ON THE POWER OF THE L_1 TEST FOR EQUALITY OF SEVERAL VARIANCES

By George W. Brown

The criterion L_1 was obtained by Neyman and Pearson[1] for testing the statistical hypothesis H_1 that k samples, known to be from normal universes, are actually from universes with equal variances, where the means are unspecified. The test seems to be of importance, when one considers the number of experiments which are concerned with the comparison of several types of treatments. The experimenter is in most cases interested in the respective means, and it is usually assumed, in order to test for significance of the difference between sample means, that the variances of the distributions are equal. At present, significance tests for justifying this assumption are rarely applied. Because of the unsatisfactory status of the problem of testing simultaneously for means and variances, the L_1 test is appropriate for justifying first the assumption of equal variances before testing for the means.

Neyman and Pearson have treated the sampling distribution of L_1 when H_1 is true, and Wilks and Thompson[2] have discussed the general distribution of the criterion when H_1 is not true. Here we shall show that the test is unbiassed when the number of observations is the same in each sample, and is in general unbiassed in the limit, in a certain sense. In addition, values of the power function have been computed for a few selected cases, when k is 2, in order to exhibit qualitatively the sharpness of the test.

Let the i-th sample ($i = 1, 2, \cdots, k$) of n_i individuals be denoted by Σ_i and suppose Σ_i has been drawn at random from a normal population with mean m_i (unknown) and variance $\sigma_i^2 = \dfrac{1}{A_i}$. Denote the observations of Σ_i by x_{ir} ($r = 1, 2, \cdots, n_i$). Then the criterion L_1 is expressible[3] in terms of the observations as follows:

$$L_1^{\frac{1}{2}n} = \frac{n^{\frac{1}{2}n} \prod_{i=1}^{k} (c_i^2)^{\frac{1}{2}n_i}}{\prod_{i=1}^{k} n_i^{\frac{1}{2}n_i} \left[\sum_{i=1}^{k} c_i^2 \right]^{\frac{1}{2}n}} \tag{1}$$

where $n = \Sigma n_i$ and $c_i^2 = \sum_{r=1}^{n_i} (x_{ir} - \bar{x}_i)^2$. For convenience we shall let $L_1^{\frac{1}{2}n} = \lambda$.

[1] [1], pp. 461–464.

[2] See [4]. Nayer [3], studied the Type I approximation to the criterion L_1 and tabulated significance limits, etc.

[3] See [1], p. 464.

The variables $A_i c_i^2$ are independently distributed according to χ^2-laws with $n_i - 1$ degrees of freedom, respectivelv, hence the joint distribution of the c_i^2, when $\frac{1}{A_i}$ is the true value of σ_i^2 $(i = 1, 2, \cdots, k)$, is given by

$$(2) \qquad \frac{1}{2^{\frac{1}{2}n} \prod_i \Gamma\left(\frac{n_i - 1}{2}\right)} \cdot \prod_i \left[A_i^{\frac{1}{2}(n_i-1)} (c_i^2)^{\frac{1}{2}(n_i-3)}\right] e^{-\frac{1}{2}\sum_i A_i c_i^2} dc_1^2 \cdots dc_k^2$$

The power function,[4] which is defined as the probability of rejecting H_1, is given by $P(\lambda < \lambda_0)$, and is a function of the true values of the parameters $A_1, \cdots, A_k$, where λ_0 is defined so that $P(\lambda < \lambda_0) = \alpha$ when H_1 is true. Thus

$$F(A_1, \cdots, A_k) = P(\lambda < \lambda_0)$$

$$(3) \qquad = \frac{1}{2^{\frac{1}{2}n} \prod_i \Gamma\left(\frac{n_i - 1}{2}\right)} \int_{\lambda<\lambda_0} \prod_i \left[A_i^{\frac{1}{2}(n_i-1)} (c_i^2)^{\frac{1}{2}(n_i-3)}\right] e^{-\frac{1}{2}\Sigma A_i c_i^2} dc_1^2 \cdots dc_k^2$$

Note that when H_1 is true $P(\lambda < \lambda_0)$ is independent of the actual common value of the parameters, because of the homogeneity of λ.

Let us now restrict ourselves to the case in which $n_i = p$, $n = kp$. (1) and (3) become

$$(1') \qquad \lambda = k^{\frac{1}{2}kp} \left\{ \frac{\Pi c_i^2}{[\Sigma c_i^2]^k} \right\}^{\frac{1}{2}p}$$

and

$$F(A_1, A_2, \cdots, A_k)$$

$$(3') \qquad = \frac{1}{\left[2^{\frac{1}{2}p} \Gamma\left(\frac{p-1}{2}\right)\right]^k} \int_{\lambda<\lambda_0} \prod_{i=1}^{k} A_i^{\frac{1}{2}(p-1)} (c_i^2)^{\frac{1}{2}(p-3)} e^{-\frac{1}{2}\Sigma A_i c_i^2} dc_1^2 \cdots dc_k^2$$

We shall prove the following

THEOREM: *If* $n_1 = n_2 = \cdots = n_k = p$, *then* $F(A_1, A_2, \cdots, A_k) \geq F(A, A, \cdots, A)$. In other words, the probability of rejecting H_1 when H_1 is true is less than or at most equal to the probability of rejecting the hypothesis when any alternative is true, that is, the test is unbiassed. It should be noted that the statement of the theorem is to hold for each value of λ_0.

It is evident that $F(A_1, A_2, \cdots, A_k)$ remains invariant under permutations of the arguments, because of the symmetry in the c_i^2 of λ and of the integrand in (3′). Moreover, by using the homogeneity of λ we obtain the following relations

$$(4) \quad F(A_1, A_2, \cdots, A_k) = F\left(\frac{A_1}{A_k}, \frac{A_2}{A_k}, \cdots, \frac{A_{k-1}}{A_k}, 1\right) = F\left(1, \frac{A_2}{A_1}, \cdots, \frac{A_{k-1}}{A_1}, \frac{A_k}{A_1}\right)$$

[4] Defined by Neyman and Pearson, [2], p. 5.

Now if we set $a_i = \frac{A_i}{A_k}$ $(i = 1, 2, \cdots, k-1)$, we may replace $F(A_1, A_2, \cdots, A_k)$ by $F(a_1, a_2, \cdots, a_{k-1}, 1) = f(a_1, \cdots, a_k)$; we must now show that $f(a_1, \cdots, a_{k-1}) \geq f(1, 1, \cdots, 1)$. From (4) we obtain

$$(4') \qquad F(a_1, a_2, \cdots, a_{k-1}, 1) = F\left(1, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \cdots, \frac{a_{k-1}}{a_1}, \frac{1}{a_1}\right)$$

and permuting the arguments we have, finally,

$$(5) \qquad f(a_1, a_2, \cdots, a_{k-1}) = f\left(\frac{1}{a_1}, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \cdots, \frac{a_{k-1}}{a_1}\right).$$

Differentiate (5) with respect to a_1,

$$(6) \qquad \begin{aligned} f_1(a_1, a_2, \cdots, a_{k-1}) = -\frac{1}{a_1^2}\Bigg[&f_1\left(\frac{1}{a_1}, \frac{a_2}{a_1}, \cdots, \frac{a_{k-1}}{a_1}\right) \\ &+ a_2 f_2\left(\frac{1}{a_1}, \frac{a_2}{a_1}, \cdots\right) + \cdots + a_{k-1} f_{k-1}\left(\frac{1}{a_1}, \frac{a_2}{a_1}, \cdots\right)\Bigg] \end{aligned}$$

and set $a_1 = a_2 = \cdots = a_{k-1} = 1$, obtaining $f_1(1, 1, \cdots, 1) = -\sum_{i=1}^{k-1} f_i(1, 1, \cdots, 1)$. But $f_i(1, 1, \cdots, 1) = f_j(1, 1, \cdots, 1)$, hence

$$(7) \qquad f_i(1, 1, \cdots, 1) = 0; \qquad i = 1, 2, \cdots, k-1.$$

Now differentiating (6) with respect to a_1 and evaluating at $a_i = 1$, we have $f_{11}(1, 1, \cdots, 1) = \sum_{ij} f_{ij}(1, 1, \cdots, 1)$, that is,

$$\begin{aligned} f_{11}(1, 1, \cdots, 1) - f_{11}(1, 1, \cdots, 1) - f_{22}(1, 1, \cdots, 1) \\ - \cdots - f_{k-1\,k-1}(1, 1, \cdots, 1) = \sum_{i \neq j} f_{ij}(1, 1, \cdots, 1), \end{aligned}$$

hence, by the symmetry of the variables,

$$(8) \qquad \begin{aligned} f_{ij}(1, 1, \cdots, 1) &= -\frac{1}{k-1} f_{11}(1, 1, \cdots, 1), \qquad i \neq j; \\ f_{ii}(1, 1, \cdots, 1) &= f_{11}(1, 1, \cdots, 1). \end{aligned}$$

It is easily verified from (8) that the $f_{ij}(1, 1, \cdots, 1)$ are coefficients of a definite quadratic form in $k-1$ variables. Therefore there is an extremum at $(1, 1, \cdots, 1)$. It remains to show that $f_{11}(1, 1, \cdots, 1) > 0$ in order to establish that the extremum is actually a minimum.

In (3′) we make the transformation $u_i = A_k \frac{c_i^2}{c_k^2}$; $i = 1, \cdots, k-1$; $u_k = A_k c_k^2$, and integrate out the variable u_k, since λ is now independent of u_k, obtaining

$$(9)\qquad f(a_1, a_2, \cdots, a_k) = B\prod_{i=1}^{k-1} a_i^{\frac{1}{2}(p-1)} \int_{\lambda<\lambda_0} \frac{\prod_{i=1}^{k-1} u_i^{\frac{1}{2}(p-3)}}{\left[1+\sum_{i=1}^{k-1} a_i u_i\right]^{\frac{1}{2}k(p-1)}}\, du_1 \cdots du_{k-1}$$

$$(10)\qquad \lambda = k^{\frac{1}{2}kp}\left\{\frac{\prod_{i=1}^{k-1} u_i}{\left[1+\sum_{i=1}^{k-1} u_i\right]^k}\right\}^{\frac{1}{2}p}; \qquad u_i > 0$$

where B is some positive constant independent of the a_i. From (9)

$$(11)\qquad f_1 = B\prod_{i=1}^{k-1} a_i^{\frac{1}{2}(p-1)} \int_{\lambda<\lambda_0} \left\{\frac{p-1}{2a_1} - \frac{k(p-1)u_1}{2\left[1+\sum_{i=1}^{k-1} a_i u_i\right]}\right\} \frac{\prod_{i=1}^{k-1} u_i^{\frac{1}{2}(p-3)}}{\left[1+\sum_{i=1}^{k-1} a_i u_i\right]^{\frac{1}{2}k(p-1)}}\, du_1 \cdots du_{k-1}$$

The last step involves differentiation under the sign of integration, which is certainly justifiable here.

Now consider λ for fixed $u_2, \cdots, u_{k-1}$, and variable u_1. $\lambda < \lambda_0$ is equivalent to the statement $\frac{u_1}{[\varphi + u_1]^k} < \theta$ where φ and θ depend on $u_2, u_3, \cdots, u_{k-1}$; $\varphi, \theta > 0$. The function $\psi(u_1) = \frac{u_1}{(\varphi + u_1)^k}$ has a maximum at $u_1 = \frac{\varphi}{k-1}$, and has no other extrema, hence the equation $\frac{u_1}{(\varphi + u_1)^k} = \theta$ has but two positive roots, x_1 and x_2, say. Let $x_2 > x_1$. Then for fixed $u_2, u_3, \cdots, u_{k-1}$ the region $\lambda < \lambda_0$ is composed of the u_1 intervals $(0, x_1)$ and (x_2, ∞). Now examining the integrand in (11) we see that it is the partial derivative with respect to u_1 of the quantity

$$\frac{1}{a_1}\frac{u_1^{\frac{1}{2}(p-1)}\prod_{2}^{k-1} u_i^{\frac{1}{2}(p-3)}}{\left[1+\sum_{1}^{k-1} a_i u_i\right]^{\frac{1}{2}k(p-1)}}.$$

This quantity vanishes at 0 and ∞, hence

$$(12)\qquad f_1 = \frac{1}{a_1}B\cdot\prod_{i=1}^{k-1} a_i^{\frac{1}{2}(p-1)} \int_G \prod_{2}^{k-1} u_i^{\frac{1}{2}(p-3)} \left[\frac{u_1^{\frac{1}{2}(p-1)}}{\left(1+\sum_{1}^{k-1} a_i u_i\right)^{\frac{1}{2}k(p-1)}}\right]_{x_2}^{x_1} du_2 \cdots du_{k-1}$$

where G is some region of positive measure in the space of the variables

$u_2, u_3, \cdots, u_{k-1}$. Now differentiating in (12) with respect to a_1, and setting $a_1 = a_2 = \cdots = a_{k-1} = 1$, we get

$$f_{11}(1, 1, \cdots, 1) = B \int_G \prod_2^{k-1} u_i^{\frac{1}{2}(p-3)} \left\{ \frac{p-3}{2} \left[\frac{u_1^{\frac{1}{2}(p-1)}}{(\varphi + u_1)^{\frac{1}{2}k(p-1)}} \right]_{x_2}^{x_1} - \left[\frac{k(p-1)u_1^{\frac{1}{2}(p+1)}}{2(\varphi + u_1)^{\frac{1}{2}k(p-1)+1}} \right]_{x_2}^{x_1} \right\} du_2 \cdots du_{k-1}$$

The first term inside the braces has the value $\theta^{\frac{1}{2}(p-1)}$ both at x_1 and x_2, hence vanishes when evaluated between those limits, so that

$$f_{11}(1, 1, \cdots, 1) = \frac{k(p-1)}{2} B \int_G \prod_2^{k-1} u_i^{\frac{1}{2}(p-3)} \left\{ \frac{x_2^{\frac{1}{2}(p+1)}}{(\varphi + x_2)^{\frac{1}{2}k(p-1)+1}} - \frac{x_1^{\frac{1}{2}(p+1)}}{(\varphi + x_1)^{\frac{1}{2}k(p-1)+1}} \right\} du_2 \cdots du_{k-1} \tag{13}$$

x_1 and x_2 are roots of the equation $\frac{u_1}{(\varphi + u_1)^k} = \theta$, hence $x_1 = \theta(\varphi + x_1)^k$ and $x_2 = \theta(\varphi + x_2)^k$. Putting these values in the numerators in (13), we have

$$f_{11}(1, 1, \cdots, 1) = \frac{k(p-1)}{2} B \int_G \theta^{\frac{1}{2}(p+1)} \prod_2^{k-1} u_i^{\frac{1}{2}(p-3)} \{(\varphi + x_2)^{k-1} - (\varphi + x_1)^{k-1}\} \, du_2 \cdots du_{k-1}.$$

The integrand is positive, since $\theta, \varphi > 0$ and $x_2 > x_1$, hence $f_{11}(1, 1, \cdots, 1) > 0$. We have shown, then, that the power function has a relative minimum, at least, when H_1 is true. We shall show that the minimum is in fact an absolute minimum.

Consider the integrand in (12). The integrand has the same sign as

$$\frac{x_1^{\frac{1}{2}(p-1)}}{\left(1 + a_1 x_1 + \sum_2^{k-1} a_i u_i\right)^{\frac{1}{2}k(p-1)}} - \frac{x_2^{\frac{1}{2}(p-1)}}{\left(1 + a_1 x_2 + \sum_2^{k-1} a_i u_i\right)^{\frac{1}{2}k(p-1)}}.$$

But $x_1 = \theta(1 + x_1 + \Sigma u_i)^k$ and $x_2 = \theta(1 + x_2 + \Sigma u_i)^k$. Hence the integrand has the same sign as

$$\frac{1 + x_1 + \sum_2^{k-1} u_i}{1 + a_1 x_1 + \sum_2^{k-1} a_i u_i} - \frac{1 + x_2 + \sum_2^{k-1} u_i}{1 + a_1 x_2 + \sum_2^{k-1} a_i u_i},$$

so that the integrand is positive or negative accordingly, as $(x_1 - x_2) \left[1 + \sum_2^{k-1} a_i u_i - a_1\left(1 + \sum_2^{k-1} u_i\right)\right]$ is positive or negative. Since $x_1 < x_2$, this last quantity is positive if $a_1 > 1$ and $a_i \leq a_1$, and negative if $a_1 < 1$ and $a_i \geq a_1$. Hence we conclude that $\frac{\partial f}{\partial a_1} > 0$ if $a_1 > 1$ and $a_i \leq a_1$, and $\frac{\partial f}{\partial a_1} < 0$

if $a_1 < 1$ and $a_i \geq a_1$. By the symmetry in the variables the same is true of $\frac{\partial f}{\partial a_i}$, i.e., $\frac{\partial f}{\partial a_i} > 0$ if $a_i > 1$ and $a_i = \max (a_j)$, and $\frac{\partial f}{\partial a_i} < 0$ if $a_i < 1$ and $a_i = \min (a_j)$. Now suppose $(a_1^0, \cdots, a_k^0) \neq (1, \cdots, 1)$. Then either $\max (a_i^0) > 1$ or $\min (a_i^0) < 1$. Hence the first partials can vanish simultaneously only at $(1, 1, \cdots, 1)$, so that f can have no other extrema. Therefore f must have an absolute minimum at $(1, 1, \cdots, 1)$. This completes the proof that the L_1 test is unbiassed when $n_1 = n_2 = \cdots = n_k$.

It is easily seen that the test is in general biassed when the samples consist of different numbers of observations. Consider the case $k = 2$, with samples of n_1 and n_2 observations respectively. In this case we have the single parameter $a = \frac{A_1}{A_2}$. As in (9) and (10),

$$f(a) = Ba^{\frac{1}{2}(n_1-1)} \int_{\lambda<\lambda_0} \frac{u^{\frac{1}{2}(n_1-3)}}{(1 + au)^{\frac{1}{2}n-1}} du \tag{14}$$

$$\lambda = \left(\frac{n^{\frac{1}{2}n}}{n_1^{\frac{1}{2}n_1} n_2^{\frac{1}{2}n_2}}\right) \frac{u^{\frac{1}{2}n_1}}{(1 + u)^{\frac{1}{2}n}}. \tag{15}$$

As before, the equation $\lambda = \lambda_0$ has but two positive roots, $x_2 > x_1 > 0$, so that, as in (12),

$$f'(a) = Ba^{\frac{1}{2}(n_1-3)} \left[\frac{u^{\frac{1}{2}(n_1-1)}}{(1+au)^{\frac{1}{2}n-1}}\right]_{x_2}^{x_1}$$

$$= Ba^{\frac{1}{2}(n_1-3)} \left[\frac{x_1^{\frac{1}{2}(n_1-1)}}{(1 + ax_1)^{\frac{1}{2}n-1}} - \frac{x_2^{\frac{1}{2}(n_1-1)}}{(1 + ax_2)^{\frac{1}{2}n-1}}\right].$$

Therefore $f'(1) = B\left[\frac{x_1^{\frac{1}{2}(n_1-1)}}{(1 + x_1)^{\frac{1}{2}n-1}} - \frac{x_2^{\frac{1}{2}(n_1-1)}}{(1 + x_2)^{\frac{1}{2}n-1}}\right]$.

Recalling that $\frac{x_1^{n_1}}{(1 + x_1)^n} = \frac{x_2^{n_1}}{(1 + x_2)^n}$ it is evident that $f'(1) = 0$ if and only if $n_1 = \frac{n}{2}$. Hence if $n_1 \neq n_2$, the power function does not have a minimum at $a = 1$. It can be shown in this case that a minimum does exist at some point, and if $n \to \infty$ so that $n_1 = \alpha_1 n$, then the minimum tends to the point $a = 1$. The proof is omitted, in view of the fact that a general result of a different nature will be obtained.

Before proceeding, we shall establish a lemma which is undoubtedly well known. However, on account of the directness of the argument, the proof is given here.

LEMMA: *If $x_1, x_2, \cdots, x_h$ have joint distribution function $f_n(x_1, x_2, \cdots, x_h)$ such that $E(x_i) \underset{n\to\infty}{\longrightarrow} m_i$ and $E[(x_i - E(x_i))^2] \underset{n\to\infty}{\longrightarrow} 0$, and if $y = \varphi(x_1, x_2, \cdots, x_h)$ is continuous in $x_1, x_2, \cdots, x_h$ at the point $(m_1, m_2, \cdots, m_h)$, then the distribution of y converges stochastically to the point $\varphi(m_1, m_2, \cdots, m_h)$.*

Proof: By Tschebycheff's Inequality,

$$P\left\{|x_i - E(x_i)| > \frac{\delta}{2}\right\} \leq \frac{4}{\delta^2} E[(x_i - E(x_i))^2].$$

Let n be large enough so that $|E(x_i) - m_i| < \frac{\delta}{2}$; $i = 1, 2, \cdots, h$. Then $|x_i - m_i| > \delta$ implies $|x_i - E(x_i)| > \frac{\delta}{2}$, hence

$$P\{|x_i - m_i| > \delta\} \leq \frac{4}{\delta^2} E[(x_i - E(x_i))^2].$$

Let w_δ denote a cube a side 2δ about the point $(m_1, \cdots, m_h)$, and let x denote the point $(x_1, \cdots, x_h)$.

$$P[x \not\subset w_\delta] \leq \sum_{i=1}^{h} P\{|x_i - m_i| > \delta\},$$

hence

$$P[x \not\subset w_\delta] \leq \frac{4}{\delta^2} \sum_{i=1}^{h} E[(x_i - E(x_i))^2],$$

therefore $P[x \not\subset w_\delta] \underset{n\to\infty}{\longrightarrow} 0$, that is $P[x \subset w_\delta] \to 1$. Given any interval w'_ϵ about the point $y = \varphi(m_1, m_2, \cdots, m_h)$, there is a cube w_δ about $(m_1, m_2, \cdots, m_h)$ such that $x \subset w_\delta$ implies $y \subset w'_\epsilon$. $P[x \subset w_\delta] \leq P[y \subset w'_\epsilon]$, but $P[x \subset w_\delta] \to 1$, therefore $P[y \subset w'_\epsilon] \to 1$. That is, y converges stochastically to the point $y = \varphi(m_1, m_2, \cdots, m_h)$.

Referring to (1), we may express λ as a function of $k - 1$ variables as follows:

$$\lambda = \frac{n^{\frac{1}{2}n} \prod_{i=1}^{k-1} u_i^{\frac{1}{2}n_i}}{\prod_{i=1}^{k} n_i^{\frac{1}{2}n_i} \left[1 + \sum_{i=1}^{k-1} u_i\right]^{\frac{1}{2}n}}$$

where $u_i = \frac{c_i^2}{c_k^2}$; $i = 1, 2, \cdots, k - 1$. Let $n \to \infty$, and let $n_i = \alpha_i n$, $\Sigma\alpha_i = 1$. Then

$$\lambda^{\frac{2}{n}} = \frac{\prod_{i=1}^{k-1} u_i^{\alpha_i}}{\prod_{i=1}^{k} \alpha_i^{\alpha_i} \left[1 + \sum_{i=1}^{k-1} u_i\right]}$$

From (2) it is seen that $E(u_i) = E\left(\frac{c_i^2}{c_k^2}\right) = \frac{A_k}{A_i}\frac{n_i - 1}{n_k - 1} = \frac{1}{a_i}\cdot\frac{n_i - 1}{n_k - 1}$, and $E(u_i^2) = \left(\frac{1}{a_i}\right)^2 \frac{(n_i - 1)(n_i + 1)}{(n_k - 1)(n_k + 1)}$. Therefore $E(u_i) \to \frac{1}{a_i}\frac{\alpha_i}{\alpha_k}$ and $E(u_i^2) \to \left(\frac{1}{a_i}\frac{\alpha_i}{\alpha_n}\right)^2$ in other words, $E[(u_i - E(u_i))^2] \to 0$. Now we apply the lemma, concluding that $\lambda^{\frac{2}{n}}$, that is, L_1 converges stochastically to the quantity

$$r = \frac{1}{\prod_{i=1}^{k-1} a_i^{\alpha_i} \left[\alpha_k + \sum_{i=1}^{k-1} \frac{\alpha_i}{a_i} \right]}.$$

TABLE I

(1) $n_1 = 5, n_2 = 5$		(2) $n_1 = 10, n_2 = 10$		(3) $n_1 = 20, n_2 = 20$	
a	$f(a)$	a	$f(a)$	a	$f(a)$
1	.05	1	.05	1	.05
4/3	.06	4/3	.07	4/3	.09
2	.09	2	.16	2	.31
3	.15	3	.33	3	.65
4	.21	4	.50	4	.84
5	.27	5	.62	5	.93
10	.52	6	.72		
20	.75	10	.90		

TABLE II

(1) $n_1 = 12, n_2 = 8$		(2) $n_1 = 15, n_2 = 10$	
a	$f(a)$	a	$f(a)$
1/10	.90	1/10	.96
1/5	.61	1/5	.74
1/4	.47	1/4	.59
1/3	.32	1/3	.40
1/2	.16	1/2	.20
3/5	.11	3/5	.11
4/5	.07	1	.05
1	.05	1.5	.09
1.2	.05	2	.17
1.5	.06	3	.38
2	.13	4	.60
3	.30	5	.70
4	.45	10	.95
5	.60		
6	.67		
10	.87		

r is the ratio of weighted geometric mean to arithmetic mean of the quantities $1, \frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_{k-1}}$, hence $r = 1$ if and only if $a_1 = a_2 = \cdots = a_{k-1} = 1$, otherwise $r < 1$. Therefore when H_1 is true $\lambda^{\frac{2}{n}}$ converges stochastically to 1, otherwise $\lambda^{\frac{2}{n}}$ converges stochastically to some value less than 1.

Let us choose $\lambda_0^{(n)}$ so that $P(\lambda < \lambda_0^{(n)}) = \alpha$ when H_1 is true. Consider some alternative hypothesis H_1^*. $\lambda^{\frac{2}{n}}$ converges stochastically to $r < 1$. Choose ζ so that $r < \zeta < 1$. $P(\lambda^{\frac{2}{n}} < \zeta) \to 0$ when H_1 is true, but $P(\lambda^{\frac{2}{n}} < \lambda_0^{(n)\frac{2}{n}}) = \alpha$

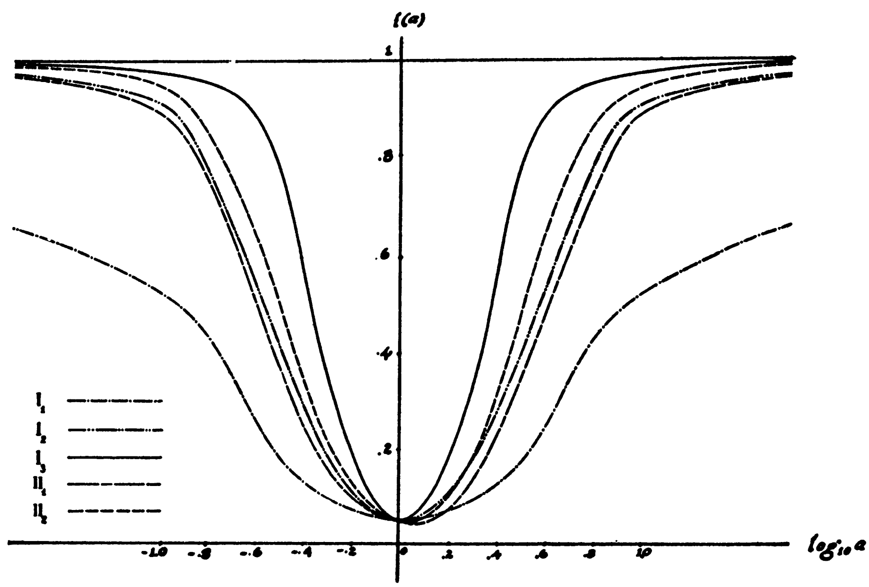

FIG. 1

when H_1 is true, thus, for n sufficiently large, $\zeta < \lambda_0^{(n)\frac{2}{n}}$, that is, $\zeta^{\frac{n}{2}} < \lambda_0^{(n)}$. Therefore $P(\lambda < \lambda_0^{(n)}) \geq P(\lambda < \zeta^{\frac{n}{2}}) = P(\lambda^{\frac{2}{n}} < \zeta)$. Now, if H_1^* is true, $P(\lambda^{\frac{2}{n}} < \zeta) \to 1$, therefore $P(\lambda < \lambda_0^{(n)}) \to 1$.

We have shown then, that *if $n \to \infty$ so that $n_i = \alpha_i n$, where the α_i are fixed, while the probability level α remains constant, then the power of the test with respect to any alternative hypothesis H_1^* tends to unity.* It is impossible, of course, to have the power function tend to unity uniformly with respect to all alternative hypotheses, since the power function is continuous for all n, and since the power with respect to H_1 is constantly α. What we can conclude, however, is that for any particular alternative hypothesis, the probability of rejecting H_1

is greater than α for sufficiently large n.[5] (We might say, then, that the test is asymptotically unbiassed.) Moreover, the fact that the power with respect to H_1^* tends to unity implies that the test becomes sharper with increasing n.

In order to illustrate the sharpness of the test, values of the power function were computed, when $k = 2$, for the cases $n_1 = n_2 = 5$; $n_1 = n_2 = 10$; $n_1 = n_2 = 20$; $n_1 = 12$, $n_2 = 8$; and $n_1 = 15$, $n_2 = 10$. The results are given in Tables I and II. The computations were made from (14) and (15) by means of Pearson's *Tables of the Incomplete Beta Function*. The roots x_1 and x_2 of the equation $\lambda = \lambda_0$ were determined, for $\alpha = .05$, by trial and error, making it possible to use the tables directly to compute as many values of the power function as desired.

When $n_1 = 12$, $n_2 = 8$, and $n_1 = 15$, $n_2 = 10$, the power functions both have minima at approximately $a = 1.1$, indicating that the bias is certainly not serious. When $n_1 = n_2$, the power function has the same value at a and $1/a$, in the other cases the values shift slightly. Note that when $n_1 = n_2 = 20$ the test is fairly delicate. For example, $f(3) = .65$, that is, if $\sigma_2 = \sqrt{3}\,\sigma_1$, the probability of rejecting H_1 is .65. In Figure 1, the power functions have been plotted against log a, because of the symmetry in the values a and $1/a$. The curves I_1, I_2, I_3, correspond to columns 1, 2, 3 respectively of Table I. Similarly, curves II_1, II_2 correspond to columns 1 and 2 of Table II.

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[5] Neyman, [5], discusses the similar property of being "unbiassed in the limit."