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LONG-RANGE DEPENDENCE AND APPELL RANK

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We study limit distributions of sums $S_N^{(G)} = \sum_{t=1}^N G(X_t)$ of nonlinear functions $G(x)$ in stationary variables of the form $X_t = Y_t + Z_t$, where $\{Y_t\}$ is a linear (moving average) sequence with long-range dependence, and $\{Z_t\}$ is a (nonlinear) weakly dependent sequence. In particular, we consider the case when $\{Y_t\}$ is Gaussian and either (1) $\{Z_t\}$ is a weakly dependent multilinear form in Gaussian innovations, or (2) $\{Z_t\}$ is a finitely dependent functional in Gaussian innovations or (3) $\{Z_t\}$ is weakly dependent and independent of $\{Y_t\}$. We show in all three cases that the limit distribution of $S_N^{(G)}$ is determined by the Appell rank of $G(x)$, or the lowest $k \geq 0$ such that $a_k = \partial^k E\{G(X_0 + c)\}/\partial c^k|_{c=0} \neq 0$.

1. Introduction and the main results. A strictly stationary time series X_t, $t \in \mathbf{Z}$ is said to be *long-range dependent* (LRD) if its covariance function $r(t) = \text{Cov}(X_0, X_t)$ is not summable and decreases as a power of the lag; more precisely, if

$$r(t) = L(t)t^{-\theta}, \tag{1.1}$$

$t \geq 1$, where $\theta \in (0, 1)$ and $L(x)$ is a function, slowly varying at infinity. In the last decade, there has been considerable interest in LRD processes and statistical inference for such processes; see, for example, Beran (1992) and the references therein. There, many problems deal with the existence and description of limit distributions of sums

$$S_N^{(G)}(t) = \sum_{s=1}^{[Nt]} G(X_s), \qquad t \geq 0, \tag{1.2}$$

where $G(x)$, $x \in \mathbf{R}$ is a (nonlinear) function with $E\{G(X_0)\} = 0$, $E\{G^2(X_0)\} < \infty$. For Gaussian LRD process X_t, this problem was first considered by Rosenblatt (1961) and later solved in full generality by Dobrushin and Major (1979) and Taqqu (1979), who showed that the limit in distribution of suitably normalized sums $S_N^{(G)}(t)$ (1.2) is determined by the *Hermite rank* $k^* = 1, 2, \ldots$ of $G(x)$, or the index of the first nonzero coefficient in the Hermite expansion

$$G(x) = \sum_{k=0}^{\infty} \frac{g_k}{k!} H_k(x; \sigma) \tag{1.3}$$

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in Hermite polynomials $H_k(x;\sigma) = (-1)^k \sigma^{2k} \exp(x^2/2\sigma^2) d^k(\exp(-x^2/2\sigma^2))/dx^k$, $k = 0, 1, \ldots$, where $\sigma^2 = E\{X_0^2\}$. Namely, if $\theta k^* < 1$, then

$$D_{N,k^*}^{-1} S_N^{(G)}(t) \Rightarrow g_{k^*} \mathscr{H}_{k^*}(t), \tag{1.4}$$

where $D_{N,k} = c(k,\theta) L(N)^{k/2} N^{1-\theta k/2}$, $c(k,\theta) = \{2/(k!(1-k\theta)(2-k\theta))\}^{1/2}$, $\Rightarrow$ stands for weak convergence of finite-dimensional distributions, and $\mathscr{H}_k(t)$ is the *Hermite process* of order $1 \le k < 1/\theta$ (see Section 2.5 for the definition). It was proved somewhat later [Breuer and Major (1983), Giraitis and Surgailis (1985)] that in the case $\theta k^* > 1$, and more generally, if $\sum_{t\in\mathbf{Z}} |\mathrm{Cov}(G(X_0), G(X_t))| < \infty$, then $S_N^{(G)}(t)$ converge to the Brownian motion, under the usual $\sqrt{N}$-normalization.

However, the "nonlinear LRD-behavior" of non-Gaussian processes is much less understood. One of the most studied models of non-Gaussian LRD processes is the *linear* (moving average) process,

$$X_t = \sum_{i\ge 0} b_i \zeta_{t-i}, \qquad t \in \mathbf{Z}, \tag{1.5}$$

where ζ_i, $i \in \mathbf{Z}$ is an i.i.d. sequence with zero mean and variance 1, and b_i, $i \ge 0$ are (deterministic) weights of the form

$$b_i = L_1(i) i^{-(1+\theta)/2}, \qquad i \ge 1, \tag{1.6}$$

where $\theta \in (0, 1)$ and where $L_1(x)$ is a function slowly varying at infinity. Linear processes of the type (1.5)–(1.6) include several important parametric classes such as fractional ARIMA and differenced fractional noise.

Limit distributions of sums $S_N^{(G)}(t)$ of polynomials $G(x)$ of linear process (1.5)–(1.6) were first studied by Surgailis (1982). It turned out that these distributions are the same as in the Gaussian case, with the only difference that the Hermite rank k^* of $G(x)$ has to be replaced by the lowest $k \ge 0$ such that

$$a_k \equiv E\{G^{(k)}(X_0)\} \ne 0, \tag{1.7}$$

where $G^{(k)}(x) = d^k G(x)/dx^k$. Later, Giraitis and Surgailis (1986, 1989), Avram and Taqqu (1987) observed that a_k are related to the Appell expansion

$$G(x) = \sum_{k\ge 0} \frac{a_k}{k!} A_k(x) \tag{1.8}$$

in Appell polynomials $A_k(x)$, $k = 0, 1, \ldots$, defined by the formal power series

$$\sum_{k\ge 0} \frac{z^k}{k!} A_k(x) = \frac{e^{zx}}{E\{e^{zX_0}\}}. \tag{1.9}$$

Recently, limit distribution of sums and quadratic forms of Appell polynomials in linear variables (1.5)–(1.6) was studied by Giraitis and Taqqu (1997), Giraitis, Taqqu and Terrin (1998). On the other hand, the asymptotics of $S_N^{(G)}(t)$ for nonsmooth $G(x)$, in particular for indicator functions $G_y(x) = \mathbf{1}(x \le y)$, $y \in \mathbf{R}$ of linear process (1.5)–(1.6) were obtained by Ho and Hsing (1996), Koul and Surgailis (1997). The above-mentioned papers provide a number of important applications of the limit theorems to statistical inference. However, the applicability of these results is limited to linear or Gaussian LRD time series, which is often unrealistic or hard to verify in practice.

The natural question is what happens if the LRD process X_t is *nonlinear*, in particular, if the Appell expansion (1.8) can be used to characterize scaling limits of $S_N^{(G)}(t)$ as above. Such a possibility is not obvious, since $A_k(x)$, $k \geq 0$ depend on the marginal distribution of X_t and hence on its "short-range dependent" behavior, which seems unrelated to long-range dependence.

In this paper we study this question in the case when the underlying process X_t can be written as

$$X_t = Y_t + Z_t, \qquad t \in \mathbf{Z}, \tag{1.10}$$

where Y_t, $t \in \mathbf{Z}$ is a *linear LRD process* of the form (1.5)–(1.6) and Z_t, $t \in \mathbf{Z}$ is a *nonlinear weakly dependent process* of a certain type. More concretely, we consider three types of the weakly dependent component Z_t in (1.10), for which we can prove the limit distribution of $S_N^{(G)}(t)$ (Theorems 1–3 below). In all three cases, this distribution is of the same type as above and is determined by

$$k^* = \min\{k \geq 0\colon a_k \neq 0\}, \tag{1.11}$$

where

$$a_k = \partial^k E\{G(X_0 + c)\}/\partial c^k|_{c=0}. \tag{1.12}$$

We call k^* (1.11) the *Appell rank of* $G(x)$. It is clear that if $G(x)$ is a polynomial, then (1.12) coincides with (1.7). It is easy to show that if $X_0 \sim \mathcal{N}(0, \sigma^2)$ then a_k (1.12) coincide with g_k in the Hermite expansion (1.3) and, in this case, Appell and Hermite ranks coincide.

Let us formulate the main results of the paper. We assume below that $Y_t, t \in \mathbf{Z}$ is a linear process

$$Y_t = \sum_{i \geq 0} b_i \zeta_{t-i}, \qquad t \in \mathbf{Z}, \tag{1.13}$$

where b_i are given by (1.6) and ζ_i, $i \in \mathbf{Z}$ are i.i.d. random variables, with zero mean, variance 1 and finite moments of arbitrary order. Note $\mathrm{Cov}(Y_0, Y_t) = L_2(t)t^{-\theta}(t \geq 1)$, where $L_2(t)$ is a slowly varying function such that $\lim_{t\to\infty} L_2(t)/L_1^2(t) = c \in (0, \infty)$. Below, $D_{N,k} = c(k, \theta)L_2(N)^{k/2}N^{1-k\theta/2}$.

THEOREM 1. *Let $X_t = Y_t + Z_t$, where $Y_t, t \in \mathbf{Z}$ is a linear LRD process of* (1.13), *and $Z_t, t \in \mathbf{Z}$ is a multilinear form*

$$Z_t = \sum_{k=1}^{n} \sum_{i_1,\ldots,i_k \in \mathbf{Z}} b^{(k)}_{i_1,\ldots,i_k} \zeta_{t-i_1} \cdots \zeta_{t-i_k}, \qquad t \in \mathbf{Z} \tag{1.14}$$

in the i.i.d. variables ζ_i, $i \in \mathbf{Z}$, with summable coefficients

$$\sum_{i_1,\ldots,i_k \in \mathbf{Z}} |b^{(k)}_{i_1,\ldots,i_k}| < \infty, \qquad k = 1, \ldots, n. \tag{1.15}$$

Let $G(x)$ *be a polynomial with* $E\{G(X_0)\} = 0$, $E\{G^2(X_0)\} < \infty$, *and let* $\theta k^* < 1$. *Then*

$$D_{N,k^*}^{-1} S_N^{(G)}(t) \Rightarrow a_{k^*} \mathscr{H}_{k^*}(t). \tag{1.16}$$

THEOREM 2. *Let* $X_t = Y_t + Z_t$, *where* $Y_t, t \in \mathbf{Z}$ *is a Gaussian LRD process of* (1.13), $\zeta_i \sim \mathscr{N}(0, 1)$ *and let*

$$Z_t = V(\zeta_t, \zeta_{t-1}, \dots, \zeta_{t-m}), \qquad t \in \mathbf{Z}, \tag{1.17}$$

where $m < \infty$ *and where* $V(z_0, \dots, z_m)$ *is an arbitrary measurable function on* $\mathbf{R}^{m+1}$.

Let $G(x)$ *be an arbitrary measurable function with* $E\{G(X_0)\} = 0$, $E\{G^2(X_0)\} < \infty$, *whose Appell rank* k^* *satisfies* $\theta k^* < 1$. *Then the convergence* (1.16) *holds true.*

THEOREM 3. *Let* $X_t = Y_t + Z_t$, *where* $Y_t, t \in \mathbf{Z}$ *is a Gaussian LRD process of* (1.13), *and* $Z_t, t \in \mathbf{Z}$ *is a strictly stationary sequence, independent of* $Y_t, t \in \mathbf{Z}$ *and such that, for any measurable function* $\lambda(x)$ *with* $E\{\lambda^2(Z_0)\} < \infty$,

$$\sum_{t \in \mathbf{Z}} \left| \mathrm{Cov}(\lambda(Z_0), \lambda(Z_t)) \right| < \infty. \tag{1.18}$$

Let $G(x)$ *satisfy the same conditions as in Theorem* 2. *Then the convergence* (1.16) *holds true.*

REMARK 1. Note $E\{Z_0^2\} < \infty$ and $E\{X_0^2\} < \infty$ in Theorem 1. Moreover, condition (1.15) guarantees that $\{Z_t\}$ is weakly dependent in the sense that $\sum_{t \in \mathbf{Z}} |\mathrm{Cov}(Z_0, Z_t)| < \infty$.

REMARK 2. Theorems 2 and 3 do not assume any moment restrictions on the marginal distribution of Z_0. In particular, Theorem 3 is valid for stationary sequences of the form $X_t = Y_t + Z_t$, where $\{Y_t\}$ is a Gaussian LRD process of (1.13), and $\{Z_t\}$ is an *arbitrary* i.i.d. sequence, independent of $\{Y_t\}$. Clearly, this allows $E\{X_0^2\} = \infty$ and even $E\{|X_0|^p\} = \infty$ for each $p > 0$. In such a case, the definition (1.1) of long-range dependence does not apply. As an indication of long-range dependence of $\{X_t\}$, one may consider the regular growth of the variance

$$\mathrm{Var}(S_N^{(G)}) \approx a_1^2 L_2(N) N^{2-\theta}, \qquad N \to \infty, \tag{1.19}$$

where $G(x)$ belongs to a certain class of square integrable functions ($E\{G^2(X_0)\} < \infty$) of Appell rank 1; the symbol $\approx$ means “asymptotically proportional to.” Relation (1.19) follows easily from the proofs of Theorems 1–3; see Corollary 4.1 below.

On the other hand, if $E\{Z_0^2\} < \infty$, then the covariance $r(t) = \mathrm{Cov}(X_0, X_t)$ is well defined, and one may ask if X_t is LRD in the sense of (1.1). While this can be easily seen to be true for X_t of Theorem 2, in the remaining two situations (Theorems 1 and 3) additional assumptions on Z_t seem necessary to guarantee the validity of (1.1). However, one can easily verify $\mathrm{Var}(\sum_{t=1}^{N} X_t) \approx L_2(N)N^{2-\theta}$, under the premises of either Theorems 1, 2 or 3.

Theorems 1–3 show that the reduction principle by Taqqu (1979) holds for a large class of non-Gaussian processes. They also suggest that some results in asymptotic inference of LRD time series may be "structurally stable" with respect to certain additive nonlinear weakly dependent "perturbations" of a Gaussian or linear model. In particular, Theorem 2 implies the following central limit theorem for the empirical distribution function $F_N(x) = N^{-1}\sum_{t=1}^{N} \mathbf{1}(X_t \le x)$:

$$ND_{N,1}^{-1}(F_N(x) - F(x)) \Rightarrow F^{(1)}(x)W, \tag{1.20}$$

where $F(x) = P\{X_0 \le x\}$ and $W \sim \mathscr{N}(0, 1)$. In (1.20), the dependence of the right-hand side in $x \in \mathbf{R}$ has exactly the same degenerate form as if $X_t, t \in \mathbf{Z}$ were Gaussian.

Theorems 1–3 are proved in Sections 3 and 4. The proof of Theorem 1 uses the formalism of (multivariate) Appell polynomials and their diagrams, which is explained in Section 2. In the case of Gaussian Y_t, $t \in \mathbf{Z}$, this formalism becomes the well-known diagram calculus of Itô–Wiener integrals; see, for example, Dobrushin (1979) or Major (1981). However, we do not want to restrict the discussion to the Gaussian case, because this case seems rather special, and the linearity of the LRD-part in (1.10) seems more important than Gaussianity.

2. Multivariate Appell polynomials and diagrams. In this section we discuss the formalism of multivariate Appell polynomials and diagrams. Most of the facts below can be found in Surgailis (1983) and Giraitis and Surgailis (1986). [See also Avram and Taqqu (1987), Giraitis and Taqqu (1997).] For the reader's convenience, we present the proofs of Lemmas 2.1–2.3 in an Appendix.

2.1. *Multivariate Appell polynomials.* From a notational point of view, it is more convenient to consider polynomials in random variables rather than in real variables. Let Y_j, $j = 1, 2, \dots, n$ be a finite system of random variables. We shall assume that all moments of $Y_1, \dots, Y_n$ are finite but no other conditions on their joint distribution is assumed. The multilinear form

$$:Y_1 Y_2 \cdots Y_n: = (-i)^n \frac{\partial^n}{\partial z_1 \cdots \partial z_n}\Big(\frac{\exp\{i\sum_{j=1}^{n} z_j Y_j\}}{E\exp\{i\sum_{j=1}^{n} z_j Y_j\}}\Big)\Big|_{z_1=\cdots=z_n=0} \tag{2.1}$$

is called the *Appell product* of random variables $Y_1, \dots, Y_n$. In particular, $:Y_1: = Y_1 - EY_1$, $:Y_1Y_2: = Y_1Y_2 - Y_1EY_2 - Y_2EY_1 + 2EY_1EY_2 - E\{Y_1Y_2\}$. Let us note some easy properties of the Appell product. From the definition (2.1) it follows that the expectation of the Appell product is zero,

$$E\{:Y_1 \cdots Y_n:\} = 0. \tag{2.2}$$

Furthermore, $:Y_1 \cdots Y_n:$ is symmetric under permutations of $Y_1, \dots, Y_n$. If $(Y_1, \dots, Y_k)$ and $(Y_{k+1}, \dots, Y_n)$ are independent, then

$$:Y_1 \cdots Y_n: = :Y_1 \cdots Y_k::Y_{k+1} \cdots Y_n:. \tag{2.3}$$

If $Y_1 = \cdots = Y_n$, then $:Y \cdots Y: \equiv :Y^{,n}: = A_n(Y)$ is the Appell polynomial relative to the distribution of Y,

$$A_n(Y) = (-i)^n \frac{\partial^n}{\partial z^n}\left(\frac{e^{izY}}{Ee^{izY}}\right)\Big|_{z=0}. \tag{2.4}$$

The multilinearity property of the Appell product means that, if $Y_i = \sum_{j=1}^m a_{ij} Z_j$, $i = 1\dots, n$ are linear combinations of random variables $Z_1, \dots, Z_m$, then

$$:Y_1 \cdots Y_n: = \sum_{j_1=1}^m \cdots \sum_{j_n=1}^m a_{1,j_1} \cdots a_{n,j_n} :Z_{j_1} \cdots Z_{j_n}:. \tag{2.5}$$

We use the compact notation $:\prod_{i\in W} Y_i: = :Y^{,W}:$ for the Appell product of random variables Y_i, $i \in W$ indexed by the elements of a finite set W. Write $Y^W = \prod_{i\in W} Y_i$ for the usual product and $E\{Y^W\} = E\{\prod_{i\in W} Y_i\}$ and $\chi(Y^{,W}) = \chi(Y_i, i \in W)$ for the expectation and the joint cumulant, respectively, of random variables Y_i, $i \in W$. We shall distinguish between the notation $U \subset W$ and $U \subseteq W$ in the sense that the former will denote *proper* inclusion, that is, such that $U \neq W$. Put also $:Y^{\varnothing}: = Y^{\varnothing} = 1$. Write $|W|$ for the number of elements of W, and $\mathscr{V}_W$ for the set of all partitions $(V_1, \dots, V_r)$ of W by nonempty subsets $V_1, \dots, V_r$, $r = 1, 2, \dots, |W|$.

LEMMA 2.1.

$$Y^W = \sum_{U\subseteq W} :Y^{,U}: \sum_{\mathscr{V}_{W\setminus U}} \chi(Y^{,V_1}) \cdots \chi(Y^{,V_r}) \tag{2.6a}$$

$$= \sum_{U\subseteq W} :Y^{,U}: E\{Y^{W\setminus U}\}, \tag{2.6b}$$

where the sum $\sum_{U\subseteq W}$ *is taken over all subsets* $U \subseteq W$ *including* $U = \varnothing$ *and* $U = W$.

2.2. *Diagrams.* Let us present a (diagram) formula which enables us to write the product $\prod_{i=1}^k :Y^{,W_i}:$ of Appell products as a sum of Appell products. Let $W_1, \dots, W_k$ be mutually disjoint finite sets and $W = \bigcup_{i=1}^k W_i$. It is convenient to imagine W as a table whose rows are $W_1, \dots, W_k$. A *diagram* is a pair $\gamma = (U, (V)_r)$, where $U \subseteq W$ is a subset, and $(V)_r \equiv (V_1, \dots, V_r) \in \mathscr{V}_{W\setminus U}$ is a partition of $W\setminus U$. The set U will be called the *free edge* of γ and the sets $V_1, \dots, V_r$ will be called the *connected edges* of γ. A connected edge $V_i \subseteq W$ is said to be *flat* if it belongs to some (row) W_i, $i = 1, \dots, k$; in particular, any $V_i \subseteq W$ with $|V_i| = 1$ is flat. A diagram $\gamma = (U, (V)_r)$ is said to be *complete* if $U = \varnothing$. Let Γ_W and $\Gamma_W^\dagger$ denote the classes of all diagrams without flat edges and all complete diagrams without flat edges, respectively.

Let, as above, Y_j, $j \in W = \bigcup_{i=1}^k W_i$ be arbitrary collection of random variables with finite moments of any order.

LEMMA 2.2.

$$\prod_{i=1}^{k} {:}Y^{,W_i}{:} = \sum_{\gamma=(U,(V)_r)\in\Gamma_W} {:}Y^{,U}{:}\chi(Y^{,V_1})\cdots\chi(Y^{,V_r}) \tag{2.7a}$$

$$= \sum_{U\subseteq W} {:}Y^{,U}{:}E\Big\{\prod_{i=1}^{k} {:}Y^{,W_i\setminus U}{:}\Big\}. \tag{2.7b}$$

Furthermore,

$$E\Big\{\prod_{i=1}^{k} {:}Y^{,W_i}{:}\Big\} = \sum_{\gamma\in\Gamma_W^{\dagger}} \chi(Y^{,V_1})\ldots\chi(Y^{,V_r}). \tag{2.8}$$

2.3. *Appell forms.* Let $\zeta_t,\ t\in\mathbf{Z}$ be a sequence of i.i.d. random variables with zero mean, variance 1 and finite moments of arbitrary order. For any collection $(s_1,\ldots,s_n)\in\mathbf{Z}^n$, the Appell product

$$:\zeta_{s_1}\cdots\zeta_{s_n}: = :(\zeta_{s'_1})^{,k_1}:\cdots:(\zeta_{s'_{k_r}})^{,k_r}: \tag{2.9}$$

provided $|\{s_i: s_i = s'_j\}| = k_j,\ j=1,\ldots,r,\ s'_i \neq s'_j (i\neq j),\ \sum_{j=1}^r k_j = n$ hold; see (2.4). The product (2.9) has the following orthogonality property: for any collections $(s_1,\ldots,s_n)\in\mathbf{Z}^n,\ (t_1,\ldots,t_{n'})\in\mathbf{Z}^{n'}$ such that $\{s_1,\ldots,s_n\}\neq\{t_1,\ldots,t_{n'}\}$ as *subsets* of $\mathbf{Z}$,

$$E\{:\zeta_{s_1}\cdots\zeta_{s_n}::\zeta_{t_1}\cdots\zeta_{t_{n'}}:\} = 0. \tag{2.10}$$

Let $L^p(\mathbf{Z}^n)$ be the space of all real sequences $q = q(s_1,\ldots,s_n), (s_1,\ldots,s_n)\in\mathbf{Z}^n$ such that $\|q\|_{L^p(\mathbf{Z}^n)} = \big(\sum_{s_1,\ldots,s_n\in\mathbf{Z}}|q(s_1,\ldots,s_n)|^p\big)^{1/p} < \infty, 1\le p\le\infty$. For any $q\in L^2(\mathbf{Z}^n)$ consider the polynomial form

$$\mathscr{A}_n(q) = \sum_{s_1,\ldots,s_n\in\mathbf{Z}} q(s_1,\ldots,s_n):\zeta_{s_1}\cdots\zeta_{s_n}:. \tag{2.11}$$

By (2.10), the last sum converges in $L^2(\Omega)$; furthermore, for any $q, q'\in L^2(\mathbf{Z}^n)$,

$$\big|E\{\mathscr{A}_n(q)\mathscr{A}_n(q')\}\big| \le C \sum_{s_1,\ldots,s_n\in\mathbf{Z}} \big|\mathrm{sym}\, q(s_1,\ldots,s_n)q'(s_1,\ldots,s_n)\big|, \tag{2.12}$$

where $\mathrm{sym}\, q(s_1,\ldots,s_n) = (n!)^{-1}\sum_{(p)_n\in\mathscr{P}_n} q(s_{p(1)},\ldots,s_{p(n)})$ is the symmetrization. We call $\mathscr{A}_n(q)$ (2.11) *Appell form* of order n. In particular,

$$\mathscr{A}_1(q) = \sum_{s\in\mathbf{Z}} q(s)\zeta_s,$$

$$\mathscr{A}_2(q) = \sum_{s_1,s_2\in\mathbf{Z}: s_1\neq s_2} q(s_1,s_2)\zeta_{s_1}\zeta_{s_2} + \sum_{s\in\mathbf{Z}} q(s,s)(\zeta_s^2 - E\zeta_s^2).$$

By (2.2), $E\{\mathscr{A}_n(q)\} = 0,\ n\ge 1$. Let $\mathscr{A}_0(q) = q\in\mathbf{R}$ be a scalar. As linear combinations of Appell polynomials $:\zeta_s^{,n}:,\ n = 0,1,\ldots$ are dense in the space of all square integrable random variables measurable with respect to ζ_s, by standard argument one can show that linear combinations of Appell forms

$\mathscr{A}_n(q),\ q \in L^2(\mathbf{Z}^n),\ n \geq 0$ are dense in the space $L^2(\Omega)$ of all square integrable variables measurable with respect to the σ-algebra $\sigma\{\zeta_s : s \in \mathbf{Z}\}$.

Let us present a diagram formula for Appell forms similar to the diagram formula of Lemma 2.2.

Let $q_i \in L^2(\mathbf{Z}^{n_i}),\ n_i \geq 1,\ i = 1, \ldots, k$ be given. Consider a function $\tilde{q} \in L^2(\mathbf{Z}^n)$ of $n = n_1 + \cdots + n_k$ variables $s_{i,j} \in \mathbf{Z},\ i = 1, \ldots, k,\ \ j = 1, \ldots, n_i$, defined by

$$\tilde{q}(s_{1,1}, \ldots, s_{k,n_k}) = q_1(s_{1,1}, \ldots, s_{1,n_1}) \cdots q_k(s_{k,1}, \ldots, s_{k,n_k}). \tag{2.13}$$

It is convenient to write down the indices of the variables in (2.13) in the form of the table:

$$W = \begin{pmatrix} (1,1), \ldots, (1, n_1) \\ (2,1), \ldots, (2, n_2) \\ \cdots \\ (k,1), \ldots, (k, n_k), \end{pmatrix} \tag{2.14}$$

whose rows are denoted by $W_i,\ i = 1, \ldots, k$. It is clear that there is a 1–1 correspondence between table W (2.14) and collection $(n)_k = (n_1, \ldots, n_k)$. Write $\Gamma_W = \Gamma_{(n)_k},\ \Gamma_W^\dagger = \Gamma_{(n)_k}^\dagger$. Given a diagram $\gamma = (U, (V)_r) \in \Gamma_{(n)_k}$, the variables $s_{i,j} : (i,j) \in U$ will be said to be *free*, while $s_{i,j} : (i,j) \in V_l$ will be said to be *connected*, $l = 1, \ldots, r$.

With each diagram $\gamma = (U, (V_1, \ldots, V_r)) \in \Gamma_{(n)_k}$ [and a given collection $q_i \in L^2(\mathbf{Z}^{n_i}), i = 1, \ldots, k$] we associate a new function $q^\gamma \in L^2(\mathbf{Z}^{n_\gamma})$, where $n_\gamma = |U|$ is the number of free variables, as follows. Replace in $\tilde{q}$ (2.13) all (connected) variables $s_{i,j} : (i,j) \in V_l$ by a single new variable $\tilde{s}_l,\ l = 1, \ldots, r$ and denote the resulting function by $\tilde{q}^\gamma = \tilde{q}^\gamma(s_{i,j}, \tilde{s}_l : (i,j) \in U, l = 1, \ldots, r)$. Then

$$q^\gamma = q^\gamma(s_{i,j} : (i,j) \in U) = \prod_{l=1}^r \chi_{|V_l|} \sum_{\tilde{s}_1, \ldots, \tilde{s}_r \in \mathbf{Z}} \tilde{q}^\gamma(s_{i,j}, \tilde{s}_l), \tag{2.15}$$

where $\chi_k = \chi_k(\zeta_0)$ is the kth cumulant of ζ_0. If $n_\gamma = 0$ then γ is complete and $q^\gamma \in \mathbf{R}$ is a scalar.

LEMMA 2.3.

$$\prod_{i=1}^k \mathscr{A}_{n_i}(q_i) = \sum_{\gamma \in \Gamma_{(n)_k}} \mathscr{A}_{n_\gamma}(q^\gamma), \tag{2.16}$$

$$E\left\{\prod_{i=1}^k \mathscr{A}_{n_i}(q_i)\right\} = \sum_{\gamma \in \Gamma_{(n)_k}^\dagger} q^\gamma. \tag{2.17}$$

Consider now a particular case $n_1 = \cdots = n_m = 1,\ 1 \leq m \leq k,\ q_1 = \cdots = q_m \equiv q \in L^2(\mathbf{Z}),\ \mathscr{A}_1(q) \equiv Y$. Let $A_k(x), k \geq 0$ be Appell polynomials corresponding to r.v. Y. From Lemma 2.3 the corollary follows.

COROLLARY 2.1.

$$
(2.18)\qquad \begin{aligned} &Y^m \prod_{i=m+1}^{k} \mathscr{A}_{n_i}(q_i)\\ &\quad = \sum_{j=0}^{m} \binom{m}{j} A_j(Y) E\Big\{Y^{m-j} \prod_{i=m+1}^{k} \mathscr{A}_{n_i}(q_i)\Big\} + \sum_{\gamma\in\Gamma_{(n)_k^m}} \mathscr{A}_{n_\gamma}(q^\gamma), \end{aligned}
$$

where $\Gamma_{(n)_k^m}$ *consists of all diagrams* $\gamma = (U, (V)_r) \in \Gamma_{(n)_k}$ *such that* $\bigcup_{i=m+1}^{k} W_i \cap U \neq \varnothing$. [In other words, such that there is at least one free variable among the variables belonging to the functions $q_{m+1}, \dots, q_k$ in (2.13).]

2.4. *Gaussian case.* Let $\zeta_t, t \in \mathbf{Z}$ be i.i.d. Gaussian $\mathscr{N}(0,1)$-distributed random variables. Then $\zeta_t = (2\pi)^{-1/2} \int_\Pi e^{itu} W(du)$, where $\Pi = (-\pi, \pi]$, $W(du) = \overline{W(-du)}$ is a complex-valued random spectral measure on the real line with zero mean and variance $E\{|W(du)|^2\} = du$ ("Gaussian white noise"). The Appell product (2.9) can be written as

$$
:\zeta_{s_1} \cdots \zeta_{s_n}: = H_{k_1}(\zeta_{s'_1}) \cdots H_{k_r}(\zeta_{s'_{k_r}}),
$$

$H_k(x)$, $k \geq 0$ being the standard Hermite polynomials, and the Appell form (2.11) as the n-tuple Itô–Wiener integral,

$$
\mathscr{A}_n(q) = \int_{\Pi^n} \hat{q}(u_1, \dots, u_n) W(du_1) \cdots W(du_n),
$$

where $\hat{q}(u_1, \dots, u_n) = (2\pi)^{-n/2} \sum_{s_1,\dots,s_n} \exp\{i \sum_j u_j s_j\} q(s_1, \dots, s_n)$ is the Fourier transform. The following orthogonality property $E\{\mathscr{A}_n(q)\mathscr{A}_{n'}(q')\} = \delta_{n-n'} n! \|q\|^2_{L^2(\mathbf{Z}^n)}$, $n, n' \geq 0$ of multiple Itô–Wiener integrals is well known. The diagram formula of Lemma 2.3 in the "frequency representation" can be found, for example, in Dobrushin (1979), Proposition 4.1, with the important simplification that all edges V_l connect only pairs, that is, $|V_l| = 2, l = 1, \dots, r$.

2.5. *Hermite processes.* Let $\theta \in (0,1)$ be a parameter. We define a Hermite process of order $1 \leq k < 1/\theta$ as the k-tuple Itô–Wiener integral,

$$
(2.19)\qquad \begin{aligned} \mathscr{H}_k(t) = d_k(\theta) \int_{\mathbf{R}^k} & \frac{\exp(it(u_1 + \cdots + u_n)) - 1}{i(u_1 + \cdots + u_n)}\\ & \times \prod_{i=1}^{k} |u_i|^{(\theta-1)/2} W(du_1) \cdots W(du_k), \qquad t \geq 0, \end{aligned}
$$

where the normalization factor $d_k(\theta)$ is chosen so that $E\{\mathscr{H}_k^2(1)\} = 1$. See Taqqu (1978, 1979) for properties of Hermite processes.

3. Proof of Theorem 1. Observe first that Z_t (1.14) can be rewritten as a sum of Appell forms,

$$
(3.1)\qquad Z_t = \sum_{k=1}^{n} \sum_{s_1,\dots,s_k \in \mathbf{Z}} q^{(k)}_{t-s_1,\dots,t-s_k} :\zeta_{s_1} \cdots \zeta_{s_k}: \equiv \sum_{k=1}^{n} Z_t^{(k)}
$$

with some (new) coefficients $q^{(k)} \in L^1(\mathbf{Z}^k)$, that is, such that

$$\sum_{s_1,\dots,s_k \in \mathbf{Z}} |q^{(k)}_{s_1,\dots,s_k}| < \infty, \qquad k = 1,\dots,n. \tag{3.2}$$

From (2.12) and (3.2) it follows that $Z^{(k)}_t$, $k = 1,\dots,n$ are weakly dependent in the sense that

$$\sum_{t \in \mathbf{Z}} \left| E\{Z^{(k)}_0, Z^{(l)}_t\} \right| < \infty, \qquad k, l = 1,\dots,n. \tag{3.3}$$

Let $A_k(x)$, $k \geq 0$ and $B_k(x)$, $k \geq 0$ be Appell polynomials relative to the distributions of X_0 and Y_0, respectively. Theorem 1 follows from Surgailis (1982) and the following lemma.

LEMMA 3.1. *Let $G(x)$ be a polynomial of order $\ell \geq 1$. Then*

$$\sum_{t=1}^{N} G(X_t) = \sum_{k=k^*}^{\ell} \frac{a_k}{k!} \sum_{t=1}^{N} B_k(Y_t) + O_P(\sqrt{N}).$$

PROOF. Consider first the case $G(x) = x^k$. By Corollary 2.1,

$$\begin{aligned} X^k_t &= \sum_{m=0}^{k} \binom{k}{m} Y^m_t Z^{k-m}_t \\ &= \sum_{m=0}^{k} \binom{k}{m} \sum_{j=0}^{m} \binom{m}{j} B_j(Y_t) E\{Y^{m-j}_t Z^{k-m}_t\} + R_{k,t}, \end{aligned} \tag{3.4}$$

where the "remainder term" $R_{k,\,t}$ is given in (3.5) below. In order to identify this term, write

$$Y^m_0 Z^{k-m}_0 = \sum_{(n)^m_k} \mathscr{A}^m_1(b) \mathscr{A}_{n_{m+1}}(q^{(n_{m+1})}) \cdots \mathscr{A}_{n_k}(q^{(n_k)}),$$

where $b = b_i$, $i \geq 0$ are given by (1.6) and the sum is taken over all collections $(n)^m_k = (n_1,\dots,n_m,n_{m+1},\dots,n_k)$, $1 \leq n_i \leq n$, $1 \leq i \leq k$ such that $n_1 = \cdots = n_m = 1$. For each such collection $(n)^m_k$, put $q_i = b$ if $1 \leq i \leq m$, $= q^{(n_i)}$ if $m+1 \leq i \leq k$. Then $Y^m_0 Z^{k-m}_0 = \sum_{(n)^m_k} \prod_{i=1}^{k} \mathscr{A}_{n_i}(q_i)$. By applying Corollary 2.1 to the last product and taking the sum over all collections $(n)^m_n$, one obtains (3.4), with

$$R_{k,\,t} = \sum_{m=0}^{k} \binom{k}{m} \sum_{(n)^m_k} \sum_{\gamma \in \Gamma_{(n)^m_k}} \mathscr{A}_{n_\gamma}(q^\gamma_t). \tag{3.5}$$

In (3.5), the last sum is the same as in (2.18) and q^γ_t are defined as in (2.15), with the difference that $\tilde{q}$ of (2.13) must be replaced by the shifted function

$$\tilde{q}_t(s_{1,\,1},\dots,s_{k,\,n_k}) = \tilde{q}(t + s_{1,\,1},\dots,t + s_{k,\,n_k}). \tag{3.6}$$

By changing the summation, (3.4) becomes

$$\begin{aligned}X_t^k &= \sum_{j=0}^{k} \frac{1}{j!} B_j(Y_t) \sum_{m=0}^{k-j} \frac{k!}{m!(k-j-m)!} E\{Y_t^m Z_t^{k-j-m}\} + R_{k,t} \\ &= \sum_{j=0}^{k} \frac{1}{j!} B_j(Y_t) \frac{k!}{(k-j)!} E\{X_0^{k-j}\} + R_{k,t}.\end{aligned}$$

Consequently, for any polynomial $G(x) = \sum_{k=0}^{\ell} d_k x^k$,

$$G(X_t) = \sum_{j=0}^{\ell} \frac{1}{j!} E\{G^{(j)}(X_t)\} B_j(Y_t) + R_t = \sum_{j=k^*}^{\ell} \frac{a_j}{j!} B_j(Y_t) + R_t, \tag{3.7}$$

where

$$R_t = \sum_{k=0}^{\ell} d_k R_{k,t} = \sum_{k=0}^{\ell} d_k \sum_{m=0}^{k} \binom{k}{m} \sum_{(n)_k^m} \sum_{\gamma \in \Gamma_{(n)_k^m}} \mathscr{A}_{n_\gamma}(q_t^\gamma). \tag{3.8}$$

Now, Lemma 3.1 follows from (3.7) and Lemma 3.2.

LEMMA 3.2. *Under the conditions of Theorem* 1,

$$\operatorname{Var}\left(\sum_{t=1}^{N} R_t\right) = O(N).$$

PROOF. As all sums on the right-hand side of (3.8) are finite, it suffices to show the lemma with R_t replaced by $\mathscr{A}_{n_\gamma}(q_t^\gamma)$, for any diagram γ as in (3.8). This follows from

$$\sum_{t \in \mathbf{Z}} \left| E\{\mathscr{A}_{n_\gamma}(q_0^\gamma), \mathscr{A}_{n_\gamma}(q_t^\gamma)\}\right| < \infty$$

or

$$\sum_{t \in \mathbf{Z}} \sum_{s_1, \ldots, s_{n_\gamma} \in \mathbf{Z}} \left| \operatorname{sym} q_0^\gamma(s_1, \ldots, s_{n_\gamma}) q_t^\gamma(s_1, \ldots, s_{n_\gamma})\right| < \infty; \tag{3.9}$$

see (2.12). We claim that, for each diagram $\gamma \in \Gamma_{(n)_k^m}$ as in (3.8), the function $q^\gamma \equiv q_0^\gamma$ has the following representation

$$\begin{aligned}&q^\gamma\big(s_{i,j}\colon (i,j) \in U\big) \\ &\quad = h^\gamma\left(s_{i,j} : (i,j) \in \bigcup_{l=m+1}^{k} W_l \cap U\right) \prod_{(i,j) \in \bigcup_{l=1}^{m} W_l \cap U} b(s_{i,j}),\end{aligned} \tag{3.10}$$

where $b(t) \equiv b_t$ are given in (1.6), $h^\gamma \in L^1(\mathbf{Z}^{\tilde{n}_\gamma})$, and where $\tilde{n}_\gamma = |\{(i,j) \in \bigcup_{l=m+1}^{k} W_l \cap U\}|$ is the number of free variables among the rows $W_{m+1}, \ldots, W_k$ of the table W.

Let us check that (3.10) implies (3.9). Indeed, with $n \equiv n_\gamma, \tilde{n} \equiv \tilde{n}_\gamma, \tilde{m} = n - \tilde{n}$, $h \equiv h^\gamma$, for any permutation $(p)_n \equiv (p_1, \dots, p_n) \in \mathscr{P}_n$, one obtains

$$\begin{aligned} I_{(p)_n} \equiv & \sum_t \sum_{s_1,\dots,s_n} \big| b(s_{p_1}) \cdots b(s_{p_{\tilde{m}}}) h(s_{p_{\tilde{m}+1}}, \dots, s_{p_n}) \\ & \qquad \times b(t+s_1) \cdots b(t+s_{\tilde{m}}) h(t+s_{\tilde{m}+1}, \dots, t+s_n) \big| \\ = & \sum_t \sum_{s_1,\dots,s_n} \big| b(s_1) \cdots b(s_r) h'(s_{r+1}, \dots, s_{r+\tilde{n}}) b(s_{r+\tilde{n}+1}) \cdots b(s_n) \\ & \qquad \times b(t+s_1) \cdots b(t+s_{\tilde{m}}) h(t+s_{\tilde{m}+1}, \dots, t+s_n) \big|, \end{aligned}$$

for some $0 \le r \le \tilde{m}$ and $h' \in L^1(\mathbf{Z}^{\tilde{n}})$, which depend on $(p)_n$. Clearly, it suffices to consider $r = 0$ only. If $\tilde{m} \le \tilde{n}$, then $I_{(p)_n} \le \|b\|^{2\tilde{m}}_{L^\infty(\mathbf{Z})} \|h\|_{L^1(\mathbf{Z}^{\tilde{n}})} \|h'\|_{L^1(\mathbf{Z}^{\tilde{n}})} < \infty$. If $\tilde{n} < \tilde{m}$, then $I_{(p)_n} \le \|b\|^{2(\tilde{m}-\tilde{n})}_{L^2(\mathbf{Z})} \|b\|^{2\tilde{n}}_{L^\infty(\mathbf{Z})} \|h\|_{L^1(\mathbf{Z}^{\tilde{n}})} \|h'\|_{L^1(\mathbf{Z}^{\tilde{n}})} < \infty$. Finally, if $\tilde{n} = \tilde{m}$, then

$$\begin{aligned} I_{(p)_n} = & \sum_{s_1,\dots,s_n} \big| h'(s_1, \dots, s_{\tilde{n}}) h(s_{\tilde{n}+1}, \dots, s_n) \big| \\ & \qquad \times \sum_t | b(t+s_1) \cdots b(t+s_{\tilde{n}}) b(s_{\tilde{n}+1} - t) \cdots b(s_n - t) |, \end{aligned}$$

where the last sum does not exceed $\|b\|^2_{L^2(\mathbf{Z})} \|b\|^{2(\tilde{n}-1)}_{L^\infty(\mathbf{Z})}$. This proves (3.9).

It remains to show the representation (3.10). Without loss of generality, let $\gamma = (U, (V)_r)$ be such that $U \cap \bigcup_{p=1}^m W_p = \{(1,1), \dots, (\tilde{m}, 1)\}$ $(\tilde{m} \le m)$. Then

$$\begin{aligned} & h^\gamma \Big(s_{i,j} \colon (i,j) \in \bigcup_{p=m+1}^k W_p \cap U \Big) \\ & \quad = \sum_{\tilde{s}_1,\dots,\tilde{s}_r} \prod_{p=\tilde{m}+1}^k q_p(s_{i,j} \colon (i,j) \in W_p) \mathbf{1}\big(s_{i,j} = \tilde{s}_l : (i,j) \in V_l, l = 1, \dots, r \big). \end{aligned}$$

Hence and from $q_p \in L^1(\mathbf{Z}^{n_p})$, $m+1 \le p \le k$, the relation $h^\gamma \in L^1(\mathbf{Z}^{\tilde{n}_\gamma})$ follows easily. This ends the proof of Lemma 3.2 and of Theorem 1, too. □

4. Proofs of Theorems 2 and 3.

4.1. *Proof of Theorem* 2. Without loss of generality, one can assume $b_i = 0$, $i = 0, 1, \dots, m$. Note for each $t \in \mathbf{Z}$, $Y_t = \sum_{i \le t-m-1} b_{t-i} \zeta_i$ is independent of $Z_t = V(\zeta_t, \dots, \zeta_{t-m})$. To simplify the notation, we shall also assume $E\{Y_0^2\} = 1$. Observe that the p.d.f. $F(x) = P\{X_0 \le x\}$ is infinitely differentiable, being the convolution of the p.d.f. $F_Z(x) = P\{Z_0 \le x\}$ with the standard Gaussian p.d.f. $F_Y(x) = P\{Y_0 \le x\}$. As

$$\begin{aligned} E\{G(X_0 + c)\} &= \int_{\mathbf{R}} \int_{\mathbf{R}} G(y+z+c) \, dF_Y(y) \, dF_Z(z) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \Big\{ \int_{\mathbf{R}} G(y+z) \, dF_Z(z) \Big\} e^{-(y-c)^2/2} \, dy, \end{aligned} \tag{4.1}$$

so the right-hand side of (4.1) is also infinitely differentiable in c, under the sign of the integral. Consequently, Appell coefficients a_k (1.12) are well defined for any $k \geq 0$ and we obtain

$$a_k = \int_{\mathbf{R}} c_k(z)\, dF_Z(z) = E\{c_k(Z_0)\}, \tag{4.2}$$

where

$$c_k(z) = \int_{\mathbf{R}} G(y+z) H_k(y)\, dF_Y(y). \tag{4.3}$$

Furthermore,

$$G(Y_t + z) = \sum_{k=0}^{\infty} \frac{c_k(z)}{k!} H_k(Y_t), \tag{4.4}$$

where the series converges in $L^2(\Omega)$, for each $z \in \mathbf{R}$ fixed. We claim that

$$G(X_t) = G(Y_t + Z_t) = \sum_{k=0}^{\infty} \frac{c_k(Z_t)}{k!} H_k(Y_t) \equiv \sum_{k=0}^{\infty} \frac{c_k(Z_t)}{k!} {:}Y_t^{,k}{:}, \tag{4.5}$$

where the series converges in $L^2(\Omega)$. Indeed,

$$\begin{aligned} E\Bigg(G(Y_t + Z_t) - \sum_{k=0}^{K} \frac{c_k(Z_t)}{k!} H_k(Y_t)\Bigg)^2 &= \int_{\mathbf{R}} \int_{\mathbf{R}} \Bigg(G(y+z) - \sum_{k=0}^{K} \frac{c_k(z)}{k!} H_k(y)\Bigg)^2 dF_Y(y)\, dF_Z(z) \\ &= \int_{\mathbf{R}} \psi_K(z)\, dF_Z(z), \end{aligned}$$

where $\psi_K(z) = \sum_{k=K+1}^{\infty} c_k^2(z)/k!$. According to (4.4), for each $z \in \mathbf{R}$, $\psi_K(z) \to 0$ ($K \to \infty$), and $\psi_K(z) \leq \bar{\psi}(z) = E\{G^2(Y_t + z)\}$, where $\bar{\psi}(z)$ is integrable: $\int_{\mathbf{R}} \bar{\psi}(z)\, dF_Z(z) = E\{G^2(X_t)\} < \infty$. Therefore, $\lim_{K\to\infty} \int_{\mathbf{R}} \psi_K(z)\, dF_Z(z) = 0$ according to the Lebesgue dominated convergence theorem, thereby proving the claim.

Let us show that

$$R_N \equiv E\Bigg(\sum_{t=1}^{N} G(X_t) - \frac{1}{k^*!} c_{k^*}(Z_t){:}Y_t^{,k^*}{:}\Bigg)^2 = o(D_{N,k^*}^2). \tag{4.6}$$

By (4.5),

$$R_N = \sum_{t,t'=1}^{N} \sum_{k,k' \neq k^*} \rho_{t,k,t',k'}/k!k'!, \tag{4.7}$$

where

$$\rho_{t,k,t',k'} = E\big\{c_k(Z_t) c_{k'}(Z_{t'}){:}Y_t^{,k}{:}{:}Y_{t'}^{,k'}{:}\big\}.$$

Let $t < t'$, $|t' - t| > m$. Put $\mathscr{F}_{[s,t]} = \sigma\{\zeta_i : s \le i \le t\}$. Then

$$\rho_{t,k,t',k'} = E\{c_k(Z_t)c_{k'}(Z_{t'})E\{:Y_t^{,k}::Y_{t'}^{,k'}:|\mathscr{G}_{t,t'}\}\}. \tag{4.8}$$

where $\mathscr{G}_{t,t'} = \mathscr{F}_{[t-m,t]} \cup \mathscr{F}_{[t'-m,t']}$. Write $Y_{t'} = \tilde{Y}_1 + \tilde{Y}_2$, where

$$\tilde{Y}_1 = \sum_{i=t-m}^{t} b_{t'-i}\zeta_i, \qquad \tilde{Y}_2 = \sum_{i=-\infty}^{t-m-1} b_{t'-i}\zeta_i + \sum_{i=t+1}^{t'-m-1} b_{t'-i}\zeta_i.$$

Note that $\tilde{Y}_1$ is measurable with respect to $\mathscr{G}_{t,t'}$, while $Y_t = \sum_{i=-\infty}^{t-m-1} b_{t-i}\zeta_i$ and $\tilde{Y}_2$ are independent of $\mathscr{G}_{tt'}$. Therefore the conditional expectation in (4.8) can be rewritten as

$$\begin{aligned} E\{:Y_t^{,k}::Y_{t'}^{,k'}:|\mathscr{G}_{t,t'}\} &= E\{:Y_t^{,k}::(\tilde{Y}_1+\tilde{Y}_2)^{,k'}:|\mathscr{G}_{t,t'}\} \\ &= \sum_{l=0}^{k'} \binom{k'}{l} :\tilde{Y}_1^{,l}: E\{:Y_t^{,k}::\tilde{Y}_2^{,k'-l}:\}. \end{aligned}$$

Here,

$$E\{:Y_t^{,k}::\tilde{Y}_2^{,k'-l}:\} = \begin{cases} 0, & \text{if } k \ne k'-l, \\ k!r_{t'-t}^k, & \text{if } k = k'-l, \end{cases}$$

where

$$r_{t'-t} = E\{Y_tY_{t'}\} = E\{Y_t\tilde{Y}_2\} = \sum_{i=m+1}^{\infty} b_ib_{t'-t+i}, \qquad t' > t+m.$$

Consequently,

$$E\{:Y_t^{,k}::Y_{t'}^{,k'}:|\mathscr{G}_{t,t'}\} = \binom{k'}{k'-k} k!r_{t'-t}^k :\tilde{Y}_1^{,k'-k}: \mathbf{1}(k \le k').$$

Substituting the last expression into (4.8) and using (4.2), the independence of $Z_{t'}$ from Z_t, $\tilde{Y}_1$, for $t' > t+m$, one obtains

$$\rho_{t,k,t',k'} = \binom{k'}{k'-k} k!r_{t'-t}^k a_{k'} E\{c_k(Z_t):\tilde{Y}_1^{,k'-k}:\}\mathbf{1}(k \le k'). \tag{4.9}$$

Denote

$$\|c_k\|^2 = \int_{\mathbf{R}} c_k^2(z)\,dF_Z(z) = E\{c_k^2(Z_0)\}, \qquad \beta_{t'-t}^2 = E\{\tilde{Y}_1^2\} = \sum_{i=t-m}^{t} b_{t'-i}^2.$$

As $|a_k| \le \|c_k\|$, from (4.8) by the Cauchy–Schwarz inequality we obtain the bound

$$|\rho_{t,k,t',k'}| \le \frac{k'!}{((k'-k)!)^{1/2}} |r_{t'-t}|^k \|c_k\|\,\|c_{k'}\||\beta_{t'-t}|^{k'-k}\mathbf{1}(k \le k'). \tag{4.10}$$

Let $M > m$ be a fixed integer which will be specified below. Write $\sum_{t,t'}^{N,M}$ for the sum over $t, t' = 1, \dots, N$, $t \le t' - M$. Then

$$\sum\nolimits_{t,t'}^{N,M} \sum_{k,k' \ne k^*} \frac{\rho_{t,k,t',k'}}{k!k'!} = \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum\nolimits_{t,\, t'}^{N,\, M} \sum_{k^* < k \le k'} \rho_{t,\, k,\, t',\, k'} / k! k'!,$$

$$\Sigma_2 = \sum\nolimits_{t,\, t'}^{N,\, M} \sum_{0 \le k < k^* < k'} \rho_{t,\, k,\, t',\, k'} / k! k'!,$$

and where we used the fact that $\rho_{t,\, k,\, t',\, k'} = 0$ for $k' < k^*$, which follows from (4.9) and $a_{k'} = 0$, $k' < k^*$. Let us show

$$\Sigma_i = o(D^2_{N,\, k^*}), \qquad i = 1, 2. \tag{4.11}$$

Consider Σ_1. By (4.10),

$$|\Sigma_1| \le \sum\nolimits_{t,\, t'}^{N,\, M} \sum_{k=k^*+1}^{\infty} \frac{|r_{t'-t}|^k \|c_k\|}{(k!)^{1/2}} \sum_{k' \ge k} \frac{\|c_{k'}\|}{(k!(k'-k)!)^{1/2}} |\beta_{t'-t}|^{k'-k}. \tag{4.12}$$

Note that $k!(k'-k)! \ge 2^{-k'} k'!$ for any $k' \ge k \ge 0$. Without loss of generality, one can assume M so large that $|2^{1/2} \beta_{t'-t}| < 2^{-1/2}$ for all $|t'-t| > M$. Then the last sum in (4.12) does not exceed

$$2^{k/2} \sum_{k' \ge k} \frac{\|c_{k'}\|}{(k'!)^{1/2}} |2^{1/2} \beta_{t'-t}|^{k'-k} \le 2^{1+k/2} \|G\|,$$

where

$$\|G\|^2 = E\{G^2(X_0)\} = \sum_{j=0}^{\infty} \|c_j\|^2 / j! < \infty.$$

Consequently,

$$|\Sigma_1| \le 2\|G\| \sum\nolimits_{t,\, t'}^{N,\, M} \sum_{k=k^*+1}^{\infty} \frac{|r_{t'-t}|^k \|c_k\| 2^{k/2}}{(k!)^{1/2}}. \tag{4.13}$$

According to (1.1), for any $\theta' < \theta$ one can find $M < \infty$ such that $|r_{t'-t}| \le |t'-t|^{-\theta'}$, $|t'-t| > M$. Let $\theta'' < \theta'$, $\delta = \theta' - \theta'' > 0$. Then

$$|r_{t'-t}|^k \le |t'-t|^{-(k^*+1)\theta''} |t'-t|^{-k\delta},$$

$k \ge k^*+1$. Choose now $\theta'' < \theta'$ so that $(k^*+1)\theta'' > k^*\theta$ and further, assume M so large that $2^{1/2}|t'-t|^{-\delta} < 2^{-1/2}$ for $|t'-t| > M$. Then from (4.13) one obtains

$$|\Sigma_1| \le 4\|G\|^2 \sum\nolimits_{t,\, t'}^{N,\, M} |t'-t|^{-(k^*+1)\theta''} = o(D^2_{N,\, k^*}).$$

Next, consider Σ_2. We have

$$|\Sigma_2| \le C \sum\nolimits_{t,\, t'}^{N,\, M} |\beta_{t'-t}| \sum_{k' > k^*} \frac{\|c_{k'}\|}{(k'!)^{1/2}} |\beta_{t'-t}|^{k'-k^*},$$

where $C < \infty$ is a constant. Choose M so that $|\beta_{t'-t}| < 2^{-1/2}$, $|t'-t| > M$ and use the fact that $|\beta_{t'-t}| = O(b^2_{t'-t})$, where $\sum_{t \ge 0} b_t^2 < \infty$, to conclude that $|\Sigma_2| \le C\|G\| N = O(N) = o(D^2_{N,\, k^*})$.

To complete the proof of (4.5), it suffices to note that for each $M < \infty$ fixed, and with $\tilde{G}_t = G(X_t) - (k^*!)^{-1} c_{k^*}(Z_t){:}Y_t^{k^*}{:}$, one has

$$\sum_{t, t'=1,\dots,N:\,|t'-t|\le M} \left|\mathrm{Cov}(\tilde{G}_t, \tilde{G}_{t'})\right| = O(N)$$

provided $E\{\tilde{G}_0^2\} < \infty$. The last inequality holds by $E\{G^2(X_0)\} < \infty$ and, Y_0, Z_0 being independent, by $E\{c_k^2(Z_0)({:}Y_0^k{:})^2\} = E\{c_k^2(Z_0)\}E({:}Y_0^k{:})^2 = k!\|c_k\|^2 < \infty$.

With (4.5) in mind, Theorem 2 follows from Taqqu (1979) and

$$E\left(\sum_{t=1}^N \bar{c}_{k^*}(Z_t) H_{k^*}(Y_t)\right)^2 = O(N), \tag{4.14}$$

where $\bar{c}_k(Z_t) = c_k(Z_t) - E\{c_k(Z_t)\} = c_k(Z_t) - a_k$. Clearly, (4.14) is a consequence of the orthogonality $\mathrm{Cov}(\bar{c}_k(Z_t)H_k(Y_t), \bar{c}_k(Z_{t'})H_k(Y_{t'})) = 0$, $|t'-t| > m$. For $t' - m > t$, the last property follows from $E\{\bar{c}_k(Z_{t'})\} = 0$ and the fact that $Z_{t'}$ is independent of Z_t, Y_t, $Y_{t'}$. This ends the proof of Theorem 2. □

4.2. *Proof of Theorem* 3. The proof is very similar to and actually simpler than the proof of Theorem 2, so we just give a brief outline.

Let $F_Y(x)$, $F_Z(x)$ be the same p.d.f. as above. Exactly as in the proof of Theorem 2, one can show that Appell coefficients a_k, $k \ge 0$ are well defined and given by (4.2)–(4.3). In a similar way, one can show the relation (4.5). The only place where we need condition (1.18) is to prove (4.14). By the independence of Y_t, $t \in \mathbf{Z}$ and Z_t, $t \in \mathbf{Z}$, one obtains

$$\begin{aligned}&\sum_{t\in\mathbf{Z}} \left|E\{\bar{c}_k(Z_0)\bar{c}_k(Z_t)H_k(Y_0)H_k(Y_t)\}\right| \\ &\qquad \le E\{H_k^2(Y_0)\} \sum_{t\in\mathbf{Z}} \left|E\{\bar{c}_k(Z_0)\bar{c}_k(Z_t)\}\right| < \infty\end{aligned}$$

[see (1.18)], as $E\{\bar{c}_k^2(Z_0)\} < \infty$. This proves (4.14) and Theorem 3. □

Below, $\alpha_N \sim \beta_N$ means $\lim_{N\to\infty} \alpha_N/\beta_N = 1$. Put $S_N^{(G)} = S_N^{(G)}(1)$.

COROLLARY 4.1. *Let* $G(x)$, $X_t = Y_t + Z_t$, Y_t, Z_t *satisfy the assumptions of either Theorems* 1, 2 *or* 3. *Then*

$$\mathrm{Var}(S_N^{(G)}) \sim a_{k^*}^2 D_{N,k^*}^2.$$

PROOF. Write $G(X_t) = G_{1,t} + G_{2,t}$, where $G_{1,t} = (k^*)^{-1} a_{k^*}{:}Y_t^{k^*}{:}$. It is well known that $\mathrm{Var}(\sum_{t=1}^N G_{1,t}) \sim a_{k^*}^2 D_{N,k^*}^2$; see, for example, Taqqu (1979) or Surgailis (1982) in the case when $\{Y_t\}$ is Gaussian or linear, respectively. Then the corollary follows from

$$\mathrm{Var}\left(\sum_{t=1}^N G_{2,t}\right) = o(D_{N,k^*}^2). \tag{4.15}$$

For $G(x)$, X_t, Y_t, Z_t as in Theorem 1 (respectively, as in Theorem 2 or 3), (4.15) follows from (3.7) and Lemma 3.2 [respectively, from (4.5) and (4.14)].

APPENDIX

We present below the proofs of Lemmas 2.1–2.3.

PROOF OF LEMMA 2.1. We prove the lemma by induction in the number $|W|$ of elements of W. To that end, we need the recursive formula

$$:Y^{,W}: = Y_{j_1}:Y^{,W_1}: - \sum_{U\subseteq W_1} :Y^{,U}:\chi(Y^{,W\setminus U}), \tag{A.1}$$

where $j_1 \in W$ is arbitrary, and $W_1 = W\setminus\{j_1\}$. (A.1) can be proved as follows. Note, by definition (2.1),

$$:Y^{,W}: = Y_{j_1}:Y^{,W_1}: - (-i)^{|W|}\partial^{|W_1|}(f_1(\mathbf{z}_1)f_2(\mathbf{z}_1))/\partial z^{W_1}|_{\mathbf{z}_1=0}, \tag{A.2}$$

where $\partial^{|U|}/\partial z^U = \prod_{j\in U}\partial/\partial z_j$, $\mathbf{z}_1 = (z_j : j \in W_1) \in \mathbf{R}^{|W_1|}$, and where

$$f_1(\mathbf{z}_1) = \exp\left\{i\sum_{j\in W_1} z_jY_j\right\}\Big/ E\exp\left\{i\sum_{j\in W_1} z_jY_j\right\},$$

$$f_2(\mathbf{z}_1) = \partial\log E\exp\left\{i\sum_{j\in W} z_jY_j\right\}\Big/\partial z_{j_1}\Big|_{z_{j_1}=0}.$$

Next, observe, that for any $U \subseteq W_1$,

$$\partial^{|U|}f_1(\mathbf{z}_1)/\partial z^U\big|_{\mathbf{z}_1=0} = i^{|U|}:Y^{,U}:, \tag{A.3}$$

$$\partial^{|U|}f_2(\mathbf{z}_1)/\partial z^U\big|_{\mathbf{z}_1=0} = i^{|U|+1}\chi(Y^{,U\cup\{j_1\}}). \tag{A.4}$$

Now, (A.1) follows from (A.2)–(A.4) and the differentiation rule $\partial^{|W_1|}(f_1f_2)/\partial z^{W_1} = \sum_{U\subseteq W_1}(\partial^{|U|}f_1/\partial z^U)(\partial^{|W_1\setminus U|}f_2/\partial z^{W_1\setminus U})$.

Let us turn to the proof of (2.6a) and (2.6b). Observe, the lemma holds for $|W| = 1$. Assume that (2.6a) is true for any subset $W_1 = W\setminus\{j_1\} \subset W$. Then, using (A.1),

$$\begin{aligned} Y^W = Y_{j_1}Y^{W_1} &= \sum_{U\subseteq W_1} Y_{j_1}:Y^{,U}: \sum_{\mathscr{V}_{W_1\setminus U}} \chi(Y^{,V_1})\cdots\chi(Y^{,V_r}) \\ &= \sum_{U\subseteq W_1} :Y^{,U\cup\{j_1\}}: \sum_{\mathscr{V}_{W_1\setminus U}} \chi(Y^{,V_1})\cdots\chi(Y^{,V_r}) \\ &\quad + \sum_{U\subseteq W_1}\sum_{\tilde{U}\subseteq U} :Y^{,\tilde{U}}: \sum_{\mathscr{V}_{W_1\setminus U}} \chi(Y^{,V_1})\cdots\chi(Y^{,V_r})\chi(Y^{,\{j_1\}\cup U\setminus\tilde{U}}). \end{aligned}$$

Hence, by changing the order of summation over U and $\tilde{U}$, (2.6a) follows, while (2.6b) follows from the well-known relation between moments and cumulants,

$$E\{Y^W\} = \sum_{(V)_r\in\mathscr{V}_W} \chi(Y^{,V_1})\cdots\chi(Y^{,V_r}). \tag{A.5}$$

This proves Lemma 2.1. □

PROOF OF LEMMA 2.2. We prove the lemma by induction in $|W|$. Applying Lemma 2.1, (2.6a) to the product $Y^W = \prod_{i=1}^k Y^{W_i}$ and to each Y^{W_i}, $i = 1, \dots, k$ separately, one obtains

$$\sum_{U \subseteq W} {:}Y^{,U}{:}E\{Y^{W\setminus U}\} = \sum_{(\tilde U)_k} \prod_{i=1}^k {:}Y^{,\tilde U_i}{:}E\{Y^{W_i \setminus \tilde U_i}\} + \prod_{i=1}^k {:}Y^{,W_i}{:},$$

where the sum $\sum_{(\tilde U)_k}$ is taken over all $\tilde U_1 \subseteq W_1, \dots, \tilde U_k \subseteq W_k$ such that $\tilde U \equiv \bigcup_{i=1}^k \tilde U_i \neq W$. As $|\tilde U| < |W|$, by the inductive assumption one obtains

$$\begin{aligned}\sum_{U \subseteq W} &{:}Y^{,U}{:}E\{Y^{W\setminus U}\} \\ &= \sum_{(\tilde U)_k} \sum_{U \subseteq \tilde U} {:}Y^{,U}{:} \sum_{(V)_r \in \mathscr{V}^*_{\tilde U \setminus U}} \chi(Y^{,V_1}) \cdots \chi(Y^{,V_r}) \prod_{i=1}^k E\{Y^{W_1 \setminus \tilde U_i}\} \qquad \text{(A.6)} \\ &\quad + \prod_{i=1}^k {:}Y^{,W_i}{:},\end{aligned}$$

where, for $U' \subseteq W$, $\mathscr{V}^*_{U'}$ is the set of all partitions $(V)_r \in \mathscr{V}_{U'}$ without flat edges. We claim that for each $U \subseteq W$,

$$\begin{aligned}\alpha_U &\equiv E\{Y^{W\setminus U}\} - \sum_{\tilde U : U \subseteq \tilde U \subset W} \sum_{(V)_r \in \mathscr{V}^*_{\tilde U \setminus U}} \chi(Y^{,V_1}) \cdots \chi(Y^{,V_r}) \prod_{i=1}^k E\{Y^{W_i \setminus \tilde U}\} \qquad \text{(A.7)} \\ &= \sum_{(V)_r \in \mathscr{V}^*_{W\setminus U}} \chi(Y^{,V_1}) \cdots \chi(Y^{,V_r}).\end{aligned}$$

This proves the induction step, as $\prod_{i=1}^k {:}Y^{,W_i}{:} = \sum_{U \subseteq W} \alpha_U {:}Y^{,U}{:}$ according to (A.6). It remains to show the claim (A.7). To do this, split the sum $\sum_{(V)_r \in \mathscr{V}_{W\setminus U}} \prod_{s=1}^r \chi(Y^{,V_s}) = E\{Y^{W\setminus U}\}$ into two parts $\sum_{(V)_r \in \mathscr{V}^*_{W\setminus U}} \prod_{s=1}^r \chi(Y^{,V_s}) + \sum_{(V)_r \notin \mathscr{V}^*_{W\setminus U}} \prod_{s=1}^r \chi(Y^{,V_s}) \equiv \sum_1 + \sum_2$. Then, rewrite

$$\sum\nolimits_2 = \sum_{\tilde U : U \subseteq \tilde U \subset W} \sum_{(V)_r \in \mathscr{V}^*_{\tilde U \setminus U}} \prod_{s=1}^r \chi(Y^{,V_s}) \prod_{i=1}^k \sum_{(V')_{r'} \in \mathscr{V}_{W_i \setminus \tilde U}} \prod_{s'=1}^{r'} \chi(Y^{,V'_{s'}}).$$

Here, the last sum on the right-hand side equals $E\{Y^{W_i \setminus \tilde U}\}$, for each $i = 1, \dots, k$, according to (A.5).

This proves (2.7a). (2.8) follows by taking expectations of both sides of (2.7a) and using (2.2). Finally, (2.7b) follows from (2.7a) and (2.8). Lemma 2.2 is proved. □

PROOF OF LEMMA 2.3. Note, by repeated use of the Cauchy–Schwarz inequality, that $q^\gamma \in L^2(\mathbf{Z}^{n_\gamma})$. Indeed, it suffices to check this property in the case when $|V_i \cap W_j| = 1$ for every $i = 1, \dots, r$, $j = 1, \dots, k$. Assume first $r = 1$, that is, that $\gamma = (U, V_1)$ has a single connected edge, say, $V_1 = \{(i, 1): 1 \le i \le k'\}$, where $2 \le k' \le k$. Then $|q^\gamma| \le |\chi_{k'}| f_1 f_2 f_3$, where

$f_1 = \big(\sum_{\tilde s} q_1^2(\tilde s, s_{1,2}, \dots, s_{1,n_1})\big)^{1/2}$, $f_2 = \big(\sum_{\tilde s} \prod_{i=2}^{k'} q_i^2(\tilde s, s_{i,1}, \dots, s_{i,n_i})\big)^{1/2}$ and $f_3 = \prod_{i=k'+1}^{k} |q_i(s_{i,1}, \dots, s_{i,n_i})|$, whence this property clearly follows. In the case $r \ge 2$, the above inequality can be used to reduce the number of connected edges by 1, and the property follows by induction in r.

It suffices to show (2.16) and (2.17) for $q_i(s_1, \dots, s_{n_i})$ vanishing outside a finite set of integer points, $i = 1, \dots, k$. In this case, $\mathscr{A}_{n_i}(q_i)$, $i = 1, \dots, k$ are finite sums, and

$$\prod_{i=1}^{k} \mathscr{A}_{n_i}(q_i) = \sum_{s_{i,j} \in \mathbf{Z}:(i,j)\in W} \tilde q(s_{i,j}:(i,j) \in W) \prod_{i=1}^{k} :\zeta^{\cdot W_i}:,$$

where $\tilde q$ is given by (2.13). By Lemma 2.2,

$$\prod_{i=1}^{k} :\zeta^{\cdot W_i}: = \sum_{\gamma=(U,(V)_p)\in\Gamma_{(n)_k}} :\zeta^{\cdot U}: \prod_{l=1}^{r} \chi(\zeta^{\cdot V_l}).$$

Hence (2.16) follows, because $\chi(\zeta^{\cdot V}) = 0$ unless all $s_{i,j}, (i,j) \in V \subseteq W$ are equal. Equation (2.17) follows from (2.16) and $E\{\mathscr{A}_n(q)\} = 0$, $n \ge 1$. Lemma 2.3 is proved. □

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