Pairman and Pearson gave a numerical example in which both the lack of high contact and the grouping introduced large errors. They started with $y_x = 100{,}000\sqrt{x}$ and from this formed ten values of A_x. From these they computed the ν'_m's and corrected them to get the μ'_m's. The exact values of the latter were already known to them through integration of the original equation.

The following table compares four values of moments from these data.

m	ν'_m	μ'_m by Sheppard's Formula	μ'_m with Pairman-Pearson Full Corrections	Method Developed Here	True Values
1	5.9880	5.9880	5.9994	5.9996	6.0000
2	42.6900	42.6067	42.8570	42.8576	42.8571
3	331.0854	329.5884	333.3349	333.3387	333.3333
4	2698.7735	2677.4576	2727.2757	2727.3555	2727.2727

Despite the use of the $\Delta^i_{A_x}$'s instead of Δ^i_x's, the results of this method are almost as good as by the older one. The method has the additional advantage of unifying the theories of the correction of moments from the two types of distribution.

REFERENCES

[1] W. F. Sheppard, "Calculation of the Most Probable Values of Frequency Constants," *Proceedings of the London Mathematical Society*, Vol. XXIX, pp. 353–380, 1898.

[2] E. Pairman and Karl Pearson, "Corrections for Moment Coefficients," *Biometrika*, Vol. XII, page 231 et seq.

[3] W. Palin Elderton, *Frequency Curves and Correlation*, pp. 24–27, Charles and Edwin Layton, London, 2d Edition, 1927.

Griffith Observatory,
and
California Institute of Technology.

FREQUENCY DISTRIBUTION OF PRODUCT AND QUOTIENT

By E. V. Huntington

The main purpose of this note is to establish Theorems 1 and 2. For the sake of completeness, the more familiar Theorems 3 and 4 are appended. All four of these theorems have numerous applications in the theory of frequency distributions. While the proofs of Theorems 1 and 2 in the elementary forms here given (and used in my class-room notes since 1934) can hardly be new, they seem not to be readily accessible in the current text-books.

Theorem 1. *Suppose a variable x is distributed in accordance with a probability law* $\int_0^\infty f(x)dx = 1$; *and a variable y in accordance with a probability law* $\int_0^\infty F(y)dy$

$= 1$, *x and y being independently distributed. Then the product, $u = xy$, will be distributed according to the law* $\int_0^\infty P(u)du = 1$, *where*

$$P(u) = \int_0^\infty f(u/y)\, F(y)(1/y)\, dy.$$

(The definite integral is a convenient representation of a probability law, since the limits on the integral sign indicate the interval over which the probability law is defined.)

PROOF. Represent the distribution of x by the density of dots along the axis of x, and the distribution of y by the density of dots along the axis of y. Since, by definition, the (relative) number of dots in an interval dx is $f(x)dx$ and the (relative) number of dots in an interval dy is $F(y)dy$, and since each dot in the interval dx is paired with each dot in the interval dy (in accordance with the hypothesis of independence), it follows that the (relative) number of dots in the corresponding area $dxdy$ will be $[f(x)dx][F(y)dy]$.

Now for fixed values of u and Δu, plot the curves $xy = u$ and $xy = u + \Delta u$ in the xy plane, as shown in Figure 1. Then the (relative) number of dots in the area bounded by these two curves is precisely what is meant by $P(u)\Delta u$. Hence the expression $P(u)\Delta u$ may be built up by integrating the expression $f(x)dx \cdot F(y)dy$ over this area, as follows.

$$P(u)\Delta u = \int_0^\infty \left[\int_{u/y}^{(u+\Delta u)/y} f(x)F(y)\, dx \right] dy$$
$$= \int_0^\infty \left[f(x')F(y) \int_{u/y}^{(u/y)+(\Delta u/y)} dx \right] dy,$$

where x' is a mean value of x between $x = u/y$ and $x = (u/y) + (\Delta u/y)$. Now at every point in the plane, $x = u/y$ (since $u = xy$). Hence we have:

$$P(u)\Delta u = \int_0^\infty [f(u'/y)F(y)(1/y)\Delta u]\, dy = \left[\int_0^\infty f(u/y)F(y)(1/y)\, dy \right] \Delta u,$$

from which the theorem follows immediately.

THEOREM 2. *Suppose a variable x is distributed in accordance with a probability law* $\int_{-\infty}^\infty f(x)dx = 1$; *and a variable y in accordance with a probability law* $\int_0^\infty F(y)dy = 1$, *x and y being independently distributed. Then the quotient, $z = x/y$, will be distributed according to the law* $\int_{-\infty}^\infty Q(z)dz = 1$, *where*

$$Q(z) = \int_0^\infty f(zy)F(y)y\, dy.$$

PROOF. As in the proof of Theorem 1, the (relative) number of dots in the area $dxdy$ will be $[f(x)dx][F(y)dy]$.

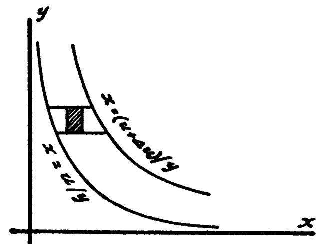

Fig. 1

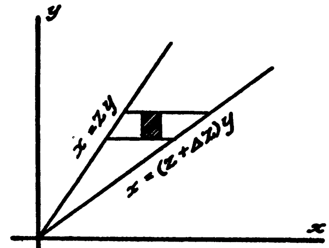

Fig. 2

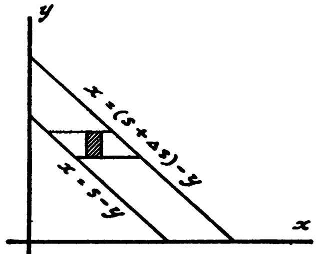

Fig. 3

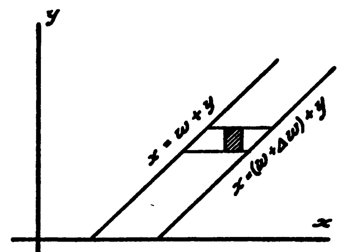

Fig. 4

Now for fixed values of z and Δz, plot the lines $x/y = z$ and $x/y = z + \Delta z$ in the xy plane, as shown in Figure 2. Then the (relative) number of dots in the area between these lines is precisely what is meant by $Q(z)\Delta z$. Hence the expression $Q(z)\Delta z$ may be built up by integrating the expression $f(x)dx \cdot F(y)dy$ over this area, as follows:

$$Q(z)\Delta z = \int_0^\infty \left[\int_{zy}^{(z+\Delta z)y} f(x)F(y)\,dx \right] dy$$

$$= \int_0^\infty \left[f(x')F(y) \int_{zy}^{zy+y\Delta z} dx \right] dy,$$

where x' is a mean value of x between $x = zy$ and $x = zy + y\Delta z$. Now at every point in the plane, $x = zy$ (since $z = x/y$). Hence we have

$$Q(z)\Delta z = \int_0^\infty [f(z'y)F(y)y\Delta z]\,dy = \left[\int_0^\infty f(zy)F(y)y\,dy \right] \Delta z,$$

from which the theorem follows immediately.

For convenience of reference, we include the corresponding theorems for the sum and difference, the proofs of which have long been well known.

THEOREM 3. *If x obeys a law $\int_0^\infty f(x)dx = 1$, and y obeys a law $\int_0^\infty F(y)dy = 1$, then the sum, $s = x + y$, will obey the law $\int_0^\infty \psi(s)ds = 1$, where*

$$\psi(s) = \int_0^\infty f(s - y)F(y)\,dy.$$

The proof consists in integrating $f(x)F(y)dxdy$ over the area bounded by the two lines $x + y = s$ and $x + y = s + \Delta s$, as shown in Figure 3.

THEOREM 4. *If x obeys a law* $\int_{-\infty}^{\infty} f(x)dx = 1$, *and y obeys a law* $\int_{-\infty}^{\infty} F(y)dy = 1$, *then the difference,* $w = x - y$, *will obey the law* $\int_{-\infty}^{\infty} R(w)dw = 1$, *where* $R(w) = \int_{-\infty}^{\infty} f(w + y)\, F(y)\, dy$.

The proof consists in integrating $f(x)F(y)dxdy$ over the area bounded by the two lines $x - y = w$ and $x - y = w + \Delta w$, as shown in Figure 4.

HARVARD UNIVERSITY.

MOMENTS ABOUT THE ARITHMETIC MEAN OF A HYPERGEOMETRIC FREQUENCY DISTRIBUTION

BY HAROLD D. LARSEN

In a recent paper[1] Kirkman has developed a method of continuation for obtaining the moments of a binomial distribution. Although other investigators[2] have found various methods which are perhaps superior from the standpoint of elegance and compactness, Kirkman's method is of some importance inasmuch as it is adaptable to use in a course in elementary statistics. With this thought in mind, we shall extend Kirkman's method to obtain the moments of the hypergeometric distribution of Table I.[3]

TABLE I

Variate v	Relative Frequency P_v
0	${}_nC_0\alpha^{(0)}\beta^{(n)}/N^{(n)}$
1	${}_nC_1\alpha^{(1)}\beta^{(n-1)}/N^{(n)}$
2	${}_nC_2\alpha^{(2)}\beta^{(n-2)}/N^{(n)}$
.	. . . . .
.	
n	${}_nC_n\alpha^{(n)}\beta^{(0)}/N^{(n)}$

[1] W. J. Kirkman, "Moments About the Arithmetic Mean of a Binomial Frequency Distribution," *Ann. Math. Statist.*, vol. vi, no. 2, June, 1935, pp. 96–101.

[2] For example, J. Riordan, "Moment Recurrence Relations for Binomial, Poisson and Hypergeometric Frequency Distributions," *Ann. Math. Statist.*, vol. viii, no. 2, June, 1937, pp. 103–111.

[3] For the Poisson distribution, this method degenerates into the application of a well-known recursion formula.