THE LIMITS OF A DISTRIBUTION FUNCTION IF TWO EXPECTED VALUES ARE GIVEN

By R. v. Mises

In a very interesting paper[1] A. Wald dealt with the following generalization of a problem started by Markoff and Tchebycheff: Denote by X a random variable, by $P(t)$ the probability of $|X| < t$ and by M_r the absolute moment of order r or the expected value of $|X|^r$; what is the sharp lower limit (limes inferior) of $P(t)$ for any point t, if $M_\alpha, M_\beta, M_\gamma, \cdots$ are given? Wald outlines an ingenious method for the general case of n given moments and adds the complete solution for the case $n = 2$. I wish to show in the following lines that the results for $n = 2$ can be deduced both in a more general and less complicated manner. Instead of two different powers of $|X|$, I shall admit largely arbitrary functions of X and I shall get the solution by a more intuitive way. Moreover the upper limit of $P(t)$ will be found too. It seems to me that my method will be applicable also to certain cases with $n > 2$.

1. **The Problem.** Without loss of generality we can restrict ourselves to a non-negative random variable X. Let $x(X)$ and $y(X)$ be two increasing functions of X with $x(0) = y(0) = 0$. We suppose that the curve defined in a Cartesian co-ordinate system by

$$x = x(t), \qquad y = y(t) \tag{1}$$

is one which is convex downwards, i.e. the slope of its chords is increasing if the co-ordinates of one or both extreme points of the chord increase (see Fig. 1). This condition is fulfilled, for instance, if $x = t^r$, $y = t^s$ and $s > r > 0$ where the indexes r, s are not necessarily integers. Another example is $x = t$, $y = t^2/1 + t$; here, however, the ratio y/x is restricted to values between 0 and 1. In a third class of examples as $x = t/1 + t$, $y = t^3/(1 + t)^3$ the curve corresponding to (1) ends at a finite point.

The probability of the inequality $X < t$ will be designated by $P(t)$, the probability of $X > t$ by $\overline{P}(t)$. The sum of $P(t)$ and $\overline{P}(t)$ is equal to 1 excepting the points t associated with a finite probability. But in any case the upper limit of $P(t)$ and the lower limit of $\overline{P}(t)$ give the sum 1.

The expected values of $x(X)$ and $y(X)$ can be defined by means of $P(t)$ or $\overline{P}(t)$

$$\begin{aligned} a &= \int_0^\infty x(t)\,dP(t) = -\int_0^\infty x(t)\,d\overline{P}(t); \\ b &= \int_0^\infty y(t)\,dP(t) = -\int_0^\infty y(t)\,d\overline{P}(t). \end{aligned} \tag{2}$$

[1] *Ann. Math. Statist.* Vol. 9 (1938) pp. 244–255.

We suppose that the values of a and b are given in a suitable manner and we ask for the lower limit of $P(t)$ and $\bar{P}(t)$ at any point t. In other words, we try to find two functions $l(t)$ and $\bar{l}(t)$ so that for all distributions associated with the given values of a and b we have

$$(3) \qquad P(t) \geqq l(t), \qquad \bar{P}(t) \geqq \bar{l}(t),$$

but that these inequalities are not valid, if $l(t)$ and $\bar{l}(t)$ are replaced by higher values. In Fig. 1 K is the curve defined by (1) and C the point with co-ordinates a, b.

We can give a more intuitive interpretation to our problem by imagining a mass distribution instead of a probability or frequency distribution. In fact, if the mass of magnitude 1 is spread along the curve K in such a way that $P(t)$ designates the sum (or integral) of masses lying to the left of the point $x(t)$, $y(t)$, then the point C will be the centre of gravity (centre of mass) of the whole mass system. By the way, it follows that C must be situated on the inner side of the convex curve K. Our question can now be stated as follows:

A mass of size 1 is distributed along a given convex curve and has its centre of gravity in a given point C. What is the least possible value of mass lying to the left or to the right of any point $x(t)$, $y(t)$ of the curve?

2. Restriction of Distributions to be Considered. The essential difficulty of our problem lies in the fact that in order to find the limits $l(t)$ and $\bar{l}(t)$ all conceivable forms of distribution functions $P(t)$ and $\bar{P}(t)$ must be taken into account. Let us now see how the field of distributions can be restricted in a decisive manner.

Two mass systems with the same total mass and the same centre of gravity will be called "equivalent systems". Then the following corollary can be stated:

If a mass system with the distribution functions P, $\bar{P}$ and a point M with co-ordinates $x(t)$, $y(t)$ are given, we can always find an equivalent system consisting of three particles or masspoints: a mass m_1 at M_1 to the left of M, a mass m at M itself and a mass m_2 at M_2 to the right of M, so that

$$(4) \qquad m_1 \leqq P(t), \qquad m_2 \leqq \bar{P}(t).$$

This proposition enables us, in asking for the lower limit values $l(t)$, $\bar{l}(t)$ at M to confine ourselves to the consideration of a special class of three-point systems and to disregard all other kinds of distributions.

In order to prove the corollary we make use of the well known laws of elementary statics. According to these laws all masses lying to the left of M (in the given system) can be replaced by a single mass of same size fixed in their centre of gravity C_2 (Fig. 1). This centre is situated in the domain between the curve K and the chord OM. The straight line MC_1 has one and only one second point of intersection M_1 with K. Any mass at C_1 can be decomposed into two masses, one of them of magnitude m_1 lying at M_1, the other of size m' at M.

In an analogous manner starting with the masses lying to the right of M in the given system, m_2, m'' and M_2 can be found. It is evident that m_1 can not exceed the sum of masses which were attached to points to left of M in the original mass system. It is the same with m_2 and the masses to the right of M. If in the original system a finite mass m_0 had been attached to the point M, the value of m in the new mass system will be defined as $m = m' + m'' + m_0$.

3. **The Extreme Distributions.** Now, in order to find the limits $l(t)$ and $\bar{l}(t)$ for a certain point M, we are concerned exclusively with a two-parameter family of mass systems, each of them consisting of three masses m_1, m, m_2 at three points M_1, M, M_2. We choose as parameters the magnitude m of the mass attached to the point M and the slope of the chord joining M_1 and M_2. If m remains constant, the chord M_1M_2 (Fig. 2) passes through a fixed point C_0 on

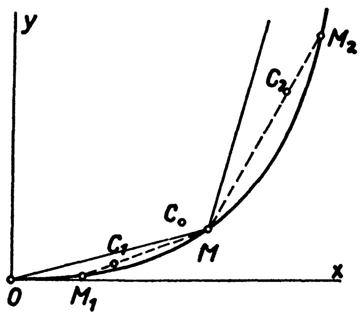

FIG. 1

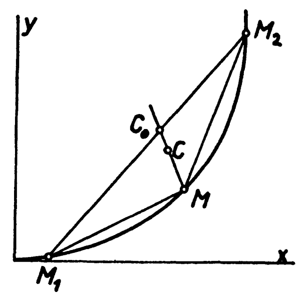

FIG. 2

the prolongation of MC where $\overline{CC_0} = \overline{MC}.m/1 - m$. The masses m_1 and m_2 vary with the direction of M_1M_2 and are determined by

$$(5) \qquad m_1 = (1 - m)\frac{\overline{C_0M_2}}{\overline{M_1M_2}}, \qquad m_2 = (1 - m)\frac{\overline{M_1C_0}}{\overline{M_1M_2}}$$

We are only interested in the least possible values of m_1 and m_2. But a convex curve for which the angle formed by its extreme tangents is not greater than 90°, has the characteristic property that the ratio of chord segments $\overline{M_1C_0} : \overline{C_0M_2}$, for an inner point C_0, is permanently increasing or decreasing when the chord turns about C_0; there is no analytical maximum or minimum. It follows that the lowest values of m_1 and m_2 can only be found in an extreme position of the chord, i.e. when M_1 coincides with O, or M_2 with the other (eventually infinite) end Q of K, or finally when one of the points M_1, M_2 coincides with M. The latter cases must be mentioned since it was one of the conditions for our three-point systems that M lies *between* M_1 and M_2. The result we have obtained until now is, that the lowest values of masses lying on one or the other side of M are to be sought in a distribution of one of the following classes: (1) The three-mass systems with one mass at M and one mass at O; (2) The three-mass systems with one mass at M and one mass at the end Q of K; (3) The two-mass system with one mass at M.

Now we must distinguish three sorts of points M or three sections of the curve K. If we trace the chord (see Fig. 3) beginning at O and passing through C we obtain the point of intersection O' and by means of the chord QC we arrive at the point Q'. The three sections of K we have to deal with are OQ', $Q'O'$ and $O'Q$.

If M is a point of OQ' there exists a chord MM' passing through C and therefore a two-mass system with masses m, m' at M and M'. In this system the mass to the left of M is zero, thus we have $l(t) = 0$ for all these points. If we consider a three-mass system with one mass at M, one mass at O and one mass at any point M_2, the value of m_2 is equal to the ratio $\overline{CC_1}/\overline{M_2C_1}$, where C_1 is the intersection of M_2C with OM. The least value of this ratio will be reached when C_1 coincides with M. Therefore $l(t)$ is equal to the ratio $\overline{CM}/\overline{M'M}$ or equal to the mass m' of the two-mass system mentioned before.

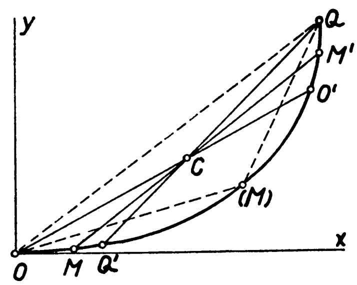

FIG. 3

Now let M be a point of the arc $Q'O'$. For such a point a two-mass system does not exist, since the straight line $M\ C$ does not meet the curve a second time. In a three-mass system $O\ M\ M_2$ the value of m is equal to $\overline{CC_1}/\overline{M_2C_1}$ as before, and the least value of this ratio is attained, if M_2 coincides with Q. It follows that $l(t)$ is equal to the ratio $\overline{CC_Q}/\overline{QC_Q}$ where C_Q denotes the point of intersection of QC with OM. In the same way we find $l(t)$ equal to $\overline{CC_O}/OC_O$, the point of intersection of OC with MQ being designated by C_O.

For a point M of the arc $O'Q$ the circumstances are the same as for the points of OQ'.

In other words the extreme distributions which furnish immediately the values of $l(t)$ and $l(t)$ are 1) the two-mass systems MM' for all points of the arcs OQ' and $O'Q$ and 2) the three-mass system OMQ for a point of the middle section $Q'O'$. The corresponding values of l and l are to be found by the elementary laws of statics in the simplest way.

4. Results. The definite results can now be stated as follows. Our data are the functions $x(t)$, $y(t)$ and the expected values a, b.

First we compute the co-ordinates p, q of the endpoint Q, i.e. $p = x(\infty)$,

$q = y(\infty)$. If q or p and q are infinite, we only need the limit value of y/x. Then the two values t_0 and t^0 corresponding to the points O' and Q' have to be found. They are determined by the equations

$$\frac{y(t_0)}{x(t_0)} = \frac{b}{a}; \qquad \frac{y(t^0) - b}{x(t^0) - a} = \frac{b - q}{a - p}. \tag{6}$$

If t belongs to one of the intervals $t \leqq t^0$ or $t \geqq t_0$, there exists one and only one value of t' different from t and satisfying the equation

$$\frac{a - x(t)}{x(t') - x(t)} = \frac{b - y(t)}{y(t') - y(t)}. \tag{7}$$

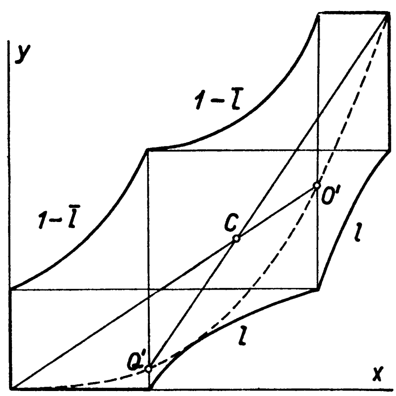

FIG. 4

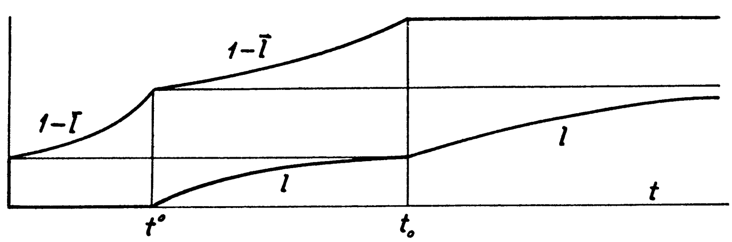

FIG. 5

The point M' with co-ordinates $x' = x(t')$, $y' = y(t')$ is the second endpoint of the chord passing through M and C. Now we have, according to the preceding considerations:

For $t \leqq t^0$: $\quad l(t) = 0, \qquad \bar{l}(t) = \dfrac{a - x}{x' - x}$

(8) " $t^0 \leqq t \leqq t_0$: $\quad l(t) = \dfrac{(a - p)(b - y) - (a - x)(b - q)}{py - qx} \qquad \bar{l}(t) = \dfrac{ay - bx}{py - qx}$

" $t \geqq t_0$: $\quad l(t) = \dfrac{a - x}{x' - x}, \qquad \bar{l}(t) = 0$

The formulae are considerably simplified, if p, q are infinite. In the case of two moments given, $x = t^r$, $y = t^s$, $s > r > 0$, we have $p = \infty$, $q = \infty$, $\lim y/x = \infty$. The second equation (6) gives $x(t^0) = a$ and (8) becomes:

$$
\begin{array}{llll}
 & \text{For } t \leqq t^0 & l(t) = 0, & \bar{l}(t) = \dfrac{a - x}{x' - x} \\[2ex]
(9) & \text{"} \quad t^0 \leqq t \leqq t_0 & l(t) = \dfrac{x - a}{x}, & \bar{l}(t) = 0 \\[2ex]
 & \text{"} \quad t \geqq t_0, & l(t) = \dfrac{a - x}{x' - x} & \bar{l}(t) = 0
\end{array}
$$

The values of $l(t)$ given here are in full accordance with the results published by Wald in his paper quoted above.

A great part of the numerical investigation is independent from the relation which joins x (or y) to t and is determined only by the values a, b, the curve K, i.e. the relation between x and y, and its endpoint Q. In the following example we have assumed $y = x^3$ and as endpoint $p = q = 1$. Fig. 4 shows for $a = 0, 6$, $b = 0, 4$ the three sections of the lines l and $1 - \bar{l}$ according to the equations (8), but with the abscissae x. The graph of any distribution function in the interval $0 \leqq x \leqq 1$ with given first moment 0, 6 and third moment 0, 4 keeps within the space between the lines l and $1 - \bar{l}$. If we now assume, e.g. $x = t^2/1 + t^2$ the abscissae x are to be transformed according to this equation and the graphs of definitive $l(t)$ and $1 - \bar{l}(t)$ functions are those given in fig. 5. Any distribution functions $P(t)$ with the expected values

$$\int \frac{t^2}{1 + t^2}\, dP(t) = 0, 6 \qquad \int \left(\frac{t^2}{1 + t^2}\right)^3 dP(t) = 0, 4$$

must keep between the two limits indicated in Fig. 5. If such a function touches the upper limit in any point, it will also attain the lower limit in another point and will correspond to a two- or three-mass system.

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