ON THE SYSTEM OF CURVES FOR WHICH THE METHOD OF MOMENTS IS THE BEST METHOD OF FITTING

by

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In Mr. R. A. Fisher's paper[1] on the mathematical foundations of theoretical statistics the following statement is found: "The method of moments applied in fitting Pearsonian curves has an efficiency exceeding 80 per cent. only in the restricted region for which β_2 lies between the limits of 2.65 and 3.42 and for which β_1 does not exceed 0.1. It was, of course, to be expected that the first two moments would have 100 per cent. efficiency for the normal curve, for they happen to be the optimum statistics for fitting the normal curve. That the moment coefficients β_1 and β_2 also tend to 100 per cent. efficiency in this region suggests that in the immediate neighborhood of the normal curve the departures from normality specified by the Pearsonian formulas agree with those of that system of curves for which the method of moments gives the solution of the method of maximum likelihood.

The system of curves for which the method of moments is the best method of fitting may easily be deduced, for if the frequency in the range dx be $y(x, \theta_1, \theta_2, \theta_3, \theta_4)dx$ then $\frac{\partial}{\partial\theta} \log y$ must involve x only as polynomials up to the fourth degree; consequently

$$y = e^{-a^2(x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4)},$$

[1] Philosophical Transactions of the Royal Society of London, vol. 222, series A (1921), p. 355.

the convergence of the probability integral requiring that the coefficient of x^4 should be negative, and the five quantities a, p_1, p_2, p_3, p_4 being connected by a single relation, representing the fact that the total probability is unity." It is with these curves having a fourth degree polynomial in the exponent that the present paper is concerned.

The first step in the study of this system of frequency functions is to find an expression for the value of the integral

$$I=\int_{-\infty}^{\infty} e^{-a^2(x^4+p_1x^3+p_2x^2+p_3x+p_4)}dx.$$

In other words, it is necessary to know how the integral depends on the parameters a, p_1, p_2, p_3, p_4.

Since a depends only on the unit of measure of x it will be sufficient for the moment to consider $a^2=1$. Furthermore a linear transformation on x leaves the value of the integral unchanged. If we replace x by $x-p_1/4$ the integral to be considered becomes

$$I=\int_{-\infty}^{\infty} e^{-(x^4+px^2+qx+r)}dx=k\int_{-\infty}^{\infty} e^{-(x^4+px^2+qx)}dx \quad \text{where } k=e^{-r}.$$

Consider now then the frequency curves $y=ke^{-(x^4+px^2+qx)}$. These curves are typically bimodal and may be classified according to the number and kind of modes. The positions of the modes are given by the solutions of the equation $\frac{dy}{dx}=0$, that is by the roots of the equation $4x^3+2px+q=0$. The discriminant of this cubic equation tells us that there will be three distinct real roots and thus two distinct maxima with a minimum between them for the curve, that is two distinct modes for the quartic exponential curve, if $-8p^3>27q^2, p<0$. Two roots will be real and equal if $-8p^3=27q^2, p<0$. The three roots will be real and equal if

$p=q=0$, the three roots being $x=0$. In the case of three real distinct roots, if two of the roots are equal in magnitude but opposite in sign then $q=0$ and the curve is symmetrical with respect to the y-axis. If $p=0, q\neq 0$ there will be one real root and two imaginary roots given by the three cube roots of $\frac{q}{4}$. That there will be a real maximum at the value of x given by the real cube root of $\frac{q}{4}$ is easily seen from the nature of the curve or by considering points at values of x on each side of this real cube root of $\frac{q}{4}$.

Hence the following classes of curves and their respective equations will be considered.

Type I: $y=ke^{-x^4}$.

The curve which is symmetrical with respect to the y-axis and has only one mode, this mode being at $x=0$.

Type II: $y=ke^{-(x^4-2bx^2)}, b>0$.

The curve which is symmetrical with respect to the y-axis and has two distinct modes at $x=\pm\sqrt{b}$.

Type III: $y=ke^{-(x^4-4cx)}, c\neq 0$.

The asymmetrical curve with one real mode at $x=\sqrt[3]{c}$.

Type IV: $y=ke^{-(x^4+px^2+qx)}$.

The general type of curve with the quartic exponent.

Type I:

First evaluate the definite integral

$$I_0=\int_{-\infty}^{\infty}e^{-x^4}dx=2\int_{0}^{\infty}e^{-x^4}dx.$$

Let $x = y^{1/4}$, $dx = (1/4)y^{-3/4}dy$. Then Then

$$I_0 = 2(1/4)\int_0^\infty y^{-3/4}e^{-y}dy = (1/2)\int_0^\infty y^{1/4-1}e^{-y}dy = \tfrac{1}{2}\Gamma(\tfrac{1}{4}).$$

Similarly it may be shown that

$$\int_0^\infty x^p e^{-x^q}dx = \frac{1}{q}\Gamma\left(\frac{p+1}{q}\right), p > -1.$$

$I_1 = \int_{-\infty}^\infty x e^{-x^4}dx = 0$ since the integrand is an odd function.

$$I_2 = \int_{-\infty}^\infty x^2 e^{-x^4}dx = \tfrac{1}{2}\Gamma(\tfrac{3}{4}).$$

$$I_3 = \int_{-\infty}^\infty x^3 e^{-x^4}dx = 0.$$

$$I_4 = \int_{-\infty}^\infty x^4 e^{-x^4}dx = \tfrac{1}{2}\Gamma(\tfrac{5}{4}) = \tfrac{1}{8}\Gamma(\tfrac{1}{4}).$$

$$I_{2n-1} = \int_{-\infty}^\infty x^{2n-1}e^{-x^4}dx = 0, \quad n = 1, 2, 3, \cdots\cdots$$

$$I_{2n} = \int_{-\infty}^\infty x^{2n}e^{-x^4}dx = \tfrac{1}{2}\Gamma\left(\frac{2n+1}{4}\right), \quad n = 1, 2, 3, \cdots\cdots$$

Hence if the total frequency is unity then $k = \dfrac{2}{\Gamma(\frac{1}{4})}$.

$$\mu_1 = I_1/I_0 = 0,$$

$$\mu_2 = I_2/I_0 = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} = \frac{\sqrt{2\pi}}{B(\frac{1}{4},\frac{1}{4})} = 0.3379891 \text{ approximately,}$$

$$\mu_3 = I_3/I_0 = 0,$$

$$\mu_4 = I_4/I_0 = 1/4,$$

$$\mu_{2n-1} = I_{2n-1}/I_0 = 0,$$

$$\mu_{2n} = I_{2n}/I_0 = \frac{\Gamma(\frac{2n+1}{4})}{\Gamma(\frac{1}{4})}.$$

Type II:

Consider the definite integral

$$I_0 = \int_{-\infty}^{\infty} e^{-(x^4-2bx^2)}dx, \quad b>0$$

Integrate by parts letting $u = e^{-(x^4 - 2bx^2)}$ and $dv = dx$. Then

$$I_0 = \int_{-\infty}^{\infty} (4x^4 - 4bx^2)\, e^{-(x^4 - 2bx^2)} dx$$

$$= 4\int_{-\infty}^{\infty} x^4 e^{-(x^4 - 2bx^2)} dx - 4b\int_{-\infty}^{\infty} x^2 e^{-(x^4 - 2bx^2)} dx.$$

Now obviously I_0 cannot be zero. Hence dividing by I_0 we find $1 = 4\mu_4 - 4b\mu_2$ and therefore

$$b = \frac{4\mu_4 - 1}{4\mu_2} = \frac{\mu_4 - .25}{\mu_2}. \qquad (\mu_2 \text{ cannot be zero}).$$

Now that b is known (calculated from the given data by this last formula) it is possible in any particular problem to find by mechanical quadrature the value of the integral I_0 to any desired degree of approximation. The simple rectangle formula with even a small number of ordinates known will give a good approximation.

Return now to the integration by parts just performed. The result takes the form

$$I_0 = \frac{d^2 I_0}{db^2} - 2b\frac{dI_0}{db} \qquad \text{or}$$

$$\frac{d^2 I_0}{db^2} - 2b\frac{dI_0}{db} - I_0 = 0$$

which is a Riccati[2] differential equation. Riccati's equation is

[2] Johnson's Differential Equations, p. 227.

$\frac{d^2v}{dx^2}+2ax^{m-1}\frac{dv}{dx}+a(m-1)x^{m-2}v=0.$ It has a solution capable of expression in finite form in terms of elementary functions if m is the reciprocal of an odd positive integer. In our equation $m=2$, $a=-1$ hence no finite solution for the differential equation is possible. That is no finite expression in terms of elementary functions can be obtained for I_0. The solution of the Ricatti equation here is

$$I_0=C_1\left(1+\frac{b^2}{2!}+\frac{5\cdot 1b^4}{4!}+\frac{9\cdot 5\cdot 1b^6}{6!}+\frac{13\cdot 9\cdot 5\cdot 1b^8}{8!}+\dots\dots\right)$$

$$+C_2b\left(1+\frac{3b^2}{3!}+\frac{7\cdot 3b^4}{5!}+\frac{11\cdot 7\cdot 3b^6}{7!}+\frac{15\cdot 11\cdot 7\cdot 3b^8}{9!}+\dots\right). \quad (1)$$

To determine C_1 and C_2 we note that when $b=0$ then

$$I_0=\frac{1}{2}\Gamma\left(\frac{1}{4}\right)=C_1$$

and that when $b=0$ then $\left.\frac{dI_0}{db}\right]_{b=0}=2\int_{-\infty}^{\infty}x^2e^{-x^4}dx=\Gamma\left(\frac{3}{4}\right)=C_2.$

It is worth while to make certain transformations on the differential equation

$$\frac{d^2I_0}{db^2}-2b\frac{dI_0}{db}-I_0=0.$$

Let $I_0=e^{b^2/2}v$. Then

$$\frac{d^2v}{db^2}-b^2v=0.$$

Let $b^2/2=t$. Then

$$\frac{d^2v}{dt^2}+\frac{1}{2t}\frac{dv}{dt}-v=0.$$

Let $v=t^{1/4}w$. Then

$$t^2\frac{d^2w}{dt^2}+t\frac{dw}{dt}-\left[t^2+(1/4)^2\right]w=0.$$

Let $t=ix$ where $i=\sqrt{-1}$. Then

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+\left[x^2-(1/4)^2\right]w=0.$$

This last equation is Bessel's differential equation[8] with $n=1/4$. Hence its solution is

$$w=AJ_{\frac{1}{4}}(x)+BJ_{-\frac{1}{4}}(x)=AJ_{\frac{1}{4}}\left(\frac{-ib^2}{2}\right)+BJ_{-\frac{1}{4}}\left(\frac{-ib^2}{2}\right)\text{where}$$

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left(1+\frac{x^2}{(n+1)\cdot 2^2}+\frac{x^4}{(n+1)(n+2)\cdot 2^4\cdot 2!}+\cdots\right)$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{\Gamma(n+1+r)\cdot r!}.$$

The above transformations give

$$I_0=(b^2/2)^{1/4}e^{b^2/2}w.$$

Hence

$$I_0=(b^2/2)^{1/4}e^{b^2/2}\left[AJ_{\frac{1}{4}}\left(\frac{-ib^2}{2}\right)+BJ_{-\frac{1}{4}}\left(\frac{-ib^2}{2}\right)\right].$$

[8] Johnson's Differential Equations, p. 235.

Setting $b=0$ in I_0 we find $B=\dfrac{\frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\sqrt[4]{2i}}$.

Setting $b=0$ in $\dfrac{dI_0}{db}$ we find $A=\dfrac{2^{3/4}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\sqrt[4]{-i}}$.

Putting in these values for A and B we find finally

$$I_0=e^{b^2/2}\left[\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left(1+\frac{b^4}{3\cdot 4}+\frac{b^8}{7\cdot 3\cdot 4^2\cdot 2!}+\cdots\cdots\right)\right.$$
$$\left.+\Gamma\left(\frac{3}{4}\right)b\left(1+\frac{b^4}{5\cdot 4}+\frac{b^8}{9\cdot 5\cdot 4^2\cdot 2!}+\cdots\cdots\right)\right]. \tag{2}$$

It is worth noting, for purposes of computation. that the expression (2) converges much more rapidly than the form (1) given above,[4] on account of the factoring out of $e^{b^2/2}$. In addition the series in (2) have the advantage that the powers increase by 4 instead of by 2 as in (1). It will be shown presently that ordinarily b is less than unity. But even for $b=1$ it will not be necessary to go further than the terms involving b^{12} to get at least seven decimal places of accuracy. For b less than one even fewer terms will suffice for this degree of accuracy.

[4] The form (1) is obtained immediately if we write

$$I_0=\int_{-\infty}^{\infty}e^{-(x^4-2bx^2)}dx=2\int_0^{\infty}e^{-(x^4-2bx^2)}dx=2\int_0^{\infty}e^{-x^4}e^{2bx^2}dx$$
$$=2\int_0^{\infty}e^{-x^4}\left(1+2bx^2+\frac{4b^2x^4}{2!}+\frac{8b^3x^6}{3!}+\cdots\cdots\right)dx,$$

assume term by term integration permissible, and make use of the fact

already mentioned that $\displaystyle\int_0^{\infty}x^p e^{-x^q}dx=\frac{1}{q}\Gamma\left(\frac{p+1}{q}\right)$.

From the point of view of the Ricatti differential equation it can be shown that $I_0 = \int_{-\infty}^{\infty} e^{-(x^4-2bx^2)}dx$ is the solution of

$\frac{d^2 I_0}{db^2} - 2b\frac{dI_0}{db} - I_0 = 0$ when the solution is sought in the form of a definite integral.[5] For the differential equation $b\phi(D)v + \psi(D)v = 0$ where $D = \frac{d}{db}$ and ϕ and ψ are polynomials in b with constant coefficients is satisfied by

$$v = c\int_{\alpha}^{\beta} e^{bt + \int\psi(t)T(t)dt}\,T(t)dt$$

where c is a constant, $T(t)$ is the reciprocal of $\phi'(t)$, and α and β are so chosen that for all values of b

$$\left[e^{bt+\int\psi(t)T(t)dt}\right]_{\alpha}^{\beta} = 0.$$

Let $b\phi(D)v + \psi(D)v = D^2v - 2bDv - v.$ Then

$$\phi(t) = -2t,\ \psi(t) = t^2 - 1,$$

$$T(t) = -\frac{1}{2t},\ \int\psi(t)T(t)dt = -\frac{1}{2}\left(\frac{t^2}{2} - \log t\right),$$

and $\left[e^{bt-\frac{1}{2}\left(\frac{t^2}{2}-\log t\right)}\right]_{\alpha}^{\beta} = 0$ for all values of b if $\alpha = 0, \beta = \infty$.

Hence

$$v = c\int_0^{\infty} e^{bt-\frac{1}{2}\left(\frac{t^2}{2}-\log t\right)}\left(-\frac{1}{2t}\right)dt$$

[5] A. R. Forsyth's Differential Equations, 6th edition. 1929. pp. 277-280.

$$= -\frac{c}{2}\int_0^\infty e^{-\frac{t^2}{4}-bt)}\frac{dt}{\sqrt{t}}\,.$$

Now let $t=2x^2$ and $c=-\sqrt{2}$. Then

$$v=2\int_0^\infty e^{-(x^4-2bx^2)}dx$$

$$=\int_{-\infty}^\infty e^{-(x^4-2bx^2)}dx$$

$$=I_0\,.$$

An idea of the variation of I_0 as a function of b can be obtained from the following table calculated from (1) for values of b at intervals of 0.1 from 0 to 1 and using $\Gamma(\frac{1}{4})=3.625610$, $\Gamma(\frac{3}{4})=1.225417$. The results are plotted in the accompanying graph, Fig. 1.

b	I_0
0.0	1.812 805
0.1	1.945 063
0.2	2.099 726
0.3	2.282 225
0.4	2.499 648
0.5	2.761 349
0.6	3.079 783
0.7	3.471 748
0.8	3.960 152
0.9	4.576 578
1.0	5.365 158

The modes are at $x=\pm\sqrt{b}$. Ordinarily the ordinate at the modes will not be greater than $e=2.7182$ times the ordinate at $x=0$. Hence ordinarily it will not be necessary to consider values of b greater than unity.

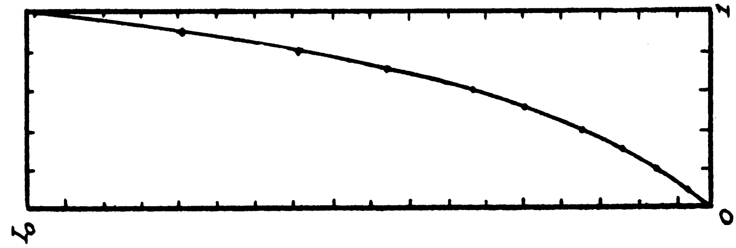


FIG. I

$$I_1 = \int_{-\infty}^{\infty} x e^{-(x^4-2bx^2)} dx = 0.$$

$$I_2 = \int_{-\infty}^{\infty} x^2 e^{-(x^4-2bx^2)} dx = \frac{1}{2}\frac{dI_0}{db}.$$

$$I_3 = \int_{-\infty}^{\infty} x^3 e^{-(x^4-2bx^2)} dx = 0.$$

$$I_4 = \int_{-\infty}^{\infty} x^4 e^{-(x^4-2bx^2)} dx = \left(\frac{1}{2}\right)^2 \frac{d^2 I_0}{db^2}.$$

$$I_{2n} = \int_{-\infty}^{\infty} x^{2n} e^{-(x^4-2bx^2)} dx = \left(\frac{1}{2}\right)^n \frac{d^n I_0}{db^n}, \quad n = 0, 1, 2, 3 \cdots$$

$$I_{2n+1} = \int_{-\infty}^{\infty} x^{2n+1} e^{-(x^4-2bx^2)} dx = 0, \quad n = 0, 1, 2, 3 \cdots$$

To find these derivatives one might use the relations[6]

$$\frac{dJ_n(x)}{dx} = \frac{n}{x}J_n(x) - J_{n+1}(x) = J_{n-1}(x) - \frac{n}{x}J_n(x).$$

But term by term differentiation is permissible and for this purpose it is simpler to use (1) rather than (2). We find

$$I_0 = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[1 + \frac{b^2}{2!} + \frac{5b^4}{4!} + \frac{9\cdot5b^6}{6!} + \frac{13\cdot9\cdot5b^8}{8!} + \cdots\right] + \Gamma\left(\frac{3}{4}\right)\left[b + \frac{3b^3}{3!} + \frac{7\cdot3b^5}{5!} + \frac{11\cdot7\cdot3b^7}{7!} + \cdots\right],$$

$$\frac{dI_0}{db} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[b + \frac{5b^3}{3!} + \frac{9\cdot5b^5}{5!} + \cdots\right] + \Gamma\left(\frac{3}{4}\right)\left[1 + \frac{3b^2}{2!} + \frac{7\cdot3b^4}{4!} + \cdots\right],$$

$$\frac{d^2I_0}{db^2} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[1 + \frac{5b^2}{2!} + \frac{9\cdot5b^4}{4!} + \cdots\right] + \Gamma\left(\frac{3}{4}\right)\left[3b + \frac{7\cdot3b^3}{3!} + \frac{11\cdot7\cdot3b^5}{5!} + \cdots\right].$$

Since $b \geqq 0$ hence I_0 and all its derivatives are greater than zero.

Now the total probability is to be unity hence take $k = \frac{1}{I_0}$.

$$\mu_1 = \frac{I_1}{I_0} = 0,$$

$$\mu_2 = \frac{I_2}{I_0} = \frac{\frac{dI_0}{db}}{2I_0}.$$

$$\mu_3 = \frac{I_3}{I_0} = 0,$$

$$\mu_4 = \frac{I_4}{I_0} = \frac{\frac{d^2I_0}{db^2}}{4I_0},$$

etc.

Type III:

$$y = ke^{-(x^4 - 4cx)}$$

[6] Whittaker and Watson, Modern Analysis, third edition, p. 360.

This curve is not symmetrical. But, obviously, changing c to $-c$ has the same effect as changing x to $-x$ or simply reversing the shape of the curve and the distribution from which it arises. Hence it will be necessary to consider only positive values of c. As stated already, it is easy to show that there is a real mode at the point given by x equal to the real cube root of c. If $c=1$ then $y=ke^3$, that is e^3 times the value of the ordinate at $x=0$. Hence usually c will not be as great as unity.

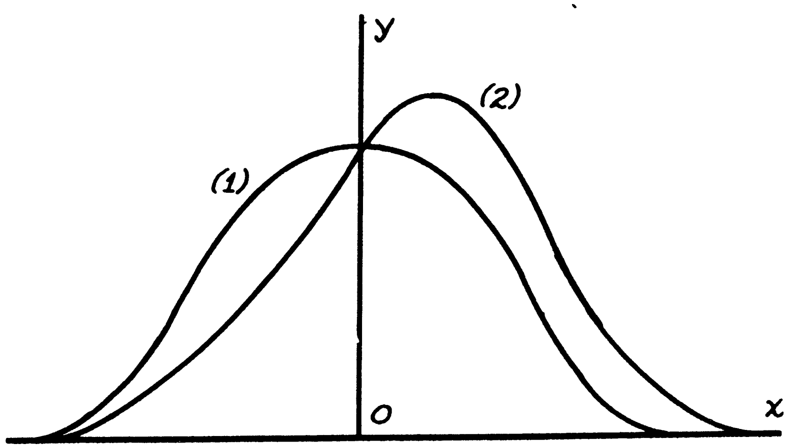


FIG. II

(1) $y=ke^{-x^4}$ (2) $y=ke^{-(x^4-4cx)}$.

Let $I_0=\int_{-\infty}^{\infty} e^{-(x^4-4cx)}dx.$

Integrate by parts letting $u=e^{-x^4}$, $dv=e^{4cx}dx.$
Then

$$I_0=\frac{1}{c}\int_{-\infty}^{\infty} x^3 e^{-(x^4-4cx)}dx.$$

Hence

$$c = \frac{\int_{-\infty}^{\infty} x^3 e^{-(x^4-4cx)}\,dx}{I_0}$$

$$= \mu'_3.$$

With this value of c calculated from the given data mechanical quadrature can be used to find an approximate value for I_0 Then let

$$K = \frac{1}{I_0}.$$

The result of the integration by parts could have been written in the form of the differential equation

$$\frac{d^3 I_0}{dc^3} - 64cI_0 = 0.$$

Conversely, it is easy to show that $I_0 = \int_{-\infty}^{\infty} e^{-(x^4-4cx)}dx$ is the definite integral form of the solution of the differential equation

$$\frac{d^3 v}{dc^3} - 64cv = 0.$$

For, here

$$\phi(D) = -64,\ \psi(D) = D^3,\ T(t) = -\frac{1}{64},\ \int \psi(t)T(t)dt = -\frac{t^4}{256},$$

and $\left[e^{ct-t^4/256}\right]_\alpha^\beta = 0$ for all values of c if $\alpha = -\infty, \beta = \infty$. Hence

$$V = -\frac{K}{64}\int_{-\infty}^{\infty} e^{ct - t^4/256}\,dt.$$

Now let $t = 4x$ and $c = -16$. Then

$$V = \int_{-\infty}^{\infty} e^{-(x^4 - 4cx)}\,dx$$

$$= I_0.$$

An expression for the value of I_0 can be obtained either by finding the series solution of the differential equation and determining the constants by setting $c = 0$ in I_0 and its derivatives, or by expanding e^{4cx} in series in the definite integral itself and then integrating term by term.

$$I_0 = \int_{-\infty}^{\infty} e^{-(x^4 - 4cx)}\,dx$$

$$= \int_{-\infty}^{0} e^{-(x^4 - 4cx)}\,dx + \int_{0}^{\infty} e^{-(x^4 - 4cx)}\,dx$$

$$= \int_{0}^{\infty} e^{-(x^4 + 4cx)}\,dx + \int_{0}^{\infty} e^{-(x^4 - 4cx)}\,dx$$

$$= \int_{0}^{\infty} e^{-x^4}(e^{-4cx} + e^{4cx})\,dx = 2\int_{0}^{\infty} e^{-x^4}\cosh(4cx)\,dx$$

$$= \int_{0}^{\infty} e^{-x^4}\left[1 + \frac{(4cx)^2}{2!} + \frac{(4cx)^4}{4!} + \frac{(4cx)^6}{6!} + \cdots\cdots\right] dx$$

$$= 2\sum_{n=0}^{\infty} \frac{(4c)^{2n}}{(2n)!} \int_0^{\infty} x^{2n} e^{-x^4} dx$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(4c)^{2n}}{(2n)!} \Gamma \frac{(2n+1)}{4}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left[1 + \frac{(4c)^4}{4\cdot 4!} + \frac{5(4c)^8}{4^2\cdot 8!} + \frac{9\cdot 5(4c)^{12}}{4^3\cdot 12!} + \cdots\cdots\right]$$

$$+ \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\left[\frac{(4c)^2}{2!} + \frac{3(4c)^6}{4\, 6!} + \frac{7\cdot 3(4c)^{10}}{4^2\cdot 10!} + \cdots\cdots\cdot\right].$$

These series may be differentiated term by term to obtain the derivatives of I_o and hence

$$I_1 = \int_{-\infty}^{\infty} x e^{-(x^4 - 4cx)} dx = \frac{1}{4}\frac{dI_o}{dc},$$

$$I_2 = \int_{-\infty}^{\infty} x^2 e^{-(x^4 - 4cx)} dx = \left(\frac{1}{4}\right)^2 \frac{d^2 I_o}{dc^2},$$

$$I_3 = \int_{-\infty}^{\infty} x^3 e^{-(x^4 - 4cx)} dx = \left(\frac{1}{4}\right)^3 \frac{d^3 I_o}{dc^3},$$

$$I_4 = \int_{-\infty}^{\infty} x^4 e^{-(x^4 - 4cx)} dx = \left(\frac{1}{4}\right)^4 \frac{d^4 I_o}{dc^4},$$

etc.

If x is replaced by $x - \sqrt[3]{c}$ the effect is to translate the modal value of x to the origin. The equation of the curve then becomes

$$y = \frac{1}{I_0} e^{-(x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4)}$$

where

$$c_1 = -4\sqrt[3]{c},$$

$$c_2 = 6\sqrt[3]{c^2},$$

$$c_3 = -8c,$$

$$c_4 = 5c\sqrt[3]{c}.$$

Type IV:

$$y = ke^{-(x^4 + px^2 + qx)}$$

Consider the definite integral

$$I_0 = \int_{-\infty}^{\infty} e^{-(x^4 + px^2 + qx)} dx.$$

If $p = q = 0$ we get Type I. If $p \neq 0, q = 0$ we get Type II. If $p = 0$, $q \neq 0$ we get Type III. Hence consider now $p \neq 0$, $q \neq 0$.

Integrate I_0 by parts with $u = e^{-(x^4 + px^2)}$ and $dv = e^{-qx} dx$. Then

$$I_0 = -\frac{1}{q}\int_{-\infty}^{\infty} (4x^3 + 2px) e^{-(x^4 + px^2 + qx)} dx$$

$$= -\frac{4}{q}\int_{-\infty}^{\infty} x^3 e^{-(x^4 + px^2 + qx)} dx - \frac{2p}{q}\int_{-\infty}^{\infty} x e^{-(x^4 + px^2 + qx)} dx.$$

Now divide by I_0 and multiply by q. Then

$$q = -4\mu_3' - 2p\mu_1'.$$

Begin again with I_0 and integrate by parts, this time with

$u = e^{-(x^4+px^2+qx)}$ and $dv = dx$. Then

$$I_0 = \int_{-\infty}^{\infty} (4x^4 + 2px^2 + qx) e^{-(x^4+px^2+qx)} dx$$

$$= 4\int_{-\infty}^{\infty} x^4 e^{-(x^4+px^2+qx)} dx + 2p\int_{-\infty}^{\infty} x^2 e^{-(x^4+px^2+qx)} dx$$

$$+ q\int_{-\infty}^{\infty} x e^{-(x^4+px^2+qx)} dx.$$

Divide by I_0. Then

$$1 = 4\mu_4' + 2p\mu_2' + q\mu_1'.$$

Now substitute $q = -4\mu_3' - 2p\mu_1'$ in $1 = 4\mu_4' + 2p\mu_2' + q\mu_1'$ and we get

$$p = \frac{1 + 4\mu_1'\mu_3' - 4\mu_4'}{2(\mu_2' - \mu_1'^2)},$$

(3)

$$q = -4\mu_3' - 2p\mu_1' = -4\mu_3' - \mu_1'\left(\frac{1 + 4\mu_1'\mu_3' - 4\mu_4'}{\mu_2' - \mu_1'^2}\right).$$

The result of the two integrations by parts can be written in the form of two simultaneous partial differential equations. They are

$$4\,\frac{\partial^2 I_0}{\partial p\,\partial q} - 2p\,\frac{\partial I_0}{\partial q} + qI_0 = 0$$

$$4\,\frac{\partial^2 I_0}{\partial p^2} - 2p\,\frac{\partial I_0}{\partial p} - q\,\frac{\partial I_0}{\partial q} - I_0 = 0.$$

Let $S_0 = \int_{-\infty}^{\infty} e^{-(x^4+px^2)}\,dx.$ Then

$$S_{2n} = \int_{-\infty}^{\infty} x^{2n} e^{-(x^4+px^2)}\,dx = (-1)^n\,\frac{d^n S_0}{dp^n},$$

$$S_{2n+1} = \int_{-\infty}^{\infty} x^{2n+1} e^{-(x^4+px^2)}\,dx = 0.$$

$$I_0 = \int_{-\infty}^{\infty} e^{-(x^4+px^2+qx)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(x^4+px^2)} e^{-qx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(x^2+px^2)}\left[1-(qx)+\frac{(qx)^2}{2!}-\frac{(qx)^3}{3!}+\cdots\right]dx$$

$$= \sum_{i=0}^{\infty} \frac{q^{2i}}{(2i)!} S_{2i} \, .$$

$$I_{2n} = \int_{-\infty}^{\infty} x^{2n} e^{-(x^4+px^2+qx)} dx \, .$$

$$= (-1)^n \frac{\partial^n I_0}{\partial p^n}$$

$$= \sum_{i=0}^{\infty} \frac{q^{2i}}{(2i)!} S_{2n+2i}$$

$$I_{2n+1} = \int_{-\infty}^{\infty} x^{2n+1} e^{-(x^4+px^2+qx)} dx$$

$$= (-1) \frac{\partial I_{2n}}{\partial q}$$

$$= \sum_{i=0}^{\infty} (-1) \frac{q^{2i-1}}{(2i-1)!} S_{2n+2i}$$

When the values of p and q are calculated from the data of any given problem by the formulas (3) then values for I_0, I_1, I_2, I_3, I_4 . etc. can be obtained by mechanical quadrature.

For two real, distinct modes $-8p^3>27q^2; (p<0)$. Hence if $-2<p<0$ then $1.54>q>1.54$. If $-8p^3=27q^2$ then one mode flattens forming a point of inflexion with a horizontal tangent at the minimum point. Changing q to $-q$ has the same effect as changing x to $-x$ and hence q is a component of skewness of the curve. If the curve is so placed that the sum of the values of x at the modes and at the minimum point is zero then the equation of the curve will be of the form

$$y = ke^{-(x^4+px^2+qx)}.$$

If now we change the scale of x by replacing x by $x\sqrt{a}$ then we are led to the functions of the form

$$y = ke^{-a^2(x^4+px^2+qx)}.$$

Performing the two integrations by parts, as before, on the integral

$$I = \int_{-\infty}^{\infty} e^{-a^2(x^4+px^2+qx)}dx$$

leads to the relations

$$
\begin{aligned}
q &= -4\mu_3' - 2p\mu_1' = -4(\mu_3 + 3M\mu_2 + M^3) - 2Mp, \\
p &= \frac{\frac{1}{a^2} + 4\mu_1'\mu_3' - 4\mu_4'}{2(\mu_2' - \mu_1'^2)} = \frac{\frac{1}{a^2} - 4(\mu_4 + 3M\mu_3 + 3M^2\mu_2)}{2\mu_2}.
\end{aligned}
\qquad (4)
$$

If

$$\int_{-\infty}^{\infty} x^n e^{-(x^4+px^2+qx)}dx = I_n'(p,q), \quad n = 0, 1, 2, 3 \cdots\cdots$$

then

$$\int_{-\infty}^{\infty} x^n e^{-a^2(x^4+px^2+qx)}dx = \frac{1}{a^{(n+1)/2}} I_n'(ap, a^{3/2}q) = I_n(ap, a^{3/2}q).$$

In particular,

$$I_0 = I_0(o, o) = \int_{-\infty}^{\infty} e^{-a^2x^4}dx = \frac{1}{2\sqrt{a}}\Gamma\left(\frac{1}{4}\right),$$

$$I_{2n-1} = I_{2n-1}(0,0) = 0,$$

$$I_{2n} = I_{2n}(0,0) = \int_{-\infty}^{\infty} x^{2n} e^{-a^2x^4} dx = \frac{1}{2a^{(2n+1)/2}} \Gamma\left(\frac{2n+1}{4}\right).$$

$$\mu_1 = I_1 / I_0 = 0,$$

$$\mu_2 = I_2 / I_0 = \frac{1}{a} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})},$$

$$\mu_3 = I_3 / I_0 = 0,$$

$$\mu_4 = I_4 / I_0 = \frac{1}{4a^2}$$

$$\mu_{2n-1} = I_{2n-1} / I_0 = 0,$$

$$\mu_{2n} = I_{2n} / I_0 = \frac{1}{a^n} \frac{\Gamma(\frac{2n+1}{4})}{\Gamma(\frac{1}{4})}.$$

In the case of Type I when

$$y = y_0 e^{-a^2x^4}, \quad y_0 = \frac{1}{2\sqrt{a}} \Gamma\left(\frac{1}{4}\right),$$

$p = q = 0$ and hence from (4), or as can be shown directly, $a^2 = \frac{1}{4\mu_4}$.

In Type II,

$$y = y_0 e^{-a^2(x^4 + px^2)}, \quad \frac{1}{y_0} = \int_{-\infty}^{\infty} e^{-a^2(x^4 + px^2)} dx,$$

$q=0$ and hence $a^2=\dfrac{1}{2p\mu_2+4\mu_4}$. In the Type III where

$y=y_0e^{-a^2(x^4+qx)}, \dfrac{1}{y_0}=\displaystyle\int_{-\infty}^{\infty}e^{-a^2(x^4+qx)}dx,\quad p=0$ and hence

$$a^2=\frac{1}{4(\mu_4+3M\mu_3+3M^2\mu_2)}.$$

In general, since p and q are determined when the modes and minimum point of the curve are known, theoretically at least, a^2 is fixed by the relations (4). In practice, however, this would mean that the accuracy in the determination of a^2 would be contingent upon the accuracy with which the modes and minimum point are determined. Hence other methods for fixing a^2 will be required in general. Now if in $I_0(ap, a^{3/2}q)$ we replace p and q by (4) which involve only a^2 and quantities calculable from the given data we have a function of a alone, say $f(a)$. It will be sufficient then if we determine a value of a such that $f(a)=N$ where N is the total given frequency. Then fix p and q by (4) and the modes and minimum point by $4x^3+2px+q=0$.

The points of inflexion are found from the equation

$$\frac{d^2y}{dx^2}=0$$

and for Type I are given by $x^4=\dfrac{3}{4a^2}$. Hence

$$x=\frac{\pm 0.930605}{\sqrt{a}}$$

approximately. For Type II they are given by $8a^2x^6+8a^2px^4+2(a^2p^2-3)x^2-p=0$. For Type III they are given by $16a^2x^6+8a^2qx^3-12x^2+a^2q^2=0$. And in general they are given by roots of the equation

$$16a^2x^6+16a^2px^4+8a^2qx^3+4(a^2p^2-3)x^2+4a^2pqx+a^2q^2-2p=0.$$

It will be noticed that the distribution given by

$$y = y_0 e^{-a^2(x^4 + px^2 + qx)}$$

can have the Mean at the origin if and only if $q = 0$. that is. if and only if the distribution is symmetrical. Now replace x by $x - m$. The area remains the same and hence also y_0 .
The equation then is

$y = y_0 e^{-a^2(x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4)}$ where

$p_1 = -4m,$

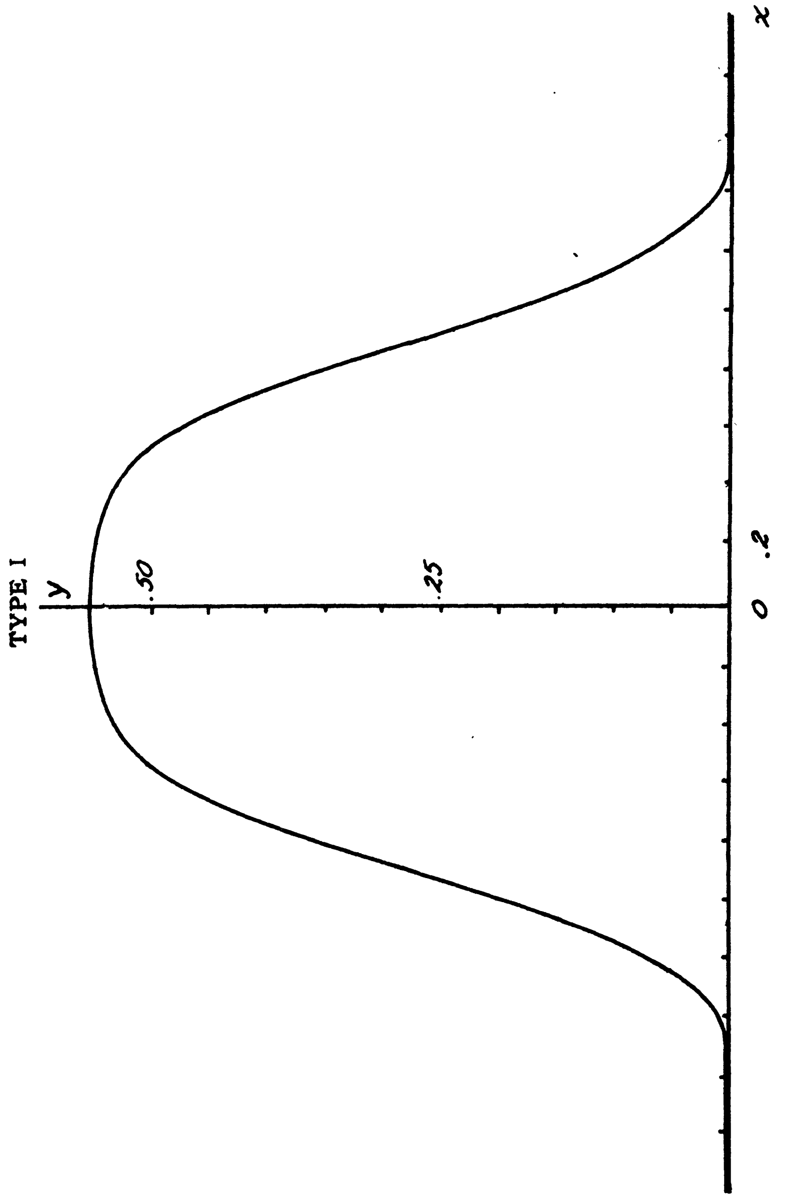
$p_2 = 6m^2 + p,$

$p_3 = q - 2mp - 4m^3,$

$p_4 = m^4 + m^2 p - mq,$

and p and q are given by the relations (4) above. An integration by parts with $u = e^{-a^2(x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4)}$ shows that

$$a^2(4\mu_4' + 3p_1\mu_3' + 2p_2\mu_2' + p_3\mu_1') = 1.$$


TYPE I
y
.50
.25
0
.2
x

This small beginning of the study of the system of frequency curves with the quartic exponent will be concluded here with the construction of artificial illustrations of Types I and II.

TYPE I

$$y=\frac{2}{\Gamma(\frac{1}{4})}e^{-x^4}, \quad \log_{10}\Gamma(\tfrac{1}{4})=0.5593811,$$

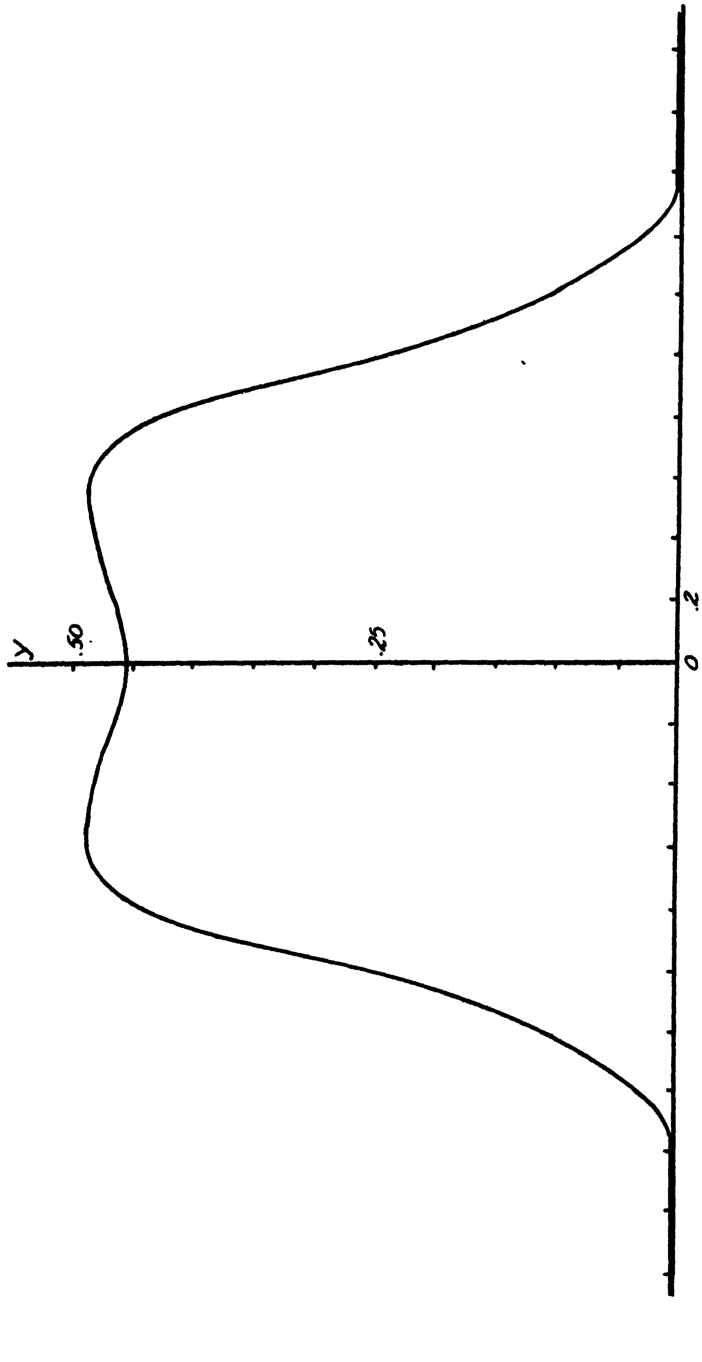
x	y	x^2y	x^4y
0.0	0.5516313	0.0000000	0.0000000
0.1	.5515762	.0055158	.0000551
0.2	.5507494	.0220300	.0008812
0.3	.5471811	.0492463	.0044322
0.4	.5376888	.0860302	.0137648
0.5	.5182096	.1295524	.0323881
0.6	.4845787	.1744483	.0628014
0.7	.4338852	.2126037	.1041758
0.8	.3662367	.2343915	.1500105
0.9	.2862255	.2318426	.1877925
1.0	.2029338	.2029338	.2029338
1.1	.1275846	.1543774	.1867966
1.2	.0693579	.0998754	.1438205
1.3	.0317147	.0535978	.0905803
1.4	.0118376	.0232017	.0454753
1.5	.0034917	.0078563	.0176767
1.6	.0007861	.0020124	.0051518
1.7	.0001301	.0003760	.0010866
1.8	.0000152	.0000492	.0001596
1.9	.0000012	.0000043	.0000156
2.0	.0000001	.0000004	.0000016
	5.2758155	1.6899455	1.2500000

$$\text{Total frequency} = \frac{2(5.2758155)-0.5516313}{10} = 1.0000000$$

$$\mu_2 = \frac{2(1.6899455)-0.0000000}{10} = 0.3379891$$

$$\mu_4 = \frac{2(1.2500000)-0.0000000}{10} = 0.2500000.$$

TYPE II



a=1, p=.5, q=0.

$$y = \frac{1}{2.187099} e^{-(x^4 - 0.5x^2)}$$

TYPE II

x	y	$x^2 y$	$x^4 y$
0.0	0.4572267	0.0000000	0.0000000
0.1	.4594725	.0045947	.0000459
0.2	.4657175	.0186287	.0007451
0.3	.4744135	.0426972	.0038427
0.4	.4827888	.0772462	.0123594
0.5	.4867153	.1216788	.0304197
0.6	.4808614	.1731101	.0623196
0.7	.4594725	.2251415	.1103193
0.8	.4180410	.2675462	.1712296
0.9	.3556970	.2881146	.2333728
1.0	.2773220	.2773220	.2773220
1.1	.1936552	.2343228	.2835306
1.2	.1181056	.1700721	.2449038
1.3	.0611958	.1034209	.1747813
1.4	.0261429	.0512401	.1004306
1.5	.0089145	.0200576	.0451297
1.6	.0023433	.0059988	.0153571
1.7	.0004575	.0013222	.0038211
1.8	.0000638	.0002067	.0006697
1.9	.0000061	.0000220	.0000795
2.0	.0000004	.0000016	.0000064
	5.2286133	2.0827448	1.7706859

Total frequency $= \dfrac{2(5.2286133) - 0.4572267}{10} = 1.000000,$

$$\mu_2 = \frac{2(2.0827448) - 0.0000000}{10} = 0.41654896$$

$$\mu_4 = \frac{2(1.7706859) - 0.0000000}{10} = 0.35413718.$$

From relations (1) or (2) it is found that when $b = 0.25$,

(i.e. $p = -0.5, q = 0$)

then $I_0 = 2.187099$. Conversely, the formula $b = \dfrac{\mu_4 - 0.25}{\mu_2}$

gives, retaining six decimal places, $b = 0.250000.$

(To be Continued in May Issue)

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