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CHARACTERISTIC FUNCTIONS OF RANDOM VARIABLES ATTRACTED TO 1-STABLE LAWS[1]

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The domain of attraction of a 1-stable law on $\mathbb{R}^d$ is characterized by the expansions of the characteristic functions of its elements.

0. Introduction. Let $X_1, X_2, \ldots$ be $\mathbb{R}^d$-valued, independent, identically distributed random variables. The distributional limits of $(S_n - A_n)/B_n$, where $A_n \in \mathbb{R}^d$, $B_n > 0$ are constants and $S_n = \sum_{k=1}^n X_k$, are given by the well-known stable laws. [Lévy (1954), Gnedenko and Kolmogorov (1954) and Ibragimov and Linnik (1971)].

A probability distribution function F on $\mathbb{R}^d$ is called *stable* if for all $a, b > 0$ there are $c > 0$ and $v \in \mathbb{R}^d$ such that

$$F_a * F_b(x) = F_c(x - v), \qquad x \in \mathbb{R}^d,$$

where $F_s(x) = F(x/s)$, $x \in \mathbb{R}^d$, $s > 0$, and *strictly stable* if this is true with $v = 0$.

In this case [Lévy (1954)] necessarily $a^p + b^p = c^p$ for some $0 < p \le 2$, and p is called the *order* of the stable law F.

A distribution G on $\mathbb{R}^d$ belongs to the *domain of attraction* of the stable law F if there are constants $A_n \in \mathbb{R}^d$ and $B_n > 0$ such that the distributions $(S_n - A_n)/B_n$ converge weakly to F where $S_n = X_1 + \cdots + X_n$ and $X_1, X_2, \ldots$ are i.i.d. with distribution G.

For $p \in (0, 2]$ and $d \in \mathbb{N}$, we let $\mathrm{DA}(p, d)$ be the collection of distribution functions in the domain of attraction of some stable law on $\mathbb{R}^d$ of order p.

In this paper, we obtain expansions of the characteristic functions of distributions on $\mathbb{R}^d$ which are in the domain of attraction of a stable law.

In Section 1 we deal with the case $d = 1$. The first partial results are in Gnedenko and Koroluk (1950). The expansions are given fully in Ibragimov and Linnik (1971) in case $p \neq 1$ (see Theorem 1).

Our main result is Theorem 2 giving the expansions in case $p = 1$.

In Section 2 we obtain as corollaries expansions in case $d \ge 2$. Other results in this case are to be found in Rvačeva (1962), Meerschaert (1986), Kuelbs and Mandrekar (1974) and Araujo and Giné (1979, 1980).

A stable law of order p on $\mathbb{R}$ has a characteristic function ψ of the form

$$\log \psi(t) = it\gamma - c|t|^p\left[1 - i\beta\,\mathrm{sgn}(t)\tan\left(\frac{p\pi}{2}\right)\right], \qquad p \neq 1,$$

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and

$$\operatorname{Re}\log\psi(t) = -c|t|, \qquad \operatorname{Im}\log\psi(t) = t\left(\gamma + \frac{2\beta c}{\pi}\log\left(\frac{1}{|t|}\right)\right), \qquad p = 1,$$

where $c > 0$, $\beta, \gamma \in \mathbb{R}$ are constants [Lévy (1954)].

The form of the characteristic functions of stable laws on $\mathbb{R}^d$ was obtained by Feldheim [see Feldheim (1937), Lévy (1954) and Samorodnitsky and Taqqu (1994), Theorem 2.3.1]:

To each stable law of order p on $\mathbb{R}^d$ there corresponds a finite measure ν on S^{d-1} (called the *spectral* measure) and $\mu \in \mathbb{R}^d$ (called the *translate*) so that the characteristic function ψ has the form

$$\text{(1a)} \qquad \log\psi(u) = i\langle u, \mu\rangle - \int_{S^{d-1}} |\langle u, s\rangle|^p \left(1 - i\,\operatorname{sgn}(\langle s, u\rangle)\tan\left(\frac{p\pi}{2}\right)\right)\nu(ds)$$

for $p \neq 1$ and

$$\text{(1b)} \qquad \log\psi(u) = i\langle u, \mu\rangle - \int_{S^{d-1}} |\langle u, s\rangle| \left(1 + i\frac{2}{\pi}\operatorname{sgn}(\langle u, s\rangle)\log(|\langle u, s\rangle|)\right)\nu(ds)$$

for $p = 1$. Evidently a stable law on $\mathbb{R}^d$ has a density if and only if the support of its spectral measure is not contained in a proper subspace of $\mathbb{R}^d$, and in this case we say that both the stable law and the spectral measure are *nondegenerate*.

Clearly, the stability of an $\mathbb{R}^d$-valued random variable Z implies that of its inner products $\langle Z, u\rangle$, $u \in \mathbb{R}^d$.

An example of Marcus (1983) shows that the converse of this is false without additional assumptions.

According to Theorems 2.1.2 and 2.1.5 in Samorodnitsky and Taqqu (1994), the $\mathbb{R}^d$-valued random variable Z is strictly stable (stable with index ≥ 1) if its inner products $\langle Z, u\rangle$, $u \in \mathbb{R}^d$, are strictly stable on $\mathbb{R}$ (stable on $\mathbb{R}$ with index ≥ 1).

The first characterizations of domains of attraction were in terms of the tails of the distributions concerned.

In the unidimensional case [Gnedenko and Kolmogorov (1954)], for $p < 2$, the (right continuous) distribution function $G \in \mathrm{DA}(p, 1)$ iff there is a function $L\colon \mathbb{R}_+ \to \mathbb{R}_+$, slowly varying at ∞ [see Feller (1971)], and constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$ such that

$$\text{(2)} \qquad \begin{aligned} L_1(x) &:= x^p(1 - G(x)) = (c_1 + o(1))L(x), \\ L_2(x) &:= x^p G(-x) = (c_2 + o(1))L(x) \quad \text{as } x \to +\infty. \end{aligned}$$

The results of Gnedenko and Kolmogorov (1954) were generalized to $\mathbb{R}^d$ in Rvačeva (1962) [see also Meerschaert (1986)], to Hilbert space in Kuelbs and Mandrekar (1974), and to Banach space in Araujo and Giné (1979).

1. Unidimensional characterization. The characteristic function ψ of $G \in \mathrm{DA}(p, 1)$ is considered in Gnedenko and Koroluk (1950) and Ibragimov and Linnik (1971).

In Gnedenko and Koroluk (1950), $\mathrm{DA}(p, 1)$ is characterized in terms of $\psi(t)$.

In Ibragimov and Linnik (1971), the asymptotic expansion of $\log \psi(t)$ around 0 is established with error small when compared to

$$\mathrm{Prob.}\left(|Z| > \frac{1}{|t|}\right) = |t|^p \left(L_1\left(\frac{1}{|t|}\right) + L_2\left(\frac{1}{|t|}\right) +\right) = |t|^p (c_1 + c_2 + o(1)) L\left(\frac{1}{|t|}\right)$$

as $t \to 0$. Here, Z is a G-distributed random variable, and $G \in \mathrm{DA}(p, 1)$, $p \neq 1$, satisfies (2) with the slowly varying functions L, L_1, L_2 and constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$. Specifically:

THEOREM 1 [Ibragimov and Linnik (1971), Theorem 2.6.5]. *Suppose that G satisfies* (2) *with $p \neq 1$. Then*

$$\log \psi(t) = it\gamma - c|t|^p L(|t|^{-1})\left[1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{p\pi}{2}\right)\right] + o(|t|^p L(|t|^{-1})),$$

where

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \qquad c = \Gamma(1 - p)(c_1 + c_2) \cos\left(\frac{p\pi}{2}\right),$$

$$\gamma = \begin{cases} 0, & 0 < p < 1, \\ \int xG(dx), & 1 < p \leq 2. \end{cases}$$

The expansion of the characteristic function when $p = 1$ is also treated in Ibragimov and Linnik (1971) for a limited class of slowly varying functions L, namely those where

$$\int_0^\lambda \frac{xL(x)\,dx}{1 + x^2} = L(\lambda)(\log \lambda + o(1))$$

as $\lambda \to \infty$ [cf. Theorem 2 here, Theorem 2.6.5 there and formula (2.6.34) there]. As can be easily checked, the functions $L(x) \sim (\log x)^a$, $a \in \mathbb{R}$, and $L(x) \sim \exp[(\log x)^a]$, $0 < a < 1$, are slowly varying functions not in this class.

THEOREM 2. *Suppose that G satisfies (2) with $p = 1$. Then*

$$\operatorname{Re} \log \psi(t) = -c|t|L(|t|^{-1}) + o(|t|L(|t|^{-1})),$$

$$\operatorname{Im} \log \psi(t) = t\gamma + \frac{2\beta c}{\pi} CtL\left(\frac{1}{|t|}\right) + t\left(H_1\left(\frac{1}{|t|}\right) - H_2\left(\frac{1}{|t|}\right)\right) + o(|t|L(|t|^{-1}))$$

as $t \to 0$, where

$$H_j(\lambda) = \int_0^\lambda \frac{xL_j(x)\,dx}{1 + x^2}, \qquad j = 1, 2,$$

$$C = \int_0^\infty \left(\cos y - \frac{1}{1 + y^2}\right) \frac{dy}{y},$$

and the constants $c > 0$, $\beta, \gamma \in \mathbb{R}$ *are defined by*

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \qquad c = \frac{(c_1 + c_2)\pi}{2},$$
$$\gamma = \int_{-\infty}^{\infty} \left(\frac{x}{1+x^2} + \operatorname{sgn}(x) \int_0^{|x|} \frac{2u^2}{(1+u^2)^2}\, du \right) G(dx).$$

REMARK 1. Note that

$$H_1(\lambda) = \int_0^{\lambda} \frac{x^2 P(Z > x)dx}{1+x^2},$$

whence

$$\begin{aligned} H_1(\lambda) - H_2(\lambda) &= E\big(\big[|Z| \wedge \lambda - \tan^{-1}(|Z| \wedge \lambda)\big] \operatorname{sgn}(Z)\big) \\ &= E((|Z| \wedge \lambda) \operatorname{sgn}(Z)) + O(1) \end{aligned}$$

as $\lambda \to \infty$, where Z is G-distributed and H_1, H_2 are as in Theorem 2.

REMARK 2. From this representation of the characteristic function of distributions in $\mathrm{DA}(p, 1)$, one deduces the existence of a p-stable random variable Y and constants $A_n, B_n \in \mathbb{R}$, $B_n > 0$ so that $(S_n - A_n)/B_n \to Y$ in distribution. These constants [unique up to $o(B_n)$ as $n \to \infty$] are given by

$$nL(B_n) = B_n^p, \qquad A_n = \begin{cases} 0, & 0 < p < 1, \\ \gamma n, & 1 < p \le 2, \\ \gamma n + n(H_1(B_n) - H_2(B_n)), & p = 1. \end{cases}$$

To see this in case $p = 1$, write

$$\log E\left(\exp\left[it\left(\frac{S_n - A_n}{B_n} \right) \right] \right) = -\frac{itA_n}{B_n} + n \log \psi\left(\frac{t}{B_n} \right) := \alpha_n(t) + i\beta_n(t).$$

Then

$$\alpha_n(t) = -c\frac{n|t|}{B_n} L\left(\frac{B_n}{|t|} \right) + o\left(\frac{n|t|L(B_n/|t|)}{B_n} \right) \to -c|t| \quad \text{as } n \to \infty$$

and

$$\begin{aligned} \beta_n(t) = {} & \frac{t(H_1(B_n/|t|) - H_1(B_n))}{L(B_n)} - \frac{t(H_2(B_n/|t|) - H_2(B_n))}{L(B_n)} \\ & + \frac{2\beta ctCL(B_n/|t|)}{\pi L(B_n)} + o\left(\frac{n|t|L(B_n/|t|)}{B_n} \right). \end{aligned}$$

Now, for $j = 1, 2$ and $k > 1$ [see (5) in Lemma 3 below],

$$H_j(k\lambda) - H_j(\lambda) = c_j L(\lambda) \log k + o(L(\lambda)) \quad \text{as } \lambda \to \infty.$$

Thus, with $k = 1/|t|$,

$$\beta_n(t) \to t(c_1 - c_2)\log\frac{1}{|t|} + \frac{2\beta cCt}{\pi} = \frac{2\beta ct}{\pi}\left(\log\frac{1}{|t|} + C\right) \quad \text{as } n \to \infty.$$

Thus, the above representation is a characterization of DA$(p, 1)$.

REMARK 3. We note that the expansion of $\psi(t)$ around 0 up to $o(|t|^p L(1/|t|))$ is determined entirely by the asymptotic equivalence class of the slowly varying function L and the constants $c_1, c_2 \ge 0$ for G satisfying (2) with $p \ne 1$.

This is not the case when $p = 1$ as shown by the following examples.

There is a distribution G so that

$$L_1(x) := x(1 - G(x)) = (\log x)^2 + (\log x)^{3/2} + O(1),$$
$$L_2(x) := xG(-x) = (\log x)^2 + O(1) \quad \text{as } x \to +\infty.$$

Here, $L(\lambda) = (\log \lambda)^2$, $p = c_1 = c_2 = 1$, and one calculates from Theorem 2 that

$$\operatorname{Im}\log\psi(t) = \frac{4t}{5\pi}L\left(\frac{1}{|t|}\right)^{5/4} + o\left(|t|L\left(\frac{1}{|t|}\right)\right) \quad \text{as } t \to 0.$$

On the other hand, there is a symmetric distribution satisfying

$$L_1(x) = L_2(x) = (\log x)^2 + O(1) \quad \text{as } x \to +\infty$$

for which also $L(\lambda) = (\log \lambda)^2$, and $p = c_1 = c_2 = 1$; but here (owing to symmetry)

$$\operatorname{Im}\log\psi(t) \equiv 0.$$

PROOF OF THEOREM 2. Assume that G is represented in the form (2).

For $x > 0$ define distribution functions G_j, $j = 1, 2$, on $\mathbb{R}_+$ by

$$G_1(x) = G(x) - G(0) \quad \text{and} \quad G_2(x) = G(0) - G(-x).$$

We have that

$$G_j(\infty) - G_j(x) = \frac{L_j(x)}{x} = \frac{(c_j + o(1))L(x)}{x}.$$

Write

$$\int\left(1 - \exp(itx) + \frac{itx}{1+x^2}\right)G(dx) = \int_0^\infty\left(1 - \exp(itx) + \frac{itx}{1+x^2}\right)G_1(dx)$$
$$+ \int_0^\infty\left(1 - \frac{itx}{1+x^2} - \exp(-itx)\right)G_2(dx)$$

and let

$$\gamma_j = \int_0^\infty \frac{2x^2}{(1+x^2)^2}(G_j(\infty) - G_j(x))\,dx = \int_0^\infty \frac{2xL_j(x)\,dx}{(1+x^2)^2}.$$

Integration by parts gives

$$\begin{aligned}
&\int_0^\infty \left(1 - \exp[-(-1)^j itx] - (-1)^j \frac{itx}{1+x^2}\right) G_j(dx) \\
&\quad = (-1)^j it \int_0^\infty \left(\exp[-(-1)^j itx] - \frac{1-x^2}{(1+x^2)^2}\right) \frac{L_j(x)\,dx}{x} \\
&\quad = |t| \int_0^\infty \sin(|t|x) \frac{L_j(x)\,dx}{x} + (-1)^j it \int_0^\infty \left(\cos(tx) - \frac{1-x^2}{(1+x^2)^2}\right) \frac{L_j(x)\,dx}{x}.
\end{aligned}$$

Changing variables, we obtain that

$$\int_0^\infty \sin(|t|x) \frac{L_j(x)\,dx}{x} = \int_0^\infty \sin(x) \frac{L_j(x/|t|)\,dx}{x},$$

$$\int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right) \frac{L_j(x)\,dx}{x} = \int_0^\infty \left(\cos(x) - \frac{1}{1+x^2}\right) \frac{L_j(x/|t|)\,dx}{x}.$$

By Lemma 1, we see that

$$\int_0^\infty \sin(|t|x) \frac{L_j(x)\,dx}{x} = (1+o(1)) L_j\left(\frac{1}{|t|}\right) \frac{\pi}{2}.$$

Now

$$\begin{aligned}
&\int_0^\infty \left(\cos(tx) - \frac{1-x^2}{(1+x^2)^2}\right) \frac{L_j(x)\,dx}{x} \\
&\quad = \int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right) \frac{L_j(x)\,dx}{x} + \int_0^\infty \frac{x(1-t^2)L_j(x)\,dx}{(1+x^2)(1+(tx)^2)} \\
&\qquad + \int_0^\infty \frac{2xL_j(x)\,dx}{(1+x^2)^2} \\
&\quad = \int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right) \frac{L_j(x)\,dx}{x} + \int_0^\infty \frac{x(1-t^2)L_j(x)\,dx}{(1+x^2)(1+(tx)^2)} + \gamma_j.
\end{aligned}$$

By Lemma 2,

$$\int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right) \frac{L_j(x)\,dx}{x} = CL_j\left(\frac{1}{|t|}\right) + o\left(L\left(\frac{1}{|t|}\right)\right).$$

Set

$$\tilde{H}_j(\lambda) := \int_0^\infty \frac{xL_j(x)\,dx}{(1+x^2)(1+x^2/\lambda^2)}.$$

By Lemma 3, $\tilde{H}_j(\lambda) = H_j(\lambda) + o(L(\lambda))$ as $\lambda \to \infty$.

Putting everything together, we obtain

$$\begin{aligned}
\int_0^\infty &\left(1+\frac{itx}{1+x^2}-\exp(itx)\right)G_1(dx)+\int_0^\infty\left(1-\frac{itx}{1+x^2}-\exp(-itx)\right)G_2(dx)\\
&= L\left(\frac{1}{|t|}\right)|t|(c_1+c_2)\frac{\pi}{2}-itL\left(\frac{1}{|t|}\right)(c_1-c_2)C\\
&\quad - it\left(\tilde{H}_1\left(\frac{1}{|t|}\right)-\tilde{H}_2\left(\frac{1}{|t|}\right)\right)-it(\gamma_1-\gamma_2)+o\left(|t|L\left(\frac{1}{|t|}\right)\right)\\
&= L\left(\frac{1}{|t|}\right)|t|(c_1+c_2)\frac{\pi}{2}-itL\left(\frac{1}{|t|}\right)(c_1-c_2)C\\
&\quad - it\left(H_1\left(\frac{1}{|t|}\right)-H_2\left(\frac{1}{|t|}\right)\right)-it(\gamma_1-\gamma_2)+o\left(|t|L\left(\frac{1}{|t|}\right)\right)
\end{aligned}$$

and hence Theorem 2. □

We conclude this section by collecting the lemmas on slowly varying functions needed for Theorem 2.

Assume that $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is locally integrable, slowly varying at infinity and such that $u \mapsto h(u)/u$ is a nonincreasing function. Recall that h has a representation

$$h(x)=\eta(x)\exp\left[\int_1^x\frac{\varepsilon(s)}{s}\,ds\right]$$

for some functions $\eta(s)\to K\in\mathbb{R}$ and $\varepsilon(s)\to 0$ as $s\to\infty$ [see Feller (1971)].

LEMMA 1.

$$\int_0^\infty\frac{\sin y}{y}h\left(\frac{y}{t}\right)dy=(1+o(1))h\left(\frac{1}{t}\right)\frac{\pi}{2}.$$

PROOF. As the proof of Lemma 2.6.1 in Ibragimov and Linnik (1971). □

LEMMA 2.

$$\int_0^\infty\left[\cos y-\frac{1}{1+y^2}\right]\frac{1}{y}h\left(\frac{y}{t}\right)dy=(1+o(1))h\left(\frac{1}{t}\right)\int_0^\infty\left[\cos y-\frac{1}{1+y^2}\right]\frac{1}{y}\,dy.$$

PROOF. We first split the region of integration into four parts: $I_1=[\Delta_1,\infty)$, $I_2=[\delta,\Delta_1)$, $I_3=[t\Delta_2,\delta)$ and $I_4=[0,t\Delta_2)$ where $\delta<1<\Delta_1=(N-\frac{1}{2})\pi$, $N\in\mathbb{N}$.

Since $|\int_{[\Delta_1+n\pi,\,\Delta_1+(n+1)\pi]}\cos y[h(y/t)\,dy/y]|$ decreases in n,

$$\left|\int_{I_1}\cos y\frac{h(y/t)\,dy}{y}\right|\le\frac{\pi h(\Delta_1/t)}{\Delta_1}\sim\frac{\pi h(1/t)}{\Delta_1}.$$

Also,

$$\int_{I_1} \frac{1}{1+y^2} \frac{h(y/t)\,dy}{y} \leq \frac{h(\Delta_1/t)}{\Delta_1}\pi \sim \frac{\pi h(1/t)}{\Delta_1}.$$

Since, for $x \in [\Delta_2 t, \delta)$,

$$\frac{h(x/t)}{h(1/t)} = (1+o(1)) \exp\left[\int_{x/t}^{1/t} \frac{\varepsilon(s)}{s}\,ds\right] = \exp[o(-\log x)] \leq x^{-1/2}$$

for t small enough and Δ_2 large enough,

$$\begin{aligned}\left|\int_{I_3}\left(\frac{1}{1+y^2} - \cos y\right) h\left(\frac{y}{t}\right)\frac{dy}{y}\right| &= O\left(h\left(\frac{1}{t}\right)\int_0^{\delta}\left|\frac{1}{1+y^2} - \cos y\right| y^{-3/2}\,dy\right)\\ &= O\left(h\left(\frac{1}{t}\right)\delta^{3/2}\right).\end{aligned}$$

Since the function h is locally integrable, it follows that for t small enough

$$\begin{aligned}\left|\int_{I_4}\left(\frac{1}{1+y^2} - \cos y\right) h\left(\frac{y}{t}\right)\frac{dy}{y}\right| &= \left|\int_0^{\Delta_2}\left(\frac{1}{1+t^2z^2} - \cos tz\right) h(z)\frac{dz}{z}\right|\\ &= O\left(t^2\Delta_2\int_0^{\Delta_2}|h(z)|\,dz\right)\\ &= O(t^2) = o\left(h\left(\frac{1}{t}\right)\right).\end{aligned}$$

For $\delta \leq x \leq \Delta_1$ we have (uniformly in x), by the slow variation property of h,

$$\lim_{t\to 0}\frac{h(x/t)}{h(1/t)} = 1.$$

It follows that

$$\begin{aligned}&\left|\int_{I_2}\left(\frac{1}{1+y^2} - \cos y\right)\left[h\left(\frac{y}{t}\right) - h\left(\frac{1}{t}\right)\right]\frac{dy}{y}\right|\\ &\quad\leq 2h\left(\frac{1}{t}\right)\left[\sup_{\delta\leq x\leq\Delta_1}\left|\frac{h(x/t)}{h(1/t)} - 1\right|\right]\int_{\delta}^{\Delta_1}\frac{dy}{y}\\ &\quad= o\left(h\left(\frac{1}{t}\right)\right).\end{aligned}$$

Applying the estimates for I_1, I_3 and I_4 with $h = 1$, it follows that

$$\int_0^{\infty}\left(\frac{1}{1+y^2} - \cos y\right)\frac{h(y/t) - h(1/t)}{y}dy = o\left(h\left(\frac{1}{t}\right)\right) + O\left(h\left(\frac{1}{t}\right)(\delta^{3/2} + \Delta_1^{-1})\right).$$

Letting $\Delta_1 \to \infty$ and $\delta \to 0$ as $t \to 0$, the lemma follows. □

LEMMA 3. *Let*

$$H(\lambda) := \int_0^\lambda \frac{xh(x)\,dx}{1+x^2};$$

then H is slowly varying at ∞,

$$\frac{h(\lambda)}{H(\lambda)} \to 0 \quad \textit{as } \lambda \to \infty, \tag{3}$$

$$\begin{aligned} \tilde{H}(\lambda) &:= \int_0^\infty \frac{xh(x)\,dx}{(1+x^2)(1+x^2/\lambda^2)} \\ &= H(\lambda) + o(h(\lambda)) \quad \textit{as } \lambda \to \infty \end{aligned} \tag{4}$$

and

$$H(k\lambda) - H(\lambda) \sim h(\lambda) \cdot \log k \quad \textit{as } \lambda \to \infty. \tag{5}$$

REMARK. Slow variation of H, (3) and (5) are established in Lemma 1 of Parameswaran (1961).

PROOF. We first show (5):

$$\begin{aligned} H(k\lambda) - H(\lambda) &= \int_\lambda^{k\lambda} \frac{xh(x)\,dx}{1+x^2} \sim \int_\lambda^{k\lambda} \frac{h(x)\,dx}{x} \\ &= \int_1^k \frac{h(\lambda x)\,dx}{x} \sim \log k\, h(\lambda). \end{aligned}$$

Next, we see that (3) follows from (5) as $\forall\ M > 1$,

$$\begin{aligned} \frac{H(\lambda)}{h(\lambda)} &= \frac{H(e^M e^{-M}\lambda)}{h(\lambda)} \\ &\geq \frac{H(e^M e^{-M}\lambda) - H(e^{-M}\lambda)}{h(\lambda)} \\ &\sim \frac{h(e^{-M}\lambda)M}{h(\lambda)} \to M \quad \text{as } \lambda \to \infty. \end{aligned}$$

It follows from (3) and (5) that H is slowly varying at ∞.

To continue, we claim that

$$\tilde{H}(\lambda) = \int_0^\lambda \frac{xh(x)\,dx}{(1+x^2)(1+x^2/\lambda^2)} + \frac{\log 2}{2} h(\lambda) + o(h(\lambda)) \quad \text{as } \lambda \to \infty. \tag{6}$$

To see this, note that

$$\begin{aligned}\int_\lambda^\infty \frac{xh(x)\,dx}{(1+x^2)(1+x^2/\lambda^2)} &= \int_1^\infty \frac{xh(\lambda x)\,dx}{(1/\lambda^2+x^2)(1+x^2)}\\ &= h(\lambda)\int_1^\infty \frac{xdx}{(1/\lambda^2+x^2)(1+x^2)}\\ &\quad + h(\lambda)\int_1^\infty \left(\frac{h(\lambda x)}{h(\lambda)}-1\right)\frac{x\,dx}{(1/\lambda^2+x^2)(1+x^2)}\\ &= \frac{\log 2}{2}h(\lambda)+o(h(\lambda))\end{aligned}$$

as $\lambda \to \infty$ by the dominated convergence theorem since $|h(\lambda x)/h(\lambda)-1| \to 0$ as $\lambda \to \infty\ \forall\ x > 1$ and $|h(\lambda x)/h(\lambda)-1| \le x\ \forall\ x$ large enough. This establishes (6).

To complete the proof of (4), we note that

$$\frac{xh(x)}{(1+x^2)(1+x^2/\lambda^2)} = \frac{\lambda^2}{\lambda^2-1}\left(\frac{xh(x)}{x^2+1}-\frac{xh(x)}{x^2+\lambda^2}\right),$$

whence, in view of (6),

$$\tilde{H}(\lambda) = \frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)\,dx}{x^2+1} - \frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)\,dx}{x^2+\lambda^2} + \frac{\log 2}{2}h(\lambda)+o(h(\lambda)).$$

Now

$$\begin{aligned}\frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)\,dx}{x^2+1} &= H(\lambda)+O\left(\frac{H(\lambda)}{\lambda^2}\right)\\ &= H(\lambda)+o(h(\lambda)) \quad \text{as } \lambda\to\infty,\end{aligned}$$

because both h and H are slowly varying at ∞; and

$$\begin{aligned}\frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)\,dx}{x^2+\lambda^2} &\sim \int_0^\lambda \frac{xh(x)dx}{x^2+\lambda^2}\\ &= \int_0^1 \frac{xh(\lambda x)dx}{x^2+1}\\ &\sim \frac{\log 2}{2}h(\lambda) \quad \text{as } \lambda\to\infty.\end{aligned}$$

Thus,

$$\tilde{H}(\lambda) = H(\lambda)+o(h(\lambda)) \quad \text{as } \lambda\to\infty,$$

which is (4). □

2. Multidimensional characterization.

COROLLARY 1. *Let* $0 < p < 2$, $p \neq 1$ *and* G *be a distribution function on* $\mathbb{R}^d$. *The following are equivalent:*

(A) G *belongs to the domain of attraction of the nondegenerate stable law of order* p, *spectral measure* ν *and translate* μ.

(B) *The characteristic function* ψ *of* G *has the form*

$$\log \psi(tu) = \begin{cases} -t^p L\left(\frac{1}{t}\right)\Phi(u) + it\langle u, \mu\rangle + o\left(t^p L\left(\frac{1}{t}\right)\right), & \text{if } p > 1, \\ -t^p L\left(\frac{1}{t}\right)\Phi(u) + o\left(t^p L\left(\frac{1}{t}\right)\right), & \text{if } p < 1 \end{cases}$$

as $t \to 0^+$, $\forall\ u \in S^{d-1}$, *where* $\mu \in \mathbb{R}^d$, L *is slowly varying at* ∞, ν *is a nondegenerate finite measure on* S^{d-1} *and*

$$\Phi(u) := \int_{S^{d-1}} |\langle u, s\rangle|^p \left(1 - i \operatorname{sgn}\langle s, u\rangle \tan\left(\frac{p\pi}{2}\right)\right)\nu(ds).$$

PROOF. (A) $\Rightarrow$ (B). Let $X_1, X_2, \ldots$ be i.i.d. with distribution G and $A_n \in \mathbb{R}^d$, $B_n > 0$ such that $(S_n - A_n)/B_n \to Z$ weakly where Z is p-stable. Let $u \in \mathbb{R}^d$. It follows from Feldheim's theorem that $\langle u, Z\rangle$ has a one-dimensional p-stable distribution with parameters $\gamma'_u = \langle u, \mu\rangle$, $c'_u = \int_{S^{d-1}} |\langle u, s\rangle|^p \nu(ds)$ and

$$\beta'_u = \frac{1}{c'_u} \int_{S^{d-1}} |\langle u, s\rangle|^p \operatorname{sgn}(\langle u, s\rangle)\nu(ds).$$

The characteristic function $\psi(tu)$ of $\langle u, X_1\rangle$ has a form

$$\log \psi(tu) = it\gamma_u - |t|^p L_u\left(\frac{1}{|t|}\right)\left(1 - i\beta_u \operatorname{sgn}(t) \tan\left(\frac{\pi p}{2}\right)\right)$$

as in Theorem 1 with some slowly varying function L_u and parameters γ_u and β_u (we normalize L_u so that $c_u = 1$). Hence,

$$it\left(\frac{n\gamma_u}{B_n} - \frac{\langle u, A_n\rangle}{B_n}\right) - |t|^p \frac{n}{B_n^p} L_u\left(\frac{B_n}{|t|}\right)\left(1 - i\beta_u \operatorname{sgn}(t) \tan\left(\frac{p\pi}{2}\right)\right)$$
$$\to it\gamma'_u - c'_u|t|^p\left(1 - i\beta'_u \operatorname{sgn}(t) \tan\left(\frac{p\pi}{2}\right)\right).$$

The parameter γ_u must be linear in u if $p > 1$, since $(n\gamma_u - \langle u, A_n\rangle)/B_n \to \langle u, \mu\rangle$ and $n/B_n \to \infty$. In case $p < 1$, γ_u can be arbitrary since $n/B_n \to 0$. Moreover, $(n/B_n^p)L_u(B_n)$ converges to c'_u and $\beta_u = \beta'_u$. Setting $L(t) = (1/c'_u)L_u(t)$ for some fixed u, we obtain, for $v \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{L(B_n)}{L_v(B_n)} = \lim_{n\to\infty} \frac{(n/B_n^p)L_u(B_n)}{c'_u(n/B_n^p)L_v(B_n)} = \frac{1}{c'_v}.$$

Hence $L_v(\lambda) \sim c'_v L(\lambda)$ as $\lambda \to \infty$.

(B) $\Rightarrow$ (A). Conversely, if the characteristic function ψ of G is as in (B), then for every $u \in \mathbb{R}^d$ the characteristic functions of $Y_n^{(u)} = B_n^{-1} \sum_{k=1}^n (\langle u, X_k \rangle - \langle A_n, u \rangle)$ converge, where $X_1, X_2, \ldots$ are i.i.d. with distribution G, where B_n is defined by $nL(B_n) = B_n^p$ and where $A_n = 0$ if $p < 1$ and $A_n = n\mu$ if $p > 1$.

It follows that the characteristic functions of $(S_n - A_n)/B_n$ converge (necessarily to a characteristic function), such that the limit variable Z has all distributions $\langle u, Z \rangle$, $u \in \mathbb{R}^d$, p-stable. Thus, Z is stable itself if $p > 1$. In case $p < 1$, we note that Z has a characteristic function of the form (1a) with $\mu = 0$ and is strictly stable. □

If G is a distribution function on $\mathbb{R}^d$, we define $G_u(\cdot)$ to be the distribution function of $\langle u, Z \rangle$, where Z is a random variable with distribution G.

COROLLARY 2. (A) *If a distribution function G on $\mathbb{R}^d$ belongs to the domain of attraction of the nondegenerate stable law of order* 1, *spectral measure ν and translate μ, then its characteristic function ψ has the form*

$$\text{(7)}\quad \begin{aligned} \operatorname{Re} \log \psi(tu) &= -tL\left(\frac{1}{t}\right) \int_{S^{d-1}} |\langle u, s \rangle| \nu(ds) + o\left(tL\left(\frac{1}{t}\right)\right), \\ \operatorname{Im} \log \psi(tu) &= tH_u\left(\frac{1}{t}\right) + tL\left(\frac{1}{t}\right) \frac{2C}{\pi} \int_{S^{d-1}} \langle u, s \rangle \nu(ds) + t\gamma_u + o\left(tL\left(\frac{1}{t}\right)\right) \end{aligned}$$

as $t \to 0^+$ $\forall$ $u \in S^{d-1}$, where L is slowly varying at ∞,

$$C = \int_0^\infty \left(\cos y - \frac{1}{1+y^2}\right) \frac{dy}{y}$$

and

$$H_u(x) = \int_0^x \frac{v(1 - G_u(v) - G_u(-v))}{1+v^2}\, dv$$

has a representation

$$\text{(8)}\qquad H_u(\lambda) = \langle u, \Gamma_\lambda \rangle - \frac{2L(\lambda)}{\pi} \int_{S^{d-1}} \langle u, s \rangle \log(|\langle u, s \rangle|) \nu(ds) - \gamma_u + o(L(\lambda))$$

for some $\Gamma_\lambda \in \mathbb{R}^d$ and satisfies

$$\text{(9)}\qquad H_u(k\lambda) - H_u(\lambda) \sim \frac{2}{\pi} L(\lambda) \int_{S^{d-1}} \langle u, s \rangle \nu(ds) \log k$$

as $\lambda \to \infty$.

(B) *Let the characteristic function ψ of a distribution G on $\mathbb{R}^d$ satisfy (7) for some $\gamma_u \in \mathbb{R}$, some finite measure ν on S^{d-1}, some slowly varying function L and some functions H_u with representation (8) and satisfying (9). Then G belongs to the domain of attraction of a nondegenerate stable law of order 1.*

PROOF. (A) As before, let $X_1, X_2, \dots$ be i.i.d. with distribution G and $A_n \in \mathbb{R}^d$, $B_n > 0$ such that $(S_n - A_n)/B_n \to Z$ weakly, where Z is 1-stable. Let $u \in \mathbb{R}^d$. It follows from Feldheim's theorem that $\langle u, Z\rangle$ has a one-dimensional 1-stable distribution with parameters

$$\gamma'_u = \langle u, \mu\rangle - \frac{2}{\pi}\int_{S^{d-1}} \langle u, s\rangle \log(|\langle u, s\rangle|)\nu(ds),$$

$$c'_u = \int_{S^{d-1}} |\langle u, s\rangle|\nu(ds), \qquad \beta'_u = \frac{1}{c'_u}\int_{S^{d-1}} \langle u, s\rangle \nu(ds).$$

By Theorem 2, the characteristic function $\psi(tu)$ of $\langle u, X_1\rangle$ has a form

$$\log\psi(tu) = -|t|L_u\left(\frac{1}{|t|}\right) + it\gamma_u + it\frac{2\beta_u C}{\pi}L_u\left(\frac{1}{|t|}\right) + it\left(H_{1u}\left(\frac{1}{|t|}\right) - H_{2u}\left(\frac{1}{|t|}\right)\right) + o\left(|t|L_u\left(\frac{1}{|t|}\right)\right),$$

where

$$H_{ju}(\lambda) = \int_0^\lambda \frac{xL_{ju}(x)}{1+x^2}\,dx,$$

$$L_{ju}(x) = \begin{cases} x(1-G_u(x)), & \text{if } j=1,\\ xG_u(-x), & \text{if } j=2,\end{cases}$$

for some parameters γ_u, β_u and slowly varying functions L_u (normalized so that $c_u = 1$), L_{ju}. Also note that, by Theorem 2, $L_{ju}(x) = (c_{ju} + o(1))L_u(x)$ with $c_{1u} + c_{2u} = 2/\pi$. Set $H_u = H_{1u} - H_{2u}$.

From the assumed convergence of characteristic functions, we have that

$$\operatorname{Re} n\log\psi\left(\frac{tu}{B_n}\right) \sim \frac{nL_u(B_n)|t|}{B_n} \to c'_u|t|.$$

As in the proof of Corollary 1, there exists a function L so that $c'_v L \sim L_v$ for all $v \in \mathbb{R}^d$. Moreover, using (5) $\forall\, t \in \mathbb{R}$, as $n \to \infty$,

$$\begin{aligned}
&\operatorname{Im} n\log\psi\left(\frac{tu}{B_n}\right) - \langle A_n, u\rangle\frac{t}{B_n}\\
&\quad= \frac{nL_u(B_n)}{B_n}(c_{1u} - c_{2u})t\log\frac{1}{|t|}\\
&\qquad + t\left(\frac{n\gamma_u}{B_n} - \frac{\langle A_n, u\rangle}{B_n} + \frac{nH_u(B_n)}{B_n} + \frac{2Cn\beta_u L_u(B_n)}{\pi B_n}\right) + o(1)\\
&\quad\to t\gamma'_u + \frac{2\beta'_u c'_u t}{\pi}\log\frac{1}{|t|}.
\end{aligned}$$

Equating coefficients of t, and $t\log 1/|t|$, we see that

$$\frac{nL_u(B_n)}{B_n}(c_{1u} - c_{2u}) \to \frac{2\beta'_u c'_u}{\pi}$$

and

$$\frac{n}{B_n}\left(H_u(B_n)+\frac{2C\beta_u}{\pi}L_u(B_n)+\gamma_u-\left\langle u,\frac{A_n}{n}\right\rangle\right)\to\gamma'_u$$

as $n\to\infty$.

Hence, $c'_u(c_{1u}-c_{2u})=c'_u\beta_u 2/\pi=c'_u 2\beta'_u/\pi$ and $\beta_u=\beta'_u$.

To conclude, we determine the conditions for H_u and γ_u. Since $c'_u L\sim L_u$ and since L_u is slowly varying,

$$\begin{aligned}H_u(B_n)&+\frac{2C\beta'_u c'_u}{\pi}L(B_n)+\gamma_u-\left\langle u,\frac{A_n}{n}\right\rangle\\&-\left\langle u,\frac{B_n\mu}{n}\right\rangle+\frac{2B_n}{n\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)=o\left(\frac{B_n}{n}\right)\end{aligned}$$

or [because $\beta'_u c'_u$ is linear in u and $nL(B_n)\sim B_n$]

$$H_u(B_n)=\langle u,\Gamma_{B_n}\rangle-\frac{2L(B_n)}{\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)-\gamma_u+o(L(B_n)),$$

where

$$\Gamma_{B_n}=\frac{A_n}{n}+\mu L(B_n)-\frac{2CL(B_n)}{\pi}\int_{S^{d-1}}\langle\cdot,s\rangle\nu(ds).$$

We obtain the expansion for $H_u(\lambda)$ ($B_n\le\lambda<B_{n+1}$) from

$$\begin{aligned}H_u(\lambda)-H_u(B_n)&=H_{1u}(\lambda)-H_{1u}(B_n)-[H_{2u}(\lambda)-H_{2u}(B_n)]\\&\sim\log\left(\frac{\lambda}{B_n}\right)(L_{1u}(\lambda)-L_{2u}(\lambda))+o(L(\lambda))=o(L(\lambda))\end{aligned}$$

and

$$\begin{aligned}H_u(\lambda)&=H_u(B_n)+H_u(\lambda)-H_u(B_n)\\&=H_u(B_n)+o(L(\lambda))\\&=\langle u,\Gamma_{B_n}\rangle-\frac{2L(\lambda)}{\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)-\gamma_u+o(L(\lambda)),\end{aligned}$$

since

$$1\le\frac{\lambda}{B_n}\le\frac{B_{n+1}}{B_n}\sim\frac{(n+1)L(B_{n+1})}{nL(B_n)}\to 1.$$

Equation (8) follows setting $\Gamma_\lambda = \Gamma_{B_n}$ if $B_n \le \lambda < B_{n+1}$. Finally, (9) holds because

$$\begin{aligned} H_u(k\lambda) - H_u(\lambda) &\sim \log(k)(L_{1u}(\lambda) - L_{2u}(\lambda)) \\ &\sim \log(k)(c_{1u} - c_{2u})L_u(\lambda) \\ &\sim \log(k)(c_{1u} - c_{2u})c'_u L(\lambda) \\ &= \frac{2}{\pi} c'_u \beta'_u \log(k) L(\lambda). \end{aligned}$$

(B) Conversely, if the characteristic function ψ of G is as in (B), then for every $u \in \mathbb{R}^d$ the characteristic functions of

$$Y_n^{(u)} = B_n^{-1} \sum_{k=1}^{n} (\langle u, X_k \rangle - \langle A_n, u \rangle)$$

converge, where $X_1, X_2, \ldots$ are i.i.d. with distribution G, where B_n is defined by $nL(B_n) = B_n$ and where

$$A_n = n\Gamma_{B_n} + \frac{2CnL(B_n)}{\pi} \int_{S^{d-1}} \langle \cdot, s \rangle \nu(ds).$$

Let $c'_u = \int_{S^{d-1}} |\langle u, s \rangle| \nu(ds)$ be defined as before. We have that

$$\begin{aligned} &\log\left(\psi\left(\frac{tu}{B_n}\right)^n \exp\left[-\frac{it\langle u, A_n \rangle}{B_n} \right] \right) \\ &\qquad \to -|t| c'_u - it\frac{2}{\pi} \int_{S^{d-1}} \langle u, s \rangle \log |\langle tu, s \rangle| \nu(ds). \end{aligned}$$ □

Example. Let $0 < p < 2$, $\nu \in \mathscr{P}(S^{d-1})$ be nondegenerate, and let L be slowly varying at ∞.

If $Y \in \mathrm{DA}(p, 1)$, $Y > 0$ with tails given by $P(Y > \lambda) = 2L(\lambda)/\pi\lambda^p$ and Z is a ν-distributed random variable on S^{d-1} independent of Y, then $X := YZ$ is in the domain of attraction of a nondegenerate stable law of order p on $\mathbb{R}^d$ and with spectral measure ν.

This follows from (and illustrates) Corollaries 1 and 2. Indeed, using the notation $\psi_U(u) := -\log(E[\exp(i\langle U, u \rangle)]$, we have that, for $u \in S^{d-1}$ and $t > 0$,

$$\begin{aligned} \psi_X(tu) &= E(\psi_Y(\langle Z, tu \rangle) + O(\psi_Y(\langle Z, tu \rangle)^2)) \\ &= E(\psi_Y(\langle Z, tu \rangle)) + o(t^p L(1/t)) \end{aligned}$$

as $t \to 0$, whence, by Ibragimov and Linnik (1971) for $p \ne 1$,

$$\begin{aligned} \psi_X(tu) &= it\gamma \langle u, E(Z) \rangle \\ &\quad - t^p L\left(\frac{1}{t}\right) \int_{S^{d-1}} |\langle u, s \rangle|^p \left(1 - i \operatorname{sgn}(\langle s, u \rangle) \tan\left(\frac{p\pi}{2}\right)\right) \nu(ds) \\ &\quad + o\left(t^p L\left(\frac{1}{t}\right)\right) \end{aligned}$$

as $t \to 0$, and, by Theorem 2 for $p = 1$,

$$\begin{aligned}\operatorname{Re}\psi_X(tu) &= -tL\left(\frac{1}{t}\right)\int_{S^{d-1}} |\langle s, u\rangle|\, d\nu(s) + o\left(tL\left(\frac{1}{t}\right)\right),\\ \operatorname{Im}\psi_X(tu) &= t\gamma\langle u, E(Z)\rangle + t\left(H\left(\frac{1}{t}\right) + \frac{2C}{\pi}L\left(\frac{1}{t}\right)\right)\int_{S^{d-1}}\langle s, u\rangle\, d\nu(s)\\ &\qquad + tL\left(\frac{1}{t}\right)\frac{2}{\pi}\int_{S^{d-1}}\langle s, u\rangle \log\frac{1}{|\langle s, u\rangle|}\, d\nu(s) + o\left(tL\left(\frac{1}{t}\right)\right)\end{aligned}$$

as $t \to 0$, where

$$H(\lambda) := \int_0^\lambda \frac{2xL(x)\, dx}{\pi(1 + x^2)}$$

and where

$$\gamma := E\left(\frac{Y}{1 + Y^2} + \int_0^Y \frac{2u^2}{(1 + u^2)^2}\, du\right).$$

If, in the example, Y was not chosen positive, but satisfying (2) with constants c, c_1, c_2, then the spectral measure of X is given by

$$\nu^*(A) = c_1\nu(A) + c_2\nu(-A), \qquad A \in \mathscr{B}(S^{d-1}).$$

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