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ASYMPTOTIC RESULTS FOR SUPER-BROWNIAN MOTIONS AND SEMILINEAR DIFFERENTIAL EQUATIONS

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Limit laws for three-dimensional super-Brownian motion are derived, conditioned on survival up to a large time. A large deviation principle is proved for the joint behavior of occupation times and their difference. These are done via analyzing the generating function and exploiting a connection between probability and differential–integral equations.

1. Introduction and statement of results. We study occupation time limit theorems for the three-dimensional super-Brownian motions (super-BM) and related processes. This is done by analyzing cumulant generating functions which satisfy some integral equations. In the case of super-BM the integral equation is equivalent to a semilinear PDE.

A sample path, $(\mu_t(dx);\ t \geq 0)$, of the super-BM, is a path of nonnegative Radon measures on $\mathbf{R}^d$. When the initial $\mu_0(dx)$ is $\mu_0 = \nu$ we denote by P_ν and E_ν the corresponding probability measure and expectation, respectively. We will omit writing the initial measure in the subscript when it is the Lebesgue measure. For a construction of the processes see, for example, [3, 7, 8].

We now state a property of the process P that is particularly important to our study. For a nonpositive integrable function φ define the φ-occupation time $D_{\varphi,T}$ (a random variable), by

$$D_{\varphi,T} = \int_0^T \int_{\mathbf{R}^d} \varphi(x)\mu_t(dx)\,dt. \tag{1.1}$$

The following connection with differential equations and integral equations is known for the cumulant generating function:

$$\log E_\nu\{\exp D_{\varphi,T}\} = \int_{\mathbf{R}^d} v(T,x;\varphi)\,\nu(dx), \qquad T \geq 0, \tag{1.2}$$

where $v(t,x;\varphi)$ is the solution of

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \triangle v + v^2 + \varphi, & \text{in } t > 0,\ x \in \mathbf{R}^d, \\ v(0,x) = 0, & x \in \mathbf{R}^d. \end{cases} \tag{1.3}$$

Note that the use of the Laplacian, as opposed to half of the Laplacian, indicates that the underlying Brownian motion is being run at twice the standard speed.

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In order to introduce integral equations let us define the heat kernel and associated operators:

$$p(t, x) = (4\pi t)^{-3/2} e^{-|x|^2/4t},$$
$$\int = \int_{\mathbf{R}^3},$$
$$(Au)(t, x) = \int_0^t \int p(t - s, x - y) u(s, y)\, dy\, ds$$
$$(Bh)(x) = \int \left(\int_0^\infty p(s, x - y)\, ds \right) h(y)\, dy,$$
$$\|f\| = \|f\|_2 = \left(\int f(x)^2\, dx \right)^{1/2}.$$

Formally, operator A is inverse to the heat operator $\partial_t - \Delta$ and B is inverse to $-\Delta$ in suitable spaces of functions $u(t, x)$ and $h(x)$. The function v in (1.2) is also the solution of the integral equation

$$v = A(v^2 + \varphi). \tag{1.4}$$

The lack of interaction in the super-BM makes formula (1.2) easy to understand. The building block is the case when the initial is a Dirac delta δ_x measure at x. For this case the shorthands P_x for the probability measure and E_x for the expectation will be used. That the cumulant generating function in (1.2) depends linearly on ν follows easily from the independence property, the lack of interaction, of the super-BM.

For three or higher dimensions, Iscoe has proved (Theorem 1 in [4]) the strong law that, as $T \to \infty$, the empirical measure $(1/T)\int_0^T \mu_s(dx)\, ds$ converges (in the vague topology) with P-probability 1 to the Lebesgue measure. When the space dimension is 2 or less, the law of large numbers fails. For critical branching Brownian motions, which is a particle analogue, Cox and Griffeath [2] investigate the large deviations from this central tendency. Their results show exponential decay of (large deviation) tail probabilities in five or more space dimensions and slower than exponential decay of tail probabilities in three and four space dimensions. Large deviation behaviors have been studied further [6, 10, 12]. In the present article we continue to investigate some fine behaviors of the three dimensional case. The problem will be approached by estimating the cumulant generating function, in contrast with estimating cumulants [2, 6].

It is known that a unique mild solution $v(t, x; \delta_0)$ of (1.4), with φ replaced by δ_0, exists up to a positive blowup time t^* [10]. The solution is a classical solution of (1.3), with φ replaced by δ_0, at $t < t^*$ and all x, except the origin. It will be proved in Proposition 3.1 that near the origin, $v(t, x; \delta_0)$ behaves like the Green function $|x|^{-1}$.

Suppose φ is continuous with a compact support, and $\int \varphi\, dx = 1$. It is worked out in [10, 12] that

$$c^2 v(c^2 t, cx; c^{-1}\varphi) \to v(t, x; \delta_0)$$

as $c \to \infty$.

In the above pointwise convergence, it is adequate to consider x as not the origin and before the blow-up time t^*. Such a convention, adopted throughout the paper, saves us from speaking of convergence to infinity.

One major result in this note is a refined limit.

THEOREM 1.1. *Let φ, ξ be continuous with compact supports and $\int \xi(x) dx = 0$. Let*

$$z = \int \varphi\, dx + \|B\xi\|^2.$$

Then, for $d = 3$,

$$\lim_{c\to\infty} c^2 v\big(c^2 t, cx; c^{-1}\varphi + c^{-1/2}\xi\big) = v(t, x; z\delta_0).$$

A probabilistic basis for Theorem 1.1 is as follows. Fix $t > 0$ and x when not the origin and consider large parameter c. A super-BM, initially the Dirac δ_{cx} measure, will charge the unit ball with probability of order c^{-2}. More precisely, the probability is asymptotically $2(c|x|)^{-2}$ for all bounded domains, not just for the unit ball. Conditioned on charging, the total charge (occupation time) up to time $c^2 t$ is of order c. Furthermore, the difference of charge to the right half of the unit ball (the first coordinate $x_1 > 0$) and to the left half ($x_1 < 0$) is of order $c^{1/2}$. So, we use the correct normalization of dividing the total occupation time by c and the difference by $c^{1/2}$. From such probabilistic thinking (see [11] for example), we anticipate the weak convergence result as follows:

Conditioned with charging the unit ball, the normalized occupation time and the difference (between the right half and the left half ball) converges in distribution to a nondegenerate random vector as c tends to infinity.

Moreover, the normalized difference, conditioned that the normalized total occupation time equals $a > 0$, converges in distribution to a normal distribution with mean 0 and a variance proportion to a (as can be guessed from the central limit theorem).

The above probabilistic reasoning uses the Brownian scaling and the central limit theorem. The choice of the particular φ, ξ (the indicator of the unit ball and the difference of the indicator of the right half ball and the left half ball) is used only as an easy-to-visualize example and can be arbitrary.

Our Theorem 1.1 is motivated by the above weak convergence result. More precisely, Theorem 1.1 states that the moment generating function converges which is sufficient, but not necessary at all for weak convergence. The stronger statement of Theorem 1.1 is, however, crucial for deriving the large deviation result. Let us state the weak convergence result as follows.

COROLLARY 1.2. *Consider P_{cx}, x not the origin. Let the supports of $\varphi \geq 0$ and ξ be contained in a compact set K. Let $\int \varphi\, dx > 0$. Then the following holds as $c \to \infty$.*

Conditioned that the super-BM charges K, the normalized random vector $(c^{-1}D_{\varphi, c^2 t}, c^{-1/2}D_{\xi, c^2 t})$ converges to the probability distribution on $(0, \infty) \times R$ whose moment generating function g is given by

$$g(\alpha, \beta) = v(t, x; z\delta_0)\frac{|x|^2}{2} + 1,$$

where $z = (\int \varphi\, dx)\alpha + (\|B\xi\|^2)\beta^2$.

PROOF. Let t, x be fixed and q_c be the probability that the super-BM P_{cx} charges K. Let g_c be the conditional moment generating function of $(c^{-1}D_{\varphi, c^2 t}, c^{-1/2}D_{\xi, c^2 t})$. That is, letting K^* be the event of charging K,

$$g_c(\alpha, \beta) = E_{cx}\big[\exp\big(\alpha(c^{-1}D_{\varphi, c^2 t}) + \beta(c^{-1/2}D_{\xi, c^2 t})\big)|K^*\big].$$

We need to establish

$$g_c(\alpha, \beta) \to g(\alpha, \beta).$$

Consider whether the super-BM charges K or not, and use (1.2) and (1.3). We have

$$\exp(v(c^2 t, cx; \alpha[c^{-1}\varphi] + \beta[c^{-1/2}\xi])) = q_c g_c(\alpha, \beta) + (1 - q_c) \times 1.$$

Thus,

$$g_c(\alpha, \beta) = \big[\big(\exp(v(c^2 t, cx; \alpha[c^{-1}\varphi] + \beta[c^{-1/2}\xi])) - 1\big)/q_c\big] + 1.$$

The proof is completed by applying Theorem 1.1 and the fact that $c^2 q_c \to 2|x|^{-2}$, as $c \to \infty$, independent of K (see, for example, [11]).

Notice that the generating function $g(\alpha, \beta)$ is a function of $c_1\alpha + c_2\beta^2$, where $c_1 = \int \varphi\, dx$, $c_2 = \|B\xi\|^2$. It can then be derived that, conditioned on $c^{-1}D_{\varphi, c^2 t} = a > 0$, the normalized $c^{-1/2}D_{\xi, c^2 t}$ converges to a normal distribution of mean 0 and variance $2c_2 a/c_1 = 2\|B\xi\|^2 a/(\int \varphi\, dx)$.

Theorem 1.1, together with the connection (1.2) with cumulant generating functions then enables us to apply the Gärtner–Ellis theorem (see, for example, [5]) to establish a large deviation theorem. We now give the rate function and then state the large deviation result,

$$(1.5)\qquad \begin{cases} \Lambda_3(\theta) \equiv \begin{cases} \int_{\mathbf{R}^3} v(1, x; \theta\delta_0)\, dx, & \text{if } -\infty < \theta < (t^*)^{1/2}, \\ +\infty, & \text{otherwise,} \end{cases} \\ K_3(a, b) \equiv \sup_{\alpha, \beta \in \mathbf{R}} \Big[a\alpha + b\beta - \Lambda_3\Big(\alpha \int \varphi\, dx + \beta^2 \|B\xi\|^2\Big)\Big]. \end{cases}$$

THEOREM 1.3. *Consider the three-dimensional super-BM process P (i.e., initially the Lebesgue measure). Suppose that φ, ξ are continuous with compact supports and $\int \xi\, dx = 0$. Let $W_{\varphi, T}$ be the average occupation time*

$$W_{\varphi, T} = \frac{1}{T} D_{\varphi, T},$$

and $W_{\xi, T}$ be similarly defined. Then $\{(W_{\varphi, T}, T^{1/4} W_{\xi, T}), T^{1/2}\}$ is a large deviation system with rate function K_3.

We are thankful to the anonymous referee for rendering the rate function K_3 more explicit (as shown next) and suggesting that we give a probabilistic interpretation. This much improves the exposition. Assume for simplicity that $\int \varphi\, dx = 1$ and $\|B\xi\|^2 = 1$. Let $a > 0$ [otherwise $K_3(a, b) = \infty$], then

$$K_3(a, b) = \sup_{\alpha, \beta \in R} [\alpha a - \Lambda_3(\alpha) + b\beta - a\beta^2] = \Lambda_3^*(a) + \frac{b^2}{4a}.$$

Note that $\Lambda_3^*(a)$ is the large-deviation rate for $W_{\varphi, T}$ being near a (see [10]). Now, from the probabilistic picture discussed following Theorem 1.1 we indeed anticipate that, given $W_{\varphi, T} = a$, the rate to see $T^{1/4} W_{\xi, T}$ close to b is $b^2/4a$, based on the normal distribution. This gives the rate function K_3 a probabilistic explanation which has motivated the research.

As long as the connection, such as (1.2)–(1.4), exists between integral–differential equations and the probability theory, it is clear that various techniques from these fields can be brought together to attack the problem. The analytic result for equations is interesting in its own right. Our proof method is based mostly on the comparison principle (maximum principle) for equations. The method reveals that the mathematical result goes somewhat beyond probability interpretations known currently. For example, no probabilistic interpretation is known at present for some of the integral–differential equations that are subject to the same technique. As an example, let us replace the quadratic nonlinearity v^2 by $|v|^p$, $p > 1$. The problem yields to the same technique of proof. There is however no simple probabilistic meaning to the case $p > 2$.

In order to understand the result better one can look at fractional dimensions as well. This can be done by replacing the Laplacian with the Bessel operator. With the quadratic nonlinearity v^2, we then can see that qualitatively similar results (as in Theorems 1.1 and 1.3) hold for $2 < d < 4$. So the investigated phenomenon is common to an interval of dimensions, as opposed to an isolated dimension. Another generalization is to replace BM with stable processes.

As the results apply to general φ, ξ functions, they can be extended to a functional level. In the case of weak convergence this is ensured by the Cramer-Wold theorem. In the case of large deviations, such higher level results can be obtained by, for example, the Dawson–Gärtner projection theorem [4]. The large deviation principle for the difference of occupation times alone was already carefully established at a functional level in [6] by analyzing cumulants. Extension from Brownian motion to stable processes is also treated there.

In order to help interested readers pursue this further, we will prove Theorems 1.1 and 1.3 in a manner readily applicable to various generalizations. For example, suppose the nonlinear term v^2 in (1.3) is replaced by $|v|^p$, $p > 1$. Then, in the regime

$$\frac{2}{p-1} < d < \frac{2}{p-1} + 2$$

of dimensions, our method will establish the qualitatively same results as Theorem 1.1 and 1.3. The correct normalizing constants depend, of course, on p, d. The following counterpart of Theorem 1.1 can be obtained.

THEOREM 1.4. *Let $u(t, x; \psi)$ be the solution of*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{d-1}{x}\dfrac{\partial u}{\partial x} + |u|^p + \psi, & \text{in } t > 0,\ x > 0, \\ u(0,x) = 0, & x > 0. \end{cases} \tag{1.6}$$

Let φ, ξ be continuous with compact supports and $\int_0^\infty \xi(x)\, x^{d-1}\, dx = 0$. Let B_d be the inverse of the Bessel operator $\frac{\partial^2}{\partial x^2} + \frac{d-1}{x}\frac{\partial}{\partial x}$, and $Z = \int_0^\infty (\varphi + |(B_d \xi)(x)|^p) x^{d-1}\, dx$. Then, for $\frac{2}{p-1} < d < \frac{2}{p-1} + 2$,

$$\lim_{c\to\infty} c^{\frac{2}{p-1}} u\left(c^2 t, cx; c^{d-\frac{2p}{p-1}}\varphi + c^{(\frac{d}{p}-\frac{2}{p-1})}\xi\right) = u(t, x; k_d Z \delta_0),$$

where k_d is a positive constant depending on dimension d. For example, $k_3 = 4\pi$, the surface area of the unit sphere in R^3.

When $1 < p \leq 2$ a connection between cumulant generating functions associated with super-Bessel process and differential equations (1.6) exists. Theorem 1.4 translates into a large deviation principle like Theorem 1.3. The exponent 1/4 of $T^{1/4}$ in Theorem 1.3 now becomes $1 - \frac{d(p-1)}{2p}$. The exponent 1/2 of $T^{1/2}$ should be $\frac{d}{2} - \frac{1}{p-1}$. If one is only interested in integer dimensions, notice that all except $p = 1 + \frac{2}{m}$, with positive integer m, admit two integer values of dimension d. Theorem 1.3, for example, concerns the case $p = 2$, that is, $m = 2$. Thus, $d = 3$ is the only integer dimension. The exponent $p = 7/4$ (then $2\frac{2}{3} < d < 4\frac{2}{3}$) represents the typical values p which admit two integer dimensions.

A challenging problem is to prove the counterpart of Theorem 1.3 for the three-dimensional voter model. It is proved that the normalizing constants are the same for occupation times [1]. One naturally anticipates that same (as the super-BM) normalization constants hold also for the difference of occupation times and that a large deviation principle like Theorem 1.3 holds. This remains to be done. Preliminary calculation reveals that such crucial ingredients as Theorem 1.1 can be extended to $d \leq 4$-dimensional super-Brownian motions as well; many details, however, need be worked out.

2. Proof of Theorems 1.1 and 1.3. For an integrable function u and a positive number c, denote by u_c the function $x \mapsto c^3u(cx)$. Recall from Theorem 1.1 the definition of $v(t, x|c)$ and fix arbitrary φ, ξ. A simple rescaling yields

$$v(t, x|c) = c^2 v(c^2 t, cx; c^{-1}\varphi + c^{-1/2}\xi) = v(t, x; \varphi_c + c^{1/2}\xi_c). \tag{2.1}$$

Comparing Theorem 1.1 with the known result [10] that

$$v(t, x; \varphi_c) \to v\Big(t, x; \Big(\int \varphi(x)\, dx\Big)\delta_0\Big) \tag{2.2}$$

as $c \to \infty$, we observe that the $c^{1/2}\xi_c$ term in the right-hand side of (2.1) contributes to the limit by $\|B\xi\|^2$, added to $(\int \varphi(x)\, dx)$ to account for the number z in Theorem 1.1.

This observation motivates us to use

$$w(t, x|c) = v(t, x|c) - c^{1/2}A\xi_c. \tag{2.3}$$

Why? First, the equation satisfied by w will no longer have the $c^{1/2}\xi_c$ term. Precisely, the equation of w is

$$w = A\big[w^2 + 2w(c^{1/2}A\xi_c) + (c^{1/2}A\xi_c)^2 + \varphi_c\big]. \tag{2.4}$$

Second, the resulting integral equation (2.4) clearly shows a way in which Theorem 1.1 can be proved: a heat-kernel calculation gives us that, for $t > 0$,

$$\int (c^{1/2}A\xi_c)(t, x)^2\, dx = \|(A\xi)(c^2 t, \cdot)\|^2 \to \|B\xi\|^2, \tag{2.5}$$

which is exactly the contribution we anticipated from ξ. So all we really need prove is

$$\lim_{c\to\infty} \int \big(wc^{1/2}A\xi_c\big)(t, x)\, dx = 0. \tag{2.6}$$

Let $\alpha = w(t, x), \beta = (c^{1/2}A\xi_c)(t, x)$. We will establish the crucial limit result (2.6) via the estimate

$$|2\alpha\beta| \le \varepsilon\alpha^r + b\beta^q, \tag{2.7}$$

where q, r is a pair of conjugate exponents ($1/q + 1/r = 1$), and $\varepsilon > 0$, $b = b(\varepsilon, r) = 2^q q^{-1}(r\varepsilon)^{-q/r}$.

Motivated by (2.4), (2.6) and (2.7), let $g(t, x; \varepsilon, c)$ be the solution of of the equation

$$g = A\big[g^2 + \varepsilon|g|^r + b|\big(c^{1/2}A\xi_c\big)(t, x)|^q + \big(c^{1/2}A\xi_c\big)^2 + \varphi_c\big]. \tag{2.8}$$

The existence of the solution g will be proved in Lemma 2.2 below. Granted its existence for now, which values of q can be used? A heat-kernel calculation yields

$$\lim_{c\to\infty} \int |(c^{1/2}A\xi_c)(t, x)|^q\, dx = 0, \tag{2.9}$$

if $\frac{3}{2} < q < 2$. So we will use an arbitrary, but fixed $\frac{3}{2} < q < 2$; thus $3 > r > 2$.

In view of (2.4), (2.7) and (2.8), the comparison principle implies that

$$w(t, x|c) \le g(t, x; \varepsilon, c). \tag{2.10}$$

We now state Lemmas 2.1 and 2.2, which are natural steps and have elementary proofs. However, in order to make manifest the main ideas, we'll postpone their proofs until the Appendix.

LEMMA 2.1. *Let $s > 0$ be less than the blow-up time of $v(t, x; z\delta_0)$. There exists $\varepsilon^* = \varepsilon^*_{s,r}$ such that for all $\varepsilon < \varepsilon^*$, the solution $f(t, x; \varepsilon)$ of*

$$f = A\big[f^2 + \varepsilon|f|^r + (z+\varepsilon)\delta_0\big] \tag{2.11}$$

exists and satisfies

$$\lim_{\varepsilon\to 0} f(t, \cdot; \varepsilon) = v(t, \cdot; z\delta_0), \qquad 0 \le t \le s, \tag{2.12}$$

pointwise. Furthermore, the integrals (on the whole space R^3) of the two functions f, v satisfy the same limit equality.

LEMMA 2.2. *For sufficiently large c, the solution $g(t, x; \varepsilon, c)$ of (2.8) exists and satisfies*

$$\limsup_{c\to\infty} g(t, \cdot; \varepsilon, c) \le f(t, \cdot; \varepsilon), 0 \le t \le s, \tag{2.13}$$

pointwise. Furthermore, the integrals (on the whole space R^3) of the two functions g, f satisfy the same limit inequality.

We are now in a position to finish the proof. Taking note of (2.3), and the fact that

$$\limsup_{c\to\infty} c^{1/2}(A\xi_c)(t, x) = 0 \tag{2.14}$$

pointwise and (2.10), we conclude

$$\limsup_{c\to\infty} v(t, x|c) \le \limsup_{c\to\infty} g(t, x; \varepsilon, c).$$

Applying (2.13), and then letting ε tend to 0, the limit result (2.12) implies

$$\limsup_{c\to\infty} v(t, x|c) \le v(t, x; z\delta_0). \tag{2.15}$$

The space integral version in the end of Lemmas 2.1, 2.2 and the fact that $\int (A\xi_c)(t, x)\,dx = 0$ due to $\int \xi(x)\,dx = 0$ imply

$$\limsup_{c\to\infty} \int v(t, x|c)\,dx \le \int v(t, x; z\delta_0)\,dx = \Lambda_3(z). \tag{2.16}$$

The other direction of inequality is similarly proved; it is in fact easier because instead of (2.11) we look at

$$f = A\big[f^2 - \varepsilon|f|^r + (z-\varepsilon)\delta_0\big]. \tag{2.17}$$

The difference is in the sign in front of the two ε's. The existence of the solution of (2.17) follows immediately from that of $v(s, x; z\delta_0)$ by the comparison principle, while the existence for f of (2.11) requires some work as will be seen in the Appendix.

Result (2.15) and the easier counterpart inequality going the opposite direction imply Theorem 1.1. Result (2.16), the easier counterpart inequality going the opposite direction and the Gärtner–Ellis theorem (see, e.g., [5]) imply Theorem 1.3. In more detail, the probability-PDE connection given in (1.2), (1.3) and (1.4) translates the limit equality into a limit result for the cumulant generating function of the occupation times. The limit Λ_3 explodes, thus steepness is required and shown in [10]. So the Gärtner–Ellis theorem concludes that the large deviation rate is the Legendre transform of the limit of the cumulant generating function, ending Theorem 1.3.

APPENDIX

We should point out that the operator A is monotonic. This is responsible for the theorems in this article and has already been utilized in Section 2. In this Appendix we will continue to take advantage of it by repeatedly using the comparison principle, also known as the supersolution–subsolution method.

Let

$$h(x) = e^{-1}|x|^{-1}, \quad |x| \le 1; \quad e^{-|x|}, \quad |x| > 1,$$

and

$$k(x) = \theta|x|^{-r}, \quad |x| \le 1; \quad \theta|x|^{-\beta}, \quad |x| > 1,$$

where $3 > r > 2$ is as in (2.9), $\beta > 3$, so that the function k is integrable, and $\theta > 0$ is such that $\int k(x)\,dx = 1$. Clearly, positive M_1, M_2 exists so that

$$h^r \le M_1 k, \tag{A.1}$$

$$(h * k)^r \le M_2 k. \tag{A.2}$$

We need the following proposition.

PROPOSITION A.1. *Let s be less than the blow-up time of $v(t, x; z^*\delta_0)$, $z^* > 0$. Then there exists a constant M_3, depending on z^*, s, such that*

$$v(s, x; z^*\delta_0) \le M_3 h(x). \tag{A.3}$$

PROOF. The function $v(t, x; z^*\delta_0)$ is the solution of

$$v = A(v^2 + z^*\delta_0). \tag{A.4}$$

It is known (see [10]) that $v(t, x; z^*\delta_0)$, $z^* > 0$, is increasing in time, and is a radial function decreasing in radius $|x|$. Also, it blows up at a finite time at the origin, and before the blow-up time it is square integrable. Let $G(x) = |x|^{-1}$ and note that the inverse operator B of $-\Delta$ is given by

$$(Bh) = cG * h$$

for some positive constant c. Therefore,

$$\begin{aligned} v(t, x; z^*\delta_0) &\le A\big(v(s, \cdot; z^*\delta_0)^2 + z^*\delta_0\big) \\ &\le B\big(v(s, \cdot; z^*\delta_0)^2 + z^*\delta_0\big) \\ &= cG * \big(v(s, \cdot; z^*\delta_0)^2 + z^*\delta_0\big) \\ &= cG * v\big(s, \cdot; z^*\delta_0\big)^2 + cz^*G, \end{aligned}$$

for $0 \le t \le s$, where s is less than the blow-up time. Let

$$q(x) = v(s, x; z^*\delta_0)^2.$$

From the inequality above, it remains to show that, near the origin, $x = 0$, the function $(G * q)(x)$ is bounded by a constant multiple of Green's function $|x|^{-1}$.

Since the function $q(x)$ is a radial function decreasing in radius $|x|$ and is integrable, $\int q(x)\,dx = I < \infty$, we can split R^3 into the ball M of radius $\frac{|x|}{2}$, centering at x and its complement and estimate being as follows:

$$\begin{aligned} (G * q)(x) &\le \int_M |x - y|^{-1} q(y)\,dy + I\left(\frac{|x|}{2}\right)^{-1} \\ &\le q\left(\frac{x}{2}\right)\int_M |x - y|^{-1}\,dy + 2I|x|^{-1} \\ &= q\left(\frac{x}{2}\right)L|x|^2 + 2I|x|^{-1} \\ &= \left[Lq\left(\frac{x}{2}\right)|x|^3 + 2I\right]|x|^{-1}, \end{aligned}$$

where L is a constant whose exact value is not important. Since function q is integrable, radial and decreasing in $|x|$, the function $q(\frac{x}{2})|x|^3$ tends to 0 as $x \to 0$. The desired property near the origin is established.

Now it remains to verify the faster-than-exponential-decay behavior near the infinity. Look at (A.4) only for $|x| > 1$, the exterior of the unit ball, and only up to time s. Let $K = v(s, (1, 0, 0); z^*\delta_0)$. That is, the v function is evaluated at the maximal (for the present purpose) time argument s and at $x = (1, 0, 0)$. Since v is increasing in time, radial and decreasing in x, the boundary ($|x| = 1, 0 \le t \le s$) value is bounded by K, and the nonlinear term v^2 in the right-hand side of (A.4) is bounded by Kv. Let T be the hitting time of the unit ball centered at the origin. By the Feynman–Kac representation for initial-boundary value problems,

$$v(s, x; z^*\delta_0) \le Ke^{Ks}P_x[T \le s].$$

Since the probability $P_x[T \le s]$ decays faster than exponentially as x tends to infinity, the proof is complete. □

PROOF OF LEMMA 2.1. Compare the defining equation (2.11) of $f(t, x; \varepsilon)$ with that of $v(t, x; z\delta_0)$. By the monotonicity of operator A, clearly

$$f(t, x; \varepsilon) \ge v(t, x; z\delta_0).$$

It remains to prove the opposite direction of the lemma. Consider $z > 0$, otherwise the lemma is easy. By scaling, there exists $\lambda > 0$ such that $v(t, x; (2\lambda + z)\delta_0)$ exists past time s, that is, s is less than the blow-up time of $v(t, x; (2\lambda + z)\delta_0)$. Define the function

$$v = v(t, x; (2\lambda + z)\delta_0),$$

and the probability measure

$$m = \left(\frac{\lambda}{2\lambda + z} k + \frac{\lambda + z}{2\lambda + z}\delta_0\right).$$

Then define

$$F(t, x; \lambda) = F = v * m. \tag{A.5}$$

Here the convolution of function v, and probability measure m, is given by $(v * m)(x) = \int v(x - y)m(dy)$, $x \in R^3$.

Why defining F? Because the equation satisfied by v and the convexity of the square function $v \to v^2$ imply

$$v * m = \big(A[v^2 + (2\lambda + z)\delta_0]\big) * m \ge A\big[(v * m)^2 + (2\lambda + z)\delta_0 * m\big].$$

Thus,

$$F \ge A\big[F^2 + (2\lambda + z)\delta_0 * m\big] = A\big[F^2 + \lambda k + (\lambda + z)\delta_0\big]. \tag{A.6}$$

In what follows we'll see that, by Proposition 3.1, λk bounds from above a multiple of F^r, so F can bound f from above. Then, by the definition of F, the desired bound of f by v is obtained. Such an idea will be used again in the proof of Lemma 2.2.

Applying (A.3) to $z^* = 2\lambda + z$, and by (A.1),

$$(v(s, \cdot; (2\lambda + z)\delta_0))^r \le M_1 M_3^r k. \tag{A.7}$$

Since v is increasing in the time parameter,

$$F(t, x)^r \le \left(\frac{\lambda}{2\lambda + z} v(s, \cdot; (2\lambda + z)\delta_0) * k + \frac{\lambda + z}{2\lambda + z} v(s, \cdot; (2\lambda + z)\delta_0)\right)^r.$$

Due to the convexity of the function $v \to v^r$ for positive v, Jensen's inequality implies

$$F(t, x)^r \le \frac{\lambda}{2\lambda + z}(v(s, \cdot; (2\lambda + z)\delta_0) * k)^r + \frac{\lambda + z}{2\lambda + z}\big(v(s, \cdot; (2\lambda + z)\delta_0)\big)^r.$$

Then (A.3), (A.2), and (A.7) imply

$$F(t,x)^r \le Mk,$$

where $M \ge (M_2 M_3^r + M_1 M_3^r) = (M_2 + M_1)M_3^r$ is a constant. Choose $M > 1$. Then (A.6) implies

$$F \ge A\left[F^2 + \frac{\lambda}{M}F^r + \left(\frac{\lambda}{M} + z\right)\delta_0\right]. \tag{A.8}$$

This integral inequality, together with the comparison principle implies that the solution f to (2.11) exists, and satisfies

$$f\left(t,\cdot;\frac{\lambda}{M}\right) \le F(t,\cdot;\lambda), \qquad 0 \le t \le s. \tag{A.9}$$

The proof is complete upon letting λ tend to 0 in (A.9) and using the definition (A.5) of $F(t,x;\lambda)$.

PROOF OF LEMMA 2.2. Let

$$h(t,x) = h(t,x;\varepsilon,c) = b\,|(c^{1/2}A\xi_c)(t,x)|^q + (c^{1/2}A\xi_c)^2 + \varphi_c, \tag{A.10}$$

which is the source term of (2.8) for g. Notice that our goal is to bound g from above by f. To achieve the goal we use the idea depicted in the paragraphs containing (A.5), (A.6). This time, we use the convexity of the function $g \to g^2 + \varepsilon|g|^r$. Decompose h as the sum of two terms: first,

$$H(t,x) = H(t,x;\varepsilon,c) = h(t,x) - \left(\int h(t,x)\,dx\right)\delta_0,$$

which has zero mass, and second,

$$\left(\int h(t,x)\,dx\right)\delta_0.$$

We need only to understand the contribution from each of these two kinds of source terms.

The contribution from the second term has been satisfactorily bounded from above in Lemma 2.1. What remains is why the first term makes no contribution at all in the limit. A short answer is that it has zero mass. A quick glance at (2.8), with the above decomposition, may give a false impression that we face again our original problem for function v which also has a positive function φ_c and a function $c^{1/2}\xi_c$ of zero mass. It is not so because $H(t,x;\varepsilon,c)$ is more like ξ_c than $c^{1/2}\xi_c$. That is, there is no $c^{1/2}$ factor. As $c \to \infty$, the lack of the $c^{1/2}$ factor makes our task rather easy. The task is to prove the solution $V(t,x;\varepsilon|c)$ of the equation

$$V = A[V^2 + \varepsilon|V|^r + H(t,x;\varepsilon,c)] \tag{A.11}$$

tends to 0 as $c \to \infty$. As in treating the original problem, we define

$$W(t, x; \varepsilon, c) = V - AH,$$

which satisfies

$$W = A\big[(W + AH)^2 + \varepsilon |W + AH|^r\big]. \tag{A.12}$$

Now, due to the lack of the $c^{1/2}$ factor, there is no need to use the more delicate estimates (2.6), (2.7) and subsequent steps. Such alternatives as

$$\begin{aligned}(W + AH)^2 &\le 2W^2 + 2(AH)^2,\\ |W + AH|^r &\le 2^{r-1}|W|^r + 2^{r-1}|AH|^r\end{aligned}$$

suffice. It can be checked that

$$\int (AH)(t, x; \varepsilon, c)^2\, dx \to 0,$$

$$\int |(AH)(t, x; \varepsilon, c)|^r\, dx \to 0,$$

as $c \to 0$ for $0 \le t \le s$; we omit its proof. This implies that the zero mass term H indeed makes no contribution. Thus the proof of the lemma is complete. $\square$

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