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LIMIT THEOREMS FOR MIXING SEQUENCES WITHOUT RATE ASSUMPTIONS

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We extend Lévy's classical criterion for a sequence of independent identically distributed random variables to belong to the domain of partial attraction of a nondegenerate Gaussian law to stationary ϕ-mixing sequences. We also extend some results of Kesten and of Kuelbs and Zinn on the LIL behavior of independent identically distributed random variables to stationary ϕ-mixing sequences. No assumptions on the rate of decay for the mixing coefficient are made.

1. Introduction. A sequence $X_1, X_2, \dots$ of random variables is called ϕ-mixing if $\phi(n) \to 0$ where

$$\phi(n) = \sup\bigl\{|P(B|A) - P(B)|: A \in \mathscr{F}_1^k,\ B \in \mathscr{F}_{k+n}^\infty,\ k \geq 1\bigr\}.$$

Here $\mathscr{F}_m^n$ denotes the σ-field generated by the random variables $\{X_j: m \leq j \leq n\}$. There is an extensive literature dealing with the asymptotic properties of ϕ-mixing sequences; in particular, it is known that, under suitable moment conditions and mixing rates, a ϕ-mixing sequence satisfies the central limit theorem, the law of the iterated logarithm and various related limit theorems. Most results in the literature require a polynomial rate of decrease for ϕ, except in the stationary case when a logarithmic rate $\phi(n) \leq C(\log n)^{-\gamma}$, $\gamma \geq \gamma_0$ is usually assumed [see, e.g., Ibragimov (1975), Berkes and Philipp (1979), Bradley (1988)]. Very little is known about what happens under weaker mixing rates or if we assume only $\phi(n) \to 0$. Ibragimov (1962) proved that if (X_n) is a stationary ϕ-mixing sequence with partial sums S_n satisfying

$$EX_1 = 0, \qquad E|X_1|^{2+\delta} < \infty \quad (\delta > 0), \qquad \sigma_n^2 = \operatorname{Var} S_n \to \infty \tag{1.1}$$

then

$$S_n/\sigma_n \to_{\mathscr{D}} N(0,1). \tag{1.2}$$

However, the question whether (1.2) remains valid if in (1.1) we assume only $EX_1^2 < \infty$ has been open for more than 35 years. In fact, we do not even know if (1.1) with $\delta = 0$ implies that (1.2) holds along an infinite sequence of n's or

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if S_n/σ_n has a subsequence with a nondegenerate limit distribution. It is also unknown if (1.1) implies the law of the iterated logarithm, that is,

$$\limsup_{n\to\infty} \frac{S_n}{(2\sigma_n^2 \log\log \sigma_n^2)^{1/2}} = 1 \quad \text{a.s.} \tag{1.3}$$

As was observed in Dehling, Denker and Philipp (1986) and in Peligrad (1990), the difficulties in these problems are connected with the fact that for very slowly decreasing ϕ the standard deviation σ_n may not be the proper norming factor in the CLT. In view of this fact, the natural question is whether for a stationary ϕ-mixing sequence (X_n) satisfying $EX_1^2 < \infty$ and $\sigma_n \to \infty$ there exist norming and centering sequences $(A_n), (B_n), (C_n), (D_n)$ such that

$$\frac{S_n - A_n}{B_n} \to_{\mathscr{D}} N(0,1) \tag{1.4}$$

and

$$\limsup_{n\to\infty} \frac{S_n - C_n}{D_n} = 1 \quad \text{a.s.,} \tag{1.5}$$

where we assume $B_n \to \infty$, $D_n \to \infty$ and in (1.5) we exclude the trivial case

$$S_n/D_n \to 0 \text{ a.s.} \quad \text{and} \quad \limsup_{n\to\infty}(-C_n)/D_n = 1$$

[when (1.5) is the result of the fluctuations of C_n and D_n and not those of S_n]. The purpose of this paper is to prove that the answer to the above question is positive in the case of the generalized LIL (1.5) and in the case of the CLT relation (1.4) holds at least along some infinite sequence (n_k) of positive integers. It will also turn out that using general norming and centering constants in the CLT and LIL, there is no need to assume finite variances and one can give optimal criteria for the LIL and subsequential CLT for arbitrary ϕ-mixing sequences without any moment conditions and without any assumptions on the rate of decay of $\phi(n)$.

THEOREM 1. *Let $\{X_j,\ j \geq 1\}$ be a strictly stationary ϕ-mixing sequence of random variables. Denote by F the common distribution function of $|X_j|$. Suppose that*

$$\liminf_{x\to\infty} \frac{x^2(1-F(x))}{\int_0^x t^2\, dF(t)} = 0 \tag{1.6}$$

and that

$$\phi(1) < 1/4. \tag{1.7}$$

Then there exist numerical sequences (A_n) and (B_n) and a sequence (n_k) of positive integers such that $B_n \to \infty$ and

$$\frac{S_{n_k} - A_{n_k}}{B_{n_k}} \to_{\mathscr{D}} N(0,1). \tag{1.8}$$

THEOREM 2. *Suppose that the sequence* $\{X_j, \ j \geq 1\}$ *satisfies the hypotheses of Theorem 1. Then there exist numerical sequences* (C_n), (D_n) *such that* $D_n \to \infty$ *and the set of limit points of*

$$\left\{\frac{S_n - C_n}{D_n}, \ n \geq 1\right\} \tag{1.9}$$

is with probability 1 precisely the interval $[-1, 1]$. *Moreover,*

$$\limsup_{n\to\infty} \frac{|S_n|}{D_n} \neq 0 \quad \text{a.s.} \tag{1.10}$$

REMARK 1. Condition (1.6) is Lévy's classical necessary and sufficient condition for an i.i.d. sequence (X_n) to belong to the domain of partial attraction of the normal law, that is, to satisfy (1.8) with suitable $(A_n), (B_n)$ and (n_k). By well-known theorems of Kesten (1972), Rogozin (1968) and Heyde (1969), (1.6) is also a necessary and sufficient condition for an i.i.d. sequence (X_n) to satisfy the LIL (1.5) with suitable (C_n) and (D_n). Thus our theorems extend the general LIL and the subsequential CLT to stationary ϕ-mixing sequences under exactly the same conditions as they require for i.i.d. random variables, without any assumption on the mixing rate. It is worth noting that Theorems 1 and 2 are new even in the case of finite second moments.

REMARK 2. We note that if (A_{n_k}), (B_{n_k}) are the sequences in (1.8) given by the proof of Theorem 1 and if (n_k) grows sufficiently rapidly (which can be guaranteed by passing to a further subsequence) then the sequences (C_n), (D_n) in Theorem 2 can be chosen as

$$C_j = A_{n_k} j/n_k, \qquad D_j = B_{n_k}(2 \log k)^{1/2}, \qquad n_{k-1} < j \leq n_k.$$

REMARK 3. From the results of Dehling, Denker and Philipp (1986) it follows easily [see also Peligrad (1993)] that if $\{X_j, \ j \geq 1\}$ is a stationary ϕ-mixing (or even strong mixing) sequence with mean 0 and finite second moments and $\sigma_n^2 = \operatorname{Var} S_n \to \infty$ then either

$$S_n/\sigma_n \to 0 \quad \text{in probability} \tag{1.11}$$

or there exists an increasing sequence (n_k) of positive integers and a constant $c > 0$ such that

$$S_{n_k}/\sigma_{n_k} \to_{\mathscr{D}} N(0, c). \tag{1.12}$$

Although the norming factor in (1.12) is more explicit than in (1.8), the problem whether (1.11) can actually hold is still an open question and thus it is unclear whether (1.12) is always valid. On the other hand, our Theorem 1 shows that with suitably chosen $(A_n), (B_n)$, (n_k) the subsequential CLT (1.8) is always valid, even if (1.11) holds.

REMARK 4. The proofs of our theorems will show that (1.7) can be weakened to $\phi(1) < 1 - 2^{-1/2}$ plus a condition on the maximal correlation coefficient ρ [in fact, $\rho(1) < 1$ will do]. On the other hand, (1.7) cannot be dropped entirely in order to avoid trivialities, such as telescoping sums. If, for instance, $\{Y_j,\ j \geq 1\}$ is the sequence of the Rademacher functions and we set $X_j = Y_{j+1} - Y_j,\ j \geq 1$, then $\{Y_j,\ j \geq 1\}$ is 1–dependent and thus ϕ-mixing, but $S_n = Y_{n+1} - Y_1$ assumes only the values 0 and ± 2 and thus the sequence $\{X_j,\ j \geq 1\}$ cannot satisfy (1.5) or (1.8) with norming factors tending to infinity. We will not pursue this issue any further, except to note that only $\phi(1) < 1$ is needed under the additional hypotheses $EX_1^2 < \infty$ and $\operatorname{Var} S_n \to \infty$. Details will appear elsewhere.

The proof of Theorem 1 is complex and will take up most of our paper. Theorem 2 will then be deduced from Theorem 1 by using an idea from Kuelbs and Zinn [(1982), pages 522 and 523]. While their argument generalizes easily to ϕ-mixing sequences in case of symmetric X_n, the general non-symmetric case is much more delicate. In order to obtain the maximal inequalities required we need the machinery developed in the proof of Theorem 1.

It should be noted that the asymptotic behavior of ϕ-mixing sequences described by our theorems does not extend to strong mixing sequences, i.e. for sequences (X_n) satisfying $\alpha(n) \to 0$ where

$$\alpha(n) = \sup\bigl\{|P(AB) - P(A)P(B)|\colon A \in \mathscr{F}_1^k,\ B \in \mathscr{F}_{k+n}^\infty,\ k \geq 1\bigr\}.$$

Indeed, Herrndorf (1983) constructed a stationary strong mixing sequence (X_n) satisfying

$$EX_1 = 0, \qquad EX_1^2 < +\infty, \qquad \sigma_n^2 = \operatorname{Var} S_n = n$$

but for which

$$S_n/b_n \to 0 \quad \text{in probability}$$

for any sequence $b_n \to \infty$. Since $EX_1^2 < +\infty$, the distribution function F of $|X_1|$ satisfies (1.6), but clearly neither (1.4) nor (1.8) can hold.

Examples of classes of ϕ-mixing and strong mixing sequences can be found in Ibragimov and Linnik (1971), Bradley (1986) and Doukhan (1994). Last, but not least, we would like to mention the basic papers by Bradley (1980a, b, 1981, 1988) and by Peligrad (1982, 1985, 1990) which created a body of methods and ideas on which we draw frequently in the present paper.

2. Preparatory lemmas.

From a technical point of view, the proofs of both theorems are much simpler under the assumption that X_1 is a continuous symmetric random variable. However, as it will turn out, we have only very limited information on the norming constants A_n, B_n and thus the usual techniques of symmetrization and convolution with a small continuous random variable to pass from the above special case to the case to the case of a general distribution pose considerable difficulties. As a consequence, we have opted to deal with the general case directly. The first lemma is trivial for continuous random variables.

LEMMA 2.1. *Let X be a nonnegative random variable with distribution function $F(x) = P(X \le x)$. Then the following three statements are equivalent:*

$$\liminf_{x\to\infty} \frac{x^2 P(X > x)}{EX^2 1\{X \le x\}} = \liminf_{x\to\infty} \frac{x^2(1 - F(x))}{\int_{(0,\,x]} t^2\, dF(t)} = 0, \tag{2.1}$$

$$\liminf_{x\to\infty} \frac{x^2 P(X \ge x)}{EX^2 1\{X < x\}} = \liminf_{x\to\infty} \frac{x^2(1 - F(x-))}{\int_{(0,\,x)} t^2\, dF(t)} = 0, \tag{2.2}$$

$$\liminf_{x\to\infty} \frac{x^2(1 - F(x-))}{\int_0^x t(1 - F(t))\, dt} = 0. \tag{2.3}$$

Here, as usual, $F(x-) = \lim_{y\uparrow x} F(y)$.

PROOF. Suppose that

$$\frac{x^2 P(X > x)}{EX^2 1\{X \le x\}} < \varepsilon.$$

Then, upon setting $t = 2x$, we have

$$\frac{t^2 P(X > \frac{1}{2}t)}{EX^2 1\{X \le \frac{1}{2}t\}} < 4\varepsilon.$$

Hence (2.1) and (2.2) are equivalent. Next, by Billingsley [(1995), page 236],

$$\int_{(0,\,x]} t^2\, dF(t) = -x^2(1 - F(x)) + 2\int_0^x t(1 - F(t))\, dt. \tag{2.4}$$

We divide this relation by $x^2(1 - F(x-))$ and conclude that (2.2) implies (2.3) which in turn implies (2.1). This proves the lemma. □

In the proof of Theorem 1 we can assume, without loss of generality, that $|X_1|$ is unbounded, that is,

$$P(|X_1| > t) > 0 \quad \text{for all } t > 0. \tag{2.5}$$

Indeed, in the opposite case, Ibragimov's (1962) Theorem 1.4, the central limit theorem for stationary ϕ-mixing sequences with finite $(2 + \delta)$th moments would yield a result much stronger than ours. [To verify the variance condition in Ibragimov's theorem, note that $EX_1^2 < \infty$ and (1.7) imply $\sigma_n^2 = ES_n^2 \to \infty$ according to Lemma 2.6 below.]

For $n \ge 1$, define a_n by

$$1 - F(a_n) \le \frac{1}{n} \quad \text{and} \quad 1 - F(a_n-) \ge \frac{1}{n}. \tag{2.6}$$

Clearly a_n is nondecreasing and $a_n \to \infty$ since if $a_n \le M$ for all $n \ge 1$ then (2.6) yields $F(M) = 1$, that is, $P(|X_1| > M) = 0$, contradicting (2.5). Put $x_0 = 0$ and

$$x_k := a_{16^k}, \qquad k \ge 1. \tag{2.7}$$

LEMMA 2.2. *Condition (1.6) implies*

$$\liminf_{k\to\infty} \frac{x_k^2 16^{-k}}{\int_{(0,\,x_k]} t^2\,dF(t)} = 0.$$

PROOF. By Lemma 2.1, condition (1.6) implies (2.3). Hence for each $0 < \varepsilon \le 1/16$ there exists $x = x(\varepsilon)$ with

$$\frac{x^2(1-F(x-))}{\int_0^x t(1-F(t))\,dt} < \varepsilon. \tag{2.8}$$

Let

$$G(x) := \frac{1}{x^2}\int_0^x t(1-F(t))\,dt, \qquad x > 0.$$

Clearly G is absolutely continuous and for any continuity point x of F (i.e., for almost every x) we have

$$x^4 G'(x) = x^3(1-F(x)) - 2x\int_0^x t(1-F(t))\,dt \le 0$$

and thus G is nonincreasing. Choose k with $x_k \le x \le x_{k+1}$. Then by (2.6), (2.7), (2.8) and the monotonicity of G we get

$$\frac{16^{-k-1}x_k^2}{\int_0^{x_k} t(1-F(t))\,dt} \le \frac{x^2(1-F(x-))}{\int_0^x t(1-F(t))\,dt} < \varepsilon.$$

But then by (2.4) and (2.6),

$$\begin{aligned}
16^{-k}x_k^2 &< 16\varepsilon\int_0^{x_k} t(1-F(t))\,dt \\
&= 8\varepsilon\int_{(0,\,x_k]} t^2\,dF(t) + 8\varepsilon x_k^2(1-F(x_k)) \\
&\le 8\varepsilon\int_{(0,\,x_k]} t^2\,dF(t) + \tfrac{1}{2}16^{-k}x_k^2.
\end{aligned}$$

This proves the lemma. □

LEMMA 2.3. *Let $\{b_n,\ n \ge 1\}$ be a sequence of nonnegative numbers, not all equal to zero, and assume that for some integer $l \ge 1$ we have*

$$\liminf_{N\to\infty} \frac{b_{N+1} + \cdots + b_{N+l}}{\sum_{n=1}^{N+l} b_n} = 0. \tag{2.9}$$

Then the inequality

$$b_{n+1} + \cdots + b_{n+l} \le 2lb_n \tag{2.10}$$

holds for infinitely many n.

PROOF. Suppose not; that is, there is an n_0 such that for all $n \ge n_0$,

$$b_{n+1} + \cdots + b_{n+l} > 2lb_n.$$

Summing these inequalities for $n_0 \le n \le N$ we obtain

$$2l \sum_{n=n_0}^{N} b_n < \sum_{n=n_0}^{N} (b_{n+1} + \cdots + b_{n+l}) \le l \sum_{n=n_0}^{N} b_n + l(b_{N+1} + \cdots + b_{N+l}),$$

whence

$$\sum_{n=n_0}^{N} b_n < b_{N+1} + \cdots + b_{N+l}$$

and thus

$$\sum_{n=1}^{N+l} b_n < 2(b_{N+1} + \cdots + b_{N+l}) + \sum_{n=1}^{n_0-1} b_n. \tag{2.11}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then letting $N \to \infty$ in (2.11) we get

$$\sum_{n=1}^{\infty} b_n \le \sum_{n=1}^{n_0-1} b_n,$$

that is, $b_n = 0$ for $n \ge n_0$; but then (2.10) also holds for $n \ge n_0$, contrary to our assumption. If $\sum_{n=1}^{\infty} b_n = \infty$, then dividing (2.11) by its left-hand side and letting $N \to \infty$, we get a contradiction in view of (2.9).

LEMMA 2.4. *Let $\{Y_j,\ j \ge 1\}$ be a strictly stationary ϕ-mixing sequence of random variables with $\phi(1) < 1/2$. Denote the distribution function of $|Y_1|$ by H. Then for any $n \ge 1$ and $t > 0$, we have*

$$P\Big(\max_{1\le i\le n} |Y_i| > t\Big) > \min\big(\tfrac{1}{6} n(1 - H(t)), \tfrac{1}{4}\big).$$

PROOF. Note that for $0 \le h \le 2/n$ we have $(1-h)^n \le 1 - nh/3$ as one can show by induction on n. Thus in the case $n(1 - H(t)) \le 2$ we get by Peligrad (1990), Proposition 3.1 and $\phi(1) < 1/2$,

$$\begin{aligned} P\Big(\max_{1\le i\le n} |Y_i| > t\Big) &\ge (1 - \phi(1))(1 - H(t)^n) \\ &> \frac{1}{2}\bigg(1 - 1 + \frac{n}{3}(1 - H(t))\bigg) = \frac{1}{6} n(1 - H(t)). \end{aligned}$$

If $n(1 - H(t)) > 2$ then the first inequality in the last chain of estimates is still valid, but now $H(t)^n \le (1 - 2/n)^n \le e^{-2} < 1/2$ and thus the statement of the lemma follows again. □

The next lemma is a special case of Bradley (1980a), Lemma 2.3.

LEMMA 2.5. *Let $\{\xi_j\}$ and $\{\eta_j\}$ be ϕ-mixing sequences of random variables with mixing coefficients $\phi(n)$ and $\phi'(n)$, respectively. Suppose that the sequences $\{\xi_j\}$ and $\{\eta_j\}$ are independent of one another. Then the sequence $\{\xi_j + \eta_j\}$ is also ϕ-mixing with mixing coefficient not exceeding $\phi(n) + \phi'(n) - \phi(n)\phi'(n)$.*

LEMMA 2.6. *Let $\{Y_k,\ k \ge 1\}$ be a strictly stationary ϕ-mixing sequence of square integrable random variables satisfying $\phi(1) < 1/4$. Let $S_n = Y_1 + \cdots + Y_n$, $\sigma_n = \operatorname{Var} S_n$. Then there exists a function $\kappa(n) \to \infty$, depending only on ϕ, such that*

$$\sigma_m/\sigma_n \ge \kappa(m/n) \tag{2.12}$$

for any $m \ge n \ge 1$ such that m/n is an integer and $\sigma_n > 0$.

In the case $n = 1$ Lemma 2.6 is contained in Lemma 2.2 of Bradley (1988). [Note that the ρ-mixing condition of Bradley's lemma is satisfied by $\phi(1) < 1/4$ and well-known bounds for covariances of ϕ-mixing random variables; see, e.g., Billingsley (1968), page 170.] Applying this statement for blocks of $\{Y_k,\ k \ge 1\}$ of length n, we get (2.12) in general.

LEMMA 2.7 [Peligrad (1985), Lemmas 3.1 and 3.2]. *Let $\{Y_n, n \ge 1\}$ be a ϕ-mixing sequence of random variables with partial sums T_n. Suppose that for some $b > 0$, $p, s \in N$ and $a_0 > 0$,*

$$\phi(s) + \max_{1 \le i \le p} P\big(|T_p - T_i| \ge \tfrac{1}{2} b a_0\big) \le \eta < 1. \tag{2.13}$$

Then for any $A \ge a_0^2$ we have

$$\begin{aligned} E_{(1+2b)^2 A} T_p^2 &\le (1+2b)^2 \frac{\eta}{1-\eta} E_A T_p^2 \\ &\quad + \left(2s\left(2 + \frac{1}{b}\right)\right)^2 \frac{1}{1-\eta} E_{A(b/2s)^2} \max_{1 \le i \le p} Y_i^2, \end{aligned} \tag{2.14}$$

where $E_A X$ denotes $EX1\{X > A\}$. Moreover, for $a \ge a_0$ and $p > s$, we have

$$\begin{aligned} P\Big(\max_{1 \le i \le p} |T_i| > (1+b)a\Big) &\le (1-\eta)^{-1} P(|T_p| > a) \\ &\quad + (1-\eta)^{-1} P\left(\max_{1 \le i \le p} |Y_i| > \frac{ba}{2(s-1)}\right). \end{aligned} \tag{2.15}$$

For $s = 1$ the last term is omitted.

The next lemma is a special case of Lemma 4.2 of Peligrad (1982).

LEMMA 2.8. *Let $\{Y_j,\ j \geq 1\}$ be a strictly stationary ϕ-mixing sequence of square integrable random variables satisfying $\phi(1) < 1/4$. Let $S_n = Y_1 + \cdots + Y_n,\ \sigma_n^2 = \operatorname{Var} S_n$. Then there exists a positive integer m_0 depending only on the sequence $\phi(n)$ such that for all $p \leq m,\ m \geq m_0$,*

$$\left(1 - 2\phi^{1/2}(1)\right)^{1/2} \sigma_p \leq 3\sigma_m.$$

PROOF. By Peligrad's (1982) Lemma 4.2 and by Lemma 2.6 the left-hand side does not exceed $\sigma_m + 2\sigma_1 \leq 3\sigma_m$. [Again, the ρ-mixing condition of Peligrad (1982), Lemma 4.2 is satisfied by $\phi(1) < 1/4$ and Lemma 1 of Billingsley (1968), page 170; see the remark after Lemma 2.6.]

LEMMA 2.9 [Petrov (1995), page 155]. *Let $Y_1, \dots, Y_n$ be independent random variables with $EY_j = 0$, $\sigma_j^2 = EY_j^2 < \infty$ and let F_j denote the distribution function of Y_j. Then setting $S_n = \sum_{j=1}^n Y_j$, $s_n^2 = \sum_{j=1}^n \sigma_j^2$, we have for any $\varepsilon > 0$,*

$$\sup_x \left| P(S_n/s_n < x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt \right| \leq C(L_n(\varepsilon) + \varepsilon),$$

where

$$L_n(\varepsilon) := s_n^{-2} \sum_{j=1}^{n} \int_{|x| \geq \varepsilon s_n} x^2\, dF_j(x)$$

and C is an absolute constant.

The next lemma is a variant of Dehling, Denker and Philipp (1986), Lemma 1, and gives a remainder term estimate in the central limit theorem for stationary ϕ-mixing sequences in terms of certain truncated moments.

LEMMA 2.10. *Let $\{Y_j,\ j \geq 1\}$ be a strictly stationary ϕ-mixing sequence of random variables with $EY_1 = 0,\ EY_1^2 < \infty$ and $\phi(1) < 1/4$. Let $S_n = \sum_{j=1}^n Y_j$ and $\sigma_n^2 = \operatorname{Var} S_n$. Let $p \geq 1$ be the fourth power of an integer and let $g \geq 6$ satisfy*

$$g \leq \phi^{-1/4}(p^{1/4}). \tag{2.16}$$

Let $\psi > 0$ and put

$$u^2 := \int_{\{|S_p| \leq g\sigma_p\}} S_p^2\, dP, \tag{2.17}$$

$$v^2 := \sigma_p^{-2} \int_{\{g\psi < |S_p|/\sigma_p \leq g\}} S_p^2\, dP \tag{2.18}$$

and

$$r = [g^2 c], \tag{2.19}$$

where c satisfies

$$\max(2\psi, g^{-1}) \le c < 1. \tag{2.20}$$

Finally, set

$$n = r(p + p^{1/4}) \tag{2.21}$$

and

$$\tau^2 := ru^2. \tag{2.22}$$

Then we have for each $0 < \delta \le 1$,

$$\begin{aligned} &\left|E\exp(it\tau^{-1}S_n) - \exp(-t^2/2)\right| \\ &\quad \le 2c + |t|u^{-1}\sigma_p c^{1/2} + C_\delta |t|^{2+\delta}(u^{-1}\sigma_p c^{1/2})^\delta \\ &\qquad + C_\delta(|t|u^{-1}\sigma_p)^{2+\delta}v^2 c^{-\delta/2} + 4\phi^{1/2}(p^{1/4}) \\ &\qquad + t^4 g^{-1} + 3|t|u^{-1}\sigma_p\kappa(p^{3/4})^{-1/2} \end{aligned} \tag{2.23}$$

where $C_\delta \ge 1$ is a constant depending on δ and κ is the function in Lemma 2.6.

PROOF. We follow the argument in Dehling, Denker and Philipp (1986). By Lemma 2.6 we have $\sigma_n \to \infty$; we also note that $u \le \sigma_p$. Hence, if $|t| > r^{1/2}$, the third term on the right-hand side of (2.23) is

$$\begin{aligned} &\ge r^{1+\delta/2}c^{\delta/2} \ge (g^2c/2)^{1+\delta/2}g^{-\delta/2} \\ &\ge \tfrac{1}{3}g^{2+\delta/2}c^{1+\delta/2} \ge \tfrac{1}{3}g^{2+\delta/2}g^{-1-\delta/2} = \tfrac{1}{3}g \ge 2 \end{aligned}$$

by (2.19), (2.20), $g \ge 6$ and $\delta \le 1$. Hence we can assume from now on that $|t| \le r^{1/2}$.

We use the standard blocking argument. We decompose S_n into r blocks of length p each, separated by blocks of length $q := p^{1/4}$ each, that is,

$$S_n = \sum_{j=1}^{r} Z_j + \sum_{j=1}^{r} Z_j^* = U_n + V_n, \tag{2.24}$$

where

$$Z_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} Y_i, \qquad Z_j^* = \sum_{i=jp+(j-1)q+1}^{j(p+q)} Y_i.$$

The blocks Z_j^* of V_n have length q and are separated by the blocks Z_j of U_n, each having length p. Thus each Z_j has q^3 times as many terms as Z_j^*. Hence by Lemma 2.6,

$$\sigma_p^2 = EZ_1^2 \ge \kappa(q^3)EZ_1^{*2} = \kappa(p^{3/4})EZ_1^{*2},$$

where κ is the function in Lemma 2.6. Also, by a well-known estimate [see Billingsley (1968), page 170] and the previous line of estimates we have

$$EV_n^2 \le rEZ_1^{*2} + 4r^2\phi^{1/2}(p)EZ_1^{*2} \le 5rEZ_1^{*2} \le 5r\kappa(p^{3/4})^{-1}\sigma_p^2 \tag{2.25}$$

since by (2.19), (2.20), (2.16) and the monotonicity of ϕ

$$r\phi^{1/2}(p) \le g^2\phi^{1/2}(p) \le \phi^{-1/2}(p^{1/4})\phi^{1/2}(p) \le 1.$$

Now by (2.22), (2.24), (2.25) and since $|\exp(iu) - 1| \le |u|$, $u \in R$, we obtain

$$\begin{aligned}\left|E\exp(it\tau^{-1}S_n) - E\exp(it\tau^{-1}U_n)\right| &\le E\left|\exp(it\tau^{-1}V_n) - 1\right| \\ &\le |t|\tau^{-1}E|V_n| \le 3|t|\frac{\sigma_p}{u}\kappa(p^{3/4})^{-1/2}.\end{aligned} \tag{2.26}$$

The blocks Z_j of U_n are separated by the blocks Z_j^* of V_n having length $q = p^{1/4}$ each. Thus by a well-known estimate [see, e.g., Billingsley (1968), page 171, Lemma 2] and by stationarity we get

$$\left|E\exp(it\tau^{-1}U_n) - (E\exp(it\tau^{-1}S_p))^r\right| \le 4r\phi(p^{1/4}) \le 4\phi^{1/2}(p^{1/4}) \tag{2.27}$$

since we have $r\phi^{1/2}(p^{1/4}) \le 1$ [proved similarly to the relation $r\phi^{1/2}(p) \le 1$ above]. Next we estimate $|E\exp(it\tau^{-1}S_p) - (1 - t^2/(2r))|$. By Chebyshev's inequality and (2.19),

$$\left|\int_{\{|S_p| > g\sigma_p\}} \exp(it\tau^{-1}S_p)\,dP\right| \le g^{-2} \le cr^{-1}. \tag{2.28}$$

For the next estimate we use Taylor's theorem. We obtain

$$\begin{aligned}&\left|\int_{\{|S_p| \le g\sigma_p\}} \exp(it\tau^{-1}S_p)\,dP - (1 - t^2/(2r))\right| \\ &\quad\le \left|P(|S_p| \le g\sigma_p) + it\tau^{-1}\int_{\{|S_p| \le g\sigma_p\}} S_p dP\right. \\ &\qquad\left. - \tfrac{1}{2}t^2\tau^{-2}\int_{\{|S_p| \le g\sigma_p\}} S_p^2\,dP - (1 - t^2/(2r))\right| \\ &\qquad + C_\delta|t|^{2+\delta}\tau^{-(2+\delta)}\int_{\{|S_p| \le g\sigma_p\}} |S_p|^{2+\delta}\,dP,\end{aligned} \tag{2.29}$$

where C_δ is a positive constant depending on δ. As in (2.28) we have

$$\left|1 - P(|S_p| \le g\sigma_p)\right| \le g^{-2} \le cr^{-1}. \tag{2.30}$$

Since $ES_p = 0$, we have by (2.22) and (2.19),

$$\begin{aligned}\left|\tau^{-1}\int_{\{|S_p| \le g\sigma_p\}} S_p\,dP\right| &= \left|\tau^{-1}\int_{\{|S_p| > g\sigma_p\}} S_p\,dP\right| \\ &\le g^{-1}\sigma_p\tau^{-1} \le u^{-1}\sigma_p r^{-1}c^{1/2}.\end{aligned} \tag{2.31}$$

The $(2+\delta)$th moment is estimated as follows. By (2.18), (2.22), (2.19), (2.17) and (2.20),

$$\begin{aligned}\tau^{-(2+\delta)}\int_{\{g\psi\sigma_p<|S_p|\le g\sigma_p\}}|S_p|^{2+\delta}\,dP &\le \tau^{-(2+\delta)}(g\sigma_p)^\delta\sigma_p^2v^2\\ &= u^{-(2+\delta)}r^{-1-\delta/2}(g\sigma_p)^\delta\sigma_p^2v^2\\ &\le 2\left(\frac{\sigma_p}{u}\right)^{2+\delta}v^2c^{-\delta/2}\frac{1}{r}\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}\tau^{-(2+\delta)}\int_{\{|S_p|\le g\psi\sigma_p\}}|S_p|^{2+\delta}\,dP &\le \tau^{-(2+\delta)}(g\psi\sigma_p)^\delta u^2\\ &= \frac{1}{r}\left(g\psi\frac{\sigma_p}{u}\frac{1}{r^{1/2}}\right)^\delta \le \frac{1}{r}\left(\frac{\sigma_p}{u}c^{1/2}\right)^\delta.\end{aligned}\tag{2.33}$$

By (2.17) and (2.22),

$$\tfrac{1}{2}t^2\tau^{-2}\int_{\{|S_p|\le g\sigma_p\}}S_p^2\,dP = t^2/(2r)$$

and hence substituting (2.30)–(2.33) into (2.29) we obtain, by (2.28),

$$\left|E\exp(it\tau^{-1}S_p)-(1-t^2/(2r))\right|\le r^{-1}\eta,$$

where

$$\eta := 2c+|t|\sigma_pc^{1/2}u^{-1}+2C_\delta|t|^{2+\delta}v^2(\sigma_p/u)^{2+\delta}c^{-\delta/2}+C_\delta|t|^{2+\delta}(\sigma_pc^{1/2}u^{-1})^\delta.$$

Hence, and since $|a^r-b^r|\le r|a-b|$ for $|a|\le 1$, $|b|\le 1$, we have for $|t|<r^{1/2}$,

$$\left|E\exp(it\tau^{-1}U_n)-(1-t^2/(2r))^r\right|\le \eta+4\phi^{1/2}(p^{1/4})\tag{2.34}$$

by (2.27). Since $|e^x-(1+x)|\le x^2$ for $|x|\le 1/2$, we obtain by the above remark,

$$\left|\exp(-t^2/2)-(1-t^2/(2r))^r\right|\le t^4/(4r)\quad\text{for } |t|\le r^{1/2}.$$

The result follows now from (2.34), (2.26), (2.19) and (2.20).

The following lemma is used in the proof of Theorem 2.

LEMMA 2.11. *Assume that the hypotheses of Lemma 2.10 hold. Let* $\varepsilon>0$ *and let* $\alpha\ge 90(1-2\phi^{1/2}(1))^{-1/2}/\varepsilon$. *Then using the same notation as in Lemma 2.10,*

$$P\Big(\max_{m\le n}|S_m|\ge(1+\varepsilon)\tau\alpha\Big)\le 10P(|S_n|\ge\tau\alpha)$$

provided that

$$(\sigma_p/\tau)^2+(\sigma_p/u)^2\kappa(p^{3/4})^{-1}+c\le 1/2\quad\textit{and}\quad c^{1/2}\sigma_p/u\le 1.\tag{2.35}$$

Here κ *is the function in Lemma 2.6.*

PROOF. We apply Lemma 2.7 with $a = a_0 = \tau\alpha$, $b = \varepsilon$ and $s = 1$. To verify (2.13) with $s = 1$ it suffices to show, in view of $\phi(1) < 1/4$ and the stationarity of $\{Y_j,\ j \geq 1\}$, that

$$\tfrac{1}{4} + P\big(|S_m| \geq \tfrac{1}{2}\tau\alpha\varepsilon\big) \leq \eta \quad \text{for any } m \leq n, \tag{2.36}$$

where

$$\eta = \frac{4}{10} + c + \frac{\sigma_p^2}{\tau^2} + \frac{\sigma_p^2}{u^2}\kappa(p^{3/4})^{-1}. \tag{2.37}$$

To prove (2.36)–(2.37) note that since $m \leq n = r(p + p^{1/4})$ we can write $m = t(p + p^{1/4}) + l$ where $0 \leq t \leq r$ and $0 \leq l < p + p^{1/4}$. Set $h = t(p + p^{1/4})$. Since $l < p + p^{1/4}$, we can write $l = l_1 + l_2$ with $0 \leq l_1,\ l_2 \leq p$ and thus using stationarity, Lemma 2.8, (2.22) and $\alpha\varepsilon(1 - 2\phi^{1/2}(1))^{1/2} \geq 90$ we get

$$\begin{aligned} P\big(|S_m - S_h| \geq \tfrac{1}{4}\varepsilon\tau\alpha\big) &= P\big(|S_l| \geq \tfrac{1}{4}\varepsilon\tau\alpha\big) \\ &\leq P\big(|S_{l_1}| \geq \tfrac{1}{8}\varepsilon\tau\alpha\big) + P\big(|S_{l_2}| \geq \tfrac{1}{8}\varepsilon\tau\alpha\big) \\ &\leq 64(\sigma_{l_1}^2 + \sigma_{l_2}^2)/(\varepsilon^2\tau^2\alpha^2) \\ &\leq 1200\sigma_p^2/(\varepsilon^2\tau^2\alpha^2(1 - 2\phi^{1/2}(1))) < \sigma_p^2/\tau^2. \end{aligned} \tag{2.38}$$

Next, by (2.24) we have $S_h = U_h + V_h$ and thus by the argument in (2.25),

$$EV_h^2 \leq 5t\sigma_p^2\kappa(p^{3/4})^{-1}.$$

Thus by (2.22) and since $t \leq r$ and $\varepsilon\alpha \geq 90$

$$P\big(|V_h| \geq \tfrac{1}{8}\varepsilon\tau\alpha\big) \leq 320t\sigma_p^2\kappa(p^{3/4})^{-1}/(\varepsilon^2\tau^2\alpha^2) \leq (\sigma_p/u)^2\kappa(p^{3/4})^{-1}. \tag{2.39}$$

Now

$$\begin{aligned} U_h &= \sum_{j\leq t}\big(Z_j 1\{|Z_j| < g\sigma_p\} - E(Z_j 1\{|Z_j| < g\sigma_p\})\big) \\ &\qquad + \sum_{j\leq t} Z_j 1\{|Z_j| \geq g\sigma_p\} + \sum_{j\leq t} E(Z_j 1\{|Z_j| < g\sigma_p\}) \\ &= I + II + III \quad \text{say.} \end{aligned} \tag{2.40}$$

By well-known estimates on ϕ-mixing random variables [Billingsley (1968), page 170],

$$\begin{aligned} EI^2 &\leq \sum_{j\leq t} EZ_j^2 1\{|Z_j| < g\sigma_p\} + 2\phi^{1/2}(p^{1/4})t^2 EZ_1^2 1\{|Z_1| < g\sigma_p\} \\ &\leq ru^2 + 2\phi^{1/2}(p^{1/4})r^2u^2 \leq ru^2 + 2cru^2 \end{aligned}$$

as $t \leq r$, by (2.17), (2.16), (2.19) and since the blocks Z_j are separated by $p^{1/4}$ random variables Y_i. Thus by (2.22), (2.20) and $\varepsilon\alpha \geq 90$,

$$P\left(|I| \geq \frac{1}{16}\varepsilon\alpha\tau\right) \leq \frac{256}{\varepsilon^2\alpha^2\tau^2}(ru^2 + 2cru^2) < \frac{768}{\varepsilon^2\alpha^2} < \frac{1}{10}. \tag{2.41}$$

Now by (2.30), $t \le r$ and (2.19),

$$P(II \neq 0) \le \sum_{j \le t} P(|Z_j| \ge g\sigma_p) \le g^{-2} t \le c. \tag{2.42}$$

Finally, by (2.31), $t \le r$ and the assumption $c^{1/2}\sigma_p/u \le 1$ we have

$$|III| \le \tau(c^{1/2}\sigma_p/u) \le \tau \le \tau\alpha\varepsilon/16. \tag{2.43}$$

Hence by (2.40), (2.41), (2.42) and (2.43) we obtain

$$P\big(|U_h| \ge \tfrac{1}{8}\varepsilon\tau\alpha\big) < \tfrac{1}{10} + c. \tag{2.44}$$

Combining (2.44) with (2.38) and (2.39) we see that (2.36) and (2.37) are satisfied. Hence the lemma is proved. □

3. Truncation. Let $p \ge 1$ and $\lambda \ge 4$ be integers. We truncate X_i, $i \ge 1$ by setting

$$\begin{aligned} X'_{i,p} &= X_i 1\{|X_i| \le a_{\lambda p}\}, \\ X''_{i,p} &= X_i 1\{|X_i| > a_{\lambda p}\} \end{aligned} \tag{3.1}$$

so that

$$X_i = X'_{i,p} + X''_{i,p}.$$

We also set for $m \ge 1$,

$$\begin{aligned} S'_{m,p} &= \sum_{i \le m} (X'_{i,p} - EX'_{i,p}), \qquad S''_{m,p} = \sum_{i \le m} X''_{i,p}, \\ \rho'_{m,p} &= E|S'_{m,p}|, \qquad \sigma'_{m,p} = E(S'^2_{m,p})^{1/2} \end{aligned} \tag{3.2}$$

and

$$S'_p = S'_{p,p}, \qquad \rho'_p = \rho'_{p,p}, \qquad \sigma'_p = \sigma'_{p,p}. \tag{3.3}$$

The following lemma is an extension of Proposition 3.3 of Peligrad (1990) to ϕ-mixing sequences without moments.

LEMMA 3.1. *We have for* $p \ge p_0$

$$15000\rho'_p \ge a_p.$$

PROOF. Let $\{X^*_{i,p},\ i \ge 1\}$ be an independent copy of $\{X'_{i,p},\ i \ge 1\}$. Clearly $\{X'_{i,p} - X^*_{i,p},\ i \ge 1\}$ is a stationary sequence of symmetric random variables; by Lemma 2.5 it is also ϕ-mixing with mixing coefficient $\phi^*(n)$ not exceeding $2\phi(n) - \phi^2(n)$. Notice that $\phi^*(1) < 7/16$. Let

$$S^*_{m,p} = \sum_{i \le m} (X^*_{i,p} - EX^*_{i,p}). \tag{3.4}$$

We have for $t > 0$,

$$P\Big(\max_{1\le i\le p}|X'_{i,\,p} - X^*_{i,\,p}| \ge t\Big) \le P\Big(\max_{1\le j\le p}|S'_{j,\,p} - S^*_{j,\,p}| \ge \tfrac{1}{2}t\Big) \tag{3.5}$$
$$\le 16P\big(|S'_p - S^*_p| \ge \tfrac{1}{2}t\big)$$
$$\le 32P\big(|S'_p| \ge \tfrac{1}{4}t\big).$$

The first inequality is trivial and the second one is an extension of Lévy's maximal inequality for independent symmetric random variables to the ϕ-mixing case. Indeed, let us follow the proof in Loéve [(1963), page 248], and observe that in the notation there,

$$P(A_k \cap B_k) \ge P(A_k)P(B_k) - \phi^*(1)P(A_k) \ge \big(\tfrac{1}{2} - \phi^*(1)\big)P(A_k) \ge \tfrac{1}{16}P(A_k)$$

upon using the fact that for any $m \le n$, the sum $\sum_{i=m}^{n}(X'_{i,\,p} - X^*_{i,\,p})$ has a symmetric distribution. Finally, the last inequality in (3.5) follows from the fact that S'_p and S^*_p have the same distribution.

Let Q_p be the distribution function of $|X'_{1,\,p} - X^*_{1,\,p}|$ and define a^*_p by

$$1 - Q_p(a^*_p) \le \frac{1}{4p}, \qquad 1 - Q_p(a^*_p-) \ge \frac{1}{4p}. \tag{3.6}$$

Clearly $a^*_p \to \infty$ since otherwise $a^*_p \le M$ and thus $Q_p(M) \ge 1 - 1/(4p)$ would hold for some $M > 0$ and infinitely many p's; but then letting $p \to \infty$ and observing that $Q_p \to H$ weakly where H is the distribution function of the symmetrized random variable $X_1 - X^*_1$ (X^*_1 is an independent copy of X_1), it follows that $H(t) = 1$ for $t \ge t_0$. That is, $X_1 - X^*_1$ is bounded a.s., but then by standard symmetrization inequalities [see, e.g., Loéve (1963), page 245], it follows that X_1 is also bounded a.s., which contradicts (2.5).

We claim that for $p \ge p_0$

$$a_p \le 2a^*_p. \tag{3.7}$$

We first show that the conclusion of the lemma follows then easily. Indeed, suppose first that

$$1 - Q_p\Big(\frac{1}{2}a^*_p\Big) \le \frac{1}{p}. \tag{3.8}$$

Then by the second relation of (3.6) the left side of (3.8) lies in the interval $[1/(4p), 1/p]$. Thus setting $t = a^*_p/2$ in (3.5) and applying Lemma 2.4 with $Y_i = X'_{i,\,p} - X^*_{i,\,p}$ we obtain

$$\tfrac{1}{24} \le \tfrac{1}{6}p\big(1 - Q_p(\tfrac{1}{2}a^*_p)\big) \le P\Big(\max_{1\le i\le p}|X'_{i,\,p} - X^*_{i,\,p}\Big| \ge \tfrac{1}{2}a^*_p\Big) \tag{3.9}$$
$$\le 32P\big(|S'_p| \ge \tfrac{1}{8}a^*_p\big) \le 256\rho'_p/a^*_p \le 512\rho'_p/a_p$$

by Markov's inequality and (3.7). On the other hand, if (3.8) is not valid then Lemma 2.4 shows that the third expression of (3.9) is greater than or equal to 1/6 and (3.9) yields again the statement of Lemma 3.1.

It remains to prove (3.7). We have for $p \geq p_0$,

$$\begin{aligned} P(|X'_{1,p}| \geq 2a^*_p) &= P(|X'_{1,p}| \geq 2a^*_p, |X'_{1,p} - X^*_{1,p}| > a^*_p) \\ &\quad + P(|X'_{1,p}| \geq 2a^*_p, |X'_{1,p} - X^*_{1,p}| \leq a^*_p) \\ &\leq \frac{1}{4p} + P(|X'_{1,p}| \geq 2a^*_p, |X^*_{1,p}| \geq a^*_p) \\ &= \frac{1}{4p} + P(|X'_{1,p}| \geq 2a^*_p) P(|X^*_{1,p}| \geq a^*_p) \\ &\leq \frac{1}{4p} + P(|X'_{1,p}| \geq 2a^*_p) P(|X_1| \geq a^*_p) \\ &\leq \frac{1}{4p} + \frac{1}{2} P(|X'_{1,p}| \geq 2a^*_p), \end{aligned}$$

where in the last step we used the fact that $a^*_p \to \infty$ and thus $X_1/a^*_p \to 0$ in probability. Hence

$$P(|X'_{1,p}| \geq 2a^*_p) \leq \frac{1}{2p}.$$

Therefore, as $\lambda \geq 4$, we have

$$P(|X_1| \geq 2a^*_p) \leq \frac{1}{2p} + P(X_1 \neq X'_{1,p}) = \frac{1}{2p} + P(|X_1| > a_{\lambda p}) \leq \frac{1}{2p} + \frac{1}{\lambda p} < \frac{1}{p},$$

which implies (3.7).

LEMMA 3.2. *There exists a constant C depending only on the sequence $\{X_j,\ j \geq 1\}$ such that for all $p \geq 1$ and $\lambda \geq 4$,*

$$E|X'_{1,p}| \leq C\sigma'_p. \tag{3.10}$$

PROOF. It is known [see, e.g., Gnedenko and Kolmogorov (1954), page 173] that if G is a distribution function with infinite second moment, then

$$\int_{|t| \leq x} |t|\, dG(t) = o\left(\int_{|t| \leq x} t^2\, dG(t)\right)^{1/2} \quad \text{as } x \to \infty.$$

Thus in the case $EX_1^2 = \infty$ we have $E|X'_{1,p}|/\sigma'_{1,p} \to 0$ as $p \to \infty$; here, and in the rest of this proof, the convergence is uniform in λ. If $EX_1^2 < \infty$ then, since X_1 is nondegenerate by (2.5), $E|X'_{1,p}|/\sigma'_{1,p}$ has a positive finite limit as $p \to \infty$. Thus in all cases $E|X'_{1,p}| \leq C\sigma'_{1,p}$ for some constant C and since $\sigma'_{1,p}/\sigma'_p \to 0$ by Lemma 2.6, (3.10) is valid. □

4. Proof of Theorem 1. We fix an integer $\lambda \geq \lambda_0$ and for each integer $p \geq 1$ we define σ'_p subject to the truncation of X_j at level $a_{\lambda p}$, as in (3.1)–

(3.3). Throughout the remainder of the paper p will be restricted to integral powers of 16, $p = 16^k$, $k = 1, 2, \ldots$. Then we have either:

CASE 1. $\liminf_{p\to\infty} \sigma'_p/a_p \le 10^{-5} \log^4 \lambda$; or

CASE 2. $\liminf_{p\to\infty} \sigma'_p/a_p > 10^{-5} \log^4 \lambda$.

In Sections 4.1 and 4.2 we will construct in each of these two cases an infinite sequence (n_k) and centering and norming constants A_{n_k}, B_{n_k} such that in (1.8) the uniform distance of the distribution functions of the two sides is at most $C(\log \lambda)^{-\beta}$ for some positive constants C and β. The proof of Theorem 1 will then be completed in Section 4.3.

4.1. *Proof of Case* 1. In this case the argument and the estimates are similar to Dehling, Denker and Philipp (1986). Choose an infinite sequence R_0 of integers $p = 16^m$ such that

$$\sup\{\sigma'_p/a_p,\ p \in R_0\} \le 2 \cdot 10^{-5} \log^4 \lambda. \tag{4.1.1}$$

We fix $p \in R_0$ and apply Lemma 2.10 to the sequence $\{X'_{i,p} - EX'_{i,p},\ i \ge 1\}$. We define nonoverlapping intervals

$$I_0 = [0, (\log \lambda)^{12}),$$

$$I_i = [(\log \lambda)^{12i}, (\log \lambda)^{12(i+1)}), \qquad 1 \le i \le N := [(\log \lambda)^{1/2}] + 1$$

and set

$$b_i = (\sigma'_p)^{-2} \int_{\{|S'_p|/\sigma'_p \in I_i\}} S'^2_p \, dP, \qquad 0 \le i \le N$$

and

$$B^*_j = \sum_{0 \le i \le j} b_i, \qquad 0 \le j \le N.$$

LEMMA 4.1. *We have*

$$\min\{b_j/B^*_j,\ 1 \le j \le N\} \le N^{-1/2}.$$

PROOF. Suppose not; that is,

$$b_j/B^*_j > N^{-1/2} \quad \text{for } 1 \le j \le N. \tag{4.1.2}$$

By (4.1.1) and Lemma 3.1, we have

$$\begin{aligned} 2(\log \lambda)^{-4} &\le \frac{\rho'_p}{\sigma'_p} \\ &= \frac{1}{\sigma'_p} \int_{\{|S'_p|/\sigma'_p \le \log^{12} \lambda\}} |S'_p| \, dP + \frac{1}{\sigma'_p} \int_{\{|S'_p|/\sigma_p > \log^{12} \lambda\}} |S'_p| \, dP \\ &\le B_0^{*1/2} + (\log \lambda)^{-12} \end{aligned} \tag{4.1.3}$$

and so

$$B_0^* \geq (\log \lambda)^{-8}.$$

By (4.1.2),

$$B_{j+1}^* = B_j^* + b_{j+1} > B_j^* + N^{-1/2} B_{j+1}^*,$$

that is,

$$B_{j+1}^* > B_j^* (1 - N^{-1/2})^{-1}$$

for $0 \leq j \leq N - 1$ and thus

$$B_N^* > B_0^* (1 - N^{-1/2})^{-N} \geq (\log \lambda)^{-8} \exp(\sqrt{N}) > 1$$

for $\lambda \geq \lambda_0$ by the choice of N. This contradicts the fact that $B_N^* \leq 1$ by definition and thus the lemma is proved. □

Choose a k with $1 \leq k \leq N$ such that

$$b_k / B_k^* \leq N^{-1/2} \leq (\log \lambda)^{-1/4}. \tag{4.1.4}$$

We now choose the parameters in Lemma 2.10. Set

$$g = (\log \lambda)^{12(k+1)} \tag{4.1.5}$$

and

$$c = 2(\log \lambda)^{-12}, \qquad \psi = \tfrac{1}{2} c. \tag{4.1.6}$$

Then

$$(\log \lambda)^{12} \leq g \leq \exp\bigl((\log \lambda)^{3/4}\bigr) \tag{4.1.7}$$

if λ is sufficiently large and thus

$$r \leq g^2 c \leq \exp\bigl(2(\log \lambda)^{3/4}\bigr) = o(\lambda^{1/2}) \quad \text{as } \lambda \to \infty. \tag{4.1.8}$$

Also, $g^2 c \geq g \geq (\log \lambda)^{24}$ and thus for $\lambda \geq \lambda_0$,

$$r \geq g^2 c - 1 \geq (\log \lambda)^{12}. \tag{4.1.9}$$

We now add the bounds provided by Lemma 2.10. In view of (4.1.6) and (4.1.7) we obtain, similarly to (4.1.3),

$$\begin{aligned} 2(\log \lambda)^{-4} \leq \frac{\rho'_p}{\sigma'_p} &= \frac{1}{\sigma'_p} \int_{\{|S'_p|/\sigma'_p \leq g\}} |S'_p| \, dP + \frac{1}{\sigma'_p} \int_{\{|S'_p|/\sigma'_p > g\}} |S'_p| \, dP \\ &\leq u/\sigma'_p + g^{-1} \leq u/\sigma'_p + (\log \lambda)^{-12}. \end{aligned}$$

Thus

$$\sigma'_p / u \leq (\log \lambda)^4. \tag{4.1.10}$$

Hence by (4.1.6), (4.1.9) and (4.1.10),

$$\frac{\sigma'_p}{u} c^{1/2} \leq 2(\log \lambda)^4 (\log \lambda)^{-6} = 2(\log \lambda)^{-2} \tag{4.1.11}$$

and

$$\frac{\sigma_p'}{\tau} = \frac{\sigma_p'}{ur^{1/2}} \le (\log\lambda)^4(\log\lambda)^{-6} = (\log\lambda)^{-2}. \tag{4.1.12}$$

Because of (4.1.4),

$$(\sigma_p' v/u)^2 \le (\log\lambda)^{-1/4}.$$

Thus the fourth term on the right-hand side of (2.23) does not exceed

$$\begin{aligned} C_\delta |t|^{2+\delta}(\sigma_p' v/u)^2\left(\frac{\sigma_p'}{u}c^{-1/2}\right)^\delta &\le C_\delta|t|^{2+\delta}(\log\lambda)^{-1/4}(\log\lambda)^{10\delta} \\ &\le C_\delta|t|^{2+\delta}(\log\lambda)^{-1/8} \end{aligned}$$

if $\delta \le 1/80$. Hence adding all the terms in (2.23) we obtain from Lemma 2.10 for

$$n = r(p + p^{1/4}) \tag{4.1.13}$$

and some $\tau_{n,p} > 0$,

$$\left|E\exp(it\tau_{n,p}^{-1}S_{n,p}') - \exp(-t^2/2)\right| \le C(|t|^4+1)(\log\lambda)^{-1/40} \tag{4.1.14}$$

for some constant $C > 0$ provided that p is so large that $\phi(p^{1/4}) \le (\log\lambda)^{-1}$ and $\kappa(p^{3/4}) \ge (\log\lambda)^{10}$. Now by (4.1.13), (4.1.8) and (2.6),

$$\begin{aligned} P(S_{m,p}'' \ne 0 \text{ for some } m \le n) &\le nP(|X_1| > a_{\lambda p}) \\ &\le 2pr(\lambda p)^{-1} \le (\log\lambda)^{-1}. \end{aligned} \tag{4.1.15}$$

Thus by (4.1.14) we get, setting $S_n = \sum_{i\le n} X_i$ and $\mu_{n,p} = \sum_{i \le n} EX_{i,p}'$,

$$\left|E\exp(it\tau_{n,p}^{-1}(S_n - \mu_{n,p})) - \exp(-t^2/2)\right| \le 2C(|t|^4+1)(\log\lambda)^{-1/40}.$$

Hence an application of Lemma 2.2 of Berkes and Philipp (1979) with $T = (\log\lambda)^{1/280}$ yields that the Prohorov distance of the distribution of $\tau_{n,p}^{-1}(S_n - \mu_{n,p})$ and the standard normal distribution is at most $C'(\log\lambda)^{-1/300}$ for some constant C'. Since the standard normal distribution function Φ satisfies $|\Phi(x+h) - \Phi(x)| \le h$ for any real x and $h > 0$, it follows that

$$\begin{aligned} \sup_x &\left| P((S_n - \mu_{n,p})\tau_{n,p}^{-1} < x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp(-t^2/2)\,dt \right| \\ &\le 2C'(\log\lambda)^{-1/300}. \end{aligned} \tag{4.1.16}$$

Note also that (4.1.6) and the estimates (4.1.10)–(4.1.12) imply that condition (2.35) of Lemma 2.11 is satisfied if $\lambda \ge \lambda_0$ and p is so large that $\kappa(p^{3/4}) \ge (\log\lambda)^{10}$. Thus Lemma 2.11 implies for each $\varepsilon > 0$ and $\alpha \ge 90(1 - 2\phi^{1/2}(1))^{-1/2}/\varepsilon$

$$P\Big(\max_{m\le n}|S_{m,p}'| \ge (1+\varepsilon)\tau_{n,p}\alpha\Big) \le 10P(|S_{n,p}'| \ge \tau_{n,p}\alpha). \tag{4.1.17}$$

Together with (4.1.15) this yields for each $\varepsilon > 0$ and $\alpha \geq 90(1-2\phi^{1/2}(1))^{-1/2}/\varepsilon$,

$$P\Big(\max_{m \leq n} |S_m - \mu_{m,\,p}| \geq (1+\varepsilon)\tau_{n,\,p}\alpha\Big) \leq 10P(|S_n - \mu_{n,\,p}| \geq \tau_{n,\,p}\alpha) + 11(\log\lambda)^{-1}. \tag{4.1.18}$$

This inequality will be used in the proof of Theorem 2.

4.2. *Proof of Case* 2. Fix $\lambda \geq \lambda_0$ and choose p so large that

$$\phi(p^{1/2}) \leq \lambda^{-2}, \qquad p \geq \lambda. \tag{4.2.1}$$

Set

$$r = [(\log\lambda)^8], \qquad n = r(p + p^{1/2}). \tag{4.2.2}$$

As in the proof of Lemma 2.10, we decompose $S'_{n,\,p} = \sum_{i \leq n}(X'_{i,\,p} - EX'_{i,\,p})$ into r blocks of length p each, separated by blocks of length $q = p^{1/2}$ each; that is, we write

$$S'_{n,\,p} = \sum_{j=1}^{r} Z_j + \sum_{j=1}^{r} Z_j^* = U_n + V_n,$$

where

$$Z_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} (X'_{i,\,p} - EX'_{i,\,p}), \qquad Z_j^* = \sum_{i=jp+(j-1)q+1}^{j(p+q)} (X'_{i,\,p} - EX'_{i,\,p}).$$

By stationarity, Lemma 2.6 and (4.2.1), (4.2.2) we have, as in (2.25),

$$EV_n^2 \leq rEZ_1^{*2} + 4r^2\phi^{1/2}(p)EZ_1^{*2} \leq 5rEZ_1^{*2} \leq 5r\kappa(p^{1/2})^{-1}\sigma_p'^2 \leq r\sigma_p'^2(\log\lambda)^{-3}$$

if p is so large that $\kappa(p^{1/2}) \geq 5(\log\lambda)^3$. [Note that the first relation of (4.2.1) implies $\phi(p) \leq (\log\lambda)^{-16}$ for $\lambda \geq \lambda_0$ and thus $r\phi^{1/2}(p) \leq 1$.] Hence

$$P\big(|V_n| \geq r^{1/2}\sigma'_p(\log\lambda)^{-1}\big) \leq (\log\lambda)^{-1}. \tag{4.2.3}$$

On the other hand, Theorem 2 of Berkes and Philipp (1979) implies the existence of independent random variables $\xi_1, \ldots, \xi_r$ with the common law $H = \mathscr{L}(S'_p)$ such that

$$P\big(|\xi_j - Z_j| \geq 6\phi(p^{1/2})\big) \leq 6\phi(p^{1/2}), \qquad 1 \leq j \leq r \tag{4.2.4}$$

and since $6r\phi(p^{1/2}) \leq \lambda^{-1}$ for $\lambda \geq \lambda_0$ by (4.2.1) and (4.2.2), we have

$$P\left(\left|U_n - \sum_{j=1}^{r}\xi_j\right| \geq \lambda^{-1}\right) \leq \lambda^{-1}. \tag{4.2.5}$$

We also observe that for $\lambda \geq \lambda_0$ and $p \geq p_0$, we have $r\sigma_p'^2 \geq 1$ as we are in Case 2 and $a_n \to \infty$. Thus (4.2.3) and (4.2.5) imply that the Lévy distance of

$\mathscr{L}(S'_{n,\,p}/r^{1/2}\sigma'_p)$ and $\mathscr{L}(\sum_{j=1}^{r}\xi_j/r^{1/2}\sigma'_p)$ is at most $2(\log\lambda)^{-1}$. Hence applying Lemma 2.9 for the ξ_j with

$$\varepsilon = \frac{1}{\log\lambda} \tag{4.2.6}$$

and noting that the standard normal distribution function Φ satisfies $|\Phi(x+h)-\Phi(x)| \le |h|$, we get

$$\begin{aligned}
&\sup_x \left| P(S'_{n,\,p}/\sigma'_p r^{1/2} \le x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-t^2/2)\,dt \right| \\
&\quad \le \sup_x \left| P\left(\sum_{j=1}^{r} \xi_j/\sigma'_p r^{1/2} \le x\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-t^2/2)\,dt \right| \\
&\qquad + 4(\log\lambda)^{-1} \\
&\quad \le C(\varepsilon + L_n(\varepsilon)) + 4(\log\lambda)^{-1} = C(5\varepsilon + L_n(\varepsilon)),
\end{aligned} \tag{4.2.7}$$

where C is an absolute constant and

$$L_n(\varepsilon) = \frac{1}{r\sigma'^2_p}\sum_{j\le r}\int_{|x|\ge\varepsilon\sigma'_p r^{1/2}} x^2\,dH(x) = (\sigma'_p)^{-2}\int_{x^2\ge\varepsilon^2\sigma'^2_p r} x^2\,dH(x). \tag{4.2.8}$$

To estimate the last quantity, we apply Lemma 2.7 to the sequence $Y_i = (X'_{i,\,p} - EX'_{i,\,p})/\sigma'_p$, $i = 1, 2, \dots$. Choose s so large that $\phi(s) \le 1/256$. By Lemma 2.8 we have $\sigma'_{p-i,\,p} \le 3\sigma'_p(1-2\phi^{1/2}(1))^{-1/2}$ for all $1 \le i \le p$, provided that p is sufficiently large. Setting $b = 1/2$, $D = 3(1-2\phi^{1/2}(1))^{-1/2}$ and $a_0 = 64D$, we see that condition (2.13) is satisfied with $\eta = 1/128$. Indeed,

$$P\left(|T_p - T_i| \ge \frac{1}{2}ba_0\right) = P(|T_{p-i}| \ge 16D) \le \frac{1}{256D^2}\left(\frac{\sigma'_{p-i,\,p}}{\sigma'_p}\right)^2 \le \frac{1}{256}.$$

Hence, for every $A \ge (64D)^2$ we have

$$E_{4A}T_p^2 \le \tfrac{4}{127}E_A T_p^2 + 64s^2\tfrac{128}{127}E_{A/16s^2}\max_{1\le i\le p} Y_i^2. \tag{4.2.9}$$

We set

$$h(A) = E_A T_p^2, \qquad \beta = \tfrac{4}{127},$$

$$g(A) = 64s^2 128/127 E_{A/16s^2}\max_{1\le i\le p} Y_i^2. \tag{4.2.10}$$

Then we can rewrite (4.2.9) in the form

$$h(4A) \le \beta h(A) + g(A).$$

Hence we obtain by induction for each $k \ge 1$ and all $A \ge (64D)^2$,

$$h(4^k A) \le \beta^k h(A) + 2g(A). \tag{4.2.11}$$

Indeed, this is true for $k=1$ and if it holds for k, then

$$\begin{aligned} h(4^{k+1}A) &\le \beta h(4^k A) + g(4^k A) \\ &\le \beta(\beta^k h(A) + 2g(A)) + g(A) \\ &\le \beta^{k+1} h(A) + 2g(A). \end{aligned}$$

Set

$$A = \log^4 \lambda \tag{4.2.12}$$

and let k be the largest integer with

$$2^k \le \tfrac{1}{2}\log\lambda.$$

Then by (4.2.12), (4.2.6) and (4.2.2),

$$4^k A \le \tfrac{1}{4}\log^6\lambda \le r\varepsilon^2.$$

Hence the last term in (4.2.8) (which equals $E_{r\varepsilon^2} T_p^2$) does not exceed, in view of (4.2.11),

$$h(4^k A) \le \beta^k h(A) + 2g(A) \le \beta^k + 2g(A) \le (\log\lambda)^{-2} + 2g(A) \tag{4.2.13}$$

for $\lambda \ge \lambda_0$. Thus it remains to estimate $g(A)$. Setting $Z_{i,p} = X'_{i,p} - EX'_{i,p}$, using the well-known identity [Billingsley (1995), page 275],

$$EX1\{X > M\} = MP(X > M) + \int_M^\infty P(X > t)\,dt, \tag{4.2.14}$$

valid for nonnegative random variables X and noting that $A/16s^2 \ge \log^2\lambda$ for $\lambda \ge \lambda_0$, we obtain

$$\begin{aligned} g(A) &\le 65 s^2 E_{\log^2\lambda} \max_{1\le i\le p} \frac{Z_{i,p}^2}{\sigma_p'^2} \\ &\le \frac{C_1 p}{\sigma_p'^2}\Big[\sigma_p'^2 \log^2\lambda P(|Z_{1,p}| \ge \sigma'_p \log\lambda) \\ &\qquad + \int_{\sigma_p'^2 \log^2\lambda}^\infty P(Z_{1,p}^2 > u)\,du\Big] \end{aligned} \tag{4.2.15}$$

for some constant C_1. Using (3.10) we can continue our estimates and obtain for $\lambda \ge \lambda_0$, using (4.2.14) once more,

$$\begin{aligned} &\le \frac{C_1 p}{\sigma_p'^2}\Big[\sigma_p'^2 \log^2\lambda P\Big(|X'_{1,p}| \ge \frac{1}{2}\sigma'_p \log\lambda\Big) \\ &\qquad + \int_{\sigma_p'^2\log^2\lambda}^\infty P\Big(|X'_{1,p}| \ge \frac{1}{2}\sqrt{u}\Big)\,du\Big] \\ &= \frac{4C_1 p}{\sigma_p'^2}\int_{\{(1/4)\sigma_p'^2\log^2\lambda \le X_{1,p}'^2\}} X_{1,p}'^2\,dP \end{aligned} \tag{4.2.16}$$

$$
\begin{aligned}
&= \frac{4C_1 p}{\sigma_p'^2} \int_{\{(1/4)\sigma_p'^2 \log^2 \lambda \le X_1^2 \le a_{\lambda p}^2\}} X_1^2 \, dP \\
&\le \frac{C_2 p}{a_p^2 \log^8 \lambda} \int_{\{C_3 a_p \log^5 \lambda \le |X_1| \le a_{\lambda p}\}} X_1^2 \, dP \\
&\le \frac{C_2 p}{a_p^2 \log^8 \lambda} \sum_{k=0}^{l} \int_{\{a_{p16^k} < |X_1| \le a_{p16^{k+1}}\}} X_1^2 \, dP
\end{aligned}
$$

since we are dealing with Case 2. Here C_2 is a positive constant and

$$l = [\log_{16} \lambda] \le \log \lambda, \tag{4.2.17}$$

where $\log_{16}$ denotes logarithm with respect to base 16. We now apply Lemma 2.3 with

$$b_n = \int_{\{x_n < |X_1| \le x_{n+1}\}} X_1^2 \, dP,$$

where x_n is defined in (2.7). To check (2.9) we observe that

$$
\begin{aligned}
b_{N+1} + \cdots + b_{N+l} &= \int_{\{x_{N+1} < |X_1| \le x_{N+l+1}\}} X_1^2 \, dP \le x_{N+l+1}^2 (1 - F(x_{N+1})) \\
&\le 16^l 16^{-(N+l+1)} x_{N+l+1}^2
\end{aligned}
$$

by (2.6). Hence by Lemma 2.2 the $\liminf$ of the fraction in (2.9) equals 0, as l is fixed. Recall now that p is of the form 16^i and for $p = 16^i$ the last integral in (4.2.16) equals b_{i+k}. Thus by (4.2.15), (4.2.16), (4.2.17), Lemma 2.3 and (2.6) we obtain for infinitely many p's of the form $p = 16^i$,

$$
\begin{aligned}
g(A) &\le \frac{C_2 p}{a_p^2 \log^8 \lambda} 2l \int_{\{a_{p/16} < |X_1| \le a_p\}} X_1^2 \, dP \\
&\le \frac{2C_2 p}{a_p^2 \log^7 \lambda} a_p^2 (1 - F(a_{p/16})) \le 32 C_2 (\log \lambda)^{-7}.
\end{aligned} \tag{4.2.18}
$$

Hence by (4.2.7), (4.2.8), (4.2.6), (4.2.13) and (4.2.18) we have

$$\sup_x \left| P(S'_{n,\,p}/\sigma'_p r^{1/2} \le x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2) \, dt \right| \le C_3 (\log \lambda)^{-1} \tag{4.2.19}$$

for some constant C_3. Now as in Section 4.1 [cf. (4.1.15)] we have by (3.1), (3.2), (2.6) and (4.2.2),

$$
\begin{aligned}
P(S''_{m,\,p} \ne 0 \text{ for some } m \le n) &\le nP(|X_1| > a_{\lambda p}) \\
&\le 2r/\lambda \le (\log \lambda)^{-1}
\end{aligned} \tag{4.2.20}
$$

and thus by (4.2.19) we get, setting $\mu_{n,\,p} = \sum_{i \le n} EX'_{i,\,p}$,

$$
\begin{aligned}
&\sup_x \left| P((S_n - \mu_{n,\,p})/\sigma'_p r^{1/2} \le x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2) \, dt \right| \\
&\quad \le C_4 (\log \lambda)^{-1}
\end{aligned} \tag{4.2.21}
$$

for some constant C_4.

Similarly to (4.1.18) we obtain in Case 2 the following maximal inequality: For each $\delta > 0$ and $\alpha \geq 60(1-2\phi^{1/2}(1))^{-1/2}/\delta$,

$$P\Big(\max_{m\leq n}|S_m-\mu_{m,\,p}|\geq(1+\delta)\sigma'_p r^{1/2}\alpha\Big) \leq 3P(|S_n-\mu_{n,\,p}|\geq\sigma'_p r^{1/2}\alpha)+4(\log\lambda)^{-1}. \tag{4.2.22}$$

This will be used in the proof of Theorem 2.

We follow the pattern of the proof of Lemma 2.11 and (4.1.18), replacing τ by $\sigma'_p r^{1/2}$ and ε by δ. First as in (2.38) we obtain by (4.2.2) and the assumption on α,

$$P\big(|S_m-S_h|\geq\tfrac14\delta\sigma'_p r^{1/2}\alpha\big)\leq\sigma'^2_p/(\sigma'^2_p r)<1/10. \tag{4.2.23}$$

Next as in (2.39) and (4.2.3) we get [see also the estimate for EV_n^2 preceding (4.2.3)]

$$P\big(|V_h|\geq\tfrac18\delta\sigma'_p r^{1/2}\alpha\big)\leq(\log\lambda)^{-3}<1/10. \tag{4.2.24}$$

Finally, since $t\leq r$,

$$EU_h^2\leq rEZ_1^2+4r^2\phi^{1/2}(p^{1/2})EZ_1^2\leq 5r\sigma'^2_p$$

since by (4.2.1) and (4.2.2), $\phi^{1/2}(p^{1/2})r\leq\lambda^{-1}(\log\lambda)^8<1$. Thus by $\alpha\delta\geq 60$,

$$P\big(|U_h|\geq\tfrac18\delta\sigma'_p r^{1/2}\alpha\big)<1/10. \tag{4.2.25}$$

Now (4.2.23), (4.2.24), (4.2.25) and $\phi(1)<1/4$ imply that condition (2.13) of Lemma 2.7 is satisfied with $s=1$, $b=\delta$, $a=a_0=\sigma'_p r^{1/2}\alpha$, $\eta=6/10$ and we obtain by Lemma 2.11 in analogy with relation (4.1.17),

$$P\Big(\max_{m\leq n}|S'_{m,\,p}|\geq(1+\delta)\sigma'_p r^{1/2}\alpha\Big)\leq 3P(|S'_{n,\,p}|\geq\sigma'_p r^{1/2}\alpha). \tag{4.2.26}$$

The inequality (4.2.22) follows now from (4.2.26) and (4.2.20). □

4.3. *Completion of the proof of Theorem* 1. The results of Sections 4.1 and 4.2 show that for each $\lambda\geq\lambda_0$ there exist, regardless whether Case 1 or Case 2 holds, infinitely many different positive integers n and corresponding norming and centering constants a_n^* and b_n^* such that $b_n^*\to\infty$ and

$$\sup_x\left|P((S_n-a_n^*)/b_n^*\leq x)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x\exp(-t^2/2)\,dt\right|\leq C_0(\log\lambda)^{-\gamma},$$

where C_0 and $\gamma\leq 1$ are positive constants [see (4.1.16) and (4.2.21)]. Relation $b_n^*\to\infty$ follows in Case 2 from the fact that $\sigma'_p\geq\rho'_p\to\infty$ by Lemma 3.1, while in Case 1 we observe that $u/\sigma'_p\geq(\log\lambda)^{-4}$ by (4.1.10) and again $\sigma'_p\to\infty$. Choosing the values $\lambda=\exp(k^{2/\gamma})$, $k=1,2,\dots$ we can get an increasing sequence (n_k) of positive integers and numbers A_{n_k}, B_{n_k} $(k = 1, 2, \ldots)$ such that $B_{n_k} \to \infty$ and

$$\sup_x \left| P((S_{n_k} - A_{n_k})/B_{n_k} \le x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt \right| \le \frac{1}{k^2}, \qquad k = 1, 2, \ldots . \tag{4.3.1}$$

This completes the proof of Theorem 1. □

5. Proof of Theorem 2. We shall deduce Theorem 2 from Theorem 1 by adapting an idea from Kuelbs and Zinn (1983), pages 522 and 523; the maximal inequalities (4.1.18) and (4.2.22) obtained in Section 4 provide the essential new ingredient required in the ϕ-mixing case. Since in (4.3.1) we have $B_{n_k} \to \infty$, by passing to a further subsequence of (n_k) we can assume that B_{n_k} is increasing and

$$P(|S_{n_{k-1}} - A_{n_{k-1}}| \ge k^{-1} B_{n_k}) \le k^{-2}, \qquad k = 1, 2, \ldots \tag{5.1}$$

and consequently

$$(S_{n_{k-1}} - A_{n_{k-1}})/B_{n_k} \to 0 \quad \text{a.s.} \tag{5.2}$$

by the Borel–Cantelli lemma. Extend the sequence (A_{n_k}) to a sequence $(A_j,\ j \ge 1)$ by defining

$$A_j = A_{n_k} j / n_k, \qquad n_{k-1} < j \le n_k.$$

Relations (4.1.18) and (4.2.22) and the linear growth of the sequence $\mu_{m,\,p}$ in these maximal inequalities imply

$$\begin{aligned} &P\Big(\max_{n_{k-1} < m \le n_k} |S_m - A_m| \ge (1+\delta) B_{n_k} \alpha \Big) \\ &\qquad \le 10 P(|S_{n_k} - A_{n_k}| \ge B_{n_k} \alpha) + 11 k^{-2} \end{aligned} \tag{5.3}$$

for all $\delta > 0$ and $\alpha \ge 90(1 - 2\phi^{1/2}(1))^{-1/2}/\delta$. Fix now $0 < \varepsilon < 1$ and note that by (5.3) and (4.3.1) we have for $k \ge k_0$

$$\begin{aligned} P\Big(&\max_{n_{k-1} < m \le n_k} |S_m - A_m| \ge (1+\varepsilon)((2+\varepsilon)\log k)^{1/2} B_{n_k} \Big) \\ &\le 10 P(|S_{n_k} - A_{n_k}| \ge ((2+\varepsilon)\log k)^{1/2} B_{n_k}) + 11 k^{-2} \\ &\le 20(1 - \Phi(((2+\varepsilon)\log k)^{1/2})) + 31 k^{-2} = O(k^{-(1+\varepsilon/2)}) \end{aligned}$$

and thus by the convergence part of the Borel–Cantelli lemma we get

$$\limsup_{k\to\infty} \frac{\max_{n_{k-1} < m \le n_k} |S_m - A_m|}{B_{n_k} (2\log k)^{1/2}} \le 1 \quad \text{a.s.} \tag{5.4}$$

On the other hand, using (4.3.1), (5.1) and $1-\Phi(x) \sim (2\pi)^{-1/2}x^{-1}\exp(-x^2/2)$, we obtain for any $0 < a < b < 1$ and sufficiently large k,

$$\begin{aligned}
&P\left(a - k^{-1} < \frac{(S_{n_k} - A_{n_k}) - (S_{n_{k-1}} - A_{n_{k-1}})}{B_{n_k}(2\log k)^{1/2}} < b + k^{-1}\right)\\
&\quad\geq P\left(a < \frac{S_{n_k} - A_{n_k}}{B_{n_k}(2\log k)^{1/2}} < b\right) - k^{-2}\\
&\quad\geq \Phi((2\log k)^{1/2}b) - \Phi((2\log k)^{1/2}a) - 3k^{-2}\\
&\quad\geq \text{const.}\frac{1}{\sqrt{\log k}}\exp(-a^2\log k) - 3k^{-2} \geq \exp(-(1-\eta)\log k),
\end{aligned}$$

where $\eta = (1-a^2)/2$. Thus applying the divergence part of the Borel–Cantelli lemma to the ϕ-mixing sequence $S_{n_k} - S_{n_{k-1}}$ we get, using also (5.2), that with probability 1 the sequence

$$\left\{\frac{S_{n_k} - A_{n_k}}{B_{n_k}(2\log k)^{1/2}},\ k \geq 1\right\} \tag{5.5}$$

has at least one limit point in the interval $[a, b]$. [Recall that the divergence part of the Borel–Cantelli lemma continues to hold for ϕ-mixing sequences; see Iosifescu and Theodorescu (1969), Corollary, page 6.] The same argument applies for intervals $[a, b]$ with $-1 < a < b < 0$ and thus almost surely the set of limit points of the sequence (5.5) contains $[-1, 1]$. Defining

$$D_m = B_{n_k}(2\log k)^{1/2}, \qquad n_{k-1} < m \leq n_k$$

and using (5.4) it follows that almost surely the set of limit points of $\{(S_m - A_m)/D_m,\ m \geq 1\}$ is identical with the interval $[-1, 1]$ and also

$$\limsup_{k\to\infty} \frac{S_{n_k} - A_{n_k}}{D_{n_k}} = 1, \qquad \liminf_{k\to\infty} \frac{S_{n_k} - A_{n_k}}{D_{n_k}} = -1 \quad \text{a.s.} \tag{5.6}$$

To complete the proof of Theorem 2 it remains to verify (1.10); we shall actually prove that $\limsup_{k\to\infty} |S_{n_k}|/D_{n_k} \neq 0$ a.s. Assume on the contrary that

$$P\Big(\lim_{k\to\infty} S_{n_k}/D_{n_k} = 0\Big) > 0 \tag{5.7}$$

holds, then by the second relation of (5.6) we have

$$A_{m_k}/D_{m_k} \to 1 \tag{5.8}$$

along some subsequence (m_k) of (n_k). Since (4.3.1) and (5.1) continue to hold along the subsequence (m_k), the proof of (5.6) yields

$$\limsup_{k\to\infty} \frac{S_{m_k} - A_{m_k}}{D^*_{m_k}} = 1 \quad \text{a.s.}, \tag{5.9}$$

where $D^*_{m_k} = B_{m_k}(2\log k)^{1/2}$. Relations (5.7), (5.8) and $(m_k) \subset (n_k)$ imply that with positive probability we have $(S_{m_k} - A_{m_k})/D_{m_k} \to -1$ and thus $S_{m_k} -$

$A_{m_k} < 0$ for $k \geq k_0$ which contradicts to (5.9). This completes the proof of Theorem 2. □

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