The Annals of Probability
2001, Vol. 29, No. 4, 1833–1858

BRANCHING EXIT MARKOV SYSTEMS AND SUPERPROCESSES[1]

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Superprocesses (under the name continuous state branching processes) appeared, first, in a pioneering work of S. Watanabe [*J. Math. Kyoto Univ.* **8** (1968) 141–167]. Deep results on paths of the super-Brownian motion were obtained by Dawson, Perkins, Le Gall and others.

In earlier papers, a superprocess was interpreted as a Markov process X_t in the space of measures. This is not sufficient for a probabilistic approach to boundary value problems. A reacher model based on the concept of exit measures was introduced by E. B. Dynkin [*Probab. Theory Related Fields* **89** (1991) 89–115]. A model of a superprocess as a system of exit measures from time-space open sets was systematically developed in 1993 [E. B. Dynkin, *Ann. Probab.* **21** (1993) 1185–1262]. In particular, branching and Markov properties of such a system were established and used to investigate partial differential equations. In the present paper, we show that the entire theory of superprocesses can be deduced from these properties.

1. Introduction.

1.1. *Exit points of Markov processes.* Suppose that $\xi = (\xi_t, \Pi_{r,x})$ is a right continuous strong Markov process in a topological space E. To every open set Q in time–space $S = \mathbb{R} \times E$ there corresponds a random point (τ, ξ_τ), where $\tau = \inf\{t: (t, \xi_t) \notin Q\}$ is the first exit time from Q. If a particle starts at time r from a point x, then the probability distribution of the exit point, given by the formula

$$k(r, x; B) = \Pi_{r,x}\{(\tau, \xi_\tau) \in B\},$$

is concentrated on the complement Q^c of Q. [If $(r, x) \notin Q$, then $k(r, x; \cdot)$ is concentrated at (r, x).] The family of random points $((\tau, \xi_\tau), \Pi_{r,x})$ has the following property: for every pre-τ $X \geq 0$ and every post-τ $Y \geq 0$,

$$\Pi_{r,x}(X 1_{\tau<\infty} Y) = \Pi_{r,x}(X 1_{\tau<\infty} \Pi_{\tau,\xi_\tau} Y). \tag{1.1}$$

Pre-τ means depending only on the part of the path before τ. Similarly, post-τ means depending on the path after τ. To every measurable $\rho \geq 0$, there correspond a pre-τ random variable

$$X = \int_{-\infty}^{\tau} \rho(s, \xi_s)\, ds$$

Received January 2000; revised March 2001.

[1]Supported in part by NSF Grant DMS-99-70942.

AMS 2000 *subject classifications*. 60J60, 60J80.

Key words and phrases. Superprocesses, exit measures, branching property, Markov property, transition operators, branching particle systems.

and a post-τ random variable

$$Y = \int_{\tau}^{\infty} \rho(s, \xi_s)\, ds.$$

Let τ and τ' be the first exit times from Q and Q'. Then $f(\tau', \xi_{\tau'})$ is a pre-τ random variable if $Q' \subset Q$ and it is a post-τ random variable if $Q' \supset Q$.

1.2. *Exit systems associated with branching particle systems.* Consider a system of particles moving in E according to the following rules:

1. The motion of each particle is described by a Markov process ξ.
2. A particle dies during time interval $(t, t+h)$ with probability $kh + o(h)$, independently on its age.
3. If a particle dies at time t at point x, then it produces n new particles with probability $p_n(t, x)$.
4. The only interaction between the particles is that the birth time and place of offspring coincide with the death time and place of their parent.

(Assumption 2 implies that the lifetime of every particle has an exponential probability distribution with mean value $1/k$.)

We denote by $P_{r,x}$ the probability law corresponding to a process started at time r by a single particle located at point x. Suppose that particles stop to move and to procreate outside an open subset Q of S. In other words, we observe each particle at the first, in the family history, exit time from Q. By the family history we mean the path of a particle and all its ancestors. If the family history starts at (r, x), then the probability law of this path is $\Pi_{r,x}$. The exit measure from Q is defined by the formula

$$X_Q = \delta_{(t_1, y_1)} + \cdots + \delta_{(t_n, y_n)},$$

where $(t_1, y_1), \ldots, (t_n, y_n)$ are the states of frozen particles and $\delta_{(t,y)}$ means the unit measure concentrated at (t, y). We also consider a process started by a finite or infinite sequence of particles that "immigrate" at times r_i at points x_i. There is no interaction between their descendants and therefore the corresponding probability law is the convolution of P_{r_i, x_i}. We denote it by P_μ, where

$$\mu = \sum \delta_{(r_i, x_i)}$$

is a measure on S describing the immigration. We arrive at a family X of random measures (X_Q, P_μ), $Q \in \mathbb{O}$, $\mu \in \mathbb{M}$, where $\mathbb{O}$ is a class of open subsets of S and $\mathbb{M}$ is the class of all integer-valued measures on S. Family X is a special case of a branching exit Markov system. A general definition of such systems is given in the next section.

1.3. *Branching exit Markov systems.* A random measure on a measurable space $(S, \mathscr{B}_S)$ is a pair (X, P), where $X(\omega, B)$ is a kernel from an auxiliary measurable space $(\Omega, \mathscr{F})$ to $(S, \mathscr{B}_S)$ and P is a probability measure on $\mathscr{F}$. A kernel from a measurable space $(E_1, \mathscr{B}_1)$ to a measurable space $(E_2, \mathscr{B}_2)$ is a function $K(x, B)$ such that $K(x, \cdot)$ is a measure on $\mathscr{B}_2$ for every $x \in E_1$ and $K(\cdot, B)$ is a $\mathscr{B}_1$-measurable function for every $B \in \mathscr{B}_2$. We assume that S is a Borel subset of a compact metric space and $\mathscr{B}_S$ is the class of all Borel subsets of S.

Suppose that the following hold:

(i) $\mathbb{O}$ is a subset of σ-algebra $\mathscr{B}_S$;

(ii) $\mathbb{M}$ is a class of measures on $(S, \mathscr{B}_S)$ which contains all measures δ_y, $y \in S$;

(iii) to every $Q \in \mathbb{O}$ and every $\mu \in \mathbb{M}$, there corresponds a random measure (X_Q, P_μ) on $(S, \mathscr{B}_S)$.

Condition (ii) is satisfied, for instance, for the class $\mathscr{M}(S)$ of all finite measures and for the class $\mathscr{N}(S)$ of all integer-valued measures.

We use the notation $\langle f, \mu \rangle$ for the integral of f with respect to a measure μ. Denote by $\mathbb{Z}$ the class of functions

$$Z = \exp\left\{ \sum_1^n \langle f_i, X_{Q_i} \rangle \right\}, \tag{1.2}$$

where $Q_i \in \mathbb{O}$ and f_i are positive measurable functions on S. We say that $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}$, $\mu \in \mathbb{M}$, is a *branching system* if the following condition holds.

1.3.A. For every $Z \in \mathbb{Z}$ and every $\mu \in \mathbb{M}$,

$$P_\mu Z = e^{-\langle u, \mu \rangle}, \tag{1.3}$$

where

$$u(y) = -\log P_y Z \tag{1.4}$$

and $P_y = P_{\delta_y}$.

Condition 1.3.A (we call it the *continuous branching property*) implies that

$$P_\mu Z = \prod P_{\mu_n} Z$$

for all $Z \in \mathbb{Z}$ if μ_n, $n = 1, 2, \ldots$, and $\mu = \sum \mu_n$ belong to $\mathbb{M}$.

A family X is called an *exit system* if the following conditions hold.

1.3.B. *For all* $\mu \in \mathbb{M}$ *and* $Q \in \mathbb{O}$,

$$P_\mu\{X_Q(Q) = 0\} = 1.$$

1.3.C. If $\mu \in \mathbb{M}$ and $\mu(Q) = 0$, then

$$P_\mu\{X_Q = \mu\} = 1.$$

Finally, we say that X is a *branching exit Markov* (*BEM*) *system*, if $X_Q \in \mathbb{M}$ for all $Q \in \mathbb{O}$ and if, in addition to 1.3.A–1.3.C, we have the following property.

1.3.D (Markov property). Suppose that $X \geq 0$ is measurable with respect to the σ-algebra $\mathscr{F}_{\subset Q}$ generated by $X_{Q'}$, $Q' \subset Q$, and $Y \geq 0$ is measurable with respect to the σ-algebra $\mathscr{F}_{\supset Q}$ generated by $X_{Q''}$, $Q'' \supset Q$. Then

$$P_\mu(YZ) = P_\mu(YP_{X_Q}Z). \tag{1.5}$$

It follows from the principles (1–4) stated at the beginning of Section 1.2 that conditions 1.3.A–1.3.D hold for the systems of random measures associated with branching particle systems. For them $S = \mathbb{R} \times E$, $\mathbb{M} = \mathscr{N}(S)$ and $\mathbb{O}$ is a class of open subsets of S.

1.4. *Transition operators.* Let $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}$, $\mu \in \mathbb{M}$, be a family of random measures. Denote by $\mathbb{B}$ the set of all bounded positive $\mathscr{B}_S$-measurable functions. Operators V_Q, $Q \in \mathbb{O}$, acting on $\mathbb{B}$ are called the transition operators of X if, for every $\mu \in \mathbb{M}$ and every $Q \in \mathbb{O}$,

$$P_\mu e^{-\langle f, X_Q\rangle} = e^{-\langle V_Q(f), \mu\rangle}. \tag{1.6}$$

If X is a branching system, then (1.6) follows from the formula

$$V_Q(f)(y) = -\log P_y e^{-\langle f, X_Q\rangle} \quad \text{for } f \in \mathbb{B}. \tag{1.7}$$

THEOREM 1.1. *Transition operators of an arbitrary system of random measures X satisfy the following condition*:

1.4.A. *for all* $Q \in \mathbb{O}$,

$$V_Q(f_n) \to 0 \quad \text{as } f_n \downarrow 0. \tag{1.8}$$

A branching system X is a branching exit system if and only if the following conditions hold:

1.4.B.

$$V_Q(f) = V_Q(\tilde{f}) \quad \text{if } f = \tilde{f} \text{ on } Q^c;$$

1.4.C. *for every* $Q \in \mathbb{O}$ *and every* $f \in \mathbb{B}$,

$$V_Q(f) = f \quad \text{on } Q^c.$$

It is a BEM system if and only if, in addition, the following condition holds:

1.4.D. *for all* $Q \subset \widetilde{Q} \in \mathbb{O}$,

$$V_Q V_{\widetilde{Q}} = V_{\widetilde{Q}}.$$

1.5. *From transition operators to BEM systems.* A real-valued function u on an Abelian semigroup G is called *negative semidefinite* if

$$\sum_{i,\, j=1}^{k} t_i t_j u(f_i + f_j) \le 0$$

for every $n \ge 2$, all $f_1, \dots, f_n \in G$ and all $t_1, \dots, t_n \in \mathbb{R}$ such that $\sum \lambda_i = 0$. We consider negative semidefinite functions on the semigroup $\mathbb{B}$. We say that a function U from $\mathbb{B}$ to $\mathbb{B}$ is negative semidefinite if the real-valued function $U(f)(y)$ is negative semidefinite for all $y \in S$.

THEOREM 1.2. *The transition operators of every BEM system are negative semidefinite. Suppose that operators V_Q acting in $\mathbb{B}$ satisfy conditions* 1.4.A–1.4.D. *They are the transition operators of a BEM system if, in addition, the following condition holds*:

1.5.A. $V_Q[U(f)]$ *is negative semidefinite for every negative semidefinite* $U(f)$.

Condition 1.5.A implies that V_Q are negative semidefinite but the converse statement is not true. Transition operators not satisfying 1.5.A can be obtained by a passage to the limit. We denote by $\mathbb{B}_c$ the set of all $\mathscr{B}_S$-measurable functions f such that $0 \le f \le c$ and we put $\|f\| = \sup_S |f(y)|$ for every f. Writing $V^k \xrightarrow{u} V$ means that V^k converges to V uniformly on each set $\mathbb{B}_c$.

THEOREM 1.3. *Suppose that X^k is a sequence of BEM systems and that V_Q^k are the transition operators of X^k. If $V_Q^k \xrightarrow{u} V_Q$ for every $Q \in \mathbb{O}$ and if V_Q satisfies the Lipschitz condition on every $\mathbb{B}_c$, then V_Q is the transition operator of a BEM system.*

1.6. *BEM systems corresponding to branching particle systems.* We return to the branching particle system and the corresponding BEM system $X = (X_Q, P_\mu)$ described in Section 1.2. We introduce an offspring generating function

$$\varphi(t, x; z) = \sum_0^\infty p_n(t, x) z^n, \qquad 0 \le z \le 1,$$

and we put

$$\Phi(t, x; z) = \varphi(t, x; z) - z.$$

The four principles stated at the beginning of Section 1.2 lead to the following result.

THEOREM 1.4. *Let V_Q be the transition operators of X. Then, for every $f \in \mathbb{B}$, the function $v = V_Q(f)$ satisfies the equation*

$$e^{-v(r,x)} = \Pi_{r,x}\left[k \int_r^\tau \Phi(s, \xi_s; e^{-v(s,\xi_s)})\, ds + e^{-f(\tau,\xi_\tau)}\right]. \tag{1.9}$$

Assuming that all particles have mass β, we get a transformed system of random measures $X^\beta = (X_Q^\beta, P_\mu^\beta)$, $\mu \in \mathbb{M}^\beta$, where

$$\mathbb{M}^\beta = \beta\mathbb{M}, \qquad X_Q^\beta = \beta X_Q, \qquad P_\mu^\beta = P_{\mu/\beta}.$$

Equation (1.9) implies a similar equation for $v^\beta = V_Q^\beta(f)$, where V_Q^β are the transition operators of X^β. By passing formally to the limit as $\beta \to 0$, we get an equation

$$u(r,x) + \Pi_{r,x} \int_r^\tau \psi(s, \xi_s; u(s, \xi_s)) = \Pi_{r,x} f(\tau, \xi_\tau), \tag{1.10}$$

where

$$\begin{gathered} u = \lim[1 - e^{-\beta v_\beta}]/\beta, \qquad f = \lim[1 - e^{-\beta f}]/\beta, \\ \psi(r,x;z) = \lim[\varphi^\beta(r,x;1-\beta u) - 1 + \beta u]k^\beta/\beta \quad \text{for } \beta u \le 1. \end{gathered} \tag{1.11}$$

[We assume that k and φ depend on β.]

Motivated by this heuristic passage to the limit, we introduce the following definition. We say that a BEM system $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}$, $\mu \in \mathscr{M}(S)$, is a (ξ, ψ)-*superprocess* if $\mathbb{O}$ is a class of open subsets of $S = \mathbb{R} \times E$, if $\xi = (\xi_t, \Pi_{r,x})$ is a right continuous strong Markov process and if the transition operators V_Q of X satisfy the following condition: for every $f \in \mathbb{B}$, $u = V_Q(f)$ is a solution of (1.10).

The uniqueness and existence problems for such systems are treated in Theorems 1.5 and 1.6. Put $Q \in \mathbb{O}_0$ if Q is an open subset of S and if $Q \subset \Delta \times E$ for some finite interval Δ.

THEOREM 1.5. *If $Q \in \mathbb{O}_0$ and if $\psi \ge 0$ is locally Lipschitz in u uniformly in (r,x), then* (1.10) *has at most one solution. All finite-dimensional distributions of a (ξ, ψ)-superprocess are defined uniquely.*

THEOREM 1.6. *A (ξ, ψ)-superprocess exists for every function*

$$\psi(r,x;u) = b(r,x)u^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) n(r,x;d\lambda), \tag{1.12}$$

where a positive Borel function $b(r,x)$ and a kernel n from $(S, \mathscr{B}_S)$ to $\mathbb{R}_+$ satisfy the condition

$$b(r,x) \quad \textit{and} \quad \int_0^\infty \lambda \wedge \lambda^2 n(r,x;d\lambda) \quad \textit{are bounded}. \tag{1.13}$$

The family (1.12) contains the functions

$$\psi(u) = \text{const. } u^\alpha, \qquad 1 < \alpha < 2, \tag{1.14}$$

that correspond to $b = 0$ and $n(d\lambda) = \text{const. } \lambda^{-(1+\alpha)}\, d\lambda$.

REMARK 1. Theorem 1.6 can be proved for a wider class of ψ (see [3]). We restrict ourselves to the most important functions.

1.7. *Organization of the paper.* A link between operators V_Q and a BEM system X is provided by a family of transition operators of higher order $V_{Q_1,\dots,Q_n}$. We call it a $\mathbb{V}$-family. Properties of $\mathbb{V}$-families are studied in Section 2. Section 3 is devoted to constructing a BEM system starting from a $\mathbb{V}$-family. BEM systems corresponding to branching particle systems are investigated in Section 4. In Section 5 we prove Theorems 1.5 and 1.6. Theorem 1.6 is proved by a passage to the limit from branching particle systems. The second proof of this theorem, based on Theorem 1.2, is given in Section 6. (This is an adaptation of Fitzsimmons' work [4].)

Theory of superprocesses is supplemented in Sections 7 and 8. In Section 7, we consider superprocesses with parameter sets $\mathbb{O}_1$ and $\mathbb{M}_1$ wider than $\mathbb{O}_0$ and $\mathbb{M}_0 = \mathscr{M}(S)$. This extension is used in Section 8 to treat the time-homogeneous case. In the same section we show how a traditional subject of investigtion—branching measure-valued Markov processes—can be derived from our general model.

2. Transition operators and $\mathbb{V}$-families.

2.1. *Transition operators of higher order.* Suppose that

$$P_\mu \exp[-\langle f_1, X_{Q_1}\rangle - \cdots - \langle f_n, X_{Q_n}\rangle] = \exp[-\langle V_{Q_1,\dots,Q_n}(f_1,\dots,f_n), \mu\rangle] \tag{2.1}$$

for all $\mu \in \mathbb{M}$ and all $f_1,\dots,f_n \in \mathbb{B}$. Then we say that operators $V_{Q_1,\dots,Q_n}$ are the *transition operators of order n for X*. Condition (2.1) is equivalent to the assumption that X is a branching system and that

$$V_{Q_1,\dots,Q_n}(f_1,\dots,f_n)(y) = -\log P_y \exp[-\langle f_1, X_{Q_1}\rangle - \cdots - \langle f_n, X_{Q_n}\rangle], \qquad f_1,\dots,f_n \in \mathbb{B},\ y \in S. \tag{2.2}$$

[For $n = 1$, formulae (2.1)–(2.2) coincide with (1.6)–(1.7).]

We use the following abbreviations. For every finite subset $I = \{Q_1,\dots,Q_n\}$ of $\mathbb{O}$, we put

$$X_I = \{X_{Q_1},\dots,X_{Q_n}\}, \qquad f_I = \{f_1,\dots,f_n\}, \qquad \langle f_I, X_I\rangle = \sum_{i=1}^k \langle f_i, X_{Q_i}\rangle. \tag{2.3}$$

In this notation, formulae (2.2) and (2.1) can be written as

$$V_I(f_I)(y) = -\log P_y e^{-\langle f_I, X_I\rangle} \tag{2.4}$$

and

$$P_\mu e^{-\langle f_I, X_I\rangle} = e^{-\langle V_I(f_I), \mu\rangle}. \tag{2.5}$$

If X satisfies condition 1.3.C, then:

2.1.A. For every $Q_i \in I$, $V_I(f_I) = f_i + V_{I_i}(f_{I_i})$ on Q_i^c, where I_i is the set obtained from I by dropping Q_i.

Indeed,

$$\langle f_I, X_I\rangle = \langle f_i, X_i\rangle + \langle f_{I_i}, X_{I_i}\rangle$$

and $\langle f_i, X_i\rangle = f_i(y)$ P_y-a.s. if $y \in Q_i^c$.

For a branching exit system X, the Markov property 1.3.D is equivalent to:

2.1.B. If $Q \subset Q_i$ for all $Q_i \in I$, then

$$V_Q V_I = V_I. \tag{2.6}$$

PROOF. It follows from (2.5) that

$$e^{-\langle V_Q V_I(f_I), \mu\rangle} = P_\mu e^{-\langle V_I(f_I), X_Q\rangle} = P_\mu P_{X_Q} e^{-\langle f_I, X_I\rangle}. \tag{2.7}$$

If $Q \subset Q_i$ for all $Q_i \in I$, then $\langle f_I, X_I\rangle \in \mathscr{F}_{\supset Q}$ and 1.3.D implies that the right-hand side of (2.7) is equal to

$$P_\mu e^{-\langle f_I, X_I\rangle} = e^{-\langle V_I(f_I), \mu\rangle}.$$

Hence 2.1.B follows from 1.3.D.

To deduce 1.3.D from 2.1.B, it is sufficient to prove (1.5) for

$$Y = e^{-\langle f_I, X_I\rangle}, \qquad Z = e^{-\langle \tilde f_I, X_{\tilde I}\rangle},$$

where $I = \{Q_1, \dots, Q_n\}$, $\widetilde{I} = \{\widetilde{Q}_1, \dots, \widetilde{Q}_m\}$ with $Q_i \subset Q \subset \widetilde{Q}_j$. Note that $YZ \in \mathbb{Z}$. By 1.3.A, the same is true for $YP_{X_Q}Z$. Therefore (1.5) will follow from 1.3.A if we check that it holds for all $\mu = \delta_y$. We use the induction on n. By (2.7), condition 2.1.B implies

$$P_\mu Z = P_\mu P_{X_Q} Z. \tag{2.8}$$

Hence, (1.5) holds for $n = 0$. Suppose it holds for $n-1$. If $y \in Q_i^c$, then, by 1.3.C, $P_y\{Y = e^{-f_i(y)} e^{-Y_i}\} = 1$, where $Y_i = e^{-\langle f_{I_i}, X_{I_i}\rangle}$, and we have

$$P_y YZ = e^{-f_i(y)} P_y Y_i Z = e^{-f_i(y)} P_y(Y_i P_{X_Q} Z) = P_y(YP_{X_Q}Z)$$

by the induction hypothesis. Hence (1.5) holds for δ_y with y not in the intersection Q_I of $Q_i \in I$. For an arbitrary y, by (2.8), $P_y YZ = P_y P_{X_{Q_I}} YZ$. By 1.3.B, X_{Q_I} is concentrated, P_y-a.s., on Q_I^c and therefore

$$P_{Q_I} YZ = P_{X_{Q_I}}(YP_{X_Q}Z).$$

We conclude that

$$P_y YZ = P_y P_{X_{Q_I}}(YP_{X_Q}Z) = P_y(YP_{X_Q}Z). \qquad \square$$

Transition operators of order n can be expressed through transition operators of order $n-1$ by the formulae

$$V_I(f_I) = f_i + V_{I_i}(f_{I_i}) \quad \text{on } Q_i^c \text{ for every } Q_i \in I, \tag{2.9}$$

$$V_I = V_{Q_I} V_I, \quad \text{where } Q_I \text{ is the intersection of all } Q_i \in I. \tag{2.10}$$

Formula (2.9) (equivalent to 2.1.A) defines the values of $V_I(f_I)$ on Q_I^c. Formula (2.10) follows from 2.1.B. By 1.3.B, it provides an expression for all values of $V_I(f_I)$ through its values on Q_I^c.

Conditions (2.9) and (2.10) can be rewritten in the form

$$V_I = V_{Q_I} \widetilde{V}_I, \tag{2.11}$$

where

$$\widetilde{V}_I(f_I) = \begin{cases} f_i + V_{I_i}(f_{I_i}), & \text{on } Q_i^c, \\ 0, & \text{on } Q_I. \end{cases} \tag{2.12}$$

2.2. *Properties of* V_Q. We need the following simple lemma.

LEMMA 2.1. *Let Y be a positive random variable and let $0 \le c \le \infty$. If $Pe^{-\lambda Y} \le e^{-\lambda c}$ for all $\lambda > 0$, then $P\{Y \ge c\} = 1$. If, in addition, $Pe^{-Y} = e^{-c}$, then $P\{Y = c\} = 1$.*

PROOF. If $c = \infty$, then, P-a.s., $e^{-\lambda Y} = 0$ and therefore $Y = \infty$. If $c < \infty$, then $Pe^{-\lambda(Y-c)} \le 1$ and, by Fatou's lemma, $P\{\lim_{\lambda\to\infty} e^{-\lambda(Y-c)}\} \le 1$. Hence, $P\{Y \ge c\} = 1$. The second part of the lemma follows from the first one. □

PROOF OF THEOREM 1.1. (i) Property 1.4.A is obvious. It is clear that 1.3.B implies 1.4.B, 1.3.C implies 1.4.C and 1.3.D implies 1.4.D because 1.4.D is a particular case of 2.1.B.

(ii) If 1.4.B holds, then $V_Q(1_Q) = V_Q(0) = 0$ and therefore $P_y e^{-X_Q(Q)} = 1$, which implies 1.3.B.

(iii) It follows from 1.4.C and (1.6) that, if $\mu(Q) = 0$, then, for all $f \in \mathbb{B}$ and all $\lambda > 0$,

$$P_\mu e^{-\lambda\langle f, X_Q\rangle} = e^{-\lambda\langle f, \mu\rangle}$$

and, by Lemma 2.1,

$$\langle f, X_Q\rangle = \langle f, \mu\rangle, \quad P_\mu\text{-a.s.} \tag{2.13}$$

Since there exists a countable family of $f \in \mathbb{B}$ which separate measures, 1.3.C follows from (2.13).

(iv) Suppose that 1.4.D is satisfied and let $Q \subset Q_i$ for all $Q_i \in I$. Then $Q \subset Q_I$ and (2.11) and 2.1.B imply

$$V_Q V_I = V_Q V_{Q_I} \widetilde{V}_I = V_{Q_I} \widetilde{V}_I = V_I.$$

We proved 2.1.B, which implies 1.3.D. □

2.3. $\mathbb{V}$-*families.* We call a collection of operators V_I a $\mathbb{V}$-*family* if it satisfies conditions (2.9)–(2.10) [equivalent to (2.11)–(2.12)] and 1.4.A. We say that a $\mathbb{V}$-family and a system of random measures correspond to each other if they are connected by formula (2.1).

THEOREM 2.1. *If* V_Q, $Q \in \mathbb{O}$, *satisfy conditions* 1.4.A–1.4.D, *then there exists a* $\mathbb{V}$-*family* $\{V_I\}$ *such that* $V_I = V_Q$ *for* $I = \{Q\}$.

PROOF. Denote by $|I|$ the cardinality of I. For $|I| = 1$, operators V_I are defined. Suppose that V_I, subject to conditions (2.9)–(2.10), are already defined for $|I| < n$. For $|I| = n$, we define V_I by (2.9)–(2.10). This is not contradictory because

$$f_i + V_{I_i}(f_{I_i}) = f_j + V_{I_j}(f_{I_j}) = f_i + f_j + V_{I_{ij}}(f_{I_{ij}}) \quad \text{on } Q_i^c \cap Q_j^c.$$

By 1.4.B it is legitimate to define $V_I(f_I)$ on Q_I by (2.10). $\square$

3. From a $\mathbb{V}$-family to a BEM system.

3.1. *Positive definite and negative semidefinite functions.* First, we prepare some tools. A real-valued function u on an Abelian semigroup G is called *positive definite* if

$$\sum_{i,\,j=1}^{k} t_i t_j u(g_i + g_j) \geq 0$$

for every $n \geq 1$, all $g_1, \dots, g_n \in G$ and all $t_1, \dots, t_n \in \mathbb{R}$. A definition of negative definite functions was given in Section 1.5. We will write $u \in$ PD for positive definite functions and $u \in$ NSD for negative semidefinite functions.

We need the following two results on these classes:

PROPOSITION 3.1. *The classes* PD *and* NSD *are closed under pointwise convergence. Moreover, they are convex cones in the following sense: if* $(A, \mathscr{A}, \eta)$ *is a measure space, if* $u_a \in$ PD (NSD) *for all* $a \in A$ *and if* $u_a(g)$ *is* η-*integrable for all* g, *then*

$$u(g) = \int u_a(g)\eta(da)$$

is also in the class PD (*respectively*, NSD).

PROPOSITION 3.2. *A real-valued function* $v \in$ NSD *if and only if* $u = e^{-\lambda v} \in$ PD *for all* $\lambda > 0$.

The first of these propositions is obvious. The second is proved, for example, in [1], page 74.

We consider positive definite and negative semidefinite real-valued functions on semigroups $G = \mathbb{B}^k$. We also consider such functions with values in $\mathbb{B}$. We say that a function U from G to $\mathbb{B}$ is negative semidefinite if the real-valued function $U(g)(y)$ is negative semidefinite for all $y \in S$. By Proposition 3.1, $\langle U(g), \mu\rangle$ is negative semidefinite for all $\mu \in \mathscr{M}(S)$. Positive definite $\mathbb{B}$-valued functions are defined in a similar way.

3.2. *Laplace functionals of random measures.* Let (X, P) be a random measure on $(S, \mathscr{B}_S)$. Its probability distribution is a measure $\mathscr{P}$ on the space $\mathscr{M}(S)$ defined on the σ-algebra generated by functions $F_B(\mu) = \mu(B)$, $B \in \mathscr{B}_S$. The corresponding Laplace functional is defined on $f \in \mathbb{B}$ by the formula

$$L_{\mathscr{P}}(f) = Pe^{\langle f, X\rangle} = \int_{\mathscr{M}(S)} e^{-\langle f, \nu\rangle}\, \mathscr{P}(d\nu).$$

THEOREM 3.1. *A functional L on $\mathbb{B}$ is the Laplace functional of a random measure if and only if it is positive definite and the following holds*:

3.2.A.

$$L(f_n) \to 1 \quad \textit{as } f_n \downarrow 0.$$

A proof of this theorem can be found in [4], A6.

COROLLARY 3.1. *Let $\mathscr{P}_k$ be probability measures on $\mathscr{M}(S)$ and let L_k be the Laplace functional of $\mathscr{P}_k$. If $L_k(f) \to L(f)$ for all $f \in \mathbb{B}$ and if L satisfies* 3.2.A, *then L is the Laplace functional of a probability measure $\mathscr{P}$ on $\mathscr{M}(S)$.*

Indeed, L satisfies all conditions of Theorem 3.1.

The Laplace functional of a probability measure $\mathscr{P}$ on $\mathscr{M}(S)^n$ is defined by the formula

$$L_{\mathscr{P}}(f_1, \dots, f_n) = \int e^{-\langle f_1, \nu_1\rangle - \cdots - \langle f_n, \nu_n\rangle} \mathscr{P}(d\nu_1, \dots, d\nu_n). \tag{3.1}$$

By identifying $\mathscr{M}(S)^n$ with the space of finite measures on the union of n copies of S, we get a multivariant version of Theorem 3.1 and its corollary:

THEOREM 3.2. *A functional $L(f_1, \dots, f_n)$ on $\mathbb{B}^n$ is the Laplace functional of a probability measure on $\mathscr{M}(S)^n$ if and only if it is positive definite and*

3.2.1.

$$L(f_1^k, \dots, f_n^k) \to 1 \quad \textit{as } f_1^k \downarrow 0, \dots, f_n^k \downarrow 0.$$

COROLLARY 3.2. *Let L_k be the Laplace functional of a probability measure $\mathscr{P}_k$ on $\mathscr{M}(S)^n$. If $L_k(f_1, \dots, f_n) \to L(f_1, \dots, f_n)$ for all $f_1, \dots, f_n \in \mathbb{B}$ and if* 3.2.1 *holds for L, then L is the Laplace functional of a probability measure $\mathscr{P}$ on $\mathscr{M}(S)^n$.*

3.3. *Constructing a BEM system starting from a $\mathbb{V}$-family.*

THEOREM 3.3. *A $\mathbb{V}$-family $\mathbb{V} = \{V_I\}$ corresponds to a BEM system if and only if V_Q satisfy conditions* 1.4.A–1.4.D *and if, for every I, $V_I(f_I^k) \to 0$ as $f_I^k \downarrow 0$ and the following holds*:

3.3.A. *V_I is negative semidefinite.*

PROOF. (i) By Theorem 1.1, the transition operator of a BEM system satisfies conditions 1.4.A–1.4.D. It follows from (2.2) [or (2.4)] that $L_I = e^{-V_I}$ is a Laplace functional of a probability measure and therefore the properties of V_I stated in the theorem follow from Theorem 3.2 and Proposition 3.2.

(ii) If V_I satisfy 3.3.A and if $|I| = n$, then, by Proposition 3.1, for every $\mu \in \mathscr{M}(S)^n$, $\langle V_I(f_I), \mu\rangle$ is negative semidefinite and, by Proposition 3.2, $L_{\mu,I}(f_I) = e^{\langle V_I(f_I),\mu\rangle}$ is positive definite. By Theorem 3.2, $L_{\mu,I}$ is the Laplace functional of a probability measure on $\mathscr{M}(S)^n$. These measures satisfy consistency conditions and, by Kolmogorov's theorem, they are probability distributions of X_I relative to P_μ for a system X of random measures (X_Q, P_μ). By Theorem 1.1, X is a BEM system. □

PROOF OF THEOREM 1.2. By Theorem 3.3, it is sufficient to demonstrate that, if operators V_Q satisfy 1.4.A–1.4.D, then condition 3.3.A follows from 1.5.A.

By taking an identity map from $\mathbb{B}$ to $\mathbb{B}$ for U in 1.5.A, we conclude that $V_Q \in \text{NSD}$. Hence condition 3.3.A holds for $|I| = 1$. Let $\widetilde{V}_I$ be given by (2.12). Clearly, if V_{I_i} satisfy 3.3.A, then so does $\widetilde{V}_I$. By (2.10) and 1.5.A, the same is true for V_I. By induction, 3.3.A holds for all I. □

3.4. *Proof of Theorem* 1.3. First, we establish that, for every I, operators V_I^k satisfy conditions similar to the conditions imposed in Theorem 1.3 on V_Q^k.

Put $\|f\| = \max\{\|f_1\|, \dots, \|f_n\|\}$ for $f = (f_1, \dots, f_n) \in \mathbb{B}^n$.

LEMMA 3.1. *Suppose V^k is a sequence of $\mathbb{V}$-families and let V_Q^k satisfy the conditions of Theorem* 1.3. *Then the following hold*:

(a) *a limit $V_I(f)$ of $V_I^k(f)$ exists for every $I = (Q_1, \dots, Q_n) \subset \mathbb{O}$ and every $f = (f_1, \dots, f_n) \in \mathbb{B}^n$*;

(b) *the convergence is uniform on every set $\mathbb{B}_c^n$*;

(c) *$V_I(f)$ satisfies the Lipschitz condition on every $\mathbb{B}_c^n$.*

PROOF. By (2.11)–(2.12),

$$V_I^k = V_{Q_I}^k \widetilde{V}_I^k, \tag{3.2}$$

where

$$\widetilde{V}_I^k = \begin{cases} f_i + \widetilde{V}_{I_i}^k(f_{I_i}), & \text{on } \widetilde{Q}_i^c, \\ 0, & \text{on } Q_I, \end{cases} \tag{3.3}$$

and therefore, for all k, m,

$$\left|\widetilde{V}_I^k(f_I) - \widetilde{V}_I^m(f_I)\right| = \begin{cases} \left|\widetilde{V}_{I_i}^k(f_{I_i}) - \widetilde{V}_{I_i}^m(f_{I_i})\right|, & \text{on } Q_i^c, \\ 0, & \text{on } Q_I. \end{cases} \tag{3.4}$$

If conditions (a)–(c) hold for $\widetilde{V}_{I_i}^k$, then, by (3.4), they hold for $\widetilde{V}_I^k$.

Because of (3.2), to finish the proof it is sufficient to show that, if conditions (a)–(c) hold for operators V^k from $\mathbb{B}$ to $\mathbb{B}$ and for operators $\widetilde{V}^k$ from $\mathbb{B}^n$ to $\mathbb{B}$, then they hold for the products $V^k\widetilde{V}^k$. Let

$$V = \lim V^k, \qquad \widetilde{V} = \lim \widetilde{V}^k.$$

By (b),

$$\begin{aligned} \|V^k(f) - V(f)\| &\le \varepsilon_k(c) \quad \text{for } f \in \mathbb{B}_c, \\ \|\widetilde{V}^k(\tilde{f}) - \widetilde{V}(\tilde{f})\| &\le \tilde{\varepsilon}_k(c) \quad \text{for } \tilde{f} \in \mathbb{B}_c^n \end{aligned} \tag{3.5}$$

with $\varepsilon_k(c) + \tilde{\varepsilon}_k(c) \to 0$ as $k \to \infty$. By (c), there exist constants $a(c)$ and $\tilde{a}(c)$ such that

$$\begin{aligned} \|V(f) - V(g)\| &\le a(c)\|f - g\| \quad \text{for all } f, g \in \mathbb{B}_c, \\ \|\widetilde{V}(\tilde{f}) - \widetilde{V}(\tilde{g})\| &\le \tilde{a}(c)\|\tilde{f} - \tilde{g}\| \quad \text{for all } \tilde{f}, \tilde{g} \in \mathbb{B}_c^k. \end{aligned} \tag{3.6}$$

By taking $g = \tilde{g} = 0$, we get

$$\|V(f)\| \le ca(c) \quad \text{for } f \in \mathbb{B}_c; \qquad \|\widetilde{V}(\tilde{f})\| \le c\tilde{a}(c) \quad for \tilde{f} \in \mathbb{B}_c^k. \tag{3.7}$$

Note that

$$\left|V^k[\widetilde{V}^k(\tilde{f})] - V[\widetilde{V}(\tilde{f})]\right| \le q(k) + h(k),$$

where

$$q(k) = \left\|V^k\big[\widetilde{V}^k(\tilde{f})\big] - V\big[\widetilde{V}^k(\tilde{f})\big]\right\|$$

and

$$h(k) = \left|V\big[\widetilde{V}^k(\tilde{f})\big] - V\big[\widetilde{V}(\tilde{f})\big]\right|.$$

By (3.7) and (b), for all $\tilde{f} \in \mathbb{B}_c^n$ and for all sufficiently large k, $\|\widetilde{V}^k(\tilde{f})\| \le \tilde{c}_1 = c\tilde{a}(c) + 1$ and, by (3.5), $q(k) \le \varepsilon_k(\tilde{c}_1)$. By (3.6) and (3.5),

$$h(k) \le a(\tilde{c}_1)\|\widetilde{V}^k(\tilde{f}) - \widetilde{V}(\tilde{f})\| \le a(\tilde{c}_1)\tilde{\varepsilon}_k(c).$$

Therefore $V^k\widetilde{V}^k$ satisfies conditions (a) and (b). It satisfies (c) because

$$|V[\widetilde{V}(\tilde{f})] - V[\widetilde{V}(\tilde{g})]\| \le a(\tilde{c}_1)\|\widetilde{V}(\tilde{f}) - \widetilde{V}(\tilde{g})\| \le a(\tilde{c}_1)\tilde{a}(c)\|\tilde{f} - \tilde{g}\|. \qquad \square$$

PROOF OF THEOREM 1.3. Clearly, operators V_I constructed in Lemma 3.1 form a $\mathbb{V}$-family. By Theorem 3.3, families V_I^k have properties 1.4.A–1.4.D and 3.3.A, which implies analogous properties for V_I. Hence, V_I correspond to a BEM system X. Clearly, V_Q are the transition operators for X. $\square$

4. BEM systems corresponding to branching particle systems.

4.1. *Evaluation of the transition operators.* Recall that, according to Section 1.2, a branching particle system is determined by a right continuous strong Markov process $\xi = (\xi_t, \Pi_{r,x})$, an offspring generating function φ and a parameter k defining the lifetime probability distribution. If X is an associated BEM system, then

$$V_Q(f) = -\log w,$$

where

$$w(r, x) = P_{r,x} e^{-\langle f, X_Q \rangle}. \tag{4.1}$$

The principles formulated in Section 1.2 imply

$$w(r, x) = \Pi_{r,x}\left[e^{-k(\tau-r)} e^{-f(\tau, \xi_\tau)} + k\int_r^\tau e^{-k(s-r)}\, ds \varphi(s, \xi_s; w(s, \xi_s))\right], \tag{4.2}$$

where τ is the first exit time of (t, ξ_t) from Q. The first term in the brackets corresponds to the case when the particle started the process is still alive at time τ, and the second term corresponds to the case when it dies at time $s \in (r, \tau)$.

We simplify (4.2) by using the following:

LEMMA 4.1. *If*

$$w(r, x) = \Pi_{r,x}\left[e^{-k(\tau-r)} u(\tau, \xi_\tau) + \int_r^\tau e^{-k(s-r)} v(s, \xi_s)\, ds\right], \tag{4.3}$$

then

$$w(r, x) + \Pi_{r,x} \int_r^\tau kw(s, \xi_s)\, ds = \Pi_{r,x}\left[u(\tau, \xi_\tau) + \int_r^\tau v(s, \xi_s)\, ds\right]. \tag{4.4}$$

PROOF. Note that

$$H(r, t) = e^{-k(t-r)}$$

satisfies the relation

$$k\int_r^t H(s, t)\, ds = 1 - H(r, t) \tag{4.5}$$

and that

$$w(r, x) = \Pi_{r,x}(Y_r + Z_r), \tag{4.6}$$

where

$$Y_s = H(s, \tau) u(\tau, \xi_\tau),$$

$$Z_s = \int_s^\tau H(s, t) v(t, \xi_t)\, dt.$$

By (4.6) and Fubini's theorem,

$$\Pi_{r,x}\int_r^\tau kw(s,\xi_s)\,ds=\int_r^\infty k\Pi_{r,x}1_{s<\tau}\Pi_{s,\xi_s}(Y_s+Z_s)\,ds.$$

By the Markov property,

$$\Pi_{r,x}1_{s<\tau}\Pi_{s,\xi_s}(Y_s+Z_s)=\Pi_{r,x}1_{s<\tau}(Y_s+Z_s)$$

and therefore

$$w(r,x)+\Pi_{r,x}\int_r^\tau kw(s,\xi_s)\,ds=\Pi_{r,x}(I_1+I_2), \tag{4.7}$$

where

$$I_1=H(r,\tau)u(\tau,\xi_\tau)+k\int_r^\tau Y_s\,ds \quad\text{and}\quad I_2=\int_r^\tau[H(r,s)v(s,\xi_s)+kZ_s]\,ds.$$

By (4.5) and Fubini's theorem

$$I_1=u(\tau,\xi_\tau),\qquad I_2=\int_r^\tau v(t,\xi_t)\,dt,$$

and (4.4) follows from (4.7). □

To prove Theorem 1.4, it is sufficient to apply Lemma 4.1 to $u(s,x)=e^{-f(s,x)}$ and $v(s,x)=k\varphi(s,x;w(s,x))$.

4.2. *Heuristic passage to the limit.* We considered a transformed system of random measures X^β described in Section 1.6. Its transition operators related to the transition operators of X by the formula $V_Q^\beta(f)=V_Q(\beta f)/\beta$. The equation (1.9) implies that, for every $f\in\mathbb{B}$, function $v^\beta=V_Q^\beta(f)$ is a solution of

$$e^{-\beta v^\beta(r,x)}=\Pi_{r,x}\left[\int_r^\tau k\Phi\big(s,\xi_s;e^{-\beta v^\beta(s,\xi_s)}\big)\,ds+e^{-\beta f(\tau,\xi_\tau)}\right]. \tag{4.8}$$

Note that (4.8) is equivalent to

$$u^\beta(r,x)+\Pi_{r,x}\int_r^\tau \psi^\beta\big(s,\xi_s;u^\beta(s,\xi_s)\big)\,ds=\Pi_{r,x}f^\beta(\tau,\xi_\tau), \tag{4.9}$$

where

$$\begin{aligned} u^\beta&=[1-e^{-\beta v^\beta}]/\beta,\quad f^\beta=[1-e^{-\beta f}]/\beta,\\ \psi^\beta(r,x;u)&=[\varphi^\beta(r,x;1-\beta u)-1+\beta u]k^\beta/\beta\quad\text{for }\beta u\le 1.\end{aligned} \tag{4.10}$$

[We assume that φ and k depend on β. Since $\beta u^\beta=1-e^{-\beta v^\beta}\le 1$, the value $\varphi^\beta(r,x;1-\beta u^\beta)$ is defined.]

Note that $F^\beta\to f$ as $\beta\to 0$. If $\psi^\beta\to\psi$, then we expect that $u^\beta\to u$, where u is a solution of (1.10). Equations (4.9) and (1.10) can be rewritten in the form

$$u^\beta+G_Q\psi^\beta(u^\beta)=K_Qf^\beta \tag{4.11}$$

and

$$u + G_Q\psi(u) = K_Q f, \tag{4.12}$$

where $\psi(u) = \psi(r, x; u(r, x))$, $\psi^\beta(u^\beta) = \psi^\beta(r, x; u^\beta(r, x))$ and where operators K_Q and G_Q are defined by the formulae

$$K_Q f(r, x) = \Pi_{r,x} f(\tau, \xi_\tau), \tag{4.13}$$

$$G_Q\rho(r, x) = \Pi_{r,x} \int_r^\tau \rho(t, \xi_t)\,dt. \tag{4.14}$$

5. Superprocesses.

5.1. *Two lemmas.* To prove Theorems 1.5 and 1.6, we need some preparations.

Note that the condition $Q \in \mathbb{O}_0$ is equivalent to the following condition:

5.1.A. There exists a constant N such that $\tau - r \le N$ for all paths of ξ starting from $(r, x) \in Q$.

The local Lipschitz condition in u uniformly in (r, x) means:

5.1.B. For every $c > 0$, there exists a constant $q(c)$ such that

$$\begin{gathered}\psi(r, x; u_1) - \psi(r, x; u_2)| \le q(c)|u_1 - u_2| \\ \text{for all } (r, x) \in S,\ u_1, u_2 \in [0, c].\end{gathered} \tag{5.1}$$

The following lemma is a modification of Gronwall's inequality.

LEMMA 5.1. *Let τ be the first exit time from $Q \in \mathbb{O}_0$. If a positive bounded function $h(r, x)$ satisfies the condition*

$$h(r, x) \le a + q\Pi_{r,x} \int_r^\tau h(s, \xi_s)\,ds \quad \textit{in } Q, \tag{5.2}$$

then

$$h(r, x) \le a\Pi_{r,x} e^{q(\tau - r)} \quad \textit{in } Q. \tag{5.3}$$

PROOF. Suppose that $h \le A$. We prove, by induction, that

$$h(r, x) \le \Pi_{r,x} Y_n(r), \tag{5.4}$$

where

$$Y_n(r) = a\sum_{k=0}^{n-1} q^k \frac{(\tau - r)^k}{k!} + Aq^n \frac{(\tau - r)^n}{n!}.$$

Clearly, (5.4) holds for $n = 1$. If it is true for n, then, by (5.2),

$$h(r, x) \le a + q\Pi_{r,x} \int_r^\tau \Pi_{s,\xi_s} Y_n(s)\,ds \quad \text{in } Q. \tag{5.5}$$

By the Markov property (1.1),

$$\Pi_{r,x} 1_{\tau>s} \Pi_{s,\xi_s} Y_n(s) = \Pi_{r,x} 1_{\tau>s} Y_n(s) \quad \text{for all } (r,x) \in Q$$

because $\{\tau > s\} \subset \{\tau = \tau_s\}$, where τ_s is the first after-s exit time from Q and τ_s is $\mathscr{F}_{\geq s}$-measurable. Hence, the right-hand side of (5.5) is equal to

$$a + q\Pi_{r,x} \int_r^\tau Y_n(s)\, ds = Y_{n+1}(r)$$

and (5.4) holds for $n+1$. Bound (5.3) follows from (5.4) and 5.1.A. $\square$

LEMMA 5.2. *Suppose that $Q \in \mathbb{O}_0$ and ψ satisfies condition* 5.1.B. *If* (4.12) *holds for u and if $\tilde{u} + G_Q\psi(\tilde{u}) = K_Q\tilde{f}$, then*

$$\|u - \tilde{u}\| \leq e^{q(c)N} \|f - \tilde{f}\| \quad \textit{for all } f, \tilde{f} \in \mathbb{B}_c, \tag{5.6}$$

where N is the constant in 5.1.A.

Suppose $\psi(z, 0)$ is bounded and $u_\beta + G_Q\psi(u_\beta) = K_Q f_\beta$. If $f \in \mathbb{B}_c$ and $\|f_\beta - f\| \to 0$ as $\beta \downarrow 0$, then there exists a solution u of (4.12) *such that*

$$\|u_\beta - u\| \leq e^{q(2c)N} \|f_\beta - f\| \quad \textit{for all sufficiently small } \beta. \tag{5.7}$$

PROOF. By (4.12), $\|u\| \leq \|f\|$, $\|\tilde{u}\| \leq \|\tilde{f}\|$ and

$$u - \tilde{u} = K_Q(f - \tilde{f}).$$

Put $h = |u - \tilde{u}|$. By (5.1), $|\tilde{\psi}(\tilde{u}) - \psi(u)| \leq \alpha(c) + q(c)h$ and therefore

$$h \leq \|f - \tilde{f}\| + q(c)G_Q h$$

and (5.6) follows from Gronwall's inequality (5.2).

If $f \in \mathbb{B}_c$, then, for all sufficiently small β, $f_\beta \in \mathbb{B}_{2c}$ and, by (4.2),

$$\|u_\beta - u_{\tilde{\beta}}\| \leq e^{q(2c)N} \|f_\beta - f_{\tilde{\beta}}\|,$$

which implies the existence of the limit $u = \lim u_\beta$ and the bound (5.7). By (5.1), $\psi(u_\beta) \leq \psi(0) + 2q(c)c$ and, by the dominated convergence theorem, u satisfies (4.12). $\square$

PROOF OF THEOREM 1.5. If u and $\tilde{u}$ are solutions of (1.10), then $u = \tilde{u}$ by (5.6).

If V_Q and $\widetilde{V}_Q$ are the transition operators of two (ξ, ψ)-superdiffusions, then $V_Q(f) = \widetilde{V}_Q(f)$ for all $f \in \mathbb{B}$. By the recursive formulae (2.11) and (2.12), the transition operators of higher order also coincide, which implies the second part of the theorem. $\square$

5.2. *Superprocesses as limits of branching particle systems.* Put

$$e(\lambda) = e^{-\lambda} - 1 + \lambda. \tag{5.8}$$

Since, for $u > 0$, $0 < e'(\lambda) = 1 - e^{-\lambda} < 1 \wedge \lambda$, we have

$$0 \le e(\lambda) \le \lambda \wedge \lambda^2 \tag{5.9}$$

and

$$|e(\lambda u_2) - e(\lambda u_1)| \le 1 \vee c(\lambda \wedge \lambda^2)|u_2 - u_1| \quad \text{for all } u_1, u_2 \in [0, c]. \tag{5.10}$$

The bounds (5.9) imply

$$0 \le 1 - \lambda + \lambda^2/2 - e^{-\lambda} \le \lambda^3 \quad \text{for all } \lambda > 0. \tag{5.11}$$

PROOF OF THEOREM 1.6. (i) We choose parameters φ_β, k_β of a branching particle system to make ψ_β given by (4.10) independent of β. To this end we put

$$\begin{aligned} k_\beta &= \frac{\gamma}{\beta}, \\ \varphi_\beta(z; u) &= u + \frac{\beta^2}{\gamma}\psi\left(z; \frac{1-u}{\beta}\right) \quad \text{for } 0 \le u \le 1, \end{aligned} \tag{5.12}$$

where γ is a strictly positive constant. We need to show that φ_β is a generating function. To simplify formulae, we drop arguments z. Clearly, $\varphi_\beta(1) = 1$. We have

$$\varphi_\beta(u) = \sum_0^\infty p_k^\beta u^k,$$

where

$$\begin{aligned} p_0^\beta &= \frac{\beta^2}{\gamma}\psi\left(\frac{1}{\beta}\right), \\ p_1^\beta &= \frac{1}{\gamma}\left[\gamma - 2b - \beta\int_0^\infty \lambda(1 - e^{-\lambda/\beta})n(d\lambda)\right], \\ p_2^\beta &= \frac{b}{\gamma} + \frac{1}{\gamma}\int_0^\infty e^{-\lambda/\beta}\lambda^2 n(d\lambda), \\ p_k^\beta &= \frac{\beta^2}{k!\gamma}\int_0^\infty e^{-\lambda/\beta}\left(\frac{\lambda}{\beta}\right)^k n(d\lambda) \quad \text{for } k > 2; \end{aligned}$$

p_0^β and p_k^β are positive for all $\beta > 0$. Function p_1^β is positive for $0 < \beta \le 1$ if γ is an upper bound of

$$2b + \int_0^\infty \lambda \wedge \lambda^2 n(d\lambda).$$

(ii) We claim that there exists a solution u of (4.12) and a function $a(c)$ such that

$$\|u - v_\beta\| \le \beta a(c) \quad \text{for all } f \in \mathbb{B}_c \text{ and all sufficiently small } \beta. \tag{5.13}$$

If A is an upper bound for the functions (1.13), then, by (5.10), ψ satisfies the condition (5.1) with $q(c) = 3A(1 \vee c)$.

Suppose $f \in \mathbb{B}_c$. Then, by (4.10) and (5.9), $f - f_\beta = e(\beta f)/\beta \leq \beta f^2 \leq \beta c^2$ and, by 5.2, there exists u such that, for sufficiently small β,

$$\|u_\beta - u\| \leq e^{q(2c)N} \beta c^2. \tag{5.14}$$

By (4.10), $v_\beta = -\beta^{-1} \log(1 - \beta u_\beta)$ and

$$v_\beta - u_\beta = F_\beta(u_\beta),$$

where $F_\beta(t) = -\beta^{-1} \log(1 - \beta t) - t$. Note that $F_\beta(0) = 0$ and, for $0 < \beta t < 1/2$,

$$0 < F'_\beta(t) = \beta t(1 - \beta t)^{-1} \leq 2\beta t,$$

which implies $0 < F_\beta(t) < \beta t^2$. We have $0 \leq f_\beta \leq f$ and $u_\beta \leq K_Q f$. Therefore $u_\beta \in \mathbb{B}_c$ and

$$|v_\beta - u_\beta| \leq \beta c^2 \quad \text{for } 0 < \beta < 1/(2c). \tag{5.15}$$

It follows from (5.14) and (5.15) that (5.13) holds with $a(c) = c^2(e^{q(c)N} + 1)$.

(iii) We conclude from (ii) that the limit V_Q of operators V_Q^β satisfies the Lipschitz condition on each set $\mathbb{B}_c$ and that $V_Q^\beta \xrightarrow{u} V_Q$. By Theorem 1.3, there exists a BEM system X with transition operators V_Q. Since $u = V_Q(f)$ satisfies (4.12), this is a (ξ, ψ)-superprocess. □

6. Direct construction of superprocesses.

6.1. *Analytic definition of operators V_Q.*

THEOREM 6.1. *Suppose that $Q \in \mathbb{O}_0$ and that ψ satisfies the condition* 5.1.B *and the following conditions*:

6.1.A. $\psi(z, 0) = 0$ *for all* z;

6.1.B. ψ *is monotone increasing in* t, *that is*, $\psi(z, t_1) \leq \psi(z, t_2)$ *for all* $z \in S$ *and all* $t_1 < t_2 \in \mathbb{R}_+$.

Then (4.12) *has a unique solution for every* $f \in \mathbb{B}$. *We denote it by* $V_Q(f)$.

PROOF. (We use the so-called monotone iteration scheme (cf., e.g., [5].) By Theorem 1.5, equation (4.12) can have no more than one solution.

Suppose that $f \in \mathbb{B}_c$. Fix a constant $k \geq q(c)$, where $q(c)$ is defined in 5.1.B, and put, for every $u \geq 0$,

$$T(u) = \Pi_{r,x}\left[e^{-k(\tau - r)} f(\tau, \xi_\tau) + \int_r^\tau e^{-k(s-r)} \Phi\big(s, \xi_s; u(s, \xi_s)\big)\, ds\right], \tag{6.1}$$

where $\Phi(u) = ku - \psi(u)$. (We do not indicate explicitly the dependence on T of k and f.) The key step is to prove that the sequence

$$\begin{aligned} &u_0 = 0, \\ &u_n = T(u_{n-1}) \quad \text{for } n = 1, 2, \dots \end{aligned} \tag{6.2}$$

is monotone increasing and bounded. Clearly, its limit u is a bounded solution of

$$u(r,x) = \Pi_{r,x}\left[e^{-k(\tau-r)} f(\tau,\xi_\tau) + \int_r^\tau e^{-k(s-r)}\Phi\big(s,\xi_s;u(s,\xi_s)\big)\,ds\right]. \tag{6.3}$$

By Lemma 4.1, (6.3) implies

$$u(r,x) + k\Pi_{r,x}\int_r^\tau u(s,\xi_s)\,ds = \Pi_{r,x}\left[f(\tau,\xi_\tau) + \int_r^\tau \Phi\big(s,\xi_s;u(s,\xi_s)\big)\,ds\right],$$

which is equivalent to (4.12).

We prove that the following hold:

(a) $T(v_1) \le T(v_2)$ if $0 \le v_1 \le v_2 \le c$ in Q;
(b) $T(c) \le c$.

To get (a), we note that

$$\Phi(t_2) - \Phi(t_1) = k(t_2 - t_1) - [\psi(t_2) - \psi(t_1)] \ge (t_2 - t_1)(k - q(c)) \ge 0.$$

Since $\psi \ge 0$, $\Phi(u) \le ku$ and therefore

$$T(c) \le \Pi_{r,x}\left[ce^{-k(\tau-r)} + ck\int_r^\tau e^{-k(s-r)}\,ds\right].$$

Since $e^{-k(\tau-r)} + k\int_r^\tau e^{-k(s-r)}k\,ds = 1$, this implies (b).

By 6.1.A, $u_1 = T(0) \ge 0$. By (a) and (b), $u_1 = T(0) \le T(c) \le c$. We use (a) and (b) to prove, by induction on n, that $0 = u_0 \le \cdots \le u_n \le c$. □

6.2. *Properties of* V_Q. We claim that the following hold:

6.2.A. If $f \le \tilde f$, then $V_Q(f) \le V_Q(\tilde f)$.

6.2.B. If $Q \subset \widetilde Q$ and if $f = 0$ on $\widetilde Q$, then $V_Q(f) \le V_{\widetilde Q}(f)$.

6.2.C. If $f_n \uparrow f$, then $V_Q(f_n) \uparrow V_Q(f)$.

To prove 6.2.A and 6.2.B, we indicate explicitly the dependence of operator (6.1) on k, Q and f and we note that $0 \le f \le \tilde f \le c$ and, if $k > q(c)$, then $T(k,Q,f;u) \le T(k,Q,\tilde f;u)$ for every function $0 \le u \le c$. This implies 6.2.A. If $Q \subset \widetilde Q$, then the first exit time $\tilde\tau$ from $\widetilde Q$ is greater than or equal to τ. If $\eta_\tau = (\tau,\xi_\tau) \in \widetilde Q$, then $f(\eta_{\tilde\tau}) = 0$, and if $\eta_\tau \notin \widetilde Q$, then $\tilde\tau = \tau$. In both cases, $e^{-k(\tau-r)}f(\eta_\tau) = e^{-k(\tilde\tau-r)}f(\eta_{\tilde\tau})$. If $k > q(c)$ and $0 \le u \le c$, then $T(k,\widetilde Q,f;u) \ge T(k,Q,f;u)$, which implies 6.2.B.

Suppose that $f_n \uparrow f$ and let $u_n = V_Q(f_n)$. By 6.2.A, $u_n \uparrow u$. By passing to the limit in the equation $u_n + G_Q\psi(u_n) = K_Q f_n$, we get $u + G_Q\psi(u) = K_Q f$. Hence $u = V_Q(f)$, which proves 6.2.C.

6.3. *An alternative construction of superprocesses.* We deduce a slightly weaker version of Theorem 1.6 by a method suggested by Fitzsimmons (see [4]).

THEOREM 6.2. *A (ξ, ψ)-superprocess exists for function ψ given by* (1.12) *if b and n satisfy condition* (1.13) *and an additional assumption*,

$$\sup_z \int_0^\beta \lambda^2 n(z; d\lambda) \to 0 \quad \text{as } \beta \downarrow 0. \tag{6.4}$$

REMARK. Condition (1.13) implies pointwise but not uniform convergence of $\int_0^\beta \lambda^2 n(z; d\lambda)$ to 0 as $\beta \downarrow 0$.

We need the following lemma.

LEMMA 6.1. *If u is a solution of* (4.12) *and if Q' is an open subset of Q, then*

$$u + G_{Q'}\psi(u) = K_{Q'}u. \tag{6.5}$$

PROOF. By the strong Markov property (1.1), $K_{Q'}K_Q = K_Q$ and $G_Q = G_{Q'} + K_{Q'}G_Q$ and therefore

$$u+G_{Q'}\psi(u) = u+G_Q\psi(u)-K_{Q'}G_Q\psi(u) = K_{Q'}(K_Q f - G_Q\psi(u)) = K_{Q'}u. \quad \square$$

PROOF OF THEOREM 6.2. (i) Operators V_Q defined in Theorem 6.1 satisfy conditions of Theorem (1.1). Indeed, by (4.12), $V_Q(f) \le K_Q f$, which implies 1.4.A. Properties 1.4.B and 1.4.C also follow easily from (4.12). Let us prove 1.4.D. Suppose $Q \subset \widetilde{Q} \in \mathbb{O}_0$. By Lemma 6.1, $v = V_{\widetilde{Q}}(f)$ satisfies the equation $v + G_Q\psi(v) = K_Q v$. On the other hand, $u = V_Q(v)$ is a solution of the equation $u + G_Q\psi(u) = K_Q v$. The equality $u = v$ follows from Lemma 5.2.

We claim that operators V_Q satisfy condition 1.5.A if the following holds:

6.3.A. There exists $k > 0$ such that $ku(f) - \psi(\cdot; u(f))$ is an N-function from $\mathbb{B}$ to $\mathbb{B}$ for every real-valued N-function $u(f)$ on $\mathbb{B}$.

Indeed, let T be the operator defined by (6.1). It follows from 6.3.A that, for all sufficiently large k, $\Phi(u(f))$ belongs to the class N if $u(f)$ is an N-function and, by Proposition 3.1, operator T preserves the class N. Therefore $V_Q(f)$, which is the limit of $T^n(f)$, has the same property.

By Theorem 1.2, V_Q are the transition operators of a BEM system X and, since $V_Q(f)$ is a solution of (4.12), X is a (ξ, ψ)-superprocess.

(ii) Condition 6.3.A holds for ψ given by (1.12) under an extra assumption,

$$b = 0, \qquad m(z) = \int_0^\infty \lambda n(z, d\lambda) \quad \text{is bounded.} \tag{6.6}$$

Indeed,

$$ku - \psi(u) = \int_0^\infty (1 - e^{-\lambda u}) n(d\lambda) + (k - m)u.$$

If $u \in N$, then $1 - e^{-\lambda u}$ belongs to N by Proposition 3.2, and $ku - \psi(u)$ is an N-function if $k > m(z)$ for all z.

(iii) To eliminate the side condition 6.3.A, we approximate ψ given by (1.12) by functions

$$\psi_\beta(u) = \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\ n_\beta(d\lambda),$$

where $0 < \beta < 1$ and

$$n_\beta(d\lambda) = 1_{\lambda > \beta} n(d\lambda) + 2b\beta^{-2}\delta_\beta.$$

Note that ψ_β satisfies (1.13). It satisfies (6.6) because

$$\int_0^\infty \lambda n_\beta(d\lambda) \le \beta^{-1} \int_0^\infty \lambda \wedge \lambda^2 n(d\lambda) + 2b/\beta.$$

Let V_Q^β be the transition operators of the (ξ, ψ_β)-superprocess. We demonstrated in the proof of Theorem 1.6 that V_Q satisfies the Lipschitz condition on each set $\mathbb{B}_c$. By Theorem 1.3, to prove the existence of a (ξ, ψ)-superprocess, it is sufficient to show that $V_Q^\beta \xrightarrow{u} V_Q$. We have

$$\psi(u) - \psi_\beta(u) = 2bR_\beta(u),$$

where

$$R_\beta(u) = \int_0^\beta (e^{-\lambda u} - 1 - \lambda u) n(d\lambda) + 2b\beta^{-2}[1 - \beta u + (\beta u)^2/2 - e^{-\beta u}].$$

By using the bounds (2.1) and (5.11), we get

$$|R_\beta(u)| \le u^2 \int_0^\beta \lambda \wedge \lambda^2 n(d\lambda) + 2b\beta u^3.$$

By conditions (1.13) and (6.4), ψ_β converges to ψ uniformly on each set $S \times [0, c]$. It follows from Lemma 5.2 that $V_Q^\beta \xrightarrow{u} V_Q$. $\square$

7. Superprocesses with extended parameter sets.

7.1. *Some properties of BEM systems.* Properties stated in Theorem 7.1 will be used in subsequent sections and they are also of independent interest.

THEOREM 7.1. *Suppose that* $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}, \mu \in \mathbb{M}$, *is a BEM system and let* $Q_1 \subset Q_2$ *be elements of* $\mathbb{O}$. *Then the following hold*:

7.1.A.

$$\{X_{Q_1} = 0\} \subset \{X_{Q_2} = 0\} \quad a.s.$$

(*Writing "a.s." means "almost sure with respect to all* $P_\mu, \mu \in \mathbb{M}$.)

7.1.B. *For every $\mu \in \mathbb{M}$ and every bounded measurable function f on $\mathbb{M} \times \mathbb{M}$,*

$$P_\mu f(X_{Q_1}, X_{Q_2}) = P_\mu F(X_{Q_1}),$$

where

$$F(\nu) = P_\nu f(\nu, X_{Q_2}).$$

7.1.C. *If $0 \le \varphi_1 \le \varphi_2$ and $\varphi_2 = 0$ on Q_2, then*

$$\langle \varphi_1, X_{Q_1} \rangle \le \langle \varphi_2, X_{Q_2} \rangle \quad a.s.$$

In particular, if $\Gamma \subset Q_2^c$, then $X_{Q_1}(\Gamma) \le X_{Q_2}(\Gamma)$ a.s.

PROOF. By 1.3.D,

$$P_\mu\{X_{Q_1} = 0, X_{Q_2} \ne 0\} = P_\mu 1_{X_{Q_1}=0} P_{X_{Q_1}}\{X_{Q_2} = 0\},$$

which implies 7.1.A.

Property 7.1.B follows from 1.3.D for $f(\nu_1, \nu_2) = f_1(\nu_1) f_2(\nu_2)$. By applying the multiplicative system theorem, we cover the general case. To prove 7.1.C, we apply 7.1.B to

$$f(\nu_1, \nu_2) = 1_{\langle \varphi_1, \nu_1 \rangle \le \langle \varphi_2, \nu_2 \rangle},$$

and we get

$$P_\mu\{\langle \varphi_1, X_{Q_1} \rangle \le \langle \varphi_2, X_{Q_2} \rangle\} = P_\mu F(X_{Q_1}),$$

where

$$F(\nu) = P_\nu\{Y \ge 0\}$$

and $Y = \langle \varphi_2, X_{Q_2} \rangle - \langle \varphi_1, \nu \rangle$. Let ν' be the restriction of ν to Q_2^c. For all $\lambda > 0$, by 1.3.A and 1.3.C,

$$P_\nu e^{-\lambda \langle \varphi_2, X_{Q_2} \rangle} \le P_{\nu'} e^{-\lambda \langle \varphi_2, X_{Q_2} \rangle} = e^{-\lambda \langle \varphi_2, \nu' \rangle} = e^{-\lambda \langle \varphi_2, \nu \rangle} \le e^{-\lambda \langle \varphi_1, \nu \rangle}$$

and 7.1.C follows from Lemma 2.1. □

7.2. *Extension of class $\mathbb{M}$.* Suppose that $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}$, $\mu \in \mathbb{M}$, is a branching exit system. We get a new branching exit system by extending class $\mathbb{M}$ to the class $\sigma(\mathbb{M})$ of all measures $\mu = \sum_1^\infty \mu_n$, where $\mu_n \in \mathbb{M}$, and by defining P_μ as the convolution of measures P_{μ_n}. For every $Z \in \mathbb{Z}$,

$$P_\mu Z = \prod P_{\mu_n} Z. \tag{7.1}$$

By using this formula, it is easy to check that 1.3.A holds for the extended system. Condition 1.3.B holds because, if $Y = X_Q(Q)$ and $\mu \in \sigma(\mathbb{M})$, then

$$P_\mu\{Y = 0\} = \lim_{\lambda \to \infty} P_\mu e^{-\lambda Y} = \lim \prod P_{\mu_n} e^{-\lambda Y} = 1.$$

If $\mu(Q) = 0$, then, by 1.3.A,

$$P_\mu e^{-\lambda \langle f, X_Q \rangle} = e^{-\langle f, \mu \rangle}$$

and property 1.3.C follows from Lemma 2.1.

7.3. *Extension of parameter sets for superprocesses.* Put $S_\Delta = \Delta \times E$, $Q_\Delta = Q \cap S_\Delta$ and let $Q_{>t} = Q_\Delta$ for $\Delta = (t, \infty)$ and $Q_{<t} = Q_\Delta$ for $\Delta = (\infty, t)$. We constructed a superprocess as a BEM system with parameter sets $\mathbb{M}_0 = \mathscr{M}(S)$ and $\mathbb{O}_0$. Now we consider wider classes: $\mathbb{O}_1$, which consists of all open sets Q such that $Q \subset S_{>t}$ for some $t \in \mathbb{R}$, and $\mathbb{M}_1$, which consists of all measures μ on S such that $\mu(S_\Delta) < \infty$ for every finite interval Δ. Note that $\mathbb{M}_0 \subset \mathbb{M}_1 \subset \sigma(\mathbb{M}_0)$. Measure P_μ is defined for $\mu \in \mathbb{M}_1$ by formula (7.1). For every Q and $k = 1, 2, \ldots$, we denote by Q^k the intersection of Q with $(-k, k) \times E$. By 7.1.C,

$$X_{Q^{k+1}}(\Gamma) \geq X_{Q^k}(\Gamma) \quad \text{a.s. for every } \Gamma \subset Q^c. \tag{7.2}$$

Therefore there exists a measure $\widehat{X}_Q$ such that

$$\widehat{X}_Q(\Gamma) = \begin{cases} \lim X_{Q^k}(\Gamma), & \text{for } \Gamma \subset Q^c, \\ 0, & \text{for } \Gamma \subset Q. \end{cases}$$

[Every X_Q is defined only up to equivalence. We choose versions of X_{Q^k} for all positive integers k in such a way that (7.2) holds for all ω and all k.] Clearly, $\widehat{X}_Q$ is a measure of class $\mathbb{M}_1$ and

$$\widehat{X}_Q = X_Q, \quad P_\mu\text{-a.s. for all } Q \in \mathbb{O}_0, \mu \in \mathbb{M}_1.$$

If $\widehat{V}_Q$ is the transition operator of $\widehat{X} = (\widehat{X}_Q, P_\mu)$, $Q \in \mathbb{O}_1$, $\mu \in \mathbb{M}_1$, then

$$\begin{aligned} \widehat{V}_{Q^k} &= V_{Q^k} \qquad \text{for all } k, \\ \widehat{V}_{Q^k}(f) &\uparrow V_{\widehat{Q}}(f) \quad \text{for every } f \in \mathbb{B}. \end{aligned} \tag{7.3}$$

By a monotone passage to the limit, we establish that 1.3.A holds for $\widehat{X}$ and that 1.4.B, 1.4.C and 1.4.D hold for $\widehat{V}_Q$. Hence, $\widehat{X}$ is a branching system and, by Theorem 1.1, it is a BEM system.

Assuming that $\psi(r, x; u)$ is continuous in u and satisfies conditions 5.1.B, 6.1.A and 6.1.B, we prove that, for every $Q \in \mathbb{O}_1$, $u = \widehat{V}_Q(f)$ is a solution of (1.10). Indeed, by 1.4.B, $u = \widehat{V}_Q(f')$, where $f' = 1_{Q^c} f$. Since $Q^k \in \mathbb{O}_0$, function $u_k = V_{Q^k}(f')$ satisfies

$$u_k(r, x) + \Pi_{r,x} \int_r^{\tau_k} \psi(s, \xi_s; u_k(s, \xi_s))\, ds = \Pi_{r,x} f'(\tau_k, \xi_{\tau_k}),$$

where τ_k is the first exit time from Q^k. For sufficiently large k, it is equal to $\tau \wedge k$, where τ is the first exit time from Q. If $\tau > k$, then $(\tau_k, \xi_{\tau_k}) \in Q$. Therefore

$$u_k(r, x) + \Pi_{r,x} \int_r^{\tau_k} \psi(s, \xi_s(u_k(s, \xi_s))\, ds = \Pi_{r,x} 1_{\tau \leq k} f(\tau, \xi_\tau) \quad \text{for } r < k.$$

By passing to the limit as $k \to \infty$, we get that u is a solution of (1.10).

8. Supplement to the concept of a superprocess.

8.1. *Time-homogeneous superprocesses.* Suppose that $\xi = (\xi_t, \Pi_x)$ is a time-homogeneous right continuous strong Markov process in a topological space E and let $\psi(x, u)$ be a positive function on $E \times \mathbb{R}_+$. Denote $\mathbb{B}(E)$ the class of all positive bounded Borel functions on E. We say that a BEM system $X = (X_D, P_\mu)$, $D \in \mathbb{O}$, $\mu \in \mathbb{M}$, is a time-homogeneous (ξ, ψ)-superprocess if $\mathbb{O}$ is the class of all open subsets of E, $\mathbb{M}$ is the class of all finite measures on E and if, for every $f \in \mathbb{B}(E)$ and all $D \in \mathbb{O}$, $\mu \in \mathbb{M}$,

$$P_\mu e^{-\langle f, X_D\rangle} = e^{-\langle V_D(f), \mu\rangle}, \tag{8.1}$$

where $u = V_D(f)$ is a solution of the equation

$$u + G_D\psi(u) = K_D f. \tag{8.2}$$

Here

$$K_D f(x) = \Pi_x f(\xi_\tau), \tag{8.3}$$

$$G_{D\rho(x)} = \Pi_x \int_0^\tau \rho(\xi_t)\, dt \tag{8.4}$$

(τ is the first exit time of ξ_t from D).

To construct such a process, we start from the superprocess $\widehat{X}$ described in Section 7.3. We imbed E into $\mathscr{R} \times E$ by identifying $x \in E$ with $(0, x) \in \mathbb{R} \times E$. We define X_D as the projection of $\widehat{X}_{\mathbb{R}\times D}$ on E and we put $P_\mu = \widehat{P}_{\delta_0\times\mu}$ for every finite measure μ on E (δ_0 is the unit mass on $\mathbb{R}$ concentrated at 0).

It follows from Theorem 1.6 that a homogeneous (ξ, ψ)-superprocess exists for every time-homogeneous right continuous strong Markov process ξ and every function

$$\psi(x, u) = b(x)u^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) n(x, d\lambda) \tag{8.5}$$

such that

$$b(x) \quad \text{and} \quad \int_0^\infty \lambda \wedge \lambda^2 n(x, d\lambda) \quad \text{are bounded.} \tag{8.6}$$

8.2. *Branching measure-valued Markov processes.* To every superprocess $X = (X_Q, P_\mu)$, $Q \in \mathbb{O}_1$, $\mu \in \mathscr{M}(S)$, there corresponds a measure-valued Markov process $\widetilde{X} = (\widetilde{X}_t, \widetilde{P}_{r,\nu})$. Here $\widetilde{X}_t$ is the restriction of $X_{S_{<t}}$ to $S_t = \{t\} \times E$ and $\widetilde{P}_{r,\nu} = P_{\delta_r\times\nu}$. Let $\widetilde{\mathscr{F}}_\Delta$ stand for the σ-algebra generated by $\widetilde{X}_t$, $t \in \Delta$. Clearly, $\mathscr{F}[r, t] \subset \mathscr{F}_{\subset S_{<t}}$ and $\mathscr{F}_{\geq t} \subset \mathscr{F}_{\supset S_{<t}}$ and the Markov property of $\widetilde{X}$ follows from 1.3.D. If $\varphi \in \mathbb{B}(E)$ and if $f(s, x) = \varphi(x)$ for all s, then, for all $r < t$,

$$\widetilde{P}_{r,\nu} e^{-\langle f, \widetilde{X}_t\rangle} = e^{-\langle u, \nu\rangle},$$

where $u_t = V_{Q_{<t}}(f)$ satisfies the equation

$$(8.7)\qquad u_t(r,x) + \Pi_{r,x}\int_r^t \psi(s,\xi_s; u_t(s,\xi_s))\,ds = \Pi_{r,x}\varphi(\xi_t) \quad \text{for } r \le t.$$

If ξ is time-homogeneous and ψ is of the form (8.5), then there exists a time-homogeneous (ξ,ψ)-superprocess (X_t, P_μ) such that, for every $\mu \in \mathscr{M}(E)$ and every $f \in \mathbb{B}(E)$,

$$(8.8)\qquad P_\mu e^{-\langle f, X_t\rangle} = e^{-\langle u_t, \mu\rangle},$$

where

$$(8.9)\qquad u_t(x) + \Pi_x \int_0^t \psi(\xi_s, u_{t-s}(\xi_s))\,ds = \Pi_x f(\xi_t).$$

Acknowledgments. I am indebted to P. Fitzsimmons for providing me the notes related to his paper [4]. I also thank S. E. Kuznetsov for fruitful discussions on the subject of this paper.

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