MOMENTS OF ANY RATIONAL INTEGRAL ISOBARIC SAMPLE MOMENT FUNCTION

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Introduction

The problem of moments of moments has been investigated by a number of authors. The assumption of an infinite universe (or that of a finite universe with replacements) permits the application of the "algebraic" method, the method of semi-invariants as introduced by Thiele (1) and developed by C. C. Craig (2) and the combinatorial analysis method introduced by R. A. Fisher (3) and used by N. St. Georgescu (4). A combinatorial analysis method has the particular advantage that it enables one to compute separate terms of a given formula.

The formulae for moments of moments have been simplified through the use of new moment functions. Thiele introduced the half-invariant (1) which resulted in considerable condensation. More recently Prof. R. A. Fisher (3) has introduced the sample function k whose expected value is a half invariant. The most compact formulization presented thus far is his formulation of the half invariants of the sample k_r in terms of the half invariants of the universe. This very compactness, however, makes it difficult to compare results with those expressed in the more conventional sample functions. Dr. Wishart has written a paper (7) in which he shows, among other things, how the Fisher results can be translated to the more conventional (Craig) results and vice versa, but such translation is in general no simple matter. It appears that the Fisher results are not immediately useful to the statistician who desires the formulae to be expressed in terms of the usual sample moment function. On the other hand the Fisher formulization is a remarkable discovery toward that harmony which must be naturally inherent in the field of moments of moments. Soper (6, 111) expressed the general situation when he wrote, "If the terrifying overgrowth of algebraic formulation accompanying this branch of statistical inquiry is destined to have a chief utility in induction and going back to causes, then perhaps Dr. Fisher's way of estimating a sample will prove to be most fertile, but if it is to be applied to problems of deduction, say to problems of successive eventuation such as propagation, then Mr. Craig's plain moments seem to have a firmer hold on the exigencies of time."

It would appear then that the Fisher formulae and the Craig formulae are both needed. Georgescu (4) showed a partial connection between them in applying to the m functions a combinatory analysis somewhat similar to that applied by R. A. Fisher to the k function. It is the purpose of the present paper to work out a combinatorial procedure for a more general sample function so that either the Fisher or Georgescu combinatorial results come out as special cases. In making such a generalization no limitation is placed on the sample function except that it be rational integral and that all terms are of the same weight. Thus the results are applicable to m_r, $m_r + k_r$, $m_r k_r$, etc. as well as to m_r and k_r although they are not applicable to $\sqrt{m_r}$ or $\frac{m_r}{k_r}$. In this way the important formulae for the moments of a new sample moment function will be available by simple substitution as soon as any such new function is defined by a rational integral isobaric expansion of power sums.

It is thus the purpose of this paper to determine the moments of a general moment function of the sample. This is done by keeping the multipliers of the various partitions of power sums indefinite until all manipulation is complete. It is then possible to assign the definite values of these multipliers which are associated with the desired sample function and to obtain the moment of the desired moment function in this way. Thus the Fisher result $\kappa(42)$, and the Craig result $S_{11}(\nu_4, \nu_2)$ are special cases of the new result $\lambda_{11}(f_4, f_2)$. It is obvious that it is not possible to carry the results using these general moment functions as far as Fisher and Wishart (3), (5), (7), have carried the results of the decidedly advantageous (from the standpoint of simplicity of result) k function and yet it is surprising to find the simplicity which can be obtained in the general case. Incidentally the introduction of the more general symbols clarifies the successive steps of the partition analysis which are somewhat confusing in any specific case because of the insertion of the value of the coefficients of the power sums in which the sample moment function is expressed.

This paper is divided into three parts. The first part includes the necessary definitions, the basic formulae, and the general development of the algebraic method. In order to facilitate the algebraic work there is inserted a table giving the expected values of all possible partition products of power sums whose weight $\leqq 8$. The second part deals with the different sample functions which might be used. The third part gives a list of the various partition formulae, of weight $\leqq 8$, which contain no unit parts and shows how these can be used in writing the chief variations of the formulae for moments of moments.

Part I

1. **General Moment Functions.** Different moment functions have been defined in various ways, but all moment functions have in common the property that they may be expressed in terms of the power sums. It appears sensible to use this expression in terms of power sums as the working algebraic definition of moment functions. For example the function k_3, which is defined by R. A. Fisher to be that function of the sample whose expected value is the third cumulant (half invariant) is to be given the working definition of

$$k_3 = \frac{n(3)}{(n-1)(n-2)} - \frac{3(2)(1)}{(n-1)(n-2)} + \frac{2(1)(1)(1)}{n(n-1)(n-2)}$$

where the numerical expressions in parentheses indicate power sums of the sample.

Every term in the definition of a sample function has a "weight" which is equal to the sum of the power sums whose product is indicated by the term. Thus the weight of each of the terms of k_3 is 3. If all the terms of a given moment function have the same weight, the function is called isobaric and the weight of the function is equal to the weight of each term. Thus k_3 is an isobaric moment function and its weight is 3. Since all the functions so far proposed are isobaric we limit this generalization of moment functions to isobaric moment functions although it is possible that a more complex analysis could be worked out for non-isobaric functions.

Generality demands the inclusion of every possible partition product of power sums. Such generality can be obtained by writing

$$
\begin{aligned}
f_1 &= a_1'(1) \\
f_2 &= a_2'(2) + a_{11}'(1)^2 \\
f_3 &= a_3'(3) + a_{21}'(2)(1) + a_{111}'(1)^3 \\
f_4 &= a_4'(4) + a_{31}'(3)(1) + a_{2^2}'(2)^2 + a_{21^2}'(2)(1)^2 + a_{1^4}(1)^4
\end{aligned}
$$

and in general

$$f_r = \sum a'_{p_1^{\pi_1} p_2^{\pi_2} p_s^{\pi_s}} (p_1)^{\pi_1} (p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$$

where $(p_1)^{\pi_1} (p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$ indicates any partition product of power sums, $a'_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is its coefficient and the summation is taken for every possible partition. The number of parts of the partition is $\rho = \Sigma\pi$. It may be assumed, without loss of generality, that the partition is ordered, i.e.

$$p_1 \geqq p_2 \geqq p_3 \geqq \cdots \geqq p_s.$$

A natural numerical coefficient of each term is the number of ways the r units can be collected to form the given partition. This value is given by

$$\begin{pmatrix} 1^r \\ p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \end{pmatrix} = \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!}.$$

If we set

$$a'_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

the definition of f_r becomes

$$f_r = \sum \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$$

In the present paper the capital letters are used to represent the corresponding functions of the universe *as defined by the corresponding power sums of the universe.* Thus

$$F_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (P_1)^{\pi_1} \cdots (P_s)^{\pi_s}$$

represents the corresponding function of the universe. In the case of the moment about the mean and the semi-invariant the Greek letters μ and λ have been used to represent the corresponding function of the universe. In the case of functions whose notation is quite widely established, it is preferable to use the conventional notation, but in introducing new functions it appears wise to use the relationship between small and capital letters since the correspondence between the English and Greek alphabets is not exactly one to one. It should be particularly noticed that this notation does not agree with a previously accepted scheme of using the small English letter to indicate the function whose expected value is indicated by the corresponding Greek letter. In the present paper it is not the expected value property which serves as the basis of notation but rather the definition of the function in terms of the partition products of power sums.

2. The Working Definition of Moments About a Fixed Point. The sample functions defined by

$$m_1' = \frac{(1)}{n}, \qquad m_2' = \frac{(2)}{n}, \qquad m_3' = \frac{(3)}{n}, \cdots, m_r' = \frac{(r)}{n}$$

are obtained from f_r by placing

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} \frac{1}{n} \text{ when } s = 1, \pi_1 = 1, \text{ and } p_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

The Greek μ' is used to indicate the corresponding function of the universe.

3. The Working Definition of Moments About the Mean. The moments about the mean are defined by

$$m_1' = \frac{(1)}{n}, \qquad m_2 = \frac{(2)}{n} - \frac{(1)(1)}{n^2},$$

$$m_3 = \frac{(3)}{n} - \frac{3(2)(1)}{n^2} + \frac{2(1)^3}{n^3}, \qquad m_4 = \frac{(4)}{n} - \frac{4(3)(1)}{n^2} + \frac{6(2)(1)^2}{n^3} - \frac{3(1)^4}{n^4}$$

and in general m_r is obtained from f_r by placing

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} \frac{1}{n} \text{ if } s = 1,\ \pi_1 = 1, \text{ and } p_1 = r. \\ \frac{(-1)^{\pi_2}}{(n)^{1+\pi_2}} \text{ if } p_1 > 1,\ \pi_1 = 1,\ s = 2, \text{ and } p_2 = 1. \\ \frac{(-1)^{r-1}(r-1)}{n^r} \text{ if } p_1 = 1,\ s = 1, \text{ and } \pi_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

The corresponding moments of the universe are indicated by the conventional μ. For conciseness moments about the mean are referred to as "moments."

4. **The Working Definition of the Half Invariants.** The half invariant moment functions of Thiele, as applied to the sample power sums are [see C. C. Craig (2, 7–10) and Frisch (12, 20–21)].

$$l_1' = \frac{(1)}{n}, \qquad l_2 = \frac{(2)}{n} - \frac{(1)(1)}{n^2}, \qquad l_3 = \frac{(3)}{n} - \frac{3(2)(1)}{n^2} + \frac{2(1)^3}{n^3}$$

$$l_4 = \frac{(4)}{n} - \frac{4(3)(1)}{n^2} - \frac{3(2)^2}{n^2} + \frac{12(2)(1)^2}{n^3} - \frac{6(1)^4}{n^4}$$

and in general

$$l_r = \sum \frac{(-1)^{\rho-1}(\rho-1)!}{n^\rho} \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1}(p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$$

so that

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \frac{(-1)^{\rho-1}(\rho-1)!}{n^\rho}.$$

The corresponding moments of the universe are indicated, after Thiele (1) and Craig (2), by λ. R. A. Fisher (3) used κ while Georgescu (4) used s.

In the present paper these functions are referred to as "Thiele moments."

5. **The k Functions of R. A. Fisher.** The k statistics of R. A. Fisher are defined in terms of the sample power sums by

$$k_1' = \frac{(1)}{n}, \qquad k_2 = \frac{(2)}{n-1} - \frac{(1)^2}{n(n-1)},$$

$$k_3 = \frac{n(3)}{(n-1)(n-2)} - \frac{3(2)(1)}{(n-1)(n-2)} + \frac{2(1)^3}{n^{(3)}}$$

$$k_4 = \frac{n(n+1)(4)}{(n-1)^{(3)}} - \frac{4(n+1)(3)(1)}{(n-1)^{(3)}} - \frac{3(2)^2}{(n-2)^{(2)}} + \frac{12(2)(1)^2}{(n-1)^{(3)}} - \frac{6(1)^4}{n^{(4)}}.$$

These values and values for k_5 and k_6 are given by R. A. Fisher (3, 203–4) while algebraic methods of attaining them are presented in sections 16, 17. They are referred to as Fisher moments. The corresponding functions of the universe, if used, would be represented by K_r.

6. **The h Function.** Just as Fisher introduced a sample function whose expected value is a Thiele moment of the universe, so it is possible to introduce a function whose expected value is a moment of the universe. Such a function is defined by

$$h_1' = \frac{(1)}{n}, \qquad h_2 = \frac{(2)}{n-1} - \frac{(1)^2}{n(n-1)},$$

$$h_3 = \frac{n(3)}{(n-1)(n-2)} - \frac{3(2)(1)}{(n-1)(n-2)} + \frac{2(1)^3}{n^{(3)}}$$

$$h_4 = \frac{(n^2-2n+3)(4)}{(n-1)^{(3)}} - \frac{4(n^2-2n+3)(3)(1)}{n^{(4)}} - \frac{3(2n-3)(2)^2}{n^{(4)}} + \frac{6(2)(1)^2}{(n-1)^{(3)}} - \frac{3(1)^4}{n^{(4)}}.$$

Methods of obtaining the expansion of this function in terms of power sums are presented in section 18. The corresponding function of the universe, if it were used, would be represented by H_r.

7. **Other Moment Functions.** It is possible to obtain an indefinite number of moment functions. For example one might define a function of weight 2 whose variance equals μ_4, (or μ_2^2). It is possible by the methods of this paper to find expressions for such moments.

For reference purposes Table I is provided showing the values of a for each partition of weight <6 for the functions m', m, l, h, k. The values of

$$\binom{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}$$

are also inserted, in the left hand column, so that it is possible to read from the table the values for $f = m_r'$, m_r, l_r, k_r when $r < 6$.

8. **Products of f Functions.** The product of two or more isobaric functions is also isobaric and of weight equal to the sum of the weights of the functions. Thus

$$f_2 f_1 = [a_2(2) + a_{11}(1)(1)][a_1(1)] = a_2 a_1(2)(1) + a_{11}a_1(1)^2$$

$$f_2 f_1^2 = a_2 a_1^2(2)(1)^2 + a_{11}a_1^2(1)^4.$$

In multiplying f_{r_1} by f_{r_2} any term of f_{r_1} is of weight r_1 and when it is multiplied by any term of weight r_2, the result is a term of weight $r_1 + r_2$.

TABLE I

Coefficients of Products of Power Sums in the Expansion of Different Moment Functions

Numerical coefficient	a	m'_r	m_r	l_r	k_r	h_r
1	a_1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
1	a_2	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$
1	a_{11}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$\frac{-1}{n^{(2)}}$	$\frac{-1}{n^{(2)}}$
1	a_3	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n}{(n-1)^{(2)}}$	$\frac{n}{(n-1)^{(2)}}$
3	a_{21}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$\frac{-1}{(n-1)^{(2)}}$	$\frac{-1}{(n-1)^{(2)}}$
1	a_{111}	0	$\frac{2}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{n^{(3)}}$	$\frac{2}{n^{(3)}}$
1	a_4	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n(n+1)}{(n-1)^{(3)}}$	$\frac{n^2-2n+3}{(n-1)^{(3)}}$
4	a_{31}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$-\frac{(n+1)}{(n-1)^{(3)}}$	$-\frac{n^2-2n+3}{n^{(4)}}$
3	a_{22}	0	0	$\frac{-1}{n^2}$	$\frac{-1}{(n-2)^{(2)}}$	$-\frac{2n-3}{n^{(4)}}$
6	a_{211}	0	$\frac{1}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{(n-1)^{(3)}}$	$\frac{1}{(n-1)^{(3)}}$
1	a_{1111}	0	$\frac{-3}{n^4}$	$\frac{-6}{n^4}$	$\frac{-6}{n^{(4)}}$	$\frac{-3}{n^{(4)}}$
1	a_5	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n^2(n+5)}{(n-1)^{(4)}}$	$\frac{n(n^2-5n+10)}{(n-1)^{(4)}}$
5	a_{41}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$-\frac{n(n+5)}{(n-1)^{(4)}}$	$-\frac{n^2-5n+10}{(n-1)^{(4)}}$
10	a_{32}	0	0	$\frac{-1}{n^2}$	$-\frac{n(n-1)}{(n-1)^{(4)}}$	$-\frac{n-2}{(n-1)^{(4)}}$

TABLE I—*Concluded*

Numerical coefficient	a	m'_r	m_r	l_r	k_r	h_r
10	a_{311}	0	$\frac{1}{n^3}$	$\frac{2}{n^3}$	$\frac{2(n+2)}{(n-1)^{(4)}}$	$\frac{n^2-4n+8}{n^{(5)}}$
15	a_{221}	0	0	$\frac{2}{n^3}$	$\frac{2(n-1)}{(n-1)^{(4)}}$	$+\frac{(2n-4)}{n^{(5)}}$
10	a_{2111}	0	$\frac{-1}{n^4}$	$\frac{-6}{n^4}$	$-\frac{6}{(n-1)^{(4)}}$	$-\frac{1}{(n-1)^{(4)}}$
1	a_{11111}	0	$\frac{4}{n^5}$	$\frac{24}{n^5}$	$\frac{24}{n^{(5)}}$	$\frac{4}{n^{(5)}}$

R. A. Fisher [3, 207] used the product $k_3^2k_2$ as an illustration of the algebraic method. The more general $f_3^2f_2$ gives

$$\begin{aligned} f_3^2f_2 &= [a_3(3) + 3a_{21}(2)(1) + a_{111}(1)^3]^2[a_2(2) + a_{11}(1)(1)] \\ &= a_3^2a_2(3)(3)(2) + a_3^2a_{11}(3)(3)(1)(1) + 6a_3a_{21}a_2(3)(2)^2(1) \\ &+ [6a_3a_{21}a_{11} + 2a_3a_2a_{111}](3)(2)(1)^3 + 9a_{21}^2a_2(2)^3(1)^2 + 2a_3a_{111}a_{11}(3)(1)^5 \\ &+ [6a_{21}a_{111}a_2 + 9a_{21}^2a_{11}](2)^2(1)^4 + [6a_{21}a_{111}a_{11} + a_2a_{111}^2](2)(1)^6 + a_{111}^3a_{11}(1)^8 \end{aligned}$$

which reduces to the value as given by him when the values of a are substituted from Table I.

9. **The Expected Value of Any Partition Product.** The expected values of partition products are well known and are indicated by

$$\begin{aligned} E(p_1) &= n\mu'_{p_1} \\ E(p_1)(p_2) &= n\mu'_{p_1+p_2} + n(n-1)\mu'_{p_1}\mu'_{p_2} \\ E(p_1)(p_2)(p_3) &= n\mu'_{p_1+p_2+p_3} + n(n-1)\,[\mu'_{p_1+p_2}\mu'_{p_3} + \mu'_{p_1+p_3}\mu'_{p_2} + \mu'_{p_2+p_3}\mu'_{p_1}] \\ &\qquad + n(n-1)\,(n-2)\,\mu'_{p_1}\mu'_{p_2}\mu'_{p_3}. \end{aligned}$$

and in general

$$E(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_s)^{\pi_s} = \sum n^{(\tau)}\begin{pmatrix} p_1^{\pi_1}\,p_2^{\pi_2}\cdots p_s^{\pi_s} \\ q_1^{\chi_1}\,q_2^{\chi_2}\cdots q_t^{\chi_t}\end{pmatrix}(\mu'_{q_1})^{\chi_1}(\mu'_{q_2})^{\chi_2}\cdots(\mu'_{q_t})^{\chi_t}$$

where $\tau = \chi_1 + \chi_2 + \chi_3 + \cdots + \chi_t$ and $\begin{pmatrix} p_1^{\pi_1}\,p_2^{\pi_2}\cdots p_s^{\pi_s} \\ q_1^{\chi_1}\,q_2^{\chi_2}\cdots q_t^{\chi_t}\end{pmatrix}$ indicates the number of ways in which the partition $p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}$ can be grouped to form the partition $q_1^{\chi_1} q_2^{\chi_2} \cdots q_t^{\chi_t}$.

The continued application of the result above leads to a large number of formulae. In order to make these results accessible I present in Table II the expected values of all partition products of weight $\leqq 8$. The essence of the table is the evaluation of the expression $\begin{pmatrix} p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \\ q_1^{\chi_1} q_2^{\chi_2} \cdots q_t^{\chi_t} \end{pmatrix}$. The numbers at the top of each column indicate the subscripts of the μ's which must, of course, be multiplied by $n^{(\tau)}$. The entries on the extreme left are the numerical coefficients associated with each row.

10. **The Expected Values of the f Functions.** With the use of Table II one is able to write expressions for the expected values of f_r when $r < 9$.

$$\begin{aligned} \mu_1'(f_1) &= E(f_1) = a_1 n\mu_1' \\ \mu_1'(f_2) &= E(f_2) = (a_2 + a_{11})n\mu_2' + a_{11}n(n-1)\mu_1'^2 \\ \mu_1'(f_3) &= E(f_3) = (a_3 + 3a_{21} + a_{111})n\mu_3' + 3(a_{21} + a_{111})n(n-1)\mu_2'\mu_1' \\ &\qquad + a_{111}n(n-1)(n-2)\mu_1'^3 \text{ etc.} \end{aligned}$$

If the expected values of the f functions are expressed in terms of the moments about the mean of the universe, these formulae become, since $\mu_1' = 0$

$$\begin{aligned} \mu_1'(f_1) &= 0 \\ \mu_1'(f_2) &= (a_2 + a_{11})n\mu_2 \\ \mu_1'(f_3) &= (a_3 + 3a_{21} + a_{111})n\mu_3 \\ \mu_1'(f_4) &= (a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111})n\mu_4 \\ &\qquad + 3(a_{22} + 2a_{211} + a_{1111})n(n-1)\mu_2^2 \text{ etc.} \end{aligned}$$

These may be written more symbolically as

$$\begin{aligned} \mu_1'(f_1) &= 0 \\ \mu_1'(f_2) &= b_2 n\mu_2 \\ \mu_1'(f_3) &= b_3 n\mu_3 \\ \mu_1'(f_4) &= b_4 n\mu_4 + 3b_{22}n(n-1)\mu_2^2 \text{ etc.} \end{aligned}$$

11. **The Expected Value of Products of f Functions.** The expected value of products of f functions may be similarly found. For example

$$\mu_2'(f_2) = E(f_2^2) = E[a_2(2) + a_{11}(1)^2]^2 = a_2^2 E(2)^2 + 2a_2a_{11}E(2)(1)(1) + a_{11}^2 E(1)^4.$$

TABLE II

Expected Values of Partition Products

weight = 1

coef.		n
	π	1
1	1	1

weight = 2

coef.		n	$n^{(2)}$
	π	2	11
1	2	1	
1	11	1	1

weight = 3

coef.		n	$n^{(2)}$	$n^{(3)}$
	π	3	21	111
1	3	1		
3	21	1	1	
1	1^3	1	3	1

weight = 4

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(4)}$
	π	4	31	22	211	1^4
1	4	1				
4	31	1	1			
3	22	1		1		
6	211	1	2	1	1	
1	1111	1	4	3	6	1

weight = 5

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(5)}$
	π	5	41	32	31^2	$2^2 1$	21^3	1^5
1	5	1						
5	41	1	1					
10	32	1		1				
10	311	1	2	1	1			
15	211	1	1	2		1		
10	2111	1	3	4	3	3	1	
1	1^5	1	5	10	10	15	10	1

TABLE II—*Continued*

weight = 6

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(6)}$
	π	6	51	42	33	411	321	2^3	31^3	$2^2\,1^2$	21^4	1^6
1	6	1										
6	51	1	1									
15	42	1		1								
10	33	1			1							
15	411	1	2	1		1						
60	321	1	1	1	1		1					
15	2^3	1		3				1				
20	31^3	1	3	3	1	3	3		1			
45	$2^2\,1^2$	1	2	3	2	1	4	1		1		
15	21^4	1	4	7	4	6	16	3	4	6	1	
1	1^6	1	6	15	10	15	60	15	20	45	15	1

weight = 7

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(5)}$	$n^{(6)}$	$n^{(7)}$
	π	7	61	52	43	51^2	421	$3^2\,1$	32^2	41^3	321^2	$2^3\,1$	31^4	$2^2\,1^3$	21^5	1^7
1	7	1														
7	61	1	1													
21	52	1		1												
35	43	1			1											
21	51^2	1	2	1		1										
105	421	1	1	1	1		1									
70	$3^2\,1$	1	1		2			1								
105	32^2	1		2	1				1							
35	41^3	1	3	3	1	3	3			1						
210	321^2	1	2	2	3	1	2	2	1		1					
105	$2^3\,1$	1	1	3	3		3		3			1				
35	31^4	1	4	6	5	6	12	4	3	4	6		1			
105	$2^2\,1^3$	1	3	5	7	3	9	6	7	1	6	3		1		
21	21^5	1	5	11	15	10	35	20	25	10	40	15	5	10	1	
1	1^7	1	7	21	35	21	105	70	105	35	210	105	35	105	21	1

TABLE II—*Concluded*

weight = 8

$f_3^2 f_2$	coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(5)}$	$n^{(5)}$	$n^{(6)}$	$n^{(6)}$	$n^{(7)}$	$n^{(8)}$
		π	8	71	62	53	44	611	521	431	42^2	$3^2\,2$	51^3	421^2	$3^2\,1^2$	$32^2\,1$	2^4	41^4	321^3	$2^3\,1^2$	31^5	$2^2\,1^4$	21^6	1^8
	1	8	1																					
	8	71	1	1																				
	28	62	1		1																			
	56	53	1			1																		
	35	44	1				1																	
	28	61^2	1	2	1			1																
	168	521	1	1	1	1			1															
	280	431	1	1		1	1			1														
	210	42^2	1		2		1				1													
$a_3^2 a_2$	280	$3^2\,2$	1		1	2						1												
	56	51^3	1	3	3	1		3	3				1											
	420	421^2	1	2	2	2	1	1	2	2	1			1										
$a_3^2 a_{11}$	280	$3^2\,1^2$	1	2	1	2	2	1		4		1			1									
$6a_3a_{21}a_2$	840	$32^2\,1$	1	1	2	3	1		2	1	1	2				1								
	105	2^4	1		4		3				6						1							
	70	41^4	1	4	6	4	1	6	12	4	3		4	6				1						
$6a_3a_{21}a_{11} + 2a_3a_{111}a_2$	560	321^3	1	3	4	5	3	3	6	9	3	4	1	3	3	3			1					
$9a_{21}^2 a_2$	420	$2^3\,1^2$	1	2	4	6	3	1	6	6	6	6		3		6	1			1				
$2a_3a_{111}a_{11}$	56	31^5	1	5	10	11	5	10	30	25	15	10	10	30	10	15		5	10		1			
$6a_{21}a_{111}a_2 + 9a_{21}^2 a_{11}$	210	$2^2\,1^4$	1	4	8	12	7	6	20	28	16	20	4	18	12	28	3	1	8	6		1		
$6a_{21}a_{111}a_{11} + a_{111}^2 a_2$	28	21^6	1	6	16	26	15	15	66	90	60	70	20	105	60	150	15	15	80	45	6	15	1	
$a_{111}^2 a_{11}$	1	1^8	1	8	28	56	35	28	168	280	210	280	56	420	280	840	105	70	560	420	56	210	28	1

Table II can now be used by indicating a_2^2 as a multiplier of $E(2)^2$, $2a_2a_{11}$ as a multiplier of $E(2)(1)(1)$ and a_{11}^2 as a multiplier of $(1)^4$. Then at once it is evident that

$$\begin{aligned}\mu_2'(f_2) &= (a_2^2 + 2a_2a_{11} + a_{11}^2)n\mu_4 + (a_2^2 + 2a_2a_{11} + 3a_{11}^2)n(n-1)\mu_2^2 \\ &= (a_2 + a_{11})^2 n\mu_4 + [(a_2 + a_{11})^2 + 2a_{11}^2]n(n-1)\mu_2^2 \\ &= b_2^2 n\mu_4 + (b_2^2 + 2b_{11}^2)n(n-1)\mu_2^2.\end{aligned}$$

Similarly

$$\begin{aligned}\mu_{11}'(f_3, f_2) &= b_3b_2n\mu_5 + (b_3b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)\mu_3\mu_2 \\ \mu_2'(f_3) &= b_3^2n\mu_6 + (9b_{21}^2 + 6b_3b_{21})n(n-1)\mu_4\mu_2 + (b_3^2 + 9b_{21}^2)n(n-1)\mu_3^2 \\ &\quad + (9b_{21}^2 + 6b_{111}^2)n(n-1)(n-2)\mu_2^3\end{aligned}$$

etc.

where $b_3 = a_3 + 3a_{21} + a_{111}$, $b_{21} = a_{21} + a_{111}$, $b_{111} = a_{111}$. The important special cases are obtained by assigning the proper values to the a's as given in Table I. Thus

$$\mu_2'(m_2) = \frac{1}{n^3}[(n-1)^2\mu_4 + (n^2 - 2n + 3)(n-1)\mu_2^2]$$

which agrees with the corrected result of "Student" in 1908 (8, 3) and Tchouproff (10, 192). Similarly

$$\mu_{11}'(m_3, m_2) = \frac{1}{n^4}[(n-1)^2(n-2)\mu_5 + (n-1)(n-2)(n^2 - 5n + 10)\mu_3\mu_2]$$

$$\begin{aligned}\mu_2'(m_3) = \frac{1}{n^5}[&(n-1)^2(n-2)^2\mu_6 + (-6n + 15)(n-1)(n-2)^2\mu_4\mu_2 \\ &+ (n^2 - 2n + 10)(n-1)(n-2)^2\mu_3^2 + (9n^2 - 36n + 60)(n-1)(n-2)\mu_2^3]\end{aligned}$$

etc.

In the same way

$$\mu_2'(k_2) = \frac{\mu_4}{n} + \frac{(n^2 - 2n + 3)\mu_2^2}{n(n-1)}$$

$$\mu_{11}'(k_3, k_2) = \frac{\mu_5}{n} + \frac{(n^2 - 5n + 10)\mu_3\mu_2}{n(n-1)}$$

$$\mu_2'(k_3) = \frac{\mu_6}{n} + \frac{(-6n + 15)\mu_4\mu_2}{n(n-1)} + \frac{(n^2 - 2n + 10)\mu_3^2}{n(n-1)} + \frac{(9n^2 - 36n + 60)\mu_2^3}{n(n-1)(n-2)}$$

etc.

and

$$\mu_2'(m_2') = \frac{1}{n}[\mu_4 + (n-1)\mu_2^2]$$

$$\mu_{11}'(m_3', m_2') = \frac{1}{n}[\mu_5 + (n-1)\mu_3\mu_2]$$

$$\mu_2'(m_3') = \frac{1}{n}[\mu_6 + (n-1)\mu_3^2]$$

etc.

12. **The Expected Value of the Products of f Functions in Terms of the Thiele Moments of the Universe.** The formulae giving the μ's in term if the λ's are

$$\mu_2 = \lambda_2$$

$$\mu_3 = \lambda_3$$

$$\mu_4 = \lambda_4 + 3\lambda_2^2$$

$$\mu_5 = \lambda_5 + 10\lambda_3\lambda_2$$

$$\mu_6 = \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3$$

. . . .

$$\mu_r = \sum \left(\frac{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} \right) (\lambda_{p_1})^{\pi_1} (\lambda_{p_2})^{\pi_2} \cdots (\lambda_{p_s})^{\pi_s}$$

where the summation holds for those partitions having no unit parts. See the results of Craig (2, 7–11) and Frisch (12, 21). It is at once possible to express the moment formulae in terms of the Thiele moments of the universe. Thus the general results above become

$$\mu_2'(f_2) = b_2^2 n\lambda_4 + [3b_2^2 n + (b_2^2 + 2b_{11}^2)n(n-1)]\lambda_2^2$$

$$\mu_{11}'(f_3, f_2) = b_3 b_2 n\lambda_5 + [10b_3 b_2 n + (b_3 b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)]\lambda_3\lambda_2$$

$$\begin{aligned}\mu_2'(f_3) = {} & b_3^2 n\lambda_6 + [15b_3^2 n + (9b_{21}^2 + 6b_3 b_{21})n(n-1)]\lambda_4\lambda_2 \\ & + [10b_3^2 n + (b_3^2 + 9b_{21}^2)n(n-1)]\lambda_3^2 \\ & + [15b_3^2 n + (27b_{21}^2 + 18b_3 b_{21})n(n-1) + (9b_{21}^2 + 6b_{111}^2)n(n-1)(n-2)]\lambda_2^3.\end{aligned}$$

13. **The Thiele Moments of the f's in terms of Thiele Moments.** It is now possible to reduce to the Thiele moments of the f's by means of the usual relations

$$\lambda_2(f_r) = \mu_2(f_r) - \mu_1'^2(f_r)$$

$$\lambda_{11}(f_{r_1}, f_{r_2}) = \mu_{11}'(f_{r_1}, f_{r_2}) - \mu_{10}'(f_{r_1}, f_{r_2})\mu_{01}'(f_{r_1}, f_{r_2})$$

$$\lambda_3(f_r) = \mu_3'(f_r) - 3\mu_2'(f_r)\mu_1'(f_r) + 2\mu_1'^3(f_r)$$

etc.

so that the results become

$$\begin{aligned}
\lambda_2(f_2) &= b_2^2 n\lambda_4 + 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2 \\
\lambda_{11}(f_3, f_2) &= b_3 b_2 n\lambda_5 + \{3[b_3 b_2 n + b_{21} b_2 n(n-1)] + 6[b_3 b_2 n + b_{21} b_{11} n(n-1)]\}\lambda_3\lambda_2 \\
\lambda_2(f_3) &= b_3^2 n\lambda_6 + \{6[b_3^2 n + b_3 b_{21} n(n-1)] + 9[b_3^2 n + b_{21}^2 n(n-1)]\}\lambda_4\lambda_2 \\
&+ 9[b_3^2 n + b_{21}^2 n(n-1)]\lambda_3^2 + \{9[b_3^2 n + 2b_3 b_{21} n(n-1) + b_{21}^2 n(n-1) + b_{21}^2 n^{(3)}] \\
&+ 6[b_3^2 n + 3b_{21}^2 n(n-1) + b_{111}^2 n(n-1)(n-2)]\}\lambda_2^3
\end{aligned}$$

etc.

The formulae as written are adapted to the partition representation of Part III.

When the f's are equal to the m's we have

$$\lambda_2(m_2) = \frac{(n-1)^2\lambda_4}{n^3} + \frac{2(n-1)\lambda_2^2}{n^2}.$$

$$\lambda_{11}(m_3, m_2) = \frac{(n-1)^2(n-2)\lambda_5}{n^4} + \frac{6(n-1)(n-2)\lambda_3\lambda_2}{n^3}$$

$$\lambda_2(m_3) = \frac{(n-1)^2(n-2)^2\lambda_6}{n^5} + \frac{9(n-1)(n-2)^2\lambda_4\lambda_2}{n^4} + \frac{9(n-1)(n-2)^2\lambda_3^2}{n^4} + \frac{6(n-1)(n-2)\lambda_2^3}{n^3}$$

etc.

which are the results as previously given by C. C. Craig (2, 55). In like manner when the $f_r = k_r$

$$\lambda_2(k_2) = \frac{\lambda_4}{n} + \frac{2\lambda_2^2}{n-1}$$

$$\lambda_{11}(k_3, k_2) = \frac{\lambda_5}{n} + \frac{6\,\lambda_3\lambda_2}{n-1}$$

$$\lambda_2(k_3) = \frac{\lambda_6}{n} + \frac{9\,\lambda_4\lambda_2}{n-1} + \frac{9\,\lambda_3^2}{n-1} + \frac{6n\,\lambda_2^3}{(n-1)(n-2)}$$

etc.

as given by R. A. Fisher [3, 210] while

$$\lambda_2(m_2') = \frac{1}{n}(\lambda_4 + 2\lambda_2^2)$$

$$\lambda_{11}(m_3', m_2') = \frac{1}{n}(\lambda_5 + 9\lambda_3\lambda_2)$$

$$\lambda_2(m_3') = \frac{1}{n}(\lambda_6 + 15\lambda_4\lambda_2 + 9\lambda_3^2 + 15\lambda_2^3).$$

etc.

14. Various Formulization of Results. Although different moment functions of the universe may be used it is customary to express the results in terms of universe moments about a fixed point, in terms of universe moments, or in terms of universe Thiele moments. It is possible to express results in any of the nine forms

$$\left.\begin{matrix}\mu'(f_r)\\ \mu(f_r)\\ \lambda(f_r)\end{matrix}\right\}\text{ in terms of }\left\{\begin{matrix}\text{moments about a fixed point }(\mu')\\ \text{moments }(\mu)\\ \text{Thiele moments }(\lambda)\end{matrix}\right.$$

where f_r represents the isobaric sample moment function of weight r. One purpose of such varied formulization is to discover the most compact form and also the one best adapted to use in the case of a normal universe or a universe whose moments obey some discoverable law. As suggested above Craig (2) has shown the relative compactness obtained by using $\lambda(m_r)$ and Thiele moments of the universe while R. A. Fisher (3) has shown the great additional compactness obtained by taking $f_r = k_r$.

15. The Application of the Algebraic Method to $\lambda_{21}(f_3, f_2)$. Before leaving the algebraic method it is perhaps wise to outline the steps in the case of a more involved problem. We take the example which R. A. Fisher (3, 207) has used in the case in which $f_r = k_r$. To find $\lambda_{21}(f_3, f_2)$.

The value of $f_3^2 f_2$ was found in section 8. To find its expected value it is only necessary to enter the coefficients of the different partition products in this expansion at the left of the corresponding rows as indicated in Table II.

The coefficient of any moment partition of the universe is found by multiplying each column entry by its corresponding left row entry and then by multiplying by $n^{(\tau)}$ as indicated at the top. Thus the coefficient of μ_8' is

$$(a_3^2a_2 + a_3^2a_{11} + 6a_3a_{21}a_2 + 6a_3a_{21}a_{11} + 2a_3a_{111}a_2 + 9a_{21}^2a_2 + 2a_3a_{111}a_{11} + 6a_2a_{21}a_{111}$$
$$+ 9a_{21}a_{21}a_{11} + 6a_{21}a_{111}a_{11} + a_{111}^2a_2 + a_{111}^2a_{11})n$$

which after some algebraic work reduces to

$$(a_3 + 3a_{21} + a_{111})^2(a_2 + a_{11})n = b_3^2b_2n.$$

In this manner it is possible to write the result either in terms of universe moments about a fixed point or in terms of universe moments. If moments are used, one may neglect all column partitions involving unity.

It should be noted that the a's defining k_r as given in Table I can be inserted here if desired. If these multipliers are introduced throughout the rows and columnar partitions involving unit parts are not used one will arrive at Table I of R. A. Fisher [3, 208] though there are some slight typographical errors in his rows for $(3)^2$ $(1)^2$ and (3) (2^2) (1).

Determining all the coefficients in this manner we find after considerable algebraic manipulation that

$$\mu'_{21}(f_3, f_2) = b_3^2 b_2 n\mu_8 + [b_3^2 b_2 + 9b_{21}^2 b_2 + 12b_3 b_{21} b_{11} + 6b_3 b_{21} b_2]n(n-1)\mu_6\mu_2$$
$$+ [2b_3^2 b_2 + 18b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11}]n(n-1)\mu_5\mu_3$$
$$+ [2b_3^2 b_{11} + 9b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 6b_3 b_{21} b_2]n(n-1)\mu_4^2 + [36b_{21}^2 b_2$$
$$+ 54b_{21}^2 b_{11} + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11} + 12b_3 b_{111} b_{11} + 72b_{21} b_{111} b_{11}$$
$$+ 18b_{111}^2 b_2]n(n-1)(n-2)\mu_4\mu_2^2 + [b_3^2 b_2 + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11}$$
$$+ 27b_{21}^2 b_2 + 90b_{21}^2 b_{11} + 36b_{21} b_{111} b_2 + 72b_{21} b_{111} b_{11} + 36b_{111}^2 b_{11}]n(n-1)(n-2)\mu_3^2\mu_2$$
$$+ [9b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 36b_{21} b_{111} b_{11} + 6b_{111}^2 b_2 + 36b_{111}^2 b_{11}]n(n-1)(n-2)(n-3)\mu_2^4.$$

If $f_r = k_r$ the proper values of b are inserted and the expression above becomes that given by R. A. Fisher (3, 208). For example the coefficient of μ_2^4 is

$$\frac{(9n^3 - 63n^2 + 240n - 420)(n-3)}{n^2(n-1)^2(n-2)}$$

when

$$b_2 = \frac{1}{n}, \qquad b_3 = \frac{1}{n}, \qquad b_{11} = -\frac{1}{n(n-1)}$$

$$b_{21} = -\frac{1}{n(n-1)}, \qquad b_{111} = \frac{2}{n(n-1)(n-2)}.$$

The algebraic results involved in changing the general formula above to other functions are too extended to present here. A symbolic means of attaining them is included in later sections of the paper.

Part II. The Determination of Specific f Functions

16. **Functions Determined by the b's.** In Part I it was shown how various f functions are defined by giving definite values to the coefficients of the power sums. It is the purpose of this part of the paper to show how functions can be specified by means of their expected values in terms of moments of the universe. This is essentially the method used by R. A. Fisher in defining his k function and it is here extended to other functions. In this case the b's are first determined and the a's are then found from them. The first moments of f_1, f_2, f_3 were given in section 10. To these we add, as shown by Table II

$$\mu'_1(f_4) = (a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111})n\mu'_4 + 4(a_{31} + 3a_{211} + a_{1111})n(n-1)\mu'_3\mu'_1$$
$$+ 3(a_{22} + 2a_{211} + a_{1111})n(n-1)\mu_2'^2 + 6(a_{211} + a_{1111})n(n-1)(n-2)\mu'_2\mu_1'^2$$
$$+ a_{1111}n(n-1)(n-2)(n-3)\mu_1'^4$$

etc.

These can be written more symbolically in terms of the b's

$$\mu_1'(f_1) = b_1 n \mu_1'$$

$$\mu_1'(f_2) = b_2 n \mu_2' + b_{11} n(n-1)\mu_1'^2$$

$$\mu_1'(f_3) = b_3 n \mu_3' + 3b_{21} n(n-1)\mu_2'\mu_1' + b_{111} n(n-1)(n-2)\mu_1'^3$$

$$\mu_1'(f_4) = b_4 n \mu_4' + 4b_{31} n(n-1)\mu_3'\mu_1' + 3b_{22} n(n-1)\mu_2'^2 + 6b_{211} n^{(3)}\mu_2'\mu_1'^2 + 6n^{(4)}\mu_1'^4,$$

and in general

$$\mu_1'(f_r) = \sum \binom{1^r}{p_1^{\pi_1}\, p_2^{\pi_2} \cdots p_s^{\pi_s}} b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}\, n^{(\rho)} (\mu_{p_1}')^{\pi_1} (\mu_{p_2}')^{\pi_2} \cdots (\mu_{p_s}')^{\pi_s}.$$

The expansion of the function in terms of the power sums of the sample demands the determination of the a's. This can be accomplished by solving the equations

$$\begin{aligned}
a_1 &= b_1 \\
a_2 + a_{11} &= b_2 \\
a_{11} &= b_{11} \\
a_3 + 3a_{21} + a_{111} &= b_3 \\
a_{21} + a_{111} &= b_{21} \\
a_{111} &= b_{111} \\
a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111} &= b_4 \\
a_{31} \qquad + 3a_{211} + a_{1111} &= b_{31} \\
a_{22} + 2a_{211} + a_{1111} &= b_{22} \\
&\text{etc.}
\end{aligned}$$

The solutions are

$$\begin{aligned}
a_1 &= b_1 \\
a_2 &= b_2 - b_{11} \\
a_{11} &= b_{11} \\
a_3 &= b_3 - 3b_{21} + 2b_{111} \\
a_{21} &= b_{21} - b_{111} \\
a_{111} &= b_{111} \\
a_4 &= b_4 - 4b_{31} - 3b_{22} + 12b_{211} - 6b_{1111} \\
a_{31} &= b_{31} - 3b_{211} + 2b_{1111} \\
a_{22} &= b_{22} - 2b_{211} + b_{1111} \\
a_{211} &= b_{211} - b_{1111} \\
a_{1111} &= b_{1111}.
\end{aligned}$$

The values of a_r, at least for $r \leqq 4$, follow the law

$$a_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)!\, b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

and

$a_{21} = \overline{a_2 a_1}$ where $\overline{a_2 a_1}$ indicates that $a_2 = b_2 - b_{11}$ is multiplied by $a_1 = b_1$, the rule of multiplication being suffixing of subscripts. Similarly $a_{22} = \overline{a_2 a_2} = \overline{(b_2 - b_{11})(b_2 - b_{11})} = b_{22} - 2b_{211} + b_{1111}$.

This statement illustrates a general theorem which will be established later in another paper by a different approach that for all cases

$$a_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)!\, b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

and that

$$a_{r_1 \cdots r_t} = \overline{a_{r_1} a_{r_2} \cdots a_{r_t}}.$$

This theorem enables one to write, with comparative ease, the coefficient of any product of power sums in a sample function whose expected values is defined. For example the functional coefficient of (3)(2) in f_5 is

$$\overline{a_3 a_2} = \overline{(b_3 - 3b_{21} + 2b_{111})(b_2 - b_{11})} = b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111}$$

while that of (3)(1)(1) is $\overline{a_3 a_1 a_1} = b_{311} - 3b_{2111} + 2b_{11111}$. If the expected value of the function is known the b's are determined and the values of the above expressions can be found by substitution.

17. **The Values of the Fisher Moments (k functions).** The k functions have been defined to be these functions whose expected values are the Thiele moments of the universe. Thus $\mu_1'(k_r) = \lambda_r$ and since

$$\lambda_r = \sum \binom{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)!\, (\mu_{p_1}')^{\pi_1} (\mu_{p_2}')^{\pi_2} \cdots (\mu_{p_s}')^{\pi_s},$$

it follows at once that by comparison with $\mu_1'(f_r)$ in the last section, that

$$b_{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} = \frac{(-1)^{\rho-1} (\rho - 1)!}{n^{(\rho)}}.$$

Thus

$$b_1 = \frac{1}{n}; \quad b_2 = \frac{1}{n}; \quad b_{11} = -\frac{1}{n^{(2)}}; \quad b_3 = \frac{1}{n}; \quad b_{21} = \frac{-1}{n^{(2)}}; \quad b_{111} = \frac{2}{n^{(3)}};$$

$$b_4 = \frac{1}{n}; \quad b_{31} = \frac{-1}{n^{(2)}}; \quad b_{22} = \frac{-1}{n^{(2)}}; \quad b_{211} = \frac{2}{n^{(3)}}; \quad b_{1111} = \frac{-6}{n^{(4)}}; \text{ etc.}$$

The insertion of these values in the formulae of section 16 gives the values of a such as those indicated in Table I and in section 5. Thus the coefficient of (3)(2) in f_5 is

$$10(b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111}) = -10\left[\frac{1}{n^{(2)}} + \frac{2}{n^{(3)}} + \frac{6}{n^{(3)}} + \frac{30}{n^{(4)}} + \frac{48}{n^{(5)}}\right]$$
$$= -\frac{10n^{(2)}}{(n-1)^{(4)}}.$$

The coefficient of (3)(1)(1) is

$$10(b_{311} - 3b_{2111} + 2b_{11111}) = 10\left[\frac{2}{n^{(3)}} + \frac{18}{n^{(4)}} + \frac{48}{n^{(5)}}\right] = \frac{10(2n+4)}{(n-1)^{(4)}}.$$

18. **The h Functions.** It is also possible to define a function whose expected value is the moment of the universe. Thus $\mu_1'(h_r) = \mu_r$ where

$$\mu_r = \sum \begin{pmatrix} & 1^r & \\ p_1^{\pi_1} & \cdots & p_s^{\pi_s} \end{pmatrix} A_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (\mu_{p_1}')^{\pi_1} (\mu_{p_2}')^{\pi_2} \cdots (\mu_{p_s}')^{\pi_s}$$

and

$$A_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} 1 \text{ if } s = 1,\ \pi_1 = 1, \text{ and } p_1 = r. \\ (-1)^{\pi_2} \text{ if } p_1 > 1,\ \pi_1 = 1,\ s = 2 \text{ and } p_2 = 1. \\ (-1)^{r-1}(r-1) \text{ if } p_1 = 1,\ s = 1, \text{ and } \pi_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

Comparing with the value of $\mu_1'(f_r)$ in section 16 we have

$$b_{b_1^{\pi_1} \cdots b_s^{\pi_s}} = \frac{A_{p_1^{\pi_1} \cdots p_s^{\pi_s}}}{n^{(\rho)}}.$$

The substitution of these values of b in the results of section 16 gives the expansions of h_r in terms of power sums as illustrated by the formulae of section 6 and Table I. Thus the coefficient of (3)(2) is

$$10(b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111})$$
$$= -10\left[0 + \frac{1}{n^{(3)}} + 0 + \frac{5}{n^{(4)}} + \frac{8}{n^{(5)}}\right] = \frac{-10(n-2)}{(n-1)^{(4)}}.$$

Similarly the coefficient of (3)(1)(1) in h_5 is

$$10(b_{311} - 3b_{2111} + 2b_{11111}) = 10\left[\frac{1}{n^{(3)}} + \frac{3}{n^{(4)}} + \frac{8}{n^{(5)}}\right] = \frac{10(n^2 - 4n + 8)}{n^{(5)}}.$$

19. **The h' Functions.** One line of attack calls for the introduction of new moment functions which will result in simpler formulae. Thus for example,

C. C. Craig wrote (2, 37) "It rather seems that the best hopes of effectively further simplifying the problem of sampling for statistical characteristics lie either in the discovery of a new kind of symmetric function of all the observations which may be used to characterize frequency functions and which will be more amenable than either moments or semi-invariants for use in sampling problems, or in, what may very well prove to be much better and more feasible, the abandonment of the method of characterizing frequency functions by symmetric functions of all the observations altogether."

R. A. Fisher has shown that it is possible to introduce symmetric functions which do simplify the resulting formula appreciably. It is the purpose of this section to introduce an additional symmetric function which simplifies the resulting formulae to a much greater extent. It is admitted that this function does not have all the properties (such as invariance with respect to change of origin) possessed by the Thiele and Fisher functions, but it does not have the property of making the resulting formulae simple. It also has the advantage that $\mu(h_r') = \mu'(h_r')$.

The basic idea is to find a sample moment function whose expected value is 0. A first attempt, placing every $b = 0$, is of no avail since every a is also equal to 0 and there is no function. A second attempt is based on the idea of finding the function h whose expected value is $\mu_1'^r$. If the universe is assumed to be measured about its mean, as is conventional, it follows at once that $\mu_1' = 0$ and $\mu_1'(h_r) = 0$ so that

$$\mu_{\mu\nu}(h_{r_1}', h_{r_2}') = \mu_{\mu\nu}'(h_{r_1}', h_{r_2}').$$

This function then has the property that its moments about a fixed point and its moments are identical.

In order to discover its expansion in terms of power sums, we note

$$\mu_1'(h_r') = \mu_1'^r$$

and it follows at once by comparison with $\mu_1'(f_r)$ in section 16 that $b_{1^r} = \frac{1}{n^{(r)}}$ and $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 0$ in all other cases. The a's are determined in the usual way. Thus

$$a_2 = b_2 - b_{11} = -\frac{1}{n(n-1)}$$

$$a_{11} = b_{11} = \frac{1}{n(n-1)}$$

so that

$$h_2' = -\frac{1}{n(n-1)}[(2) - (1)(1)].$$

Similarly

$$h_3' = \frac{1}{n^{(3)}}[2(3) - 3(2)(1) + (1)^3]$$

$$h_4' = -\frac{1}{n^{(4)}}[6(4) - 8(3)(1) - 3(2)(2) + 6(2)(1)(1) - (1)^4]$$

and in general

$$h_r' = \frac{(-1)^{r-1}}{n^{(r)}}\left\{\sum (-1)^{\rho-1}[(p_1-1)!]^{\pi_1}[(p_2-1)!]^{\pi_2}\cdots[(p_s-1)!]^{\pi_s} \binom{1^r}{p_1^{\pi_1}\cdots p_s^{\pi_s}}(p_1)^{\pi_1}\cdots(p_s)^{\pi_s}\right\}.$$

In order to show the simple form in which results can be given we substitute the values of the b's in the results obtained above. Not only does $\mu_1'(h_r') = 0$, but by section 11

$$\lambda_2(h_2') = \mu_2(h_2') = \mu_2'(h_2') = \frac{2}{n(n-1)}\mu_2^2$$

$$\lambda_{11}(h_3', h_2') = \mu_{11}(h_3', h_2') = \mu_{11}'(h_3', h_2') = 0$$

$$\lambda_2(h_3') = \mu_2(h_3') = \mu_2'(h_3') = \frac{6}{n(n-1)(n-2)}\mu_2^3$$

while from section 15

$$\lambda_{21}(h_3', h_2') = \mu_{21}(h_3', h_2') = \mu_{21}'(h_3', h_2') = \frac{36\,\mu_3^2\mu_2}{n^2(n-1)^2(n-2)} + \frac{36(n-3)\,\mu_2^4}{n^2(n-1)^2(n-2)}.$$

It is to be noticed that these formulae contain very few terms and that the terms themselves involve very low moments of the universe. This simplicity has been attained without making any assumption such as normality, regarding the nature of the universe.

20. **Table of Values of b for Different Functions When $r < 6$.** This process of defining functions by means of expected values could be extended indefinitely. Perhaps it has been applied to enough functions to suggest the breadth of the applicability of the theory developed in Part I and Part III.

As the b's are the quantities which are used in the formulae I have provided Table III giving their values for the six functions, m_r', m_r, l_r, k_r, h_r, h_r' when $r = 1, 2, 3, 4, 5$. When the a's are known, the b's are computed from them according to the formulae of section 16.

TABLE III

Values of the b's for $r \leqq 5$

Num. coef.	b	m'_r	m_r	l_r	k_r	h_r	h'_r
1	b_1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
1	b_2	$\frac{1}{n}$	$\frac{n-1}{n^2}$	$\frac{n-1}{n^2}$	$\frac{1}{n}$	$\frac{1}{n}$	0
1	b_{11}	0	$-\frac{1}{n^2}$	$-\frac{1}{n^2}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	$\frac{1}{n^{(2)}}$
1	b_3	$\frac{1}{n}$	$\frac{(n-1)(n-2)}{n^3}$	$\frac{(n-1)(n-2)}{n^3}$	$\frac{1}{n}$	$\frac{1}{n}$	0
3	b_{21}	0	$-\frac{(n-2)}{n^3}$	$-\frac{n-2}{n^3}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
1	b_{111}	0	$\frac{2}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{n^{(3)}}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$
1	b_4	$\frac{1}{n}$	$\frac{(n-1)(n^2-3n+3)}{n^4}$	$\frac{(n-1)(n^2-6n+6)}{n^4}$	$\frac{1}{n}$	$\frac{1}{n}$	0
4	b_{31}	0	$-\frac{(n^2-3n+3)}{n^4}$	$-\frac{(n^2-6n+6)}{n^4}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
3	b_{22}	0	$\frac{2n-3}{n^4}$	$-\frac{(n^2-4n+6)}{n^4}$	$-\frac{1}{n^{(2)}}$	0	0
6	b_{211}	0	$\frac{n-3}{n^4}$	$\frac{2(n-3)}{n^4}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$	0
1	b_{1111}	0	$-\frac{3}{n^4}$	$-\frac{6}{n^4}$	$-\frac{6}{n^{(4)}}$	$-\frac{3}{n^{(4)}}$	$\frac{1}{n^{(4)}}$
1	b_5	$\frac{1}{n}$	$\frac{(n-1)(n-2)(n^2-2n+2)}{n^5}$	$\frac{(n-1)(n-2)(n^2-12n+12)}{n^5}$	$\frac{1}{n}$	$\frac{1}{n}$	0
5	b_{41}	0	$-\frac{(n^3-4n^2+6n-4)}{n^5}$	$-\frac{(n^3-14n^2+36n-24)}{n^5}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
10	b_{32}	0	$\frac{n^2-4n+4}{n^5}$	$-\frac{(n^3-8n^2+24n-24)}{n^5}$	$-\frac{1}{n^{(2)}}$	0	0
10	b_{311}	0	$\frac{n^2-3n+4}{n^5}$	$\frac{2n^2-18n+24}{n^5}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$	0
15	b_{221}	0	$-\frac{2(n-2)}{n^5}$	$\frac{2n^2-12n+24}{n^5}$	$\frac{2}{n^{(3)}}$	0	0
10	b_{2111}	0	$-\frac{n-4}{n^5}$	$-\frac{6(n-4)}{n^5}$	$-\frac{6}{n^{(4)}}$	$-\frac{1}{n^{(4)}}$	0
1	b_{11111}	0	$\frac{4}{n^5}$	$\frac{24}{n^5}$	$\frac{24}{n^{(5)}}$	$\frac{4}{n^{(5)}}$	$\frac{1}{n^{(5)}}$

Part III. Combinatory Methods

21. Partition Representation of Expected Value of f Functions. The formulae

$$\mu_1'(f_1) = b_1 n \mu_1'$$
$$\mu_1'(f_2) = b_2 n \mu_2' + b_{11} n(n-1)\mu_1'^2$$
$$\mu_1'(f_3) = b_3 n \mu_3' + 3b_{21} n(n-1)\mu_2'\mu_1' + b_{111} n(n-1)(n-2)\mu_1'^3$$
$$\mu_1'(f_4) = b_4 n \mu_4' + 4b_{31} n(n-1)\mu_2'\mu_1' + 3b_{22} n(n-1)\mu_2'^2 + 6b_{211} n(n-1)(n-2)\mu_2'\mu_1'^2 + b_{1111} n^{(4)}\mu_1'^4$$

are "synthetically" given by the column partitions

$$\begin{array}{ccccc}
1 & & & & \\
 & & & & \\
2 & 1 & & & \\
 & 1 & & & \\
 & & & & \\
3 & 2 & 1 & & \\
 & 1 & 1 & & \\
 & & 1 & & \\
 & & & & \\
4 & 3 & 2 & 2 & \\
 & 1 & 2 & 1 & \\
 & & & 1 & 1 \\
 & & & & 1
\end{array}$$

The partition parts represent both the subscripts of the moments and the subscripts of the b's. If ρ indicates the number of parts, the n multiplier is $n^{(\rho)}$. The numerical coefficient is obtained by taking the sum of the entries in the column (the weight) and dividing it by the factorials of all entries times the factorials of all repeated entries as indicated by

$$\begin{pmatrix} 1^r \\ p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \end{pmatrix} = \frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s}\, \pi_1!\, \pi_2! \cdots \pi_s!}.$$

The translation from the synthetic partition form to the expanded form is accelerated if the coefficients are known. These are provided in the following partition representation of the formula for $\mu_1'(f_r)$ when $r \leqq 8$ and the results are expressed in terms of the moments of the universe

$\mu_1(f_1)$: $\quad 0$

$\mu_1'(f_2)$: $\quad \begin{array}{c} \mathbf{1} \\ 2 \end{array}$

$\mu_1'(f_3)$: $\quad \begin{array}{c} \mathbf{1} \\ 3 \end{array}$

$\mu_1'(f_4)$:	**1**	**3**
	4	2
		2

$\mu_1'(f_5)$:	**1**	**10**
	5	3
		2

$\mu_1'(f_6)$:	**1**	**15**	**10**	**15**
	6	4	3	2
		2	3	2
				2

$\mu_1'(f_7)$:	**1**	**21**	**35**	**105**
	7	5	4	3
		2	3	2
				2

$\mu_1'(f_8)$:	**1**	**28**	**56**	**35**	**210**	**280**	**105**
	8	6	5	4	4	3	2
		2	3	4	2	3	2
					2	2	2
							2

The proper formula can be stated immediately from its synthetic representation. Thus for example

$$\mu_1'(f_6) = b_6 n\mu_6 + 15b_{42}n(n-1)\mu_4\mu_2 + 10b_{33}n(n-1)\mu_3^2 + 15b_{222}n(n-1)(n-2)\mu_2^3.$$

22. **Partition Representation of the Expected Value of a Product of f Functions.** Two column partitions may be used similarly to represent the expected values of the products of two f's, three column partitions for the expected value of the triple product, etc. In order to obtain all terms it is only necessary to combine every partition of each f in every possible way. The synthetic representation of $E(m_2, m_1)$ is

1	**1**	**2**	**1**
21	20	11	10
	01	10	10
			01

The sum of the entries in each row indicates the proper moment while the number of rows indicates the number of parts as in the preceding section. The n coefficient associated with a ρ rowed partition is then $n^{(\rho)}$. The b coefficient is indicated by the columnar entries. Thus

$$\mu_{11}'(f_2, f_1 = b_2b_1n\mu_3' + [b_2b_1 + 2b_{11}b_1]n(n-1)\mu_2'\mu_1' + b_{11}b_1n(n-1)(n-2)\mu_1'^3.$$

We verify this by the algebraic method

$$\begin{aligned}
\mu'_{11}(f_2, f_1) &= E\{[a_2(2) + a_{11}(1)(1)][a_1(1)]\} \\
&= E[a_2a_1(2)(1) + a_{11}a_1(1)^3] \\
&= a_2a_1[n\mu'_3 + n(n-1)\mu'_2\mu'_1 \\
&\qquad + a_{11}a_1[n\mu'_3 + 3n(n-1)\mu'_2\mu'_1 + n(n-1)(n-2)\mu_1'^3] \\
&= (a_2 + a_{11})a_1n\mu'_3 + (a_2 + a_{11})a_1n(n-1)\mu'_2\mu'_1 \\
&\qquad + 2a_{11}a_1\mu'_2\mu'_1 + a_{11}a_1n(n-1)(n-2)\mu_1'^3 \\
&= b_2b_1n\mu'_3 + b_2b_1n(n-1)\mu'_2\mu'_1 + 2b_{11}b_1n(n-1)\mu'_2\mu'_1 \\
&\qquad + b_{11}b_1n(n-1)(n-2)\mu_1'^3
\end{aligned}$$

as indicated.

It thus appears that the partition representation is a mnemonic device for indicating the solution as obtained by the algebraic method. A more formal justification is based upon the property that if

$$E(f_2) = b_2(2) + b_{11}(1)(1) \quad \text{and} \quad E(f_1) = b_1(1)$$

then $E(f_2, f_1)$ can be obtained by a symbolic multiplication of $b_2(2) + b_{11}(1)(1)$ by $b_1(1)$ where the b's are multiplied but the power sums are collected in all possible ways. Thus

$$E(f_2, f_1) = b_2b_1[(3) + (2)(1)] + b_{11}b_1[2(2)(1) + (1)^3]$$

which gives

$$E(f_2, f_1) = b_2b_1n\mu'_3 + b_2b_1n(n-1)\mu'_2\mu'_1 + 2b_{11}b_1n(n-1)\mu'_2\mu'_1 + b_{11}b_1n^{(3)}\mu_1'^3$$

as before.

This symbolic multiplication is generally true and serves as the real algebraic justification of the partition representation. It will be established in a later paper dealing with the more general case of a finite population. The general type of partition analysis has been used previously by Fisher (3) and Georgescu (4). Each has established it through analytic rather than algebraic means.

23. **Determination of the Coefficients.** Methods of determining the numerical coefficient have previously been given by such authors as Fisher (3), Wishart (5) (7) and Georgescu (4). If the f's are of different weight, the coefficients of any partition (an interchange of rows is not looked upon as changing the partition) is given by writing in the numerator the factorials of the different r's and in the denominator the factorials of all the different entries and the factorials of all repeated rows. Thus the coefficient of

$$\begin{matrix} 210 \\ 111 \\ 111 \end{matrix} \quad \text{is} \quad \frac{4!\,3!\,2!}{2!\,(1!)^7\,2!} = 72.$$

In case two or more functions have the same weight additional equivalent partitions are formed by interchange of columns. The reader is referred to the above papers for rules for determining the coefficients in the more involved cases though the coefficients are presented for all the two way partitions of the next section.

An alternative method of finding the coefficients is that given by C. C. Craig (2, 24–25) since it appears that the symbolic formulae used in the present paper are essentially his formulae for ν's in terms of λ's. For example his formula for ν_{44} (2, 22) is given symbolically by the formula for 44 in the next section. The only difference revealed is that the subscripts of the λ's are read by rows rather than by columns and that they are sometimes interchanged. The more precise formulization is needed for the present interpretation although it is not needed for Prof. Craig's purpose.

A third method utilizes the symbolic multiplication process stated in section 22. Subscripts of the b's are used to indicate which power sums are collected. Thus $[b_2(2) + b_{11}(1)(1)]^2$ gives

$$\underline{b_2b_2(4) + b_{20}b_{02}(2)(2)} + \underline{2[2b_{20}b_{11}(3)(1) + b_{200}b_{011}(2)(1)(1)]} + \underline{2b_{11}b_{11}(2)(2)}$$
$$\underline{+ 4b_{110}b_{101}(2)(1)(1) + b_{1100}b_{0011}(1)(1)(1)(1)}$$

where the underscored terms indicate the products given by $[b_2(2)]^2$, $2[b_2(2)]$ $[b_{11}(1)(1)]$, and $[b_{11}(1)(1)]^2$ respectively. This is represented by

1	**1**	**4**	**2**	**2**	**4**	**1**
22	20	21	20	11	11	10
	02	01	01	11	10	10
			01		01	01
						01
—	—			—		

The underscored terms are the only ones remaining when $\mu_1' = 0$.

This method is especially useful when a large number of formulae are to be computed, as in the next section.

24. The Partition Representation of Formulae of Total Weight $\leqq$ 8. The partition representation of $\mu_1'(f_r)$ when $r \leqq 8$ are given in section 21. The partition representation of the remaining formulae of total weight $\leqq 8$, which do not contain unit parts, are given below.

22	**1**	**1**	**2**	
	22	20	11	
		02	11	
32	**1**	**1**	**3**	**6**
	32	30	12	21
		02	20	11

42	**1**	**1**	**8**	**6**	**4**	**6**	**3**	**12**
	42	40	31	22	30	21	20	20
		02	11	20	12	21	20	11
							02	11

33	**1**	**6**	**9**	**1**	**9**	**9**	**6**
	33	31	22	30	21	20	11
		02	11	03	12	11	11
						02	11

222	**1**	**3**	**12**	**6**	**4**	**1**	**6**	**8**
	222	220	211	201	111	200	200	110
		002	011	021	111	020	011	011
						002	011	101

52	**1**	**1**	**10**	**10**	**5**	**10**	**20**	**10**	**20**	**15**	**60**
	52	50	41	32	40	22	31	30	30	12	21
		02	11	20	12	30	21	20	11	20	20
								02	11	20	11

43	**1**	**3**	**12**	**6**	**1**	**4**	**12**	**18**	**12**	**3**	**18**	**36**	**36**
	43	41	32	23	40	13	31	22	30	03	21	12	21
		02	11	20	03	30	12	21	11	20	20	20	11
									02	20	02	11	11

322	**1**	**2**	**4**	**12**	**3**	**1**	**4**	**6**	**12**	**12**
	322	320	311	221	122	022	301	220	121	211
		002	011	101	200	300	021	102	201	111

	1	**2**	**6**	**12**	**12**	**12**	**24**	**12**	**24**
	300	300	102	021	201	111	210	120	111
	020	011	020	101	020	011	101	101	101
	002	011	200	200	101	200	011	101	110

62	**1**	**1**	**12**	**15**	**6**	**30**	**20**	**15**	**20**
	62	60	51	42	50	41	32	40	31
		02	11	20	12	21	30	22	31

	15	**30**	**120**	**45**	**10**	**60**	**120**	**90**		**15**	**90**
	40	40	31	22	30	30	30	21		20	20
	20	11	20	20	30	12	21	21		20	20
	02	11	11	20	02	20	11	20		20	11
										02	11

53	**1**	**3**	**15**	**10**	**1**	**15**	**30**	**10**	**5**	**30**		
	53	51	42	33	50	41	32	23	40	31		
		02	11	20	03	12	21	30	13	22		
	15	**60**	**90**	**15**	**30**	**10**	**30**	**60**	**90**	**90**	**45**	**60**
	40	31	22	13	31	30	30	30	12	21	20	20
	11	11	20	20	20	03	21	12	21	21	20	11
	02	11	11	20	02	20	02	11	20	11	11	11
											02	11

44	**1**	**12**	**16**	**8**	**48**	**1**	**16**	**18**			
	44	42	33	41	32	40	31	22			
		02	11	03	12	04	13	22			
	6	**96**	**36**	**72**	**48**	**16**	**72**	**144**	**9**	**72**	**24**
	40	31	22	22	30	30	21	21	20	20	11
	02	11	20	11	12	03	21	12	20	11	11
	02	02	02	11	02	11	02	11	02	11	11
									02	02	11

422	**1**	**2**	**4**	**16**	**6**	**4**	**8**	**4**	**24**	**16**
	422	420	411	321	222	401	320	122	212	311
		002	011	101	200	021	102	300	210	111
	1	**16**	**6**	**12**						
	400	310	220	211						
	022	112	202	211						
	1	**2**	**16**	**32**	**12**	**3**	**24**	**24**	**48**	**48**
	400	400	310	310	202	022	211	220	211	121
	020	011	110	101	200	200	200	101	101	200
	002	011	002	011	020	200	011	101	110	101
	8	**16**	**12**	**24**	**12**	**16**	**48**	**96**	**24**	**24**
	300	300	210	021	120	300	201	210	111	210
	120	021	210	201	102	111	120	111	111	201
	002	101	002	200	200	011	101	101	200	011
	3	**24**	**6**	**48**	**24**					
	200	200	200	200	110					
	200	110	200	110	110					
	020	110	011	101	101					
	002	002	011	011	101					

332	**1**	**1**	**9**	**12**	**6**	**2**	**18**	**18**	**6**	**12**
	332	330	222	321	312	302	212	221	320	311
		002	110	011	020	030	120	111	012	021

2	**9**	**18**	**6**
301	220	211	310
031	112	121	022

9	**18**	**6**	**12**	**12**	**18**	**9**	**72**	**18**	**36**
220	220	310	301	310	202	112	211	112	211
110	101	020	020	011	110	200	110	110	101
002	011	002	011	011	020	020	011	110	020

1	**6**	**12**	**9**	**18**	**36**	**36**	**18**	**36**	**72**	**36**
300	300	300	210	210	210	201	201	210	210	111
030	012	021	120	102	012	111	021	101	111	111
002	020	011	002	020	110	020	110	021	011	110

9	**18**	**36**	**6**	**36**
200	200	200	110	110
110	101	110	110	110
020	011	011	110	101
002	020	011	002	011

2222	**1**	**4**	**24**	**24**	**32**	**3**	**24**	**8**
	2222	2220	2211	2201	2111	2200	2011	1111
		0002	0011	0021	0111	0022	0211	1111

6	**12**	**48**	**96**	**48**
2200	2200	2011	2011	1111
0020	0011	0011	0101	1100
0002	0011	0200	0110	0011

24	**48**	**96**	**16**	**48**	**16**	**32**
2001	2010	2100	0111	1011	1011	0111
0201	0201	0111	0111	1110	0111	1101
0020	0011	0011	2000	0101	1100	1010

1	**12**	**32**	**12**	**48**
2000	2000	2000	1100	1100
0200	0200	0101	1100	0110
0020	0011	0110	0011	0011
0002	0011	0011	0011	1001

25. The Formulae for the Sample Moments about a Fixed Point in Terms of the Moments of the Universe. The partitions of section 21 and section 24 can be immediately interpreted to give the formulae for the moments of the sample function. For example

$$\mu'_{11}(f_3, f_2) = b_3 b_2 n\mu_5 + (b_3 b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)\mu_3\mu_2$$

and the value of $\mu'_{21}(f_3, f_2)$ as given in section 15 can be read by inspection. The value of the b's are to be inserted for any specific function. The coefficient of μ_2^3 in the expansion of $\mu'_3(f_2)$ is

$$(b_2^3 + 6b_2 b_{11}^2 + 8b_{11}^3)n(n-1)(n-2).$$

In case $f_2 = m_2$, $b_2 = \dfrac{n-1}{n^2}$, and $b_{11} = \dfrac{-1}{n^2}$ so that the coefficient is

$$\frac{(n-1)(n-2)(n^3 - 3n^2 + 9n - 15)}{n^5}$$

as indicated previously by Tchouproff (10, 192) and Church (9, 82).

The partitions of section 21 give the 8 formulae $\mu_{r,(N)}$ which Tchouproff gave (10, 155). In this case $f_r = m'_r$ and every b is 0 except those having single subscripts and these equal $\dfrac{1}{n}$.

The partitions of section 21 give the formulae $\nu_{r,(N)}$ which were given by Tchouproff (10, 186). In this case it is only necessary to take $f_r = m_r$ and to give the b's the proper values. Tchouproff has arranged his results according to decreasing powers of n. As an illustration we derive his result for $\nu_{4,(N)} = \mu'_1(m_4)$. From section 21

$$\mu'_1(f_4) = b_4 n\mu_4 + 3b_{22}n(n-1)\mu_2^2$$

and from Table II

$$b_4 = \frac{(n-1)(n^2-3n+3)}{n^4} \quad \text{and} \quad b_{22} = \frac{2n-3}{n^4}$$

so that

$$\mu'_1(m_4) = \left(1 - \frac{4}{n} + \frac{6}{n^2} - \frac{3}{n^3}\right)\mu_4 + \left(\frac{6}{n} - \frac{15}{n^2} + \frac{9}{n^3}\right)\mu_2^2$$

$$= \mu_4 + \frac{1}{n}(6\mu_2^2 - 4\mu_4) - \frac{1}{n^2}(15\mu_2^2 - 6\mu_4) + \frac{1}{n^3}(9\mu_2^2 - 3\mu_4)$$

as indicated by him.

The partitions of section 24 also give formulae which have appeared before. For example the partitions

1	1	2
22	20	11
	02	11

which symbolize the formula

$$\mu_2'(f_2) = b_2^2 n\mu_4 + (b_2^2 + 2b_{11}^2)n(n - 1)\mu_2^2$$

become

$$\mu_2'(m_2) = \frac{(n - 1)}{n^3}\left[(n - 1)\mu_4 + (n^2 - 2n + 3)\mu_2^2\right]$$

which was early derived by "Student" (8, 3) and Tchouproff (10, 192). Similarly the partitions of 222 and 2222 give the formula for and $\mu_3'(m_2)$ and $\mu_4'(m_2)$ which were given by Tchouproff (10, 192–193) and Church (9, 82).

Sections 21 and 24 can then be used to write the moments about a fixed point of a sample function in terms of the moments of the universe. In the case of new functions the b's must first be determined. Formulae involving unit columnar partitions are not included. If the formulae were desired in terms of moments about a fixed point of the universe, it would be necessary to write in addition all possible partitions. See for example the last formula of section 23.

26. **The Formulae For Moments of Any Sample Function in Terms of Moments of the Universe.** The partitions of sections 21 and 24 are also useful in writing the formulae for the moments of the sample moments. It is necessary to make the usual adjustments in changing from moments about a fixed point to moments:

$$\mu_2(f_r) = \mu_2'(f_r) - \mu_1'^2(f_r)$$

$$\mu_{11}(f_{r_1}, f_{r_2}) = \mu_{11}'(f_{r_1}, f_{r_2}) - \mu_{10}'(f_{r_1}, f_{r_2})\mu_{01}'(f_{r_1}, f_{r_2}).$$

The particular two way partitions which are involved in this adjustment are immediately recognizable. They are the ones which have an entry which is the only entry in the row and in the column in which it is. Thus $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ gives one of the terms contributing to $\mu_2'(f_2)\,\mu_1'(f_2)$. In addition its coefficient is the same, if sign is not considered, as the coefficient of $\mu_2'(f_2)\,\mu_1'(f_2)$ in the expansion of $\mu_3(f_2)$ in terms of moments of f_2. This has to be true since each is the number of ways of forming $\begin{matrix}220\\002\end{matrix}$. And so in general the remaining function of n accompanying this adjustment is the product of the coefficient associated with 22 and that associated with 2. The sign is plus when odd numbers of moments are multiplied and minus when even numbers of moments are multiplied. Hence $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ contributes $-3n^2b_2^3$ to the adjustment to moments and the total contribution of $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ to the value of $\mu_3(f_2)$ is $3b_2^3[n(n - 1) - n^2] = -3b_2^3 n$. More extensive study leads to the following general method of using the formulae of section 24.

A. Write the coefficient of every two way partition according to section 25.

B. Block off each single entry by drawing a line through its row and column. For example

$$\begin{array}{c} 6 \\ 2200 \\ 0020 \\ 0002 \end{array}$$

The resulting partitions, 22, 2, 2 are called component parts.

C. Form new partitions by eliminating component parts one at a time, two at a time, three at a time, etc. from the original partition in all possible ways.

D. Form the coefficient of the resulting parts according to the methods of section 25. Multiply by $(-1)^{s-1}$ where s is the number of resulting parts. The values of b will not change.

E. Multiply in addition by $s - 1$ when the component parts are all taken separately.

As an example we find the contribution of the partition $\begin{array}{c} 6 \\ 2200 \\ 0020 \\ 0002 \end{array}$ to the value of $\mu_4(f_2)$. It gives

$$6b_2^4[n(n-1)(n-2) - 3n^2(n-1) + 2n^3]\mu_4\mu_2\mu_2 = 12nb_2^4\mu_4\mu_2^2.$$

Similarly $\begin{array}{c} 1 \\ 2000 \\ 0200 \\ 0020 \\ 0002 \end{array}$ contribu

$$b_2^4[n^{(4)} - 4nn^{(3)} + 6n^3(n-1) - 3n^4]\mu_2^4 = 3b_2^4(n-2)\mu_2^4.$$

We use the method in finding the coefficient of μ_2^3 in the expansion of $\mu_3(m_2)$. We find first the coefficient of μ_2^3 in the expansion of $\mu_3(f_2)$. It is indicated by the partitions

$$\begin{array}{ccc} 1 & 6 & 8 \\ 200 & 200 & 110 \\ 020 & 011 & 011 \\ 002 & 011 & 101 \end{array}$$

so that the coefficient of μ_2^3 is

$$b_2^3[n(n-1)(n-2) - 3n^2(n-1) + 2n^3] + 6b_2b_{11}^2[n(n-1)(n-2) - n^2(n-1)]$$
$$+ 8b_{11}^3n(n-1)(n-2) = b_2^3(2n) + 6b_2b_{11}^2(-2n^2 + 2n)$$
$$+ 8b_{11}^3n(n-1)(n-2).$$

When $b_2 = \frac{n-1}{n^2}$ and $b_{11} = \frac{-1}{n^2}$ this becomes $\frac{2(n-1)(n^2-12n+15)}{n^5}$ as previously given by such authors as Tchouproff (10, 194), Church (9, 82), Carver (Richardson) (11, 271).

The general Tchouproff-Church formulae for the third and fourth moments of the variance may be written out in this way as may many other moment formulae which have not been printed.

27. **The Thiele Moments of the Sample Function in Terms of the Moments of the Universe.** It is possible also to write the Thiele moments of the sample function in terms of the moments of the universe. The technique is very similar to that of the previous section. The basis of the transformation is now the formula for Thiele moments in terms of moments about a fixed point rather than moments in terms of moments about a fixed point. The results are the same as those of the last section when a double or a triple product of f's is involved, but they differ with the introduction of a larger number of products. The partitions having component parts are broken up into these component parts as before but the parts are combined in all possible ways. Multipliers are determined as before with the exception that there is a multiplication by $(-1)^{s-1}(s-1)!$ where s is the number of resultant parts. Thus the term $\begin{matrix}2000\\0200\\0020\\0002\end{matrix}$ contributes $b_2^4[n^{(4)} - 4nn^{(3)} - 3n^2(n-1)^2 + 12n^3(n-1) - 6n^4]\mu_2^4 = -6b_2^4n\mu_2^4$ to the value of $\lambda_4(f_2)$.

28. **The Moments About a Fixed Point of the Sample Function in Terms of the Thiele Moments of the Universe.** We return to the problem of section 25, only we wish to express the results in terms of the Thiele moments of the universe. We must use the formulae of section 12.

$$\mu_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (\lambda_{p_1})^{\pi_1} \cdots (\lambda_{p_s})^{\pi_s}$$

where $p_i \gtreqless 1$.

Thus μ_r will contribute to all partitions of r and inversely the contributions to a given partition are composed only of these terms which are obtained by combining the different elements of the partition. Since the numerical coefficient in the expansion of μ_r is the number of ways in which the r units can be collected to form the partition, it follows at once that the complete λ coefficient can be obtained by grouping the parts of the partition in all possible ways, determining the coefficient of each according to the methods of section 25, and adding. In this way the formulae of section 21 can be used to give expansions in terms of partition moments. For example the representation of $\mu_1'(f_6)$

1	**15**	**10**	**15**
6	4	3	2
	2	3	2
			2

gives at once

$$b_6 n\lambda_6 + 15[b_6 n + b_{42}n(n-1)]\lambda_4\lambda_2 + 10[b_6 n + b_{33}n(n-1)]\lambda_3^2 \\ + 15[b_6 n + 3b_{42}n(n-1) + b_{222}n(n-1)(n-2)]\lambda_2^3.$$

The partitions of section 21 can be made to give the formula $\mu_1'(l_r)$ which were given by Thiele (1, 45–46). For example the formula for $\mu_1'(f_4)$ is indicated by

1	**3**
4	2
	2

so that

$$\mu_1'(f_4) = b_4 n\lambda_4 + 3[b_4 n + b_{22}n(n-1)]\lambda_2^2$$

and since

$$b_4 = \frac{(n-1)(n^2-6n+6)}{n^4} \quad \text{and} \quad b_{22} = \frac{2n-3}{n^4}$$

$$\mu_1'(l_4) = \frac{(n-1)(n^2-6n+6)\lambda_4}{n^3} - \frac{6(n-1)\lambda_2^2}{n^2},$$

which agrees with the result as given by him (1, 45).

The two way partitions of section 24 can be used similarly. This device for changing to the λ's is due to the ingenuity of R. A. Fisher who applied it to the case where $f_r = k_r$.

As an illustration we write from section 24 the value of $\mu_2'(f_2)$ in terms of λ's. The partition representation

1	**1**	**2**
22	20	11
	02	11

gives at once

$$b_2^2 n\lambda_4 + [b_2^2 n + b_2^2 n(n-1)]\lambda_2^2 + 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2$$

which agrees with the result of section 12. The other illustrations of that section may be written out similarly.

As a final illustration of this technique we find the coefficient of λ_4^2 in the expansion of $\mu_{21}'(f_3, f_2)$. The partitions are

2	**9**	**18**	**6**
301	220	211	310
031	112	121	022

and the coefficient is

$$2[b_3^2b_2n + b_3^2b_{11}n(n - 1)] + 9[b_3^2b_2n + b_{21}^2b_2n(n - 1)] \\ + 18[b_3^2b_2n + b_{21}^2b_{11}n(n - 1)] + 6[b_3^2b_2n + b_3b_{21}b_2n(n - 1)].$$

If the b's are inserted to form the k's, the first and last terms become 0 and the others give $\frac{27n - 45}{n(n - 1)^2}$. This agrees with the value as given by R. A. Fisher (3, 208).

29. The Moments of the Sample Function in Terms of the Thiele Moments of the Universe. The partition representations of section 21 and section 24 can be used similarly to write formulae for the moments of the sample function in terms of the Thiele moments of the universe. It is only necessary to use the general plan of section 26, but to write the coefficient of every resulting partition according to the method of section 28. For example the partition

$$\begin{array}{cccc} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&2 \end{array}$$

gives the coefficient

$$b_2^4[n + 4n^{(2)} + 3n^{(2)} + 6n^{(3)} + n^{(4)}] - 4b_2^4[n^2 + 3n^2(n - 1) + n^2(n - 1)(n - 2)] \\ + 6b_2^4[n^3 + n^3(n - 1)] - 3b_2^4n^4 = b_2^4[n^4 - 4n^4 + 6n^4 - 3n^4] = 0.$$

30. The Thiele Moments of the Sample Function in Terms of the Thiele Moments of the Universe. The partition representations of section 21 and section 24 can also be interpreted to give the Thiele moments of the sample function in terms of the Thiele moments of the universe. The scheme is similar to that of section 29 except that the formulae for changing to Thiele moments are used as in section 27. For example the partition $\begin{matrix} 2000\\ 0200\\ 0020\\ 0002 \end{matrix}$ has now associated with it

$$b_2^4[n + 4n^{(2)} + 3n^{(2)} + 6n^{(3)} + n^{(4)}] - 4b_2^4[n^2 + 3n^2(n - 1) + n^2(n - 1)(n - 2)] \\ - 3b_2^4n^2(n - 1)^2 + 12b_2^4\,[n^3 + n^3\,(n - 1)] - 6b_2^4n^4 = 0.$$

The application of this method enables one to write the formulae of section 13 (and others which they typify) with relative ease. It is now possible to complete the task left unfinished in section 15. We do not take the space necessary to write all the terms of $\lambda_{21}(f_3, f_2)$ since the lengthy expression can be obtained quite readily from the representation of section 24. One term, say the coefficient of $\lambda_6\lambda_2$, is represented by

1	**9**	**12**	**6**
330	222	321	312
002	110	011	020

and gives

$$9[b_3^2b_2n + b_{21}^2b_2n(n-1)] + 12[b_3^2b_2n + b_3b_{21}b_{11}n(n-1)] + 6[b_3^2b_2n + b_3b_{21}b_2n(n-1)]$$

which becomes $\frac{21}{n(n-1)}$ when $b_3 = b_2 = \frac{1}{n}$ and $b_{21} = b_{11} = \frac{-1}{n(n-1)}$. This agrees with the result given by R. A. Fisher (3, 209).

For simplicity of form it is logical to use this formulization of results, Thiele moments in terms of Thiele moments, and it has been used by Thiele (1), Craig (2), Fisher (3) and Georgescu (4). They however have used different sample moment functions. Thiele and Georgescu used the Thiele moments of the sample, Craig and Georgescu the moments while Fisher introduced the k function.

The present discussion deals with the corresponding partition moments of any rational integral isobaric moment function of the sample. The results indicated here give many of the results of the previous authors as special cases. For example the symbolic formula 44 of section 24 gives the $m^7\lambda_2(\mu_4)$ of Thiele (1, 45), the $S_{02}(\nu_2, \nu_4)$ of Craig (2, 57), the $\kappa(44)$ of R. A. Fisher (3, 210) as special cases when the formula 44 is given the interpretation of this section.

Some may prefer the Craig attack (2, 21–35) to the partition method. It should be noted that the formulae of sections 21 and 24 can be used in place of part of the Craig method. Thus his formulae (2, 22)

$$\nu_{80} = \lambda_{80} + 28\,\lambda_{60}\lambda_{20} + 56\,\lambda_{50}\lambda_{30} + \text{etc.}$$

$$\nu_{44} = \lambda_{44} + (12\,\lambda_{42}\lambda_{02} + 16\,\lambda_{33}\lambda_{11}) + \text{etc.}$$

are immediately obtainable from the symbolic formulae by writing λ's in place of b's and by using row, rather than column, subscripts. It is then necessary to compute the values of $\lambda_{k_1k_2}\cdots$ as given by him (2, 16–17, 40) and to insert in his expansions of $S_{kl}(\nu_m, \nu_n)$ in terms of ν's. For example

$$S_{11}(\nu_3, \nu_2) = \frac{1}{n}[\nu_{50} + (n-1)\nu_{32} - n\nu_{30}\nu_{02}] \qquad (2, 32)$$

and from the symbolic formulae of sections 21 and 24

$$\nu_{50} = \lambda_{50} + 10\lambda_{30}\lambda_{20}$$

$$\nu_{32} = \lambda_{32} + \lambda_{30}\lambda_{02} + 3\lambda_{12}\lambda_{20} + 6\lambda_{21}\lambda_{11}$$

$$\nu_{30} = \lambda_{30}$$

$$\nu_{20} = \lambda_{20}$$

so that

$$S_{11}(\nu_3, \nu_2) = \frac{1}{n}[\lambda_{50} + (n-1)\lambda_{32} + 9\lambda_{30}\lambda_{20} + (n-1)(6\lambda_{21}\lambda_{11} + 3\lambda_{21}\lambda_{20})] \tag{2, 30}$$

which agrees with that given by Prof. Craig (aside from an obvious typographical error). The insertion of the values of λ gives the value as indicated by $\lambda_{11}(m_3, m_2)$ of section 13 and by the first method of the present section.

31. **Special Rules for the Determination of the Coefficients in the Case of the Fisher and Georgescu Analyses.** R. A. Fisher (3) gave a number of simple rules which assist greatly in the determination of the coefficients accompanying the partitions. Georgescu (4) also introduced special rules for the evaluation of the coefficients of the different partitions he used. It is not to be expected that all these rules are applicable in the more general case under present consideration, but the vanishing of such coefficients as that of $\begin{matrix}2000\\0200\\0020\\0002\end{matrix}$ leads one to suspect that there might be some rules which are applicable to this general case. A sensible method of procedure is to examine the rules of Fisher and Georgescu and determine if they hold in the more general analysis. The special rules of R. A. Fisher might be given somewhat as follows.

A. If a partition has a column with a single entry, that column may be eliminated and the factor n^{-1} introduced.

B. Any partition having a row with a single entry may be neglected.

C. "We may exclude any partition in which any set of rows is connected to its complementary set by a single column only."

D. In determining the algebraic coefficient of a partition the "pattern" is sufficient and precise entries are not needed. Thus the partitions $\begin{matrix}21\\11\end{matrix}$ and $\begin{matrix}35\\42\end{matrix}$, although they have different numerical factors, have associated with them the same function of n. This value is indicated by the pattern $\begin{matrix}xx\\xx\end{matrix}$ which has associated with it the function $\frac{1}{n-1}$. As a result of this property Fisher was able to provide a table (3, 223–226) of useful patterns which is of great assistance in writing the value of the coefficients.

E. Formulae of moments of k functions involving k_1 can be derived from corresponding formulae not involving k_1. "The effect upon the corresponding formula of adding a new unit part to the partition is (1) to modify every term in the formula by increasing the suffix of one of its κ functions by unity in every possible way, and (2) to divide the whole by n." (3, 206).

Two of the important Georgescu rules may be stated.

A′. The numerator function (aside from numerical coefficient) is not **altered** if columns are changed to rows and vice versa. Thus the coefficient of s_2^3 in $S(3^2) = \dfrac{6N(N-1)}{(N+1)^3}$ and the coefficient of s_3^2 in $S(2^3)$ is $\dfrac{4N(N-1)}{(N+1)^4}$. Georgescu has replaced n by $N+1$.

B′. All partitions which can be broken up into component parts have coefficients of 0. This is extended to include all partitions which have as component parts other partitions. Thus

$$\begin{matrix} 2100 \\ 1100 \\ 0012 \\ 0034 \end{matrix}$$

has a coefficient 0 as does the equivalent

$$\begin{matrix} 2010 \\ 1010 \\ 0102 \\ 0304 \end{matrix}$$

32. Special Rules for the Determination of the Coefficients in the More General Case. In the more general case we have

A. If a partition has a single column with a single entry, c, that column may be eliminated and the value b_c inserted as a multiplier. This is immediately evident since the contribution of that column to each term in the expansion is b_c times its value if the column were eliminated.

B. The coefficient of any partition having an entry which is the only entry in its row and column, is 0.

This rule, which saves considerable labor in that it makes unnecessary the computation of the coefficients of many of the partitions of section 24, is established in this way. Without loss of generality the partition may be represented by

$$\pi_{u+1,\,v+1} = \begin{matrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1v} & 0 \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2v} & 0 \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3v} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{u1} & c_{u2} & c_{u3} & \cdots & c_{uv} & 0 \\ 0 & 0 & 0 & & 0 & c_{u+1,\,v+1} \end{matrix}$$

and $\pi_{u,v}$ may represent the partition containing the first u rows and the first v columns. We determine the coefficient of $\pi_{u+1,\,v+1}$ in terms of the coefficient of $\pi_{u,v}$. Consider first any grouping of the u rows of $\pi_{u,v}$ into w rows. There will be w corresponding groupings of $\pi_{u+1,\,v+1}$ in which the last row is added, in turn, to each of the w rows and another $w+1$ rowed term in which it is not added. In each of the first w cases the coefficient by rule A is multiplied by $b_{c_{u+1}, v+1}$. In the case of the $w + 1$ rowed partition the coefficient is multiplied by $b_{c_{u+1}, v+1}$, and $n^{(w)}$ is replaced by $n^{(w+1)}$. A final adjustment takes care of the transition from the moment about a fixed point of the sample function to the Thiele moment of the sample function. This adjustment demands the multiplication of the coefficient of $\pi_{u, v}$ by $b_{c_{u+1}, v+1}\, n$ and the subtraction from the sum of the other terms. If B_w is the coefficient of the w rowed form, it follows at once that the corresponding coefficient is

$$B_w b_{c+1, v+1} [wn^{(w)} + n^{(w+1)} - n\, n^{(w)}] = 0.$$

This holds for the expansion of any term of $\pi_{u, v}$ and hence the coefficient of $\pi_{u+1, v+1}$ is 0. Of course the argument holds if the partition has more than 2 component parts.

It thus appears that this rule holds not only for k_r and m_r as Fisher and Georgescu have noted, but for f_r.

C. The coefficient of any partition which can be broken into component parts is 0. In this sense a component part is any group of rows or columns which have no entry in common with any other group of rows or columns. It corresponds in matrix language to a matrix which results when one matrix is zero bordered by another matrix although rows and columns may thereafter be interchanged.

The proof of this more general case follows the general line of the simpler case although the reasoning is more complicated. For example the coefficient of

$$\begin{matrix} c_{11} & c_{12} \cdots c_{1v} & 0 & 0 \\ c_{21} & c_{22} \cdots c_{2v} & 0 & 0 \\ c_{31} & c_{32} \cdots c_{3v} & 0 & 0 \\ & & & \\ c_{u1} & c_{u2} \cdots c_{uv} & 0 & 0 \\ 0 & 0 \quad \cdots 0 & c_{u+1, v+1} & c_{u+1, v+2} \\ 0 & 0 \quad \cdots 0 & c_{u+2, v+1} & c_{u+2, v+2} \end{matrix}$$

is 0 since any w rowed term of the $\pi_{u, v}$ contributes

$$\begin{aligned} &B_w b_{c_{u+1}, v+1 + c_{u+2}, v+1}\, b_{c_{u+2}, v+2 + c_{u+2}, v+2} [wn^{(w)} + n^{(w+1)} - nn^{(w)}] \\ &\quad + B_w b_{c_{u+1}, v+1 c_{u+2}, v+2}\, b_{c_{u+1}, v+2 c_{u+2}, v+2} [w(w-1)\, n^{(w)} + 2wn^{(w+1)} + n^{(w+2)} \\ &\qquad - n(n-1)\, n^{(w)}] = 0. \end{aligned}$$

Other special rules of Fisher and Georgescu do not hold in the general case. Thus Fisher rule B is not generally true since the partitions

$$\begin{matrix} 12 \\ 30 \end{matrix} \quad \text{and} \quad \begin{matrix} 22 \\ 20 \end{matrix}$$

have respective algebraic coefficients of $b_4b_2n + b_{31}b_2n(n - 1)$ and

$$b_4b_2n + b_{22}b_2n(n - 1)$$

and these are not in general equal to 0.

The Fisher rule C is replaced by the somewhat less general C of the present section.

The Fisher rule D is not applicable in the general case. The Fisher rule D is applicable in all cases in which the value of the $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is completely determined by the number of parts for in this case the particular value of each part is not pertinent. We may say then that the Fisher rule D is applicable to all cases in which $b_{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}}$ is a function of ρ, n where ρ is the number of parts. This condition is satisfied by $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \dfrac{(-1)^\rho (\rho - 1)!}{n^{(\rho)}}$ and the coefficients are worked out for it in Fisher's paper. The same method is applicable to other functions satisfying the general condition although the values of the coefficients will of course vary with the definition of b.

The Fisher rule E is not applicable to the general case. Its validity, from an algebraic standpoint, depends upon the Fisher property B which is not generally applicable. The Fisher rule E as applied to the more general case gives correct terms but it does not give all the terms. For example the Fisher rule E applied to $\lambda_2(k_2)$ gives

$$\lambda_2(k_2) = \frac{\lambda_4}{n} + \frac{2\lambda_2^2}{n - 1}$$

$$\lambda_{21}(k_2, k_1) = \frac{\lambda_5}{n^2} + \frac{4\lambda_3\lambda_2}{n(n - 1)}.$$

The application of a corresponding rule to

$$\lambda_2(f_2) = b_2^2n\lambda_4 + 2[b_2^2n + b_{11}n(n - 1)]\lambda_2^2$$

would give

$$\lambda_{21}(f_2,f_1) = b_2^2b_1n\lambda_5 + 4[b_2^2b_1n + b_{11}^2b_1n(n - 1)]\lambda_3\lambda_2$$

while the correct result is indicated by

1	4	2	4
221	210	201	111
	011	020	110

and is

$$\lambda_{21}(f_2f_1) = b_2^2b_1n\lambda_5 + 4[b_2^2b_1n + b_2b_{11}b_1n(n-1)]\lambda_3\lambda_2 + 2[b_2^2b_1n + b_2^2b_1n(n - 1)]\lambda_3\lambda_2$$
$$+ 4[b_2^2b_1n + b_{11}^2b_1n(n - 1)]\lambda_3\lambda_2.$$

The difference is due to the vanishing of the two middle terms in the case of the k functions.

The rule B′, which Georgescu found most useful in computing and checking his formulae, is not generally true. It is not even true in the case of the k function, as can be discovered by using it on the list given by R. A. Fisher (3, 210). It is interesting to note that the Georgescu method, while not being able to utilize many of the special rules of the Fisher method, does use this rule which is not in general adaptable to the Fisher method.

33. **Special Rules in the Case of the h' Functions.** Special rules can be worked out for other sample functions. As an illustration we examine the function h_r' which was defined in section 19. It is recalled that $b_{1^\rho} = \frac{1}{n^{(\rho)}}$ and that $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 0$ for all other cases. It follows at once that

A. Any partition having any entry other than unity (or zero) may be neglected.

B. The value of b_{1^ρ} is $\frac{1}{n^{(\rho)}}$.

As an illustration we write the value $\lambda_{21}(h_3', h_2')$. From the partitions of section 24 we select

36		**36**
111		110
111	and	110
110		101
		011

as being the only partitions making a contribution. The result of section 19 follows at once.

34. **The Case of a Normal Universe.** A normal universe is characterized by the relationship that $\lambda_r = 0$ when $r > 2$. It follows that it is only necessary to compute the coefficients of those partitions giving powers of λ_2.

Wishart (5) (7) has developed the partition analysis of the k function in the case of a normal parent while Georgescu has studied the corresponding m function. It is not the purpose of this section to make extensive study of the case of the normal parent but simply to indicate that the results of section 24 are immediately applicable. As an illustration we write the values of $\lambda_1(f_2)$, $\lambda_2(f_2)$, $\lambda_3(f_2)$ and $\lambda_4(f_2)$ in the case of a normal universe. The terms are given successively, by

1	**2**	**8**	**48**
2	11	110	1100
	11	011	0110
		101	0011
			1001

and hence

$$\lambda_1(f_2) = b_2 n \lambda_2$$

$$\lambda_2(f_2) = 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2$$

$$\lambda_3(f_2) = 8[b_2^3 n + 3b_2 b_{11}^2 n(n-1) + b_{11}^3 n(n-1)(n-2)]\lambda_2^3$$

$$\begin{aligned}\lambda_4(f_2) = 48[b_2^4 n &+ 6b_2^2 b_{11}^2 n(n-1) + b_{11}^4 n(n-1) + 4b_2 b_{11}^3 n(n-1)(n-2) \\ &+ 2b_{11}^4 n(n-1)(n-2) + b_{11}^4 n(n-1)(n-2)(n-3)]\lambda_2^4.\end{aligned}$$

It is only necessary to substitute the b's to obtain the results for different values of f. This is done in Table IV.

TABLE IV

The first four Thiele moments of f_2 for various sample functions in the case of a normal universe

Sample function	$\lambda_1(f_2)$	$\lambda_2(f_2)$	$\lambda_3(f_2)$	$\lambda_4(f_2)$
m_2	$\frac{(n-1)}{n}\lambda_2$	$\frac{2(n-1)}{n^2}\lambda_2^2$	$\frac{8(n-1)}{n^3}\lambda_2^3$	$\frac{48(n-1)\lambda_2^4}{n^4}$
k_2	λ_2	$\frac{2\lambda_2^2}{n-1}$	$\frac{8\lambda_2^3}{(n-1)^2}$	$\frac{48\lambda_2^4}{(n-1)^3}$
l_2	$\frac{(n-1)}{n}\lambda_2$	$\frac{2(n-1)}{n^2}\lambda_2^2$	$\frac{8(n-1)\lambda_2^3}{n^3}$	$\frac{48(n-1)\lambda_2^4}{n^4}$
m_2'	λ_2	$\frac{2\lambda_2^2}{n}$	$\frac{8\lambda_2^3}{n^3}$	$\frac{48\lambda_2^4}{n^4}$
h_2	λ_2	$\frac{2\lambda_2^2}{n-1}$	$\frac{8\lambda_2^3}{(n-1)^2}$	$\frac{48\lambda_2^4}{(n-1)^3}$
h_2'	0	$\frac{2\lambda_2^2}{n(n-1)}$	$\frac{8(n-2)\lambda_2^3}{n^2(n-1)^2}$	$\frac{48(n^2-3n+3)\lambda_2^4}{n^3(n-1)^3}$

One surmises that the general value of

$$\lambda_r(f_2) \text{ is } 2^{r-1}(r-1)!\,\lambda_2^r B\colon \begin{array}{l} 11000\cdots 0 \\ 01100\cdots 0 \\ 00110\cdots 0 \\ \cdots\cdots\cdots\cdots \\ 00000\cdots 11 \\ 10000\cdots 01 \end{array}$$

where B represents the b coefficient of the r rowed partition. This induction appears consistent with the fact that

$$\lambda_{r+1}(k_2) = \frac{2^r r!\, \lambda_2^{r+1}}{(n-1)^r}$$

as shown by John Wishart (7). The whole subject of the Thiele moments of the general function in the case of a normal universe would make an interesting subject of investigation.

35. **Summary and Conclusion.** The contributions of this paper include

1. The definitions of specific moment functions in terms of power sums.
2. The use of indeterminate multipliers in representing a general isobaric moment function.
3. The finding of the expected value of products of these functions by algebraic methods.
4. The use of tables in writing these expected values in terms of moments (or of moments about a fixed point) of the universe.
5. The finding of the expected values of specific moment functions by substitution.
6. Means of establishing the expansion of new moment functions which are defined by their expected values.
7. The introduction of the sample function of weight r whose expected value is μ_r.
8. The introduction of the sample function of weight r whose expected value is $\mu_1'^r$.
9. The two way partition formulae of weight $\leqq 8$ which do not involve unit parts.

The use of these partition formulae in writing:

10. The moments about a fixed point of f_r in terms of moments.
11. The moments of f_r in terms of moments.
12. The Thiele moments of f_r in terms of moments.
13. The moments about a fixed point of f_r in terms of Thiele moments.
14. The moments of f_r in terms of Thiele moments.
15. The Thiele moments of f_r in terms of Thiele moments.
16. Special rules in the case of Thiele moments.
17. The applicability of these results to a given sample moment function and hence the derivation of varied results, of such authors as Thiele, Tchouproff, Church, Fisher, Craig, and Georgescu, from the same partition formulae.
18. The simplicity of the formulae when h_r' is used as the sample function.
19. The application of the synthetic formulae to the Craig method.
20. The applicability of the theory to a normal universe.

The introduction of such general procedure opens up a wide field for future study. It is impossible in a single paper dealing with so broad a subject to do more than to outline the general scheme by which two way partitions can be used as a central formulization of the various formulae for moments of moments. More detailed proofs and more extensive analysis of the more important of the special cases will undoubtedly be supplied by later writers.

In later papers the author will show how the partition representation can be used in the case of multivariate distributions and how it can also be used, in connection with the sampling polynomials introduced by H. C. Carver (11), to represent the more complex formulae obtained in the case of finite sampling.

It is obvious that the author is indebted to the classical moment studies of Fisher and Craig. He also wishes to acknowledge his indebtedness to Prof. Craig and to Prof. Carver who have read the manuscript and have made valuable suggestions.

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