \mathscr{E}_n(u_\eta) + \mathscr{E}_n(u_f) \quad \text{a.e. on } Q_n$$

and, by Lemma 5.9,

$$v_n \le u_\eta + u_f \quad \text{a.e. on } Q_n. \tag{5.25}$$

3°. By Lemma 5.7,

$$A^n_f(0, \infty) = \langle u_\eta + u_f, X_{Q_n} \rangle \quad \text{a.s. on } Q_n \tag{5.26}$$

and, by Lemma 5.8,

$$A^n_\eta \uparrow A_\eta \quad \text{a.s.} \tag{5.27}$$

4°. We claim that

$$\langle u_\eta + u_f, X_{Q_n} \rangle \to A_f(0, \infty) \quad \text{a.s.} \tag{5.28}$$

Indeed, let τ_n be the first exit time of ξ from Q_n. For almost all (r, x), by (2.13), (1.6) and (2.8),

$$P_{r,x}\langle u_\eta, X_{Q_n} \rangle = \Pi_{r,x} u_\eta(\tau_n, \xi_{\tau_n}) \le P_{r,x} G\eta(\tau_n, \xi_{\tau_n}) = G\eta(r, x) - G_n\eta(r, x) \to 0.$$

By Fatou's lemma, this implies

$$\lim \langle u_\eta, X_{Q_n} \rangle = 0 \quad \text{a.s.} \tag{5.29}$$

By similar arguments,

$$\lim \langle f - u_f, X_{Q_n} \rangle = 0 \quad \text{a.s.} \tag{5.30}$$

By Theorem 5.1,

$$\lim \langle f, X_{Q_n} \rangle = A_f(0, \infty) \quad \text{a.s.} \tag{5.31}$$

By combining (5.29)–(5.31), we get (5.28).

5°. It follows from (5.23), (5.26), (5.27), (5.14), (5.28) that $v_n(r, x)$ tends a.e. to the log-potential v of $A_\eta + A_f$. By (5.19) and (4.3), $\mathscr{E}(u_\eta + u_f) < \infty$ on $\{h < \infty\}$ and (5.20) follows from (5.24) and the dominated convergence theorem. □

5.9. *Proof of Theorem* 1.2. It follows from Lemmas 5.4, 5.5 and 5.10 that $H^* \cup H^p \subset H^{p*}$. On the other hand $H^{p*} \subset H^* \cap H^p$. □

6. The $\mathscr{E}$-equation in a simple cylinder.

6.1. The main part of this section is devoted to proving Theorem 1.4. (At the end, we prove Theorem 1.6.) We fix a bounded smooth domain D and a simple cylinder $Q = [0, b) \times D$. To simplify notation, we set $S = Q$ and we write $H, H^*, \dots$ for classes of functions with domain Q, and $G, \mathscr{E}, \dots$ for operators acting on these classes.

By 1.5.B, equation (1.13) has a unique solution which can be represented by the formula

$$u(r, x) = -\log P_{r,x} e^{-A(r,b]} \quad \text{on } S^*. \tag{6.1}$$

[To apply 1.5.B, one can continue L to $\mathbb{R}_+ \times D$ preserving properties 2.1.A and 2.1.B and continue h by setting $h = 0$ on $[b, \infty) \times D$.]

Put

$$\|f\| = \int_Q |f(r, x)|\, dr\, dx.$$

Our first goal is to prove Theorem 6.1.

THEOREM 6.1. *Suppose $h = G\eta$, $\eta(Q) < \infty$, A is an NLA functional with potential h and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. Then the log-potential u [given by (6.1)] satisfies the condition*

$$\|u^\alpha\| \le C_1 \eta(Q) + C_2 \tag{6.2}$$

where the constants C_1, C_2 depend on the operator L but not on h.

Let A_λ be given by (1.23) and let

$$u_\lambda(r, x) = -\log P_{r,x} \exp(-A_\lambda(r, b]). \tag{6.3}$$

We have

$$\lim_{\lambda \to \infty} \langle u_\lambda, \mu \rangle = \langle u, \mu \rangle \tag{6.4}$$

for every $\mu \in \mathscr{M}^*$. In particular,

$$u_\lambda(r, x) \to u(r, x) \quad \text{on } S^*. \tag{6.5}$$

6.2. *Properties of G.* First, we establish a few properties of the operator G given by (1.31).

6.2.A. There exists a constant C such that

$$\int_{Q_r} p(r, x; t, y)\, dx \le C \quad \text{for all } (t, y) \in Q,\ r < t.$$

6.2.B. If f_n is a bounded sequence in $L^1(Q)$, then the sequence Gf_n contains a subsequence which converges a.e.

6.2.C. Let $f \in L^1(Q)$ and let $u = Gf$. Then

(6.6) $$\int_Q f \operatorname{sign} u \, ds\, dx \geq -\theta \|u\|.$$

Here

$$\theta = \sup_{x \in D} c^*(x),$$

where

(6.7) $$c^* = \sum_{i,j=1}^d \nabla_i \nabla_j a_{ij} - \sum_{i=1}^d \nabla_i b_i.$$

Properties 6.2.A and 6.2.B hold for any bounded p-open set Q. Moreover 6.2.C holds for finite unions of simple cylinders and, more generally, for every p-open set Q such that each point $(r, c) \in \partial Q$ which can be touched from inside of Q by a vertical segment, is regular (that is $\Pi_{r,c}\{(t, \xi_t) \in Q$ for all $t \in (r, r')\} = 0$ for every $r' > r$.)

Property 6.2.A follows from well-known bounds for $p(r, x; t, y)$ ([21], Chapter 1).

Proof of 6.2.B. Denote by φ_δ a function equal to 0 for $|t| < \delta/2$, equal to 1 for $|t| > \delta$ and linear on $[-\delta, -\delta/2]$ and on $[\delta/2, \delta]$. Formula

$$p_\delta(s, x; t, y) = \varphi_\delta(t-s) p(s, x; t, y)$$

defines a continuous kernel on $\bar{Q}$. The corresponding operator G^δ is compact in $L^1(Q)$ because the functions $G^\delta f_n$ are equicontinuous for every sequence f_n bounded in $L^1(Q)$.

By 6.2.A and Fubini's theorem,

$$\begin{aligned}\|Gf - G^\delta f\| &= \int_Q ds\, dx \int_Q [1 - \varphi_\delta(t-s)] p(s, x; t, y) |f(t, y)|\, dt\, dy \\ &\leq \int_Q dt\, dy |f(t, y)| \int_{(t-\delta)\vee 0}^t ds\, dx\; p(s, x; t, y) \leq C\delta \|f\|.\end{aligned}$$

Therefore G is a compact operator in $L^1(Q)$. □

Proof of 6.2.C. 1°. Let $Q = [a, b) \times D$ and let φ be a bounded increasing continuously differentiable function on $\mathbb{R}$ such that $\varphi(0) = 0$. Suppose that

(6.8) $$u \in C^2(\bar{Q}), \quad u = 0 \text{ on } \partial Q.$$

Put $\Phi(t) = \int_0^t \varphi(s)\, ds$. For every $r \in \mathbb{R}_+$, we get by integration by parts,

(6.9) $$-\int_D \varphi(u) Lu\, dx = \int_D \left[\sum a_{ij} \varphi'(u) \nabla_i u \nabla_j u - c^* \Phi(u)\right] dx$$

and therefore

(6.10) $$-\int_D dx \varphi(u) Lu \geq -\theta \int_D \Phi(u)\, dx.$$

2°. Suppose $u = Gf$ with $f \in C^2(\bar{Q})$. Then u satisfies (6.8) and $Lu = -(f + \dot{u})$. By (6.10),

$$\int_D \varphi(u)(\dot{u} + f)\,dx \ge -\theta \int_D \Phi(u)\,dx. \tag{6.11}$$

Note that $u(x, b) = 0$ for all $x \in D$. Hence $\int_r^b \varphi(u)\dot{u}\,dr = \Phi(u(x, b)) - \Phi(u(x, r)) \le 0$ and therefore (6.11) implies

$$\int_r^b \int_D \varphi(u) f\,ds\,dx \ge -\theta \int_r^b \int_D \Phi(u)\,ds\,dx. \tag{6.12}$$

An arbitrary $f \in L^1(Q)$ is the strong limit of a sequence $f_n \in L^1(Q) \cap C^2(\bar{Q})$. Let $u_n = Gf_n$, $u = Gf$. Formula (6.12) holds for f_n and u_n. By 6.2.A, $\|u_n - u\| \to 0$ and $\int_D |u_n(r, x) - u(r, x)|\,dx \to 0$. We have

$$\begin{aligned} &\int \varphi(u) f\,ds\,dx - \int \varphi(u_n) f_n\,ds\,dx \\ &\quad = \int \varphi(u_n)(f - f_n)\,ds\,dx + \int (\varphi(u) - \varphi(u_n)) f\,ds\,dx. \end{aligned} \tag{6.13}$$

A subsequence u_{n_k} converges to u a.e. and the second term in the right-hand side of (6.13) converges to 0 along this subsequence. The first term also converges to 0. Since (6.12) holds for f_n, u_n, it holds also for f, u.

3°. By applying (6.12) to a sequence of functions φ_n which converge boundedly to sign u and by passing to the limit, we get (6.6). □

6.3. *Proof of Theorem* 6.1. 1°. Note that

$$G\rho_\lambda(r, x) = G\eta(r, x) - \int_Q e^{-\lambda(t-r)} p(r, x; t, z)\eta(dt, dz)$$

and therefore the functions $h_\lambda = G\rho_\lambda$ have the properties

$$h_\lambda \le h \quad \text{and} \quad h_\lambda \uparrow h \quad \text{as } \lambda \to \infty. \tag{6.14}$$

By (2.16), u_λ given by (6.3) satisfies equation

$$u_\lambda + \mathscr{E}(u_\lambda) = h_\lambda. \tag{6.15}$$

We have

$$u_\lambda = GF_\lambda, \tag{6.16}$$

where

$$F_\lambda = \rho_\lambda - u_\lambda^\alpha. \tag{6.17}$$

By 6.2.C,

$$\int_Q F_\lambda \operatorname{sign} u_\lambda\,ds\,dx = \int_Q F_\lambda \operatorname{sign} GF_\lambda\,ds\,dx \ge -\theta\|u_\lambda\|$$

and, since $\operatorname{sign} u_\lambda^\alpha = \operatorname{sign} u_\lambda$, we have

$$\|u_\lambda^\alpha\| = \int_Q u_\lambda^\alpha \operatorname{sign} u_\lambda^\alpha\,ds\,dx \le \|\rho_\lambda\| + \theta\|u_\lambda\|. \tag{6.18}$$

By 6.2.A and (1.24),

$$\|\rho_\lambda\| \le C\eta(Q). \tag{6.19}$$

Note that, if $\alpha > 1$, then for every $\delta > 0$, there exists a constant C_δ such that

$$|b-a| \le \delta|b^\alpha - a^\alpha| + C_\delta \tag{6.20}$$

for all reals a, b. It follows from (6.18), (6.19) and (6.20) that

$$\|u_\lambda^\alpha\| \le \theta\delta\|u_\lambda^\alpha\| + C\eta(Q) + \theta C_\delta. \tag{6.21}$$

If $\delta\theta \le 1/2$, then

$$\|u_\lambda^\alpha\| \le 2C\eta(Q) + 2\theta C_\delta, \tag{6.22}$$

which implies (6.2) with $C_1 = 2C, C_2 = 2\theta C_\delta$.

2°. By (6.14), $h_\lambda \uparrow h$. By (6.15), (6.2) and 6.2.B, every sequence u_{λ_n} contains a subsequence which converges, a.e. We claim that, if a sequence u_{λ_n} converges a.e., then u_{λ_n} converges on S^* to the log-potential u of A. Suppose $u_{\lambda_n} \to v$ a.e. By (6.15) and (6.14), $u_\lambda \le h$ for all λ and, by the dominated convergence theorem,

$$\mathscr{E}(u_{\lambda_n}) \to \mathscr{E}(v) \quad \text{on } S^*. \tag{6.23}$$

By (6.15) and Fatou's lemma, $v + \mathscr{E}(v) \le h$ and therefore $\bar{v} = h - \mathscr{E}(v) \ge 0$. It follows from (6.15), (6.14) and (6.23) that $u_{\lambda_n} \to \bar{v}$ on S^*. Clearly, $\bar{v} = v$ a.e. and therefore $\bar{v} + \mathscr{E}(\bar{v}) = \bar{v} + \mathscr{E}(v) = h$. By Theorem 1.2, $u + \mathscr{E}(u) = h$ on S^*. By the uniqueness Theorem 1.1, $\bar{v} = u$ on S^*.

Formula (6.5) holds because, otherwise, $|u_{\lambda_n} - u| > \delta$ for an $(r, x) \in S^*$, $\delta > 0$ and a sequence $\lambda_n \to \infty$. By applying once more the dominated convergence theorem, we get (6.4). □

6.4. Theorem 6.1 can be modified as follows.

THEOREM 6.2. *Let Q, h, A and A_λ be as in Theorem 6.1. For every $\lambda > 0$, we put*

$$\tilde{u}_\lambda(r, x) = -\log P_{r,x} \exp\left(-\tfrac{1}{2}\tilde{A}_\lambda(r, b]\right), \tag{6.24}$$

where

$$\tilde{A}_\lambda = \tfrac{1}{2}(A_\lambda + A). \tag{6.25}$$

If $\mu \in \mathscr{M}^*$, then

$$\lim_{\lambda\to\infty} \langle \tilde{u}_\lambda, \mu\rangle = \langle u, \mu\rangle, \tag{6.26}$$

where u is given by (6.1).

PROOF. Put

$$\eta_\lambda(ds, dx) = \rho_\lambda^s(x)\,ds\,dx, \qquad \tilde{\eta}_\lambda = \tfrac{1}{2}(\eta_\lambda + \eta). \tag{6.27}$$

Clearly, $\tilde{A}_\lambda$ is an NLA functional with potential $\tilde{h}_\lambda = \frac{1}{2}(h_\lambda + h)$ and determining set $\mathscr{M}^*$. By 1.5.B,

$$\tilde{u}_\lambda + \mathscr{E}(\tilde{u}_\lambda) = G\tilde{\eta}_\lambda \quad \text{on } S^*.$$

By 6.2.A and (1.23), $\tilde{\eta}_\lambda(Q) \le C\eta(Q)$ and (6.2) implies that $\sup_\lambda \|\tilde{u}_\lambda^\alpha\| < \infty$. The same arguments as in proof of Theorem 6.1 show that $u = \lim \tilde{u}_\lambda$ exists on S^* and that it satisfies (1.13). Since $\tilde{u}_\lambda \le h$, the dominated convergence theorem implies (6.26). □

6.5. *Proof of Theorem* 1.4. 1°. Consider functionals $\tilde{A}_\lambda$ and measures $\eta_\lambda, \tilde{\eta}_\lambda$ defined by (6.25) and (6.27) and denote by $\eta_r, \tilde{\eta}_{\lambda r}$ the restrictions of η and $\tilde{\eta}_\lambda$ to $S_{>r}$. By (2.16),

$$P_\mu \exp(-A_\lambda(r, b]) = \exp(-\langle u_{\lambda r}, \mu\rangle),$$

where

$$u_{\lambda r} + \mathscr{E}(u_{\lambda r}) = G\eta_{\lambda r}.$$

Suppose that $\mu \in \mathscr{M}^*$. By Theorem 6.1 (applied to η_r), $\langle u_{\lambda r}, \mu\rangle \to \langle u_r, \mu\rangle$ as $\lambda \to \infty$ where

$$u_r + \mathscr{E}(u_r) = G\eta_r \quad \text{on } S^*. \tag{6.28}$$

Therefore

$$\lim_{\lambda\to\infty} P_\mu \exp(-A_\lambda(r, b]) = \exp(-\langle u_r, \mu\rangle) \tag{6.29}$$

Analogously, by Theorem 6.2,

$$\lim_{\lambda\to\infty} P_\mu \exp(-\tilde{A}_\lambda(r, b]) = \exp(-\langle u_r, \mu\rangle) \tag{6.30}$$

By (6.29) and (6.30),

$$\begin{aligned} &P_\mu\{\exp(-A_\lambda(r, b]/2) - \exp(-A(r, b]/2)\}^2 \\ &\quad = P_\mu \exp(-A_\lambda(r, b]) + P_\mu \exp(-A(r, b]) - 2P_\mu \exp(-\tilde{A}_\lambda(r, b]) \\ &\quad \to 0 \quad \text{as } \lambda \to \infty. \end{aligned} \tag{6.31}$$

Therefore $\exp(-A_\lambda(r, b])$ converges to $\exp(-A(r, b])$ in $L^2(P_\mu)$ and $A_\lambda(r, b]$ converges in P_μ-probability to $A(r, b]$. □

6.6. *Proof of Theorem* 1.6. 1°. Fix $t > 0$. Formula $\tilde{A}(I) = A(I \cap (0, t])$ defines a NLA functional with the same determining set $\mathscr{M}^*$ as A and with the potential

$$\tilde{h}(r, x) = \begin{cases} 0, & \text{for } r \geq t, \\ h(r, x) - \Pi_{r, x} h(t, \xi_t), & \text{for } r < t. \end{cases} \tag{6.32}$$

Clearly, $\tilde{h} \leq h$. By using the strong Markov property of ξ, we check that $\tilde{h}$ has the mean value property on every simple cylinder and therefore it is parabolic. By Theorem 5.1,

$$A(0, t] = \tilde{A}(0, \infty) = \tilde{Z} \quad P_\mu\text{-a.s.}, \tag{6.33}$$

where

$$\tilde{Z} = \lim \langle \tilde{h}, X_{Q_n} \rangle. \tag{6.34}$$

2°. We have

$$\langle \tilde{h}, X_{Q_n} \rangle = \langle h, (X^n)'_t \rangle - \langle \hat{h}, X_{Q_n} \rangle \tag{6.35}$$

where $\hat{h} = (h - \tilde{h}) 1_{S_{\leq t}}$. By the Remark to Lemma 5.1, the limit

$$Y = \lim \langle \hat{h}, X_{Q_n} \rangle$$

exists P_μ-a.s. for every $\mu \in \mathscr{M}(h)$. By (2.13), (2.10) and the strong Markov property of ξ,

$$P_\mu \langle \hat{h}, X_{Q_n} \rangle = \Pi_\mu \hat{h}(\tau_n, \xi_{\tau_n}) = \Pi_\mu 1_{\tau_n \leq t < \zeta} h(t, \xi_t).$$

The right-hand side tends to 0 as $n \to \infty$ and, by Fatou's lemma, $P_\mu Y = 0$. Formula (1.26) follows from (6.33), (6.34) and (6.35). □

7. Bibliographical notes and concluding remarks.

7.1. Additive functionals of a super-Brownian motion (with $\alpha = 2$) of the form

$$A(I) = \int_I \langle f^s, X_s \rangle \, ds$$

have been introduced and studied, first, by Iscoe [23] under the name "weighted occupation times." In [8] a continuous linear additive functional A with potential h was constructed in the case: ξ is an arbitrary right process, $\alpha = 2$ and h is a bounded exit rule. The construction was based on integration with respect to a martingale measure. The case of a superdiffusion with an arbitrary $\alpha \in (1, 2]$ was investigated in [9]. There a continuous linear additive functional with potential $h = G\eta$ was constructed for every η which vanishes on sets Γ with $CM(\Gamma) = 0$.

Le Gall [25] investigated recently equation $\Delta u = u^2$ in a bounded domain E with smooth boundary by using additive functionals of a Brownian snake. His functionals correspond to NLA functionals of a super-Brownian motion ($\alpha = 2$) with potential $h(x) = \int_{\partial D} k(x, y) \nu(dx)$ [here k is the Poisson kernel and ν does not charge sets Γ with $CM(\Gamma) = 0$].

7.2. Homogeneous linear additive functionals of a time-homogeneous superdiffusion X were studied in [16]. For a stationary transition density $p(r, x; t, y) = p_{t-r}(x, y)$ and for time-independent u, formula (1.7) takes the form

$$\mathscr{E}(u)(x) = \int_E g(x, y)u(y)^\alpha \, dy,$$

where

$$g(x, y) = \int_0^\infty p_s(x, y)\, ds$$

is the Green's function of ξ. The class of time-independent exit rules coincides with the class of excessive functions and time-independent parabolic functions are L-harmonic functions, that is, solutions of the equation $Lu = 0$. The Martin kernel $k(x, y)$ and the Martin exit space are defined in terms of $g(x, y)$ [not $p(r, x; s, y)$ as in inhomogeneous case]. In this setting, all our results remain valid with the word "functionals" replaced by "homogeneous functionals." Theorems 1.1—1.4 in [16] follow from Theorem 1.3. Theorems 2.1 and 2.1* in [16] are particular cases of Theorems 3.1 and 3.2 and Theorem 2.2 there is very close to the homogeneous version of Theorem 1.4.

Linear additive functionals of superdiffusions were constructed in [16] by passing to the limit from functionals of the form $A(dt) = \langle \rho, X_t \rangle \, dt$ and from absorption processes. It was not clear that functionals constructed this way were natural. A new approach in the present paper was made possible by general results obtained in [17].

7.3. A particular case of problem (1.34) has been studied in [3]. The equation $\dot{u} + Lu - u^\alpha = -\eta$ with a zero boundary condition was considered in a cylinder $[0, b) \times D$ where D is a bounded smooth domain. It was proved that the problem has a solution if and only if η does not charge sets Γ with $CM(\Gamma) = 0$. [We can get this by applying Theorem 1.3 to h with the spectral measure concentrated on $[0, b) \times D$ and by taking into account 1.6.A.] Baras and Pierre have also treated the problem

$$\begin{aligned} \dot{u} + Lu - u^\alpha &= -\eta \quad && \text{in } [0, b) \times E, \\ u &= 0 && \text{on } [0, b) \times \partial E, \\ u &= \gamma && \text{on } \{b\} \times E. \end{aligned} \tag{7.1}$$

by reducing it to a problem with 0 boundary condition in a larger domain $[0, b') \times E$ with a modified measure η.

The boundary value problem (0.15) was investigated in [2] and [18]. The case $\nu = 0$ was treated in [2] and the case $\eta = 0$ was considered in [18]. [Even earlier, Gmira and Véron [22] have investigated a class of functions ψ such that the problem

$$\begin{aligned} \Delta u &= \psi(u) \quad && \text{in } D, \\ v &= \nu && \text{on } \partial D, \end{aligned}$$

has a solution for every finite measure ν.] The results of [18] (modified for the time-inhomogeneous setting) are substantially used in the proof of Theorem 1.2.

7.4. In conclusion we state a challenging open problem.

7.4.A. Let Γ be a compact subset of S^e. For which domains E does the condition $CM(\Gamma) = 0$ imply that Γ is $\mathscr{G}$-polar?

The converse implication—$\{\Gamma$ is $\mathscr{G}$-polar$\} \Longrightarrow \{CM(\Gamma) = 0\}$—follows for an arbitrary domain E from Theorem 1.3 (cf. proof of 1.5.A in [16]). In a homogeneous setting, 7.4.A is proved for bounded domains with smooth boundaries (see [19], Theorem 1.2). In an inhomogeneous setting, the problem is open even for this class of domains.

The following problem is closely related to 7.4.A.

7.4.B. Describe the class $\mathscr{N}$ of pairs (η, ν) for which there exists a solution of the boundary value problem (1.34).

By the Remark to Theorem 1.2, $(\eta, \nu) \in \mathscr{N}$ if and only if $(\eta, 0) \in \mathscr{N}$ and $(0, \nu) \in \mathscr{N}$. By 1.6.A, the following two conditions are equivalent: (i) η does not charge sets of CM-capacity 0; (ii) η does not charge $\mathscr{G}$-polar sets. By Theorem 1.3, each of these conditions is necessary and sufficient for $(\eta, 0)$ to belong to $\mathscr{N}$. Analogous tests are valid for $\nu \in \mathscr{N}$ and for all domains E for which the answer to 7.4.A is positive.

REFERENCES

[1] ADAMS, D. R. and HEDBERG, L. I. (1996). *Function Spaces and Potential Theory*. Springer, Berlin.

[2] BARAS, P. and PIERRE, M. (1984). Singularités éliminable pour des équations semi-linéares. *Ann. Inst. Fourier* (Grenoble) 34 185–206.

[3] BARAS, P. and PIERRE, M. (1984). Problems paraboliques semi-linéares avec donnees measures. *Appl. Anal.* 18 111–149.

[4] DAWSON, D. A. (1991). Measure-valued Markov processes. *Ecole d'Eté de Probabilités de Saint-Flour XXI Lecture Notes in Math.* 1464 1–260. Springer, Berlin.

[5] DELLACHERIE, C. and MEYER, P.-A. (1975, 1980, 1983, 1987). *Probabilités et potentiel*. Hermann, Paris.

[6] DYNKIN, E. B. (1965). *Markov Processes*. Springer, Berlin.

[7] DYNKIN, E. B. (1978). Sufficient statistics and extreme points. *Ann. Probab.* 6 705–730. (Reprinted in E. B. Dynkin. (1982). Markov processes and related problems of analysis. *London Math. Soc. Lecture Note Ser.* 54. Cambridge Univ.)

[8] DYNKIN, E. B. (1989). Superprocesses and their linear additive functionals. *Trans. Amer. Math. Soc.* 314 255–282.

[9] DYNKIN, E. B. (1991). Additive functionals of superdiffusion processes. In *Random Walks, Brownian Motion and Interacting Particle Systems* (R. Durrett and H. Kesten, eds.) 269–281. Birkhäuser, Boston.

[10] DYNKIN, E. B. (1991). Branching particle systems and superprocesses. *Ann. Probab.* 19 1157–1194.

[11] DYNKIN, E. B. (1991). Path processes and historical superprocesses. *Probab. Theory Related Fields* 90 89–115.

[12] DYNKIN, E. B. (1992). Superdiffusions and parabolic nonlinear differential equations. *Ann. Probab.* 20 942–962.

[13] DYNKIN, E. B. (1993). On regularity of superprocesses. *Probab. Theory Related Fields* 95 263–281.

[14] DYNKIN, E. B. (1993). Superprocesses and partial differential equations. *Ann. Probab.* 21 1185–1262.

[15] DYNKIN, E. B. (1994). *An Introduction to Branching Measure-Valued Processes*. Amer. Math. Soc., Providence, RI.

[16] DYNKIN, E. B. and KUZNETSOV, S. E. (1996). Linear additive functionals of superdiffusions and related nonlinear P.D.E. *Trans. Amer. Math. Soc.* 348 1959–1987.

[17] DYNKIN, E. B. and KUZNETSOV, S. E. (1997). Natural linear additive functionals of superprocesses. *Ann. Probab.* 25 640–661.

[18] DYNKIN, E. B. and KUZNETSOV, S. E. (1996). Solutions of $Lu = u^\alpha$ dominated by L-harmonic functions. *J. Analyse Math.* 68 15–37.

[19] DYNKIN, E. B. and KUZNETSOV, S. E. (1996). Superdiffusions and removable singularities for quasilinear partial differential equations. *Comm. Pure Appl. Math.* 49 125–176.

[20] FITZSIMMONS, P. J. (1988). Construction and regularity of measure-valued Markov branching processes. *Israel J. Math.* 64 337–361.

[21] FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Englewood Cliffs, NJ.

[22] GMIRA, A. and VÉRON, L. (1991). Boundary singularities of solutions of some nonlinear elliptic equations. *Duke Math. J.* 64 271–324.

[23] ISCOE, I. (1986). A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Related Fields* 71 85–116.

[24] KUZNETSOV, S. E. (1974). Construction of Markov processes with random birth and death. *Theoret. Probability Appl.* 18 571–574.

[25] LE GALL, J.-F. (1995). The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Theory Related Fields* 102 393–432.

DEPARTMENT OF MATHEMATICS
WHITE HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7901
E-MAIL: ebd1@cornell.edu
sk47@cornell.edu